Efficient quantized techniques for consensus algorithms

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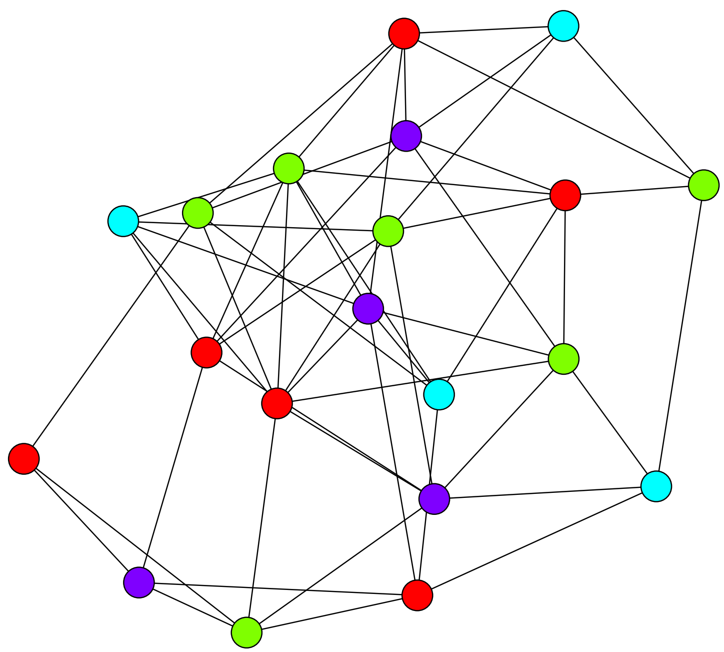
Average consensus over networks

Several agents in a network have to communicate in order to achieve an agreement about the average of their states.



Average consensus problem

Linear dynamical system on $\mathbb{R}^N$

$$x(t+1) = x(t) + K\,x(t),$$

or componentwise $x_i(t+1) = x_i(t) + \sum_{i=1}^{N} K_{ij}\,x_j(t), \quad i = 1 \dots N$

The matrix K has to depend on the communication network:
if j does *not* communicate with i, then $K_{ij} = 0$

Goal: design K so that all agents tend to share the same state

$$\lim_{t \to +\infty} x_i(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(0) \quad \forall\, i.$$

Applications & motivations

In distributed control and information theory.

- Data fusion in sensor networks
- Coordination and rendezvous of UAV and robots
- Load balancing between processors

Our problem

We suppose that the agents can exchange information through a

- time-invariant
- strongly connected
- digital

communication network.

Exchanged information has to be symbolic, i.e. quantized.

Ideal links vs digital links

If links are not digital, it's sufficient to choose K such that $P = I + K$ satisfies

- P is non negative
- P is doubly stochastic, i.e. $\sum_{j=1}^{N} P_{ij} = \sum_{i=1}^{N} P_{ij} = 1,$

to achieve the goal of average consensus.

Adaptation to digital links

Take a slightly different evolution map:

$$x(t+1) = x(t) + K\,\hat{x}(t),$$

where

- $P = I + K$ satisfies the properties above;
- $\hat{x}(t)$ is a vector of estimates of $x(t)$ constructed by a coder-decoder scheme with memory.

Coder-Decoder scheme

The j-th agent sends the symbol s_j to the i-th agent.
They are synchronous: they share the coder state.

Agent j: $\begin{cases} \xi_j(t+1) = F_j(\xi_j(t), s_j(t)) \\ s_j(t) = Q_j(\xi_j(t), x_j(t)) \end{cases}$ Encoder

