# Continuous and discontinuous opinion dynamics with bounded confidence

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**Abstract**

In this paper we study of a continuous-time version of the Hegselmann-Krause opinion dynamics, which models bounded confidence by a discontinuous interaction. Intending solutions in the sense of Krasowskii, we provide results of existence, completeness and convergence to *clusters* of agents sharing a common opinion. For a deeper understanding of the role of the mentioned discontinuity, we study a class of continuous approximating systems, and their convergence to the original one. Our results indicate that their qualitative behavior is similar, and we argue that discontinuity is not an essential feature in bounded confidence opinion dynamics.

## 1 Introduction and Preliminaries

A crucial point in modeling opinion dynamics resides in how to specify interactions between agents. This can be accomplished by graph-theoretical representations which are popular in describing social networks. Most successful models include a bounded confidence constraint, so that agents do not interact with fellow agents if their opinions are too far apart. This feature may be described by a state-dependent discontinuous interaction function: if the difference between opinions is larger than the chosen threshold, the "strength" of the interaction drops to zero. In this paper, we focus on a continuous-time version of the Hegselmann-Krause model –a simple and celebrated opinion dynamics system with a discontinuity threshold– which was recently introduced in [3]. The resulting ODE system has a discontinuous right-hand side, which makes the analysis more difficult from a mathematical point of view. We want to give a thorough description of the properties of the ODE solutions and to understand the role of the discontinuity of the right-hand side of the system. To this end, we introduce and study sequences of continuous systems which approximate the original one.

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## 1.1 Problem Statement

Let us consider a population of $N$ agents, indexed in a set $\mathcal{I} = \{1, \ldots, N\}$. Each of them has a time-dependent real-valued "opinion" $x_i(t)$, which obeys the following dynamics

$$\dot{x}_i = \sum_{j \in \mathcal{I}} s(x_j - x_i)(x_j - x_i), \qquad i \in \mathcal{I}, \tag{1}$$

where $s : \mathbb{R} \to \mathbb{R}$ is defined by

$$s(\tau) = \begin{cases} 1 & \text{if } |\tau| < 1 \\ 0 & \text{if } |\tau| \geq 1. \end{cases}$$

Notice that the function $s$, which encodes the coupling between the agent opinions, is discontinuous: for this reason, we refer to (1) as to the Discontinuous Hegselmann-Krause system (DHK).

In addition, we also consider related Continuous Hegselmann-Krause systems (CHK)

$$\dot{x}_i = \sum_{j \in \mathcal{I}} s^n(x_j - x_i)(x_j - x_i), \qquad i \in \mathcal{I}, \tag{2}$$

where the function $s^n : \mathbb{R} \to [0, 1]$ is Lipschitz, even and supported on an interval containing zero. The index $n \in \mathbb{N}$ is introduced in order to consider, when needed, suitable sequences of CHK systems which approximate (1). A significant example is the following.

**Example 1** ($\varepsilon_n$ approximations)**.** Let $\varepsilon_n > 0$ and

$$s^n(t) = \begin{cases} 1 & \text{if} \quad |t| \leq 1 - \epsilon_n \\ 0 & \text{if} \quad |t| \geq 1 \\ -\frac{1}{\epsilon_n}(t-1) & \text{if} \quad 1 - \epsilon_n < t < 1 \\ \frac{1}{\epsilon_n}(t+1) & \text{if} \quad -1 < t < -1 + \epsilon_n \end{cases}$$

We remark that if $\epsilon_n \to 0$ as $n \to \infty$, then $s^n(\tau) \to s(\tau)$ for all $\tau \in \mathbb{R}$. $\square$

## 1.2 Contribution and Paper Overview

The first contribution of this paper consists of studying the properties of solutions to system (1). As this system has a discontinuous right-hand side, solutions have to be intended in an extended sense. The notion of Krasovskii solution, recalled in Section 1.4, was chosen from a number of alternatives. The study of Krasovskii solutions to (1) is carried out in Section 3, where we show that these solution exist for any initial condition, are complete, preserve the average of the initial conditions and asymptotically converge to certain equilibrium points. These equilibria can be described as collections of *clusters* of agents which share a common opinion. The robustness of such clusters to small perturbations is also investigated.

The second contribution consists of investigating the role of the discontinuity by studying sequences of Continuous Hegselmann-Krause systems (2), which approximate (1). Firstly, the properties of the solutions to (2) are presented in Section 2: note that the analysis of (2) is presented earlier in the paper because it is technically simpler and allows us to introduce ideas which are used again in the analysis of (1). In Section 4 the relationship between solutions of Continuous and Discontinuous Hegselmann-Krause models is discussed. Our results show that the qualitative properties of the solutions to (2) and (1) are very similar, in terms of both finite-time and limit behavior. We believe that this is good news from the point of view of mathematical modeling, as it means that the discontinuity can be smoothed out to reduce technical difficulties, without loosing significant phenomena. We also show that solutions to (2) approximate solutions to (1), with a *caveat*: the approximation is uniform on bounded intervals, but one can not infer the time-limit behavior of a single solution to (1) by looking at its approximations.

### 1.3 Relationship with Literature

We now examine the references about opinion dynamics which are most pertinent to our work and refer to the survey papers [6], [16] for a more comprehensive literature review. In the perspective of our research, the most interesting interaction models include an idea of "bounded confidence". Krause and Hegselmann [13] and Deffuant [10] are credited to have introduced the bounded confidence idea in opinion dynamics models. Starting from these works, others have contributed in recent years to this line of research [15, 2]. Such studies are to some extent able to explain real-world phenomena such as the "persistence of disagreement" between opinions, in spite of opinion aggregation during interactions. Hence, typical steady-states feature some phenomenon of partial agreement as the emergence of groups of agents who share the same opinion.

Many models developed by physicists [15, 16] involve difference equations, because they were intended to be simulated by computers. This modeling approach has the disadvantage of assuming the interactions to be instantaneously effective on the opinions, and to happen either synchronously or according to some schedule. Similar assumptions may be questionable in social science applications, and suggest the alternative approach using ODE dynamics. A simple model describing the opinion evolution on a single topic, with no autonomous dynamics for each individual, no external influences, and a linear way to aggregate opinions combined with a bounded confidence rule, leads to the continuous-time version of the Hegselmann-Krause model (1) which was introduced in [3].

The present paper studies the model (1) and how it relates to the continuous counterparts (2). We anticipate that our results about (1) are qualitatively similar to the convergence results presented in [3, Section 2]. However, the results we present in Section 3 are more general, as the set of Krasovskii solutions to (1) is strictly greater than the sub-set of Carathéodory solutions considered in [3], and includes the study of some pathological behaviors which where not covered by [3].

Finally, we note that an incomplete and preliminary account of our work has appeared in the Proceedings of the 18th World Congress of the International Federation of Automatic Control as [7]. This short paper contains partial results about the discontinuous dynamics, and does not consider the continuous model or the relationship between them.

