Ergodic Dynamics in Social Networks

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Outline

1. Interactions in social networks
2. Opinion dynamics leading to consensus: Opinion diffusion
3. More realistic opinion dynamics: Obstinacy and prejudices
   - Randomization and ergodic oscillations
4. Other (non-social) ergodic dynamics
   - Algorithms for estimation from relative measurements

Models of opinion dynamics

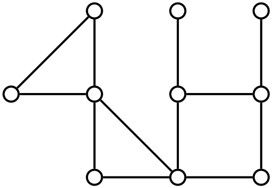
A population $\mathcal{I}$ of *individuals* is given

Individuals have **opinions** $x_i(k)$ in $\mathbb{R}$

Opinions evolve through **interactions** between agents

then, we have to model

- the set of allowed interactions: the **social network**
  - nodes are individuals $i \in \mathcal{I}$
  - edges are potential interactions, *i.e.*, pairs $(i, j) \in \mathcal{I} \times \mathcal{I}$
- the **interaction process**: discrete-time, deterministic/randomized
- the **effects of interactions**



Diffusive coupling: Deterministic updates

Assumption: interactions bring opinions closer to each other

$\Longrightarrow$ (discrete-time) dynamics: **local averaging**

$$x_i(k+1) = \sum_{j \in \mathcal{I}} C_{ij}\, x_j(k)$$

positive couplings $C_{ij} \geq 0$, $\sum_j C_{ij} = 1$, $C_{ij} = 0$ if (i,j) is not an edge

Result:

- $x(k)$ converges to a **consensus** on one opinion

M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974

Diffusive coupling: Randomized updates

Synchronous rounds of updates are a poor description of real interaction processes: we can instead study sparse randomized interactions

Gossip approach: at each time t, interaction and update occur across one random edge (i, j)

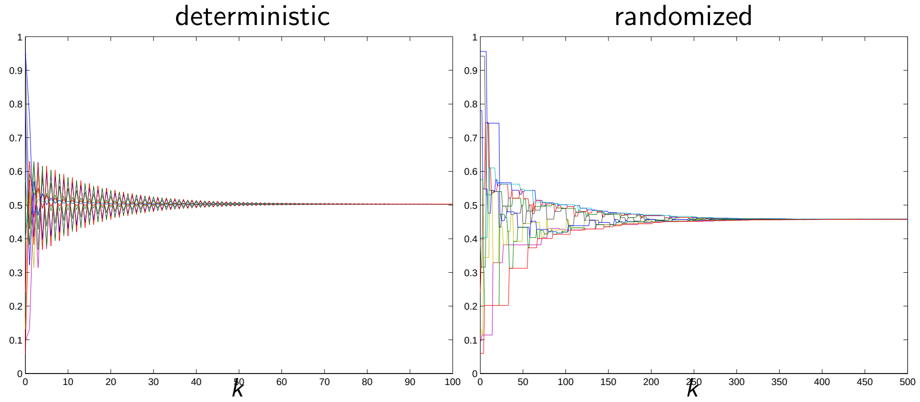
$$
\begin{aligned}
x_i(k+1) =& \frac{1}{2} x_i(k) + \frac{1}{2} x_j(k) \\
x_j(k+1) =& \frac{1}{2} x_i(k) + \frac{1}{2} x_j(k) \\
x_\ell(k+1) =& x_\ell(k) \quad \text{if } \ell \notin \{i, j\}
\end{aligned}
$$

Result:

- $x(k)$ almost surely converges to a **consensus** on one opinion

S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Transactions on Information Theory*, 52(6):2508–2530, 2006

Diffusive coupling: Examples and discussion



+ easy, well understood
– societies do not exhibit consensus

We need to model the **reasons for persistent disagreement in societies**

Prejudices and stubborn agents

Assumption: interactions bring opinions closer to each other, but the initial opinions are never forgotten

$p \in \mathbb{R}^{\mathcal{I}}$ is a vector of **prejudices**
$w \in [0,1]^{\mathcal{I}}$ is a vector of **obstinacies**

$$x_i(0) = p_i$$
$$x_i(k+1) = (1-w_i) \sum_{j \in \mathcal{I}} C_{ij}\, x_j(k) + w_i p_i$$

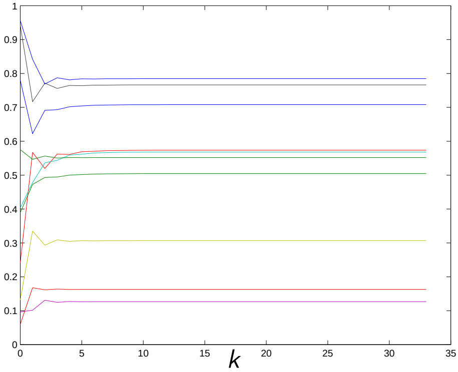
Result:

- $x(k)$ converges to a non-trivial opinion profile

$$x(k) \to x^{\star} = \big(I - (I - \mathrm{diag}(w))C\big)^{-1} \mathrm{diag}(w) p$$