$\Downarrow$ s_j Symbol $\Downarrow$

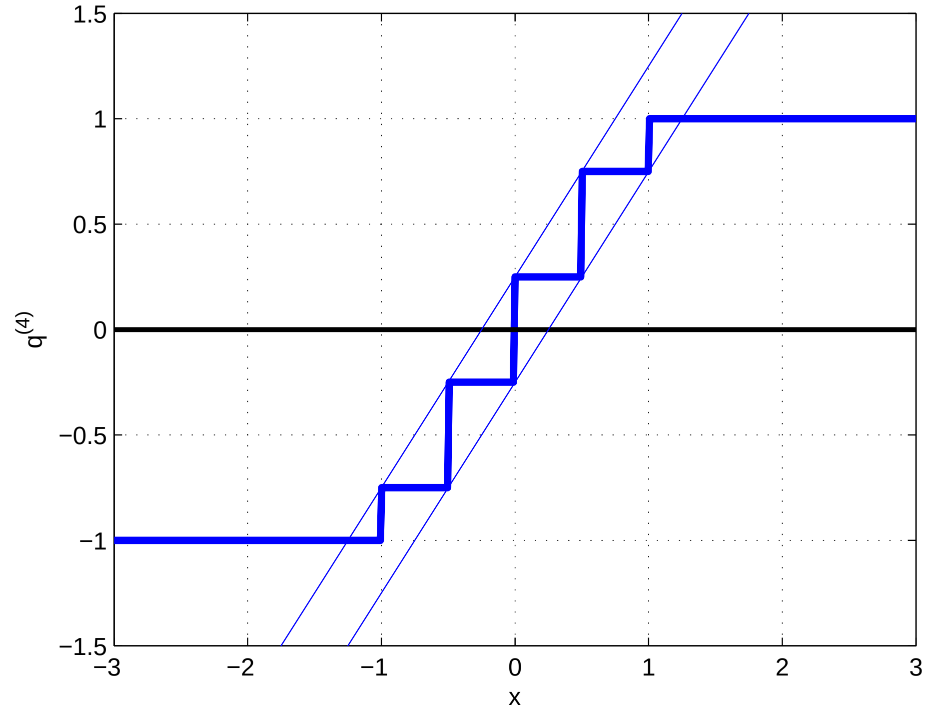
Agent i: $\begin{cases} \xi_j(t+1) = F_j(\xi_j(t), s_j(t)) \\ \hat{x}_j(t) = H_j(\xi_j(t), s_j(t)) \end{cases}$ Decoder

Zooming in-out quantizers I

The uniform quantizer with m levels $q^{(m)} : \mathbb{R} \to \mathcal{S}_m$ is like:

Information is concentrated in a (unitary) sensitivity interval.

A scale factor is chosen to fit $q^{(m)}$ to data.



The scale factor will shrink while approaching consensus.

Zooming in-out coding scheme

- Parameters:
  - $m \in \mathbb{N}$ number of levels,
  - $k_{in} \in]0,1[$, $k_{out} \in]1,+\infty[$ zooming rates
- coder/decoder dynamics

$$\begin{cases} \xi_{j,1}(t+1) = \hat{x}_j(t) = \xi_{j,1}(t) + \xi_{j,2}(t)\, s_j(t) & \text{Estimate} \\ \xi_{j,2}(t+1) = \begin{cases} k_{in}\, \xi_{j,2}(t) & \text{if } |s_j(t)| < 1 \\ k_{out}\, \xi_{j,2}(t) & \text{if } |s_j(t)| = 1 \end{cases} & \text{Scaling factor} \\ s_j(t) = q_j^{(m)} \left(\dfrac{x_j(t) - \xi_{j,1}(t)}{\xi_{j,2}(t)} \right) & \text{Symbol} \end{cases}$$

Convergence result

Under some technical assumptions on the parameters of the zooming scheme,

$\rho := \max_{i=0\dots N-1} \{|\lambda_i| \text{ such that } \lambda_i \text{ is an eigenvalue of } P, \lambda_i \neq 1\}$

- $\rho < k_{in} < 1, \quad k_{out} = \frac{1}{k_{in}}$,
- $m \geq \frac{(4+3k_{in})\sqrt{N}}{k_{in}(k_{in}-\rho)}$,
- $\xi_{i,2}(0) = \bar{\xi}_2 \geq \frac{2(\rho+2)\|x(0)\|}{k_{in} - \frac{3\sqrt{N}}{m}}$ and $\xi_{i,1}(0) = 0 \quad \forall\, i.$

one can prove that, for any initial condition $x(0) \in \mathbb{R}^N$, average consensus is reached:

$$\lim_{t\to+\infty} x_i(t) = \frac{1}{N}\sum_{i=1}^{N} x(0) \quad \forall\, i.$$