## 1.4 Preliminaries

### 1.4.1 Graphs

In this paper, we shall make use of some notions from graph theory, and in particular from algebraic graph theory. Indeed, graph theory provides an effective tool to model interactions between agents and its use is becoming common both in engineering [4, 17] and in economics and social sciences [11]. A (weighted) graph $G$ is a triple $(V, E, A)$ where $V$ is a finite set of vertices or nodes, $E \subset V \times V$ is a set of edges and the adjacency matrix $A$ is a matrix of weights, such that for any $u, v \in V$, $A_{uv} > 0$ only if $(u, v) \in E$. The Laplacian matrix of $G$ is defined as $L_{uv} = -A_{uv}$ when $u \neq v$ and $L_{uu} = \sum_{v \in V} A_{uv}$. If $(u, v) \in E$, then $v$ is said to be a neighbor of $u$ in $G$. A path (of length $l$) from $u$ to $v$ in $G$ is an ordered list of edges $(e_1, \dots, e_l)$ in the form $((u, w_1), (w_1, w_2), (w_2, w_3), \dots, (w_{l-1}, v))$. Two nodes $u, v \in V$ are said to be connected if there exists a path from $u$ to $v$, and disconnected otherwise. A graph is said to be connected if every two nodes are connected, and disconnected otherwise. A graph is said to be symmetric when $(u, v) \in E$ implies $(v, u) \in E$ and the matrix $A$ is symmetric. In a symmetric graph, being neighbors is an equivalence relation between nodes: the corresponding equivalence classes are said to be the connected components of the graph.

### 1.4.2 Solutions to ODEs

As already remarked, in order to deal with possibly discontinuous ODEs, we need to take different notions of solutions into consideration. We give the definitions of classical, Carathéodory and Krasovskii solutions (see [12, 9]). Let us consider the differential equation

$$\begin{cases} \dot{x} = g(x) \\ x(t_0) = \bar{x} \end{cases} \tag{3}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $g : \mathbb{R}^N \to \mathbb{R}^N$.

A *classical solution* to (3) on an interval $I \subset \mathbb{R}$ containing $t_0$, is a map $\phi : I \to \mathbb{R}^N$ such that

1. $\phi$ is differentiable on $I$,
2. $\phi(t_0) = \bar{x}$,
3. $\dot{\phi}(t) = g(\phi(t))$ for all $t \in I$.

A *Carathéodory solution* to (3) on an interval $I \subset \mathbb{R}$ containing $t_0$, is a map $\phi : I \to \mathbb{R}^N$ such that

1. $\phi$ is absolutely continuous on $I$,
2. $\phi(t_0) = \bar{x}$,
3. $\dot{\phi}(t) = g(\phi(t))$ for almost every $t \in I$.

Equivalently, a Carathéodory solution to (3) is a solution to the integral equation

$$x(t) = \bar{x} + \int_{t_0}^{t} g(x(s)) ds.$$

A *Krasovskii solution* to (3) on an interval $I \subset \mathbb{R}$ containing $t_0$, is a map $\phi : I \to \mathbb{R}^N$ such that

1. $\phi$ is absolutely continuous on $I$,
2. $\phi(t_0) = \bar{x}$,
3. $\dot{\phi}(t) \in \mathcal{K}g(\phi(t))$ for almost every $t \in I$, where

$$\mathcal{K}g(x) = \bigcap_{\delta>0} \overline{\mathrm{co}}(\{g(y) \ : \ y \text{ such that } \|x-y\| < \delta\})$$

and given a set $A$, by $\overline{\mathrm{co}}(A)$ we denote the closed convex hull of $A$.

From the above definitions, it is clear that classical solutions are Carathéodory solutions and, in turns, Carathéodory solutions are Krasovskii solutions. Note also that Carathéodory solutions coincide with solutions in the classical sense when $g$ is continuous.

In the context of bounded confidence opinion dynamics, a subset of Carathéodory solutions has been considered in [3]: the main drawback of the approach taken there is difficulty in studying existence and continuation properties of those solutions. On the other hand Krasovskii solutions are easier to be treated as far as existence and continuation properties are considered. Moreover they give rise to more general results, in the sense that results on Carathéodorysolutions can be a posteriori obtained as particular cases of those for Krasovskii solutions.

## 1.5 Interaction Graphs

It is useful and suggestive to rewrite systems (1) and (2) as dynamics over a suitable state-dependent weighted graph, which represents the coupling between the opinions of different agents. In such a graph the agents are the nodes, and the opinions of two agents depend on each other whenever the agents are neighbors in the graph. By the way systems (1) and (2) are defined, such interaction graph depends on the opinion states, via the functions $s$ and $s^n$, respectively. More precisely, for any $x \in \mathbb{R}^N$ we define an interaction graph $\mathcal{G}(x) = (\mathcal{I}, \mathcal{E}(x), A(x))$ where the edge set is

$$\mathcal{E}(x) = \{(i,j),\, i,j \in \mathcal{I} \ : \ |x_i - x_j| < 1\},$$

that is, $(i,j) \in \mathcal{E}(x)$ if and only if $s(x_j - x_i) > 0$, and the adjacency matrix $A(x)$ is defined by

$$A(x)_{ij} = \begin{cases} s(x_i - x_j) & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases} \qquad i,j \in \mathcal{I},$$

that is, $A(x)_{ij} = 1$ if and only if $|x_i - x_j| < 1$ and $j \neq i$. The Laplacian matrix $L(x)$ associated to $\mathcal{G}(x)$ is then given by

$$L(x)_{ij} = \begin{cases} -s(x_i - x_j) & \text{if } j \neq i \\ \sum_{k \neq i} s(x_k - x_i) & \text{if } j = i \end{cases} \qquad i,j \in \mathcal{I}.$$

With these notations system (1) can be written as

$$\dot{x} = -L(x)x.$$

In order to deal with the discontinuity, it is also useful to identify "border" configurations by the following definitions of border edge set

$$\partial\mathcal{E}(x) = \{(i,j),\, i,j \in \mathcal{I} \,:\, |x_i - x_j| = 1\}\,.$$

and graph $\bar{\mathcal{G}}(x) = \left(\mathcal{I}, \bar{\mathcal{E}}(x), \bar{A}\right)$, with $\bar{\mathcal{E}}(x) = \mathcal{E}(x) \cup \partial\mathcal{E}(x)$ and $\bar{A}(x)_{ij} = 1$ if and only if $|x_i - x_j| \leq 1$ and $j \neq i$.

Similarly, for systems (2) we can define the state-dependent weighted graphs $\mathcal{G}^n(x) = (\mathcal{I}, \mathcal{E}^n(x), A^n(x))$, where $(i,j) \in \mathcal{E}^n$ if and only if $s^n(x_j - x_i) > 0$ and the adjacency matrix $A^n(x)$ is defined by

$$A^n(x)_{ij} = \begin{cases} s^n(x_i - x_j) & \text{if } j \neq i \\ 0 & \text{if } j = i \end{cases} \qquad i,j \in \mathcal{I}.$$

The Laplacian matrix $L^n(x)$ associated to $\mathcal{G}^n(x)$ is then given by

$$L^n(x)_{ij} = \begin{cases} -s^n(x_i - x_j) & \text{if } j \neq i \\ \sum_{k\neq i} s^n(x_k - x_i) & \text{if } j = i \end{cases} \qquad i,j \in \mathcal{I}.$$

With these notations system (2) can be written as

$$\dot{x}^n = -L^n(x^n)x^n.$$

**Remark 1** (Symmetry and translation invariance)**.** We remark that the graphs $\mathcal{G}$, $\bar{\mathcal{G}}$, and $\mathcal{G}^n$ are symmetric and invariant with respect to the translation $x + \alpha\mathbf{1}$, where $\alpha \in \mathbb{R}$ and $\mathbf{1} = (1, ..., 1)^T$, i.e. $\mathcal{G}(x) = \mathcal{G}(x + \alpha\mathbf{1})$, $\bar{\mathcal{G}}(x) = \bar{\mathcal{G}}(x + \alpha\mathbf{1})$, $\mathcal{G}^n(x) = \mathcal{G}^n(x + \alpha\mathbf{1})$.

