Continuous-Time Discontinuous Equations in Bounded Confidence Opinion Dynamics

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Outline

1. Opinion dynamics: motivations and models
2. Results: existence, completeness, robustness
3. Conclusion: contribution and further research

Opinion dynamics: not only consensus

Opinion dynamics is a main topic in social network analysis (and design), attracting interest by

- physicists
- applied mathematicians
- control theorists

C. Castellano, S. Fortunato, and V. Loreto. Statistical physics of social dynamics. *Reviews of Modern Physics*, 81(2):591–646, 2009

Agent interacting on a network rarely reach an opinion consensus, but instead a fragmentation in a few different opinions.

Explanation from two concurrent phenomena:

1. agents' opinions become more similar after interaction (attraction);
2. interaction is more likely if opinions are similar, and less likely of opinions are very different (homophily).

Bounded confidence (Discrete-time)

Homophily can in particular be modeled as bounded confidence: there is no interaction if opinions differ by more than a sharp confidence threshold.

Hegselmann-Krause model:

Consider N agents, indexed in a set $\mathcal{I}$ of cardinality N, each of them having a time dependent "opinion" $x_i(t) \in \mathbb{R}$, with dynamics

$$x_i(t+1) = \frac{1}{|\{j \, : \, |x_i - x_j| < 1\}|} \sum_{j:|x_i - x_j|<1} x_j(t), \qquad i \in \mathcal{I} \tag{HK}$$

R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artifical Societies and Social Simulation*, 5(3), 2002

We want to study a **continuous-time version of (HK)**

Continuous-time bounded-confidence dynamics

$$\dot{x}_i(t) = \sum_{j:|x_i - x_j|<1} (x_j(t) - x_i(t)), \qquad i \in \mathcal{I} \qquad \text{(CT-HK)}$$

V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis. Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *SIAM JCO*, 48(8):5214–5240, 2010

Advantages:

- Considering duration of interactions;
- No need for synchronous updates.

Difficulty: equation (CT-HK) has discontinuous right-hand side

- Need for a suitable definition of "solution"
- We choose **Krasowskii solutions**

Krasowskii Solutions: Definition

A *Krasowskii solution* to

$$\begin{cases} \dot{x} = g(t,x) \\ x(t_0) = \bar{x} \end{cases}$$

on an interval $I \subset \mathbb{R}$ containing t_0, is a map $\phi : I \to \mathbb{R}^N$ such that

1. ϕ is absolutely continuous on I,
2. $\phi(t_0) = \bar{x}$,
3. $\dot{\phi}(t) \in \mathcal{K}g(\phi(t))$ for almost every $t \in I$, where

$$\mathcal{K}g(x) = \bigcap_{\delta>0} \overline{\text{co}}(\{g(t,y) \ : \ y \text{ such that } \|x-y\| < \delta\})$$

and given a set A, by $\overline{\text{co}}(A)$ denotes the closed convex hull of A.

Existence & Completeness

Proposition (Basic properties)

Let $x(\cdot)$ be a Krasowskii solution to (CT-HK), on its domain of definition.

- (Existence). *For any initial condition $\bar{x} \in \mathbb{R}^N$, there exists a local Krasowskii solution to (CT-HK).*
- (Order preservation). *For any $i, j \in \mathcal{I}$, if $x_i(t_1) < x_j(t_1)$, then $x_i(t_2) < x_j(t_2)$, for any $t_2 > t_1$.*
- (Contractivity). *For any $t_2 > t_1$, $\overline{\text{co}}(\{x_i(t_2)\}_{i\in\mathcal{I}}) \subset \overline{\text{co}}(\{x_i(t_1)\}_{i\in\mathcal{I}})$.*
- (Completeness). *The solution $x(\cdot)$ is complete.*
- (Average preservation). *For every $t > 0$, $N^{-1}\sum_{i=1}^{N} x_i(t) = N^{-1}\sum_{i=1}^{N} x_i(0)$.*

Convergence & Fragmentation

Opinions split in separate groups (clusters)

Theorem (Convergence to clusters)

The set of Krasowskii equilibria of (CT-HK) is

$$F = \left\{ x \in \mathbb{R}^N \; : \text{for every } (i,j) \in \mathcal{I} \times \mathcal{I}, \; \text{either } x_i = x_j \text{ or } |x_i - x_j| \geq 1 \right\}$$

and if $x(\cdot)$ is a Krasowskii solution to (CT-HK), then $x(t)$ converges to a point $x_* \in F$ as $t \to +\infty$.

Note: Clusters are separated by at least 1.

However, simulations and previous results suggest that the distance between clusters is typically at least 2. **Why?**

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Robustness of equilibria

An equilibrium is robust if the addition of one perturbing agent does not make any two clusters merge (in the subsequent evolution).

Proposition (Stability for 2 clusters (no loss of generality))

Consider a configuration $x^* \in F$ *consisting of two clusters of* n_A *and* n_B *agents* ($n_A \leq n_B$) *having opinions* x_A *and* x_B*, respectively. Then there exists* $\bar{T}_{n_A,n_B} > 0$ *such that* x^* *is robust*

if $|x_A - x_B| \geq \left(1 + \frac{1}{n_A+n_B}\right)\left(1 + \frac{n_A}{n_B}\right) e^{\bar{T}_{n_A,n_B}} \sim 1 + \frac{n_A}{n_B}$ *for* $n_A \to \infty$,

and only if $|x_A - x_B| \geq 1 + \frac{n_A+1}{n_B} \sim 1 + \frac{n_A}{n_B}$ *for* $n_A \to \infty$.

Conclusion:

- in practice, we observe only robust configurations
- when N is large and the opinions are "uniform", robust equilibria have inter-cluster distance larger than 2

Contribution & Further Research

Advantages of Krasowskii solutions in opinion dynamics

- Completeness of solutions;
- Intuitive convergence and robustness proofs;
- Interpretation as smoothing out the sharp confidence threshold

Open problems

- Relationship with non-discontinuous dynamics (non-sharp thresholds)

  F. Ceragioli and P. Frasca. Continuous and discontinuous opinion dynamics with bounded confidence. *Nonlinear Analysis: Real World Applications*, 2011. Submitted
- Extension to graph-limited interaction
- Include limited verbalization
- Non-reciprocal interactions

  A. Mirtabatabaei and F. Bullo. Opinion dynamics in heterogeneous networks: Convergence conjectures and theorems. *SIAM JCO*, March 2011. Submitted