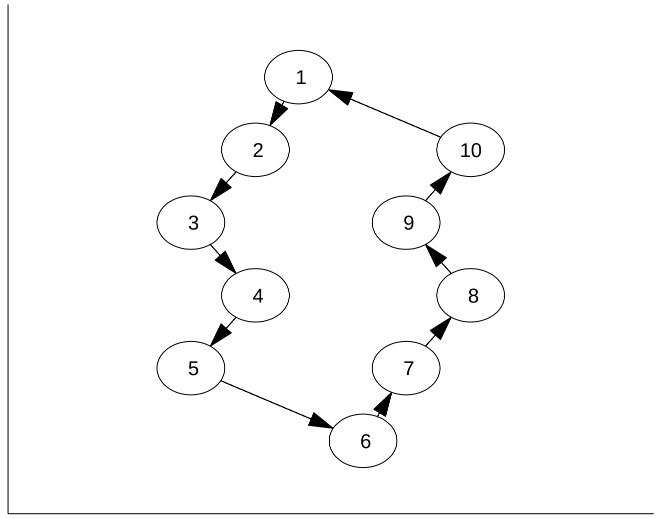
Simulations: Summary

Simulations show the method as effective and fast for average consensus, also outside the scope of the above theorem:

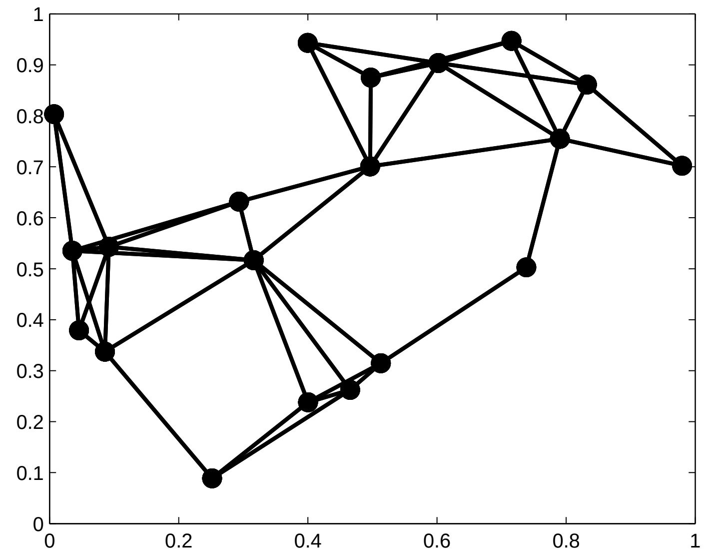
- over different network topologies,
- for wide ranges of the parameters.