As we said, the graphs introduced above are interaction graphs in the following sense: if two nodes are disconnected, they can not influence each other opinions.

## 2 Continuous Hegselmann-Krause Model

We start by proving some basic properties of the solutions of the continuous Hegselmann-Krause model.

**Proposition 1** (Basic properties of CHK)**.** *Let $x^n(\cdot)$ be a solution*[1] *to* (2) *such that $x^n(0) = \bar{x}$, on its domain of definition.*

*(i)* (Uniqueness). *$x^n(\cdot)$ is the unique solution to* (2) *such that $x^n(0) = \bar{x}$.*

*(ii)* (Order preservation). *For any $i,j \in \mathcal{I}$, if $x^n_i(t_1) < x^n_j(t_1)$, then $x^n_i(t_2) < x^n_j(t_2)$, for any $t_2 > t_1$.*

[1] Here solutions are intended as classical solutions. Note, however, that classical, Carathéodory, and Krasovskii solutions coincide for (2) because $s^n$ is continuous.

*(iii)* (Contractivity). *For any $t_2 > t_1$, we have that $\overline{\mathrm{co}}(\{x_i^n(t_2)\}_{i\in\mathcal{I}}) \subset \overline{\mathrm{co}}(\{x_i^n(t_1)\}_{i\in\mathcal{I}})$.*

*(iv)* (Completeness). *The solution $x^n(\cdot)$ is complete.*

*(v)* (Average preservation). *Let $x^n_{\mathrm{ave}}(t) = N^{-1}\sum_{i\in\mathcal{I}} x_i^n(t)$. Then $x^n_{\mathrm{ave}}(t) = x^n_{\mathrm{ave}}(0)$, for all $t > 0$.*

*Proof.* i) The solution is unique because the right-hand side of (2) is locally Lipschitz.

ii) It is not restrictive to assume that $x_1^n(0) \le x_2^n(0) \le ... \le x_N^n(0)$. We then prove that $x_i^n(0) \le x_{i+1}^n(0)$ implies $x_i^n(t) \le x_{i+1}^n(t)$ for all $t \ge 0$, which is equivalent to the statement. First of all we remark that $x_i^n(0) = x_{i+1}^n(0)$ then $x_i^n(t) = x_{i+1}^n(t)$ for all $t \ge 0$ thanks to the fact that the dynamics of $x_i^n$ and $x_{i+1}^n$ are the same and we have uniqueness of the solution for a given initial condition by statement (i). Let us now consider the case $x_i^n(0) < x_{i+1}^n(0)$. Assume by contradiction that there exists $T > 0$ such that $x_i^n(T) = x_{i+1}^n(T)$ and let $T^* = \sup\{t^* : x_i^n(t) < x_{i+1}^n(t)\,\forall t \in (0, t^*)\}$ . One has $x_i^n(t) = x_{i+1}^n(t)$ for all $t \ge T^*$. We consider the equation

$$\dot{x}_{i+1}^n(t) - \dot{x}_i^n(t) = \sum_{j\in\mathcal{I}} s^n(x_j(t) - x_{i+1}(t))(x_j(t) - x_{i+1}(t)) - \sum_{j\in\mathcal{I}} s^n(x_j(t) - x_i(t))(x_j(t) - x_i(t)) \tag{4}$$

The state $x_{i+1}^n(T^*) - x_i^n(T^*) = 0$ is an equilibrium for the equation (4), then $x_{i+1}^n(t) - x_i^n(t) \equiv 0$ is the unique solution to (4), which contradicts the fact that $x_i^n(t) < x_{i+1}^n(t)$ if $t < T^*$.

iii) By statement (ii), we can assume with no loss of generality that $x_1^n(t) \le x_2^n(t) \le ... \le x_N^n(t)$ for all $t \ge 0$. This implies that $\dot{x}_1^n(t) = \sum_{j\in\mathcal{I}} s^n(x_j^n(t) - x_1^n(t))(x_j^n(t) - x_1^n(t)) \ge 0$ for all $t \ge 0$ and, analogously, that $\dot{x}_N^n(t) \le 0$ for all $t \ge 0$, and this completes the proof of the statement.

iv) Thanks to statement (iii), any local solution is bounded. Standard arguments guarantee that it can then be extended for all $t > 0$.

v) By differentiating $x^n_{\mathrm{ave}}(t)$ we get

$$\dot{x}^n_{\mathrm{ave}}(t) = \frac{1}{N}\sum_{i\in\mathcal{I}} \dot{x}_i^n(t) = \frac{1}{N}\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} s^n(x_j^n(t) - x_i^n(t))(x_j^n(t) - x_i^n(t)) = 0,$$

where we have used the fact that $s^n$ is even in order to get the last equality. □

Next, we show that solutions to a Continuous Hegselmann-Krause model converge to equilibria, which can be described as clusters of agents sharing the same opinion.

**Theorem 2** (Convergence of CHK)**.** *The set of the equilibria of* (2) *is*

$$F_n = \{x \in \mathbb{R}^N : \forall (i,j) \in \mathcal{I}\times\mathcal{I},\ \text{either } x_i = x_j \text{ or } s^n(x_i - x_j) = 0\}$$

*and if $x^n(\cdot)$ is a solution to* (2)*, then $x^n(t)$ converges to a point $x^n_* \in F_n$ as $t \to +\infty$.*

*Proof.* The proof is in three steps. We first describe the set of equilibria, then prove convergence to this set, and finally prove convergence to one equilibrium. Note that by Proposition 1, statement (ii), we can assume with no loss of generality that the agents are sorted so that $x_1^n(t) \leq x_2^n(t) \leq ... \leq x_N^n(t)$ for all $t \geq 0$.

i) Clearly the points in $F_n$ are equilibria of (2). To prove that there are no other equilibria, note that by the sorting assumption $s^n(x_j^n - x_1^n) \geq 0$ and $x_j^n - x_1^n \geq 0$ for all $j \in \mathcal{I}$. For the right-hand side of (2) to be equal to zero it is then necessary that for every $j \in \mathcal{I}$ either $s^n(x_j^n - x_1^n) = 0$ or $x_j^n - x_1^n = 0$. Repeating the reasoning for all $i = 2, 3, \dots,$ we obtain that there are no other equilibria.

ii) We define the Lyapunov function $V(x) = \frac{1}{2}\sum_{i\in\mathcal{I}} x_i^2$ and compute, using the symmetry of $s^n$ as done in [5],

$$\begin{aligned}\frac{d}{dt}V(x^n(t)) &= \sum_{i\in\mathcal{I}} x_i^n(t)\dot{x}_i^n(t)\\ &= \sum_{i\in\mathcal{I}} x_i^n(t) \sum_{j\in\mathcal{I}} s^n\left(x_j^n(t) - x_i^n(t)\right)\left(x_j^n(t) - x_i^n(t)\right)\\ &= -\frac{1}{2}\sum_{i,j} s^n\left(x_j^n(t) - x_i^n(t)\right)\left(x_j^n(t) - x_i^n(t)\right)^2 \leq 0.\end{aligned}$$