N. E. Friedkin and E. C. Johnsen. *Social Influence Network Theory: A Sociological Examination of Small Group Dynamics*. Cambridge University Press, 2011

Prejudices: Example and discussion



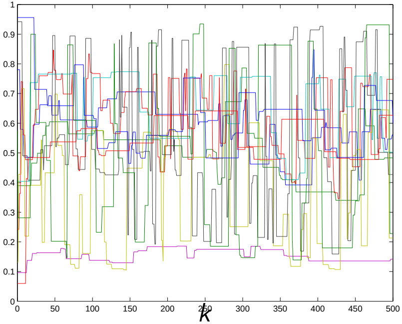
- linear dynamics $\rightarrow$ easy to study
- complex limit opinion profiles (no consensus)
- supported by experimental evidence

We can also define sparse random interactions:

for a randomly chosen edge (i, j)

$$x_i(k+1) = (1 - w_i)\left(\frac{1}{2}x_i(k) + \frac{1}{2}x_j(k)\right) + w_i p_i$$
$$x_j(k+1) = (1 - w_j)\left(\frac{1}{2}x_j(k) + \frac{1}{2}x_i(k)\right) + w_j p_j$$
$$x_\ell(k+1) = x_\ell(k) \quad \text{if } \ell \notin \{i, j\}$$

Result:
$x(k)$ persistently oscillates



Intermediate steps:

1. $\exists$ random variable x_∞ such that $x(k) \to x_\infty$ in distribution
2. the distribution of x_∞ is the unique invariant distribution of $x(k)$
3. x(k) is **ergodic**

   sample averages $\Longleftrightarrow$ time averages $\bar{x}(k) := \frac{1}{k+1}\sum_{h=0}^{k} x(h)$

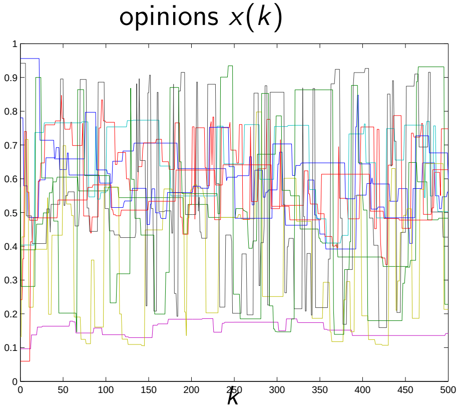
   $$\bar{x}(k) \to \mathbb{E}[x_\infty] \qquad \text{as } k \to \infty$$

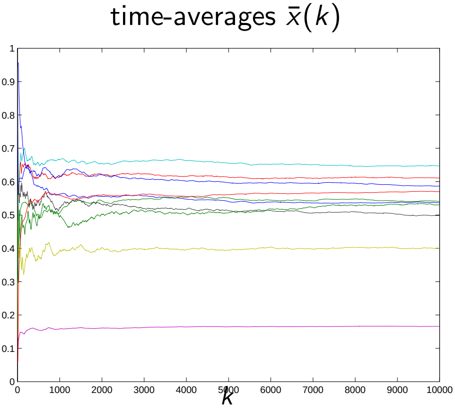
4. $\mathbb{E}[x_\infty] = x^\star$

Consequence: *oscillations occur around the average dynamics and can be smoothed away by time-averaging*

Proof tool: studying the time-reversed process

Effectiveness of averaging





D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar. Opinion fluctuations and disagreement in social networks. *Mathematics of Operations Research*, 38(1):1–27, 2013

P. Frasca, C. Ravazzi, R. Tempo, and H. Ishii. Gossips and prejudices: Ergodic randomized dynamics in social networks. In *IFAC Workshop on Estimation and Control of Networked Systems*, pages 212–219, Koblenz, Germany, September 2013

Individual time-averages

To compute time-averages each node needs to know the absolute time k

We can overcome this drawback by defining two auxiliary dynamics:

- local times $\kappa_\ell(0) = 0$ for all $\ell \in \mathcal{I}$

$$
\begin{aligned}
\kappa_i(k+1) &= \kappa_i(k) + 1 \\
\kappa_j(k+1) &= \kappa_j(k) + 1 \\
\kappa_\ell(k+1) &= \kappa_\ell(k) \qquad \text{if } \ell \notin \{i,j\}
\end{aligned}
$$

- "local" time-averages $\widetilde{x}_\ell(0) = 0$ for all $\ell \in \mathcal{I}$

$$
\begin{aligned}
\widetilde{x}_i(k+1) &= \frac{1}{\kappa_i(k+1)}\big(\kappa_i(k)\widetilde{x}_i(k) + x_i(k+1)\big) \\
\widetilde{x}_j(k+1) &= \frac{1}{\kappa_j(k+1)}\big(\kappa_j(k)\widetilde{x}_j(k) + x_j(k+1)\big) \\
\widetilde{x}_\ell(k+1) &= \widetilde{x}_\ell(k) \qquad \text{if } \ell \notin \{i,j\}
\end{aligned}
$$