Simulations: Test Graphs

Directed circuit graph

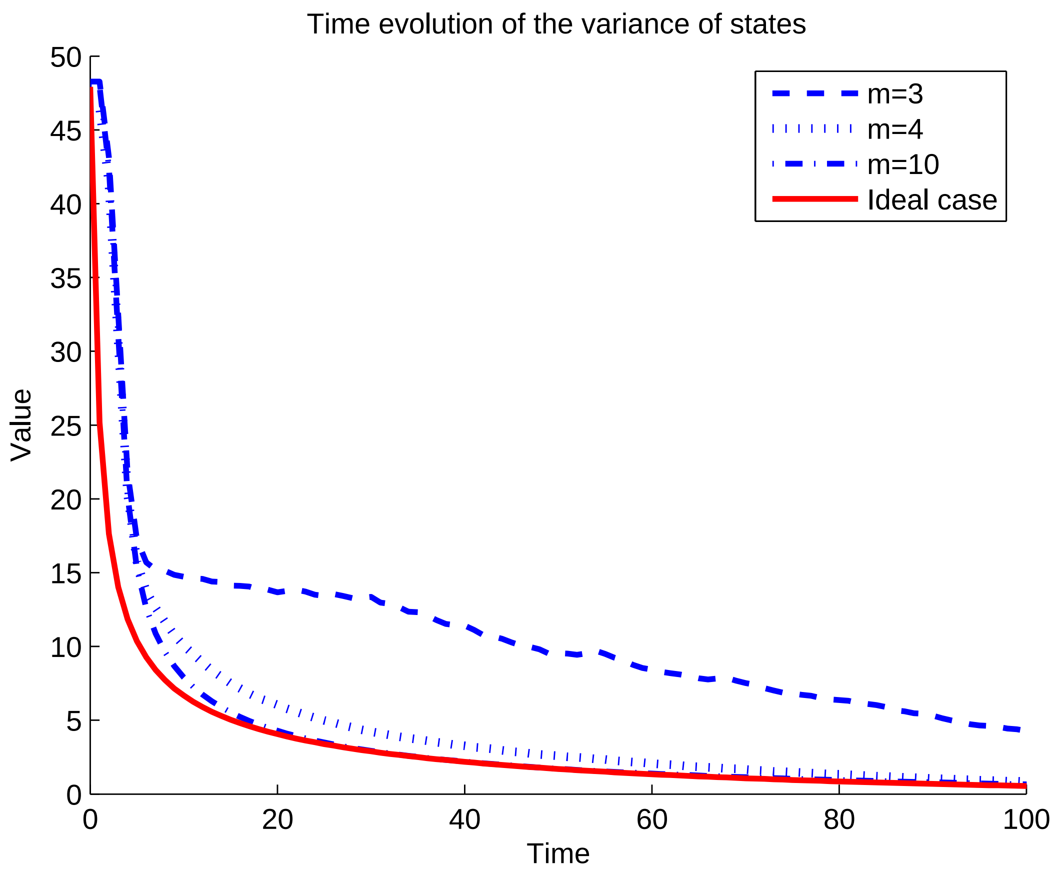


Random geometric graph



Simulations

Even few levels are enough for the method to converge.

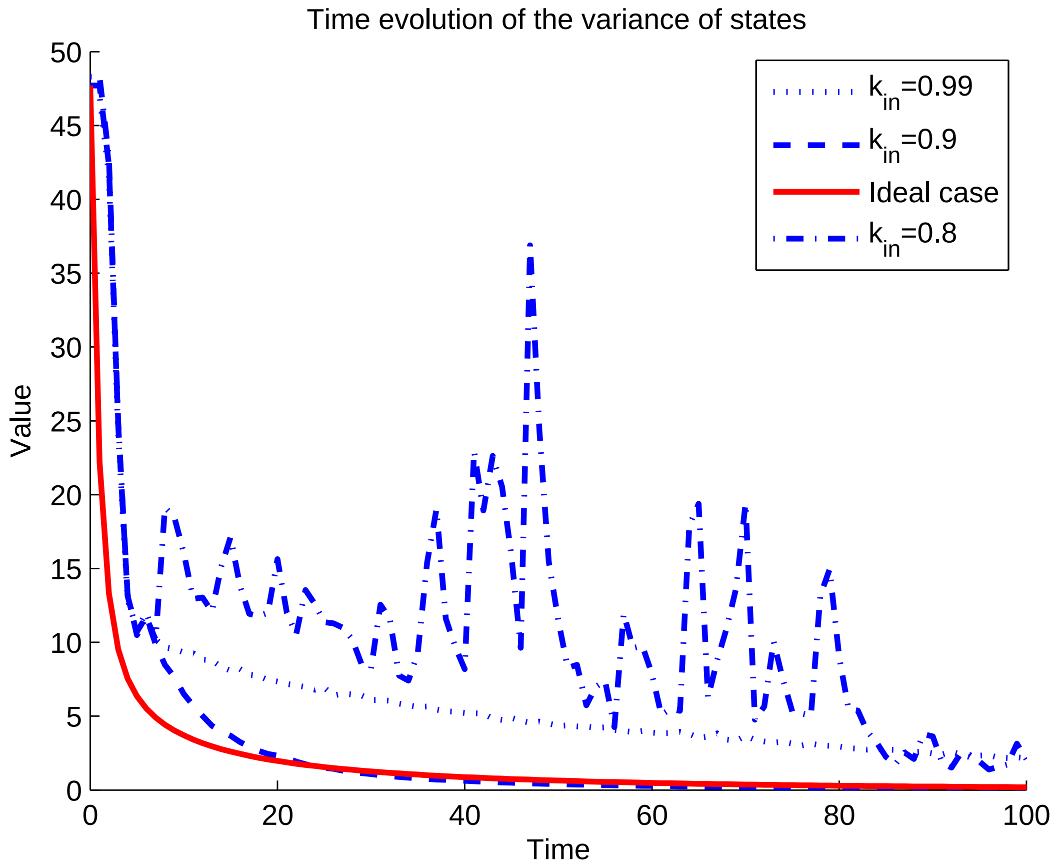


Directed circuit

(N=20, $k_{in} = 0.9$, $k_{out} = 2$).

Simulations II

The effectiveness of the method depends on the zooming rates.



Random geometric graph

(N=20, $m = 4$, $k_{out} = 4$).

Perspective research

- Improve the sufficiency convergence theorems to explain the good experimental results.
- Evaluate the speed of convergence of these algorithms.
- Design coding schemes to deal with digital noisy channels.