Since the inequality is strict if $x(t) \notin F_n$, and $F_n$ is closed and invariant, we can apply the LaSalle invariance principle to conclude convergence of $x^n(\cdot)$ to the set $F_n$.

iii) We observe that the set $F_n$ is the union of a finite number of sets $F_P$, where $P = \{P_1, \dots, P_k\}$ is a partition of $\mathcal{I}$ in $1 \leq k \leq N$ subsets, and

$$F_P = \left\{x \in \mathbb{R}^N \,:\, \forall\, i, j \in \mathcal{I}, \text{ if } \exists\, h \text{ s.t. } i, j \in P_h, \text{then } x_i = x_j, \text{else } s^n(x_i - x_j) = 0\right\}.$$

Note that, since $s^n$ is an even function, $s^n(x_i - x_j) = 0$ if and only if $|x_i - x_j| \geq r_n$, where $2r_n$ is the diameter of the support of $s^n$. As the sets $F_P \subset F_n$ are closed and disjoint, we argue that each solution converges towards one of them. We are thus left to show that convergence to a certain $F_P$ implies convergence to a point in $F_P$. Let us describe the convergence of $x^n(\cdot)$ when $k$, the number of parts in $P$, is given. When $k = 1$, the only partition is the trivial one, corresponding to equilibria in which the states of all agents coincide. In this case, average preservation implies that $x_i^n(t) \to x_{\text{ave}}^n(0)$ for all $i$ as $t \to \infty$. When $k = 2$, for every partition $P$ there exists $a \in \{1, \dots, N\}$ such that for every $x \in F_P$ it holds that $x_i = x_a$ for every $i \leq a$, $x_i = x_{a+1}$ for every $i > a$, and $x_{a+1} - x_a \geq r_n$. Let $T_a = \inf\left\{t \geq 0 \,:\, x_{a+1}^n(t) - x_a^n(t) > r_n\right\}$. If $T_a < +\infty$, then there is disconnection at finite time and when $t \to +\infty$ we have that $x_i^n(t) \to \frac{1}{a}\sum_{j\leq a} x_j^n(T)$ if $i \leq a$, whereas $x_i^n(t) \to \frac{1}{N-a}\sum_{j>a} x_j^n(T)$ if $i > a$. If instead $T = \infty$, then $x_{a+1}^n(t) - x_a^n(t) \to r_n$ as $t \to +\infty$ and the preservation of the average implies that $\sum_{i\leq N} x_i^n(t) \to N x_{\text{ave}}^n(0)$. Hence we argue that $x_i^n(t) \to x_{\text{ave}}^n(0) - r_n\frac{N-a}{N}$ if $i \leq a$ and $x_i^n(t) \to x_{\text{ave}}^n(0) + r_n\frac{a}{N}$ otherwise. When $k \geq 3$, the above argument can be extended by defining $k-1$ appropriate disconnection times $T_{a_1}, \dots, T_{a_{k-1}}$: we conclude that $x^n(\cdot)$ converges to a point in $F_n$. □

The set of equilibria $F_n$ in Theorem 2 has the following feature: its points are such that the agent opinions either coincide or their distance is larger than the size of the support[2] of $s^n$. Equivalently, the opinions of two agents are equal if and only if the agents are connected in the interaction graph. Following the opinion dynamics literature, we refer to such groups of agents as *clusters*, and to the corresponding values as *cluster values*. More formally, one can consider for a given $x \in F_n$, the map $\mathcal{I} \ni i \mapsto x_i \in \mathbb{R}$: the image of such map consists of the cluster values and the clusters are the preimages of the cluster values. The size of a cluster is its cardinality.

In [3], the authors propose for clusters a definition of robustness with respect to small perturbations, suggesting that opinion dynamics models are more likely to converge to robust clusters. In this paper, we adopt a similar definition of robustness, which is discussed in the following remark.

**Remark 2** (Robustness of CHK equilibria)**.** An equilibrium is said to be robust if *no perturbation consisting in adding one agent to the configuration can cause two of the former clusters to coalesce* in the resulting evolution.

Indeed, note that the added agent may be connected to two, one or no cluster. In the third case, there is no evolution. If the added agent has one neighbor cluster, the subsystem consisting of the cluster and the added agent converges to a new single cluster. In both of these cases no former clusters merge. This observation immediately implies that a sufficient condition for an equilibrium to be robust is that the clusters be at a distance which is larger than twice the size of the support. On the other hand, any necessary condition would depend on the specific interaction function $s^n$ at hand. In the case of Example 1, the following necessary condition holds[3]. For any pair of clusters, denote them by $A$ and $B$, having values $x_A$ and $x_B$, and sizes $n_A \leq n_B$. Then, for an equilibrium $x$ to be robust it is necessary that $|x_B - x_A| \geq (1 - \varepsilon_n)(1 + \frac{n_A}{n_B})$ for every pair $A, B$. □

The interest for robust equilibria is motivated by the following intuition. Those equilibria which are robust against the action of isolated agents are more suitable to be limit points of "real" opinion dynamics system, which would be subject to uncertainties and disturbances. Furthermore, as we demonstrate later in Section 4, simulated solutions usually converge to robust equilibria.

# 3 Discontinuous Hegselmann-Krause Model

We have seen in Section 1.5 that system (1) can be written as

$$\dot{x} = -L(x)x, \tag{5}$$

being $L(x)$ the Laplacian matrix of the state-dependent graph $\mathcal{G}(x)$ and

$$(-L(x)x)_i = \sum_{j \in \mathcal{I}} s(x_j - x_i)(x_j - x_i)$$

[2]By size of the support of an $s^n$ function we intend half its diameter. For instance, in Example 1 the size of the support is 1.

[3]We omit the simple proof which follows the lines of the proof of Proposition 5.

the components of the right-hand side.

As the differential equation (5) has a discontinuous right-hand side, we consider Krasovskii solutions to (1), which we characterize as follows. For any $H \subset \partial\mathcal{E}(x)$ we let $L^H(x)$ be the Laplacian matrix associated to the graph $\mathcal{G}^H(x)$ with edges $\mathcal{E}(x) \cup H$, and correspondingly

$$(-L^H(x)x)_i = \sum_{j:(i,j)\in\mathcal{E}(x)\,\cup\, H} (x_j(t) - x_i(t)).$$

By the definition, it is clear that a Krasovskii solution to (1) satisfies at almost every time the inclusion

$$\dot{x} \in \overline{\mathrm{co}}(\{-L^H(x)x \,:\, H \subset \partial\mathcal{E}(x)\}),$$

or equivalently the inclusion

$$\dot{x} \in \Big\{ -\sum_{H\subset\partial\mathcal{E}(x)} \alpha_H L^H(x)x \,:\, \alpha_H \geq 0, \sum_{K\subset\partial\mathcal{E}(x)} \alpha_K = 1 \Big\}.$$

Namely, for a given Krasovskii solution $\phi(\cdot)$,

$$\dot{\phi}(t) = -\sum_{H\subset\partial\mathcal{E}(\phi(t))} \alpha_H^{\phi}(t) L^H(\phi(t))\phi(t) \quad \text{for almost every } t,$$

where the time-dependent coefficients $\alpha_H^{\phi}$ depend on the solution $\phi(\cdot)$ itself.