These individual averages $\widetilde{x}(k)$ have the same properties as the global ones

A sociological puzzle: interaction process

The original Friedkin's model postulates **synchronous** interactions

but

his experiments involved pairwise discussions

Our work is filling the gap:

asynchronous interactions + *time-averaging* $\Longleftrightarrow$ *synchronous dynamics*

Non-social ergodic dynamics on networks

Other ergodic dynamics on networks

Several algorithms based on randomized updates produce ergodic oscillations

- PageRank computation

  H. Ishii and R. Tempo. Distributed randomized algorithms for the PageRank computation. *IEEE Transactions on Automatic Control*, 55(9):1987–2002, 2010

- Estimation from relative measurements

  C. Ravazzi, P. Frasca, H. Ishii, and R. Tempo. A distributed randomized algorithm for relative localization in sensor networks. In *European Control Conference*, pages 1776–1781, Zürich, Switzerland, July 2013

Problem statement: sensor relative localization

- $\mathcal{I}$ is a set of sensors
- $\xi \in \mathbb{R}^{\mathcal{I}}$ is an unknown vector
- each sensor u obtains noisy relative measurements with some other nodes j,

$$b_{ij} = \xi_i - \xi_j + \eta_{ij} \qquad \eta_{ij} \text{ are noises}$$

Goal: *for each sensor $i \in \mathcal{I}$, estimate the scalar value ξ_i*

Applications:

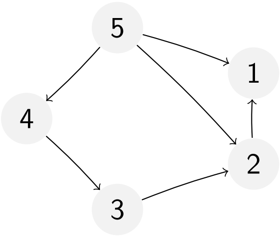
- self-localization of robotic networks
- clock synchronization
- ranking problems (Netflix)

Relative localization as a graph problem

Measurements $\longrightarrow$ edges $\mathcal{E}$ of an oriented connected graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$

incidence matrix $A \in \{0, \pm 1\}^{\mathcal{E} \times \mathcal{I}}$

$$A_{ej} = \begin{cases} +1 & \text{if } e = (i, j) \\ -1 & \text{if } e = (j, i) \\ 0 & \text{otherwise} \end{cases}$$



$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Laplacian matrix $L = A^\top A$

Relative localization as a least-squares problem

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$$\min_z ||Az - b||_2^2$$

has unique minimum-norm solution $x^\star = L^\dagger A^\top b$

Can the sensor network effectively compute the solution?

We take a pairwise "gossip" approach
At every time instant k, an edge $(i,j) \in \mathcal{E}$ is selected, according to

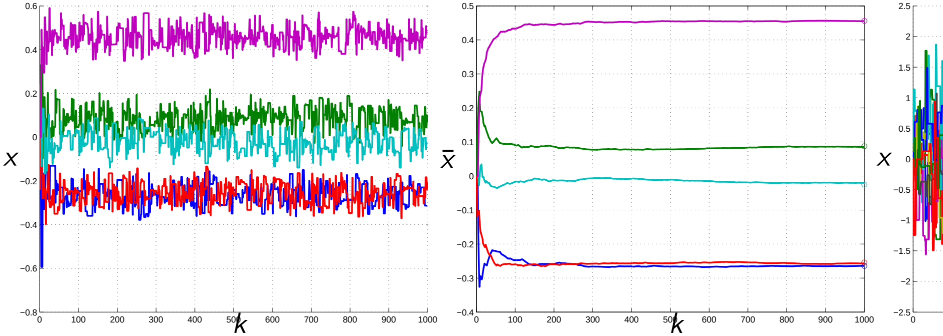
$$\mathbb{P}[(i,j) \text{ is selected at time } k] = \frac{1}{|\mathcal{E}|}$$

and the states are updated according to $\quad (\gamma \in (0,1))$

$$x_i(k+1) = (1-\gamma)x_i(k) + \gamma x_j(k) + \gamma b_{(i,j)}$$
$$x_j(k+1) = (1-\gamma)x_j(k) + \gamma x_i(k) - \gamma b_{(i,j)}$$
$$x_\ell(k+1) = x_\ell(k) \qquad \text{if } \ell \notin \{i,j\}$$

from initial condition $x_\ell(0) = 0$ for all $\ell \in \mathcal{I}$

The states $x(k)$ persistently oscillate, but



time-averages $\bar{x}(k)$ smooth out oscillations $\Longrightarrow$ $\bar{x}(k) \to x^{\star}$ as $k \to +\infty$

Concluding remarks

Studying ergodicity of network dynamics allows to

- understand social processes
- design distributed asynchronous algorithms for relevant problems

C. Ravazzi, P. Frasca, R. Tempo, and H. Ishii. Ergodic randomized algorithms and dynamics over networks, September 2013. Submitted for publication

Open issues & current research

in the social sciences:

- dialogue with social scientists
- from opinion dynamics to opinion control

in engineering:

- broaden the scope of the randomization + averaging approach
- averaging implies slow convergence: is there a fix?