Using this graph-theoretical characterization, we now prove some basic properties of Krasovskii solutions to (1).

**Proposition 3** (Basic properties of DHK)**.** *Let $x(\cdot)$ denote a Krasovskii solution to* (1)*, on its domain of definition.*

*(i)* (Existence). *For any initial condition $\bar{x} \in \mathbb{R}^N$, there exists a Krasovskii solution starting from $\bar{x}$.*

*(ii)* (Order preservation). *For any $i, j \in \mathcal{I}$, if $x_i(t_1) < x_j(t_1)$, then $x_i(t_2) < x_j(t_2)$, for any $t_2 > t_1$.*

*(iii)* (Contractivity). *For any $t_2 > t_1$, $\overline{\mathrm{co}}(\{x_i(t_2)\}_{i\in\mathcal{I}}) \subset \overline{\mathrm{co}}(\{x_i(t_1)\}_{i\in\mathcal{I}})$.*

*(iv)* (Completeness). *The solution $x(\cdot)$ is complete.*

*(v)* (Average preservation). *Let $x_{\mathrm{ave}}(t) = N^{-1}\sum_{i=1}^N x_i(t)$. Then $x_{\mathrm{ave}}(t) = x_{\mathrm{ave}}(0)$, for $t > 0$.*

*Proof.* In the proof, the following notation will be useful. For every $i \in \mathcal{I}$, and every $x \in \mathbb{R}^N$, we let

$$\mathcal{N}_i(x) := \{k \in \mathcal{I} \,:\, |x_i - x_k| < 1\},$$

and for any $H \subset \partial\mathcal{E}(x)$, we let

$$N_i^H = \{k \in \mathcal{I} \,:\, (i,k) \in H\}.$$

Clearly, $N_i^H \subset \partial\mathcal{N}_i(x) := \{k \in \mathcal{I} \,:\, |x_i - x_k| = 1\}$. With this notation,

$$(-L(x)x)_i = \sum_{j \in \mathcal{N}_i(x)} (x_j(t) - x_i(t))$$

and

$$(-L^H(x)x)_i = \sum_{j \in \mathcal{N}_i(x) \cup N_i^H} (x_j(t) - x_i(t)).$$

We are now ready to prove our statements.

i) Since the right-hand side of (1) is locally essentially bounded, local existence of a Krasovskii solution is guaranteed (see for instance [12]).

ii) To prove the claim, we study the dynamics of the difference between $x_j$ and $x_i$, similarly to what is done in [14, Section 10.1] for Carathéodory solutions. By continuity of Krasowskii solutions, we can assume with no loss of generality that $x_j$ and $x_i$ are close, for instance that $x_j - x_i < 1$. For brevity, in the following we omit the explicit dependence of $\mathcal{N}_i$ on $x$. For almost every time $t$, we have

$$\begin{aligned}
\frac{d}{dt}(x_j - x_i) &= \sum_{H \in \partial\mathcal{E}(x)} \alpha_H \left[ \sum_{h \in \mathcal{N}_j \cup N_j^H} (x_h - x_j) - \sum_{h \in \mathcal{N}_i \cup N_i^H} (x_h - x_i) \right] \\
&= \sum_{H \in \partial\mathcal{E}(x)} \alpha_H \Bigg[ \sum_{h \in \mathcal{N}_i \cap \mathcal{N}_j} ((x_h - x_j) - (x_h - x_i)) + \sum_{h \in \mathcal{N}_i \cap N_j^H} ((x_h - x_j) - (x_h - x_i)) \\
&\qquad + \sum_{h \in N_i^H \cap \mathcal{N}_j} ((x_h - x_j) - (x_h - x_i)) - \sum_{h \in (\mathcal{N}_i \cup N_i^H) \setminus (\mathcal{N}_j \cup N_j^H)} (x_h - x_i) \\
&\qquad + \sum_{h \in (\mathcal{N}_j \cup N_j^H) \setminus (\mathcal{N}_i \cup N_i^H)} (x_h - x_j) \Bigg] \\
&= \sum_{H \in \partial\mathcal{E}(x)} \alpha_H \Bigg[ \sum_{h \in \mathcal{N}_i \cap \mathcal{N}_j} -(x_j - x_i) + \sum_{h \in \mathcal{N}_i \cap N_j^H} (-x_j + x_i) + \sum_{h \in N_i^H \cap \mathcal{N}_j} (-x_j + x_i) \\
&\qquad - \sum_{h \in (\mathcal{N}_i \cup N_i^H) \setminus (\mathcal{N}_j \cup N_j^H)} (x_h - x_i) + \sum_{h \in (\mathcal{N}_j \cup N_j^H) \setminus (\mathcal{N}_i \cup N_i^H)} (x_h - x_j) \Bigg] \\
&= -|\mathcal{N}_i \cap \mathcal{N}_j|(x_j - x_i) + \sum_{H \subset \partial\mathcal{E}(x)} \alpha_H \Big[ -\left(\left|\mathcal{N}_i \cap N_j^H\right| + \left|N_i^H \cap \mathcal{N}_j\right|\right)(x_j - x_i) \\
&\qquad - \sum_{h \in (\mathcal{N}_i \cup N_i^H) \setminus (\mathcal{N}_j \cup N_j^H)} (x_h - x_i) + \sum_{h \in (\mathcal{N}_j \cup N_j^H) \setminus (\mathcal{N}_i \cup N_i^H)} (x_h - x_j) \Big].
\end{aligned}$$

Since if $h \in (\mathcal{N}_i \cup N_i^H) \setminus (N_j^H \cup \mathcal{N}_j)$, then $x_h - x_i < 0$, whereas if $h \in (\mathcal{N}_j \cup N_j^H) \setminus$

$(N_i^H \cup \mathcal{N}_i)$, then $x_h - x_j > 0$, and since $\left|\mathcal{N}_i \cap N_j^H\right| \leq |\mathcal{N}_i|$, we get that

$$\frac{d}{dt}(x_j - x_i) \geq -\big(|\mathcal{N}_i \cap \mathcal{N}_j| + |\mathcal{N}_i| + |\mathcal{N}_j|\big)(x_j - x_i).$$

The obtained inequality ensures that $x_j - x_i$ can not reach zero in finite time, and yields our claim.

iii) To prove the claim we show that the leftmost agent can only move to its right. To this goal, we need a recall the proof of statement (ii). While our argument shows that strict inequalities between agents' states are preserved by the dynamics, we have to remark that equalities are not. It is not in general true that if $x_i(t_1) = x_j(t_1)$, then $x_i(t_2) = x_j(t_2)$ for any $t_2 > t_1$. Indeed, we can observe that if $x_i(t_1) = x_j(t_1)$, then $\dot{x}_i(t_1)$ and $\dot{x}_j(t_1)$ have to satisfy to the same differential inclusion, but need not to be equal. However, it can be proven that it is always possible, given a solution $x(\cdot)$, to sort the states so that $x_{i_1}(t) \leq x_{i_2}(t) \leq \ldots \leq x_{i_N}(t)$, for every $t$. Note that this mapping $i_{(\cdot)} : \{1, \ldots, N\} \to \mathcal{I}$ depends on the solution and needs not to be unique. Nevertheless, it allows us to define $x_{\min}(t) := x_{i_1}(t)$, and $x_{\max}(t) := x_{i_N}(t)$. This fact is useful because it allows us to observe that

$$x_i(t) - x_{\min}(t) \geq 0$$

for every $t$ and every $i \in \mathcal{I}$ and then, for almost every time $t$, $\frac{d}{dt}x_{\min}(t) \in [0, +\infty)$. Repeating an analogous argument for $x_{\max}$ implies the claim.

iv) Claim (iii) ensures that solutions are bounded. By standard arguments, this is enough to guarantee that local solutions can be extended for all $t > 0$.

v) For every $x \in \mathbb{R}^N$, every $H \subset \partial\mathcal{E}(x)$ and every $i, j \in \mathcal{I}$, it holds that $j \in \mathcal{N}_i(x) \cup N_i^H$ if and only if $i \in \mathcal{N}_j(x) \cup N_j^H$; that is, the graph $\mathcal{G}^H(x)$ is symmetric. This key remark allows us to argue that for almost every time $t$,

$$\begin{aligned}
\frac{d}{dt}x_{\text{ave}}(t) =& N^{-1}\sum_{i\in\mathcal{I}}\dot{x}_i(t)\\
=& N^{-1}\sum_{i\in\mathcal{I}}\sum_{H\subset\partial\mathcal{E}(x(t))}\alpha_H(t)\sum_{j\in\mathcal{N}_i(x(t))\,\cup\,N_i^H}(x_j(t) - x_i(t))\\
=& N^{-1}\sum_{H\subset\partial\mathcal{E}(x(t))}\alpha_H(t)\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{N}_i(x(t))\,\cup\,N_i^H}(x_j(t) - x_i(t)) = 0.
\end{aligned}$$

This ensures $x_{\text{ave}}(t) = x_{\text{ave}}(0)$ for every $t > 0$. □

We are now ready to prove convergence to a configuration in which agents are separated into clusters of agents which share the same opinion. We first recall that a point $\tilde{x}$ is said to be a Krasovskii equilibrium of (1) if the function $x(t) \equiv \tilde{x}$ is a Krasovskii solution to (1), i.e. $0 \in \overline{\text{co}}\big(\{-L^H(\tilde{x})\tilde{x} \,:\, H \subset \partial\mathcal{E}(\tilde{x})\}\big)$.

**Theorem 4** (Convergence of DHK). *The set of Krasovskii equilibria of* (1) *is*

$$F = \Big\{x \in \mathbb{R}^N \ : \textit{for every } (i,j) \in \mathcal{I}\times\mathcal{I},\ \textit{either } x_i = x_j \ \textit{or } |x_i - x_j| \geq 1\Big\}$$

*and if $x(\cdot)$ is a Krasovskii solution to* (1)*, then $x(t)$ converges to a point $x_* \in F$ as $t \to +\infty$.*

*Proof.* The proof follows the lines of the proof of Theorem 2 and is done in three steps. We first describe the set of equilibria, then prove convergence to this set, and finally prove convergence to one equilibrium.

i) It is clear that every point in $F$ is an equilibrium. To prove that there are no other equilibria, we proceed as follows. Without loss of generality we can sort the components of $\tilde{x}$ so that $\tilde{x}_{i_1} \leq ... \leq \tilde{x}_{i_N}$. For a vector $v \in \overline{\text{co}}(\{-L^H(\tilde{x})\tilde{x} \,:\, H \subset \partial\mathcal{E}(\tilde{x})\})$ to be equal to zero, it is necessary that $v_{i_1} = 0$. But since $\tilde{x}_k - \tilde{x}_{i_1} \geq 0$ for every $k \in \mathcal{I}$, it is necessary that $\tilde{x}_j - \tilde{x}_{i_1} \in \{0\} \cup [1, +\infty)$, for every $j \in \mathcal{I}$. Repeating this reasoning for $i_2, \ldots,$ we have that the set of equilibria actually coincides with $F$.

ii) We define the Lyapunov function $V(x) = \frac{1}{2}\sum_{i\in\mathcal{I}} x_i^2$ and compute, using the symmetry of the graph $\mathcal{G}(x)$,

$$\begin{aligned}
\frac{d}{dt}V(x(t)) =& \sum_{i\in\mathcal{I}} x_i(t)\dot{x}_i(t) \\
=& \sum_{i\in\mathcal{I}} x_i(t) \left( \sum_{j\in\mathcal{N}_i(x)} (x_j(t) - x_i(t)) + \sum_{H\subset\partial\mathcal{E}(x)} \alpha_H \sum_{j\in N_i^H} (x_j(t) - x_i(t)) \right) \\
=& -\frac{1}{2} \sum_{(i,j)\in\mathcal{E}(x)} (x_j(t) - x_i(t))^2 - \frac{1}{2} \sum_{H\subset\partial\mathcal{E}(x)} \alpha_H \sum_{(i,j)\in H} (x_j(t) - x_i(t))^2 \leq 0.
\end{aligned}$$

Since the inequality is strict if $x(t) \notin F$, and $F$ is closed and weakly invariant, we can apply a LaSalle invariance principle [1, Theorem 3] to conclude convergence to the set $F$.

iii) We observe that the set $F$ is the union of a finite number of sets $F_P$, where $P = \{P_1, \ldots, P_k\}$ is a partition of $\mathcal{I}$ in $1 \leq k \leq N$ subsets, and

$$F_P = \left\{ x \in \mathbb{R}^N \,:\, \forall\, i, j \in \mathcal{I},\ \text{if}\ \exists\, h \ \text{s.t.}\ i, j \in P_h, \text{then}\ x_i = x_j, \text{else}\ |x_i - x_j| \geq 1 \right\}.$$

As the sets $F_P \subset F$ are closed and disjoint, each solution converges towards one of them. Without loss of generality, we relabel the states so that, for the solution at hand, $x_1(t) \leq \ldots \leq x_N(t)$ for every $t \geq 0$. When $k = 1$, the only partition is the trivial one, corresponding to equilibria in which the states of all agents coincide. In this case, average preservation implies that $x_i(t) \to x_{\text{ave}}(0)$ for all $i$ as $t \to \infty$. When $k = 2$, there exists $a \in \{1, \ldots, N\}$ such that for every $x \in F_P$ it holds that $x_i = x_a$ for every $i \leq a$, $x_i = x_{a+1}$ for every $i > a$, and $x_{a+1} - x_a \geq 1$. Let $T_a = \inf\{t \geq 0 \,:\, x_{a+1}(t) - x_a(t) > 1\}$. If $T_a < +\infty$, then there is disconnection at finite time and $x_i(t) \to \frac{1}{a}\sum_{j\leq a} x_j(T)$ if $i \leq a$ whereas $x_i(t) \to \frac{1}{N-a}\sum_{j>a} x_j(T)$ otherwise. If instead $T = \infty$, then $x_{a+1}(t) - x_a(t) \to 1^-$ as $t \to +\infty$. By the average preservation, we argue that $x_i(t) \to x_{\text{ave}}(0) - \frac{N-a}{N}$ if $i \leq a$ and $x_i(t) \to x_{\text{ave}}(0) + \frac{a}{N}$ otherwise. As the argument can be extended to $k \geq 3$ by defining $k - 1$ appropriate disconnection times, we conclude that every solution converges to a point in $F$. □

Similarly to Theorem 2, this result proves that also solutions to (1) converge to a state in which *clusters* of agents, corresponding to the limit connected components of the graph, share the same opinion.

**Remark 3.** (Weak and strong equilibria) According to the definition of Krasovskii equilibrium, Krasovskii solutions which have Krasovskii equilibria as initial conditions may leave the equilibria. For example, if $N = 2$, $\overline{x} = (1, 0) \in F$, there are two Krasovskii solutions issuing from $\overline{x}$: $x^1(t) \equiv (1, 0)$ and $x^2(t) = (1/2 + 1/2e^{-2t}, 1/2 - 1/2e^{-2t})$. In other words, the set $F$ is weakly invariant but not strongly invariant. A subset of $F$ which is strongly invariant is $\mathring{F} = \Big\{x \in \mathbb{R}^N \ : \text{ for every } (i, j) \in \mathcal{I} \times \mathcal{I}, \text{ either } x_i = x_j \text{ or } |x_i - x_j| > 1\Big\}$.

As for the continuous Hegselmann-Krause model, we define an equilibrium to be robust if no perturbation consisting in adding one agent to the configuration can cause two of the former clusters to coalesce in the resulting evolution.

**Proposition 5** (Robustness of DHK equilibria)**.** *Let $x \in F$, and when considering any pair of clusters in $x$, denote them by $A$ and $B$, having values $x_A$ and $x_B$, and cardinalities $n_A \le n_B$. Then, for the equilibrium $x \in F$ to be* robust *it is* sufficient *that $|x_B - x_A| > 2$ for every pair $A, B$, and it is* necessary *that $|x_B - x_A| > 1 + \frac{n_A}{n_B}$ for every pair $A, B$.*

*Proof.* If $|x_B - x_A| > 2$, then the added agent can only be connected to one cluster. This implies that no pair of the former clusters can merge, and proves the sufficient condition.

To prove the necessary condition, assume that for a pair of clusters, $|x_B - x_A| \le 1 + \frac{n_A}{n_B}$, and without loss of generality that $x_A < x_B$. Then, if the perturbing agent –whose value is denoted by $x_0$– is added with value $x_0(0) = x_A + \frac{n_B}{n_A}(x_B - x_A)$, then $\dot{x}_0(t) = 0$ for every $t \ge 0$, while the agents in $x_A$ (resp., in $x_B$) experience a positive (resp., negative) derivative, so that the two clusters converge into each other. □

As we have already noted about the continuous Hegselmann-Krause model, and in accordance with the findings in [3], simulated solutions typically converge to robust equilibria (cf. Figure 1).

**Remark 4** (Robustness for large populations)**.** Whenever not all clusters have the same cardinality, Proposition 5 leaves a gap between the necessary and the sufficient condition. For large groups of agents, this gap can actually be filled by showing that in the limit for $n_A \to \infty$, the condition

$$|x_B - x_A| > 1 + \frac{n_A}{n_B}$$

is both *necessary and sufficient.* Details about this more refined analysis are not included in this paper: the interested reader can find them in the report [8].

### 3.1 Krasovskii and Carathéodory Solutions

As a consequence of their definitions, the set of Krasovskii solutions may be larger than the set of solutions intended in a Carathéodory sense. We now provide an example of a solution sliding on a discontinuity surface, proving that there are Krasovskii solutions to (1) which are not Carathéodory solutions.

**Example 2** (Sliding mode)**.** Let $N = 3$ and consider a configuration $x$ in which $1 > x_2 - x_1 > 0$ and $x_3 - x_2 = 1$. Then, $x$ is on a discontinuity surface due to the disconnection between

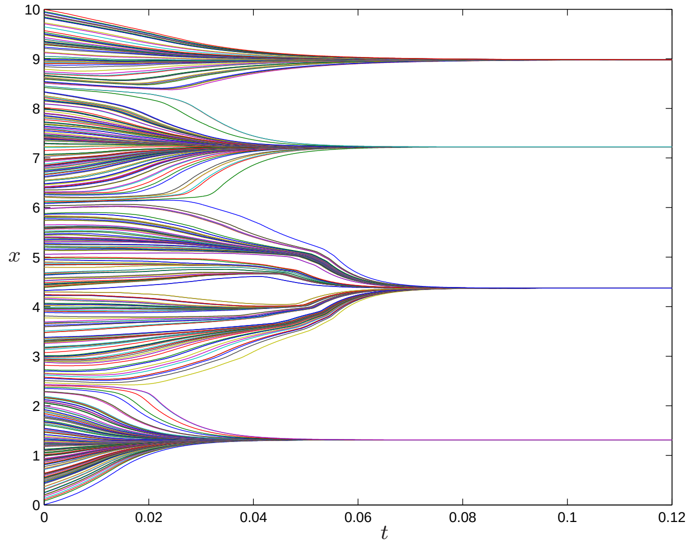


Figure 1: A sample solution to (1). The initial condition has been randomly generated from $N = 500$ independent uniform distributions, and the solution has been computed by an explicit Euler solver with time step equal to $10^{-4}$. We remark that the final clusters satisfy the necessary condition in Proposition 5.

agents 2 and 3. Then, for almost every time

$$\dot{x} \in \left\{ \alpha \begin{bmatrix} x_2 - x_1 \\ 1 + x_1 - x_2 \\ -1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 - x_1 \\ x_1 - x_2 \\ 0 \end{bmatrix} \ : \ \alpha \in [0,1] \right\}.$$

Since the normal vector to the discontinuity plane is $v_\perp = [0, -1, 1]$, we have that

$$v_\perp \cdot \dot{x} = -2\alpha + x_2 - x_1$$

is equal to zero if $\alpha = \frac{1}{2}(x_2 - x_1)$. Namely, the Krasovskii solution corresponding to such $\alpha$ does not exit the discontinuity plane $x_3 - x_2 = 1$ at time 0, but it slides on it. The sliding solution takes into account the fact that opinions $x_3$ and $x_2$ may remain for a while at the threshold distance before reaching an equilibrium configuration. □

It is an open question whether sliding mode solutions can be attractive for the dynamics. However, we know from [3] that a unique complete Carathéodory solution exists for almost every initial condition. This implies that the set of initial conditions such that the corresponding solutions converge to a sliding mode has measure zero, because Carathéodory solutions corresponding to those initial conditions would not be complete.

# 4 Comparing the Continuous and Discontinuous H-K Models

The results in Sections 2 and 3 show that continuous and discontinuous H-K models have similar qualitative properties. These similarities increase the interest for considering sequences of systems (2) which approximate a dynamic (1). Do solutions of the former system converge to a solution of the latter? The following result provides a partial answer to this question.

**Theorem 6** (Limits of solutions)**.** *Let $\{\bar{x}^n\}_{n\in\mathbb{N}}, \bar{x}^n \in \mathbb{R}^N$, let $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ be the solutions to (2) with initial condition $\bar{x}^n$. Assume that $\bar{x}^n \to \bar{x}$ and $s^n(\tau) \to s(\tau)$ for all $\tau \in \mathbb{R}$ as $n \to \infty$. Then there exists a subsequence $x^{n_k}(\cdot)$ of $x^n(\cdot)$ and a function $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^N$ such that*

*i) $x^{n_k}(\cdot) \to x(\cdot)$ pointwise for every $t \geq 0$ and uniformly on intervals $[0,T]$ for any $T > 0$;*

*ii) $x(\cdot)$ is a Carathéodory solution to (1) such that $x(0) = \bar{x}$.*

*Proof.* i) Let $T > 0$ be fixed. First of all we remark that, since the sequence $\{\bar{x}^n\}$ is convergent, then it is bounded. Let $x_m = \inf\{\bar{x}_i^n, i \in \mathcal{I}, n \in \mathbb{N}\}$ and $x_M = \sup\{\bar{x}_i^n, i \in \mathcal{I}, n \in \mathbb{N}\}$. Thanks to contractivity (Proposition 1, statement (iii)), we have that

$$x_i^n(t) \in [x_m, x_M] \ \forall t \in [0,T], \forall i \in \mathcal{I},$$

then the functions $x_i^n$ are uniformly bounded for all $i \in \mathcal{I}$. Moreover

$$|\dot{x}_i^n(t)| = |\sum_{j\in\mathcal{I}} s^n(x_j^n(t) - x_i^n(t))(x_j^n(t) - x_i^n(t))| \leq \sum_{j\in\mathcal{I}} |x_j^n(t) - x_i^n(t))| \leq N|x_M - x_m|$$

then the functions $x^n(\cdot)$ are also equicontinuous. By Ascoli-Arzelà Theorem we deduce that there exists a subsequence $x^{n_k}(\cdot)$ of $x^n(\cdot)$ and a continuous function $x : [0,T] \to \mathbb{R}^N$ such that $x^n(\cdot) \to x(\cdot)$ uniformly on $[0,T]$.

Now, we want to show that the function $x$ can be extended to $[0,+\infty)$. Since $x^{n_k}(T) \to x(T)$, we can consider the subsequence $x^{n_k}$ on the interval $[T, 2T]$ and repeat the same reasoning as in $[0,T]$. As $\mathbb{R}_{\geq 0} = [0,T] \cup (\cup_{K\in\mathbb{N}}[KT,(K+1)T])$, we argue that $x$ can be iteratively extended.

ii) We now prove that $x = (x_1, ..., x_N)$ is a Carathéodory solution to (1), i.e. for any $i \in \mathcal{I}$ and $t \in \mathbb{R}_{\geq 0}$

$$x_i(t) = \bar{x}_i + \int_0^t \sum_{j\in\mathcal{I}} s(x_j(\tau) - x_i(\tau))(x_j(\tau) - x_i(\tau))d\tau. \tag{6}$$

Let $t$ be fixed and let $T > t$. For any $i \in \mathcal{I}$ one has

$$x_i^{n_k}(t) = \bar{x}_i^{n_k} + \int_0^t \sum_{j\in\mathcal{I}} s^{n_k}(x_j^{n_k}(\tau) - x_i^{n_k}(\tau))(x_j^{n_k}(\tau) - x_i^{n_k}(\tau))d\tau.$$

Since $x_i^{n_k}$ converges to $x_i$ and $s_n(\tau) \to s(\tau)$ for all $\tau \in \mathbb{R}$, we get that

$$\sum_{j\in\mathcal{I}} s^{n_k}(x_j^{n_k}(t) - x_i^{n_k}(t))(x_j^{n_k}(t) - x_i^{n_k}(t)) \to \sum_{j\in\mathcal{I}} s(x_j(t) - x_i(t))(x_j(t) - x_i(t)) \ \forall t \in [0,T].$$

Morever for any $i \in \mathcal{I}$, $n_k$ and $t \in [0,T]$ we have that

$$|\sum_{j\in\mathcal{I}} s^{n_k}(x_j^{n_k}(t)-x_i^{n_k}(t))(x_j^{n_k}(t)-x_i^{n_k}(t))| \leq \sum_{j\in\mathcal{I}} |x_j^{n_k}(t)-x_i^{n_k}(t)| \leq N|x_M - x_m|$$

then, by Lebesgue's dominated convergence theorem, we get (6), i.e. $x = (x_1, ..., x_N)$ is a Carathéodory solution to (1) with initial condition $\bar{x}$. □

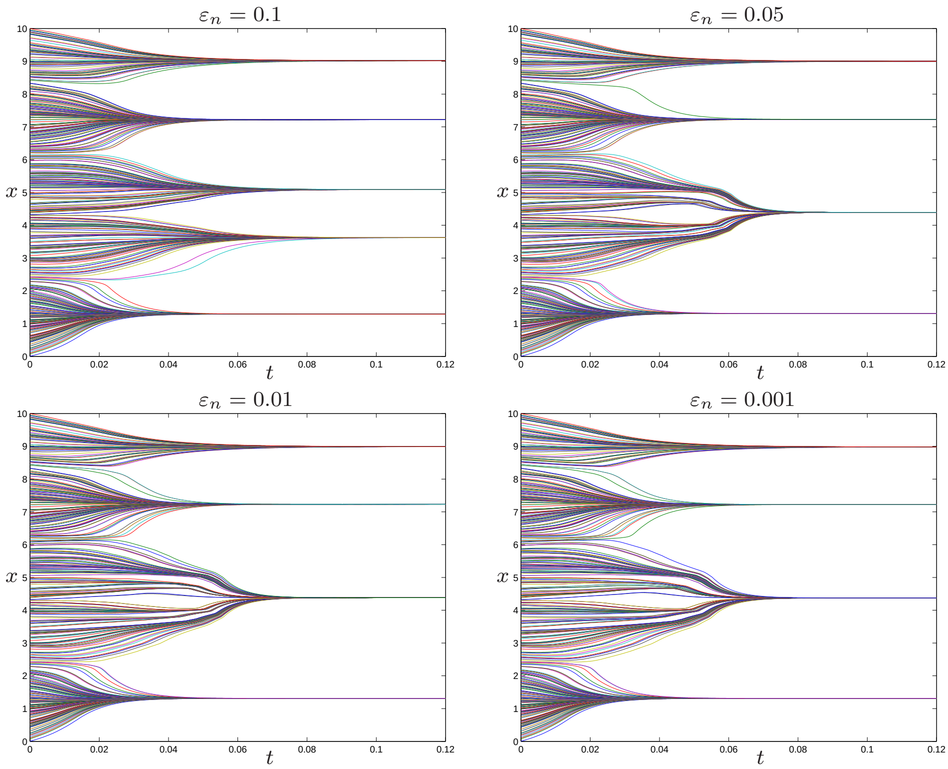


Figure 2: Sample solutions to (2) when the initial condition is the same as in Figure 1 and $s^n$ is defined as in Example 1, with different values of $\varepsilon_n$. When $\varepsilon_n = 0.001$, the evolution is qualitatively the same as the one of (1) in Figure 1. Only when $\varepsilon_n = 0.1$, the limit configuration is not a robust equilibrium, because there is pair of clusters such that $x_B - x_A \simeq 1.47$, $n_A = 87$, $n_B = 96$. This observation, considering that the necessary condition in Remark 2 becomes more restrictive as $\varepsilon_n$ decreases, gives an intuitive explanation for the different number of clusters formed by these evolutions.

Theorem 6 proves that –up to subsequences– $x^n(\cdot) \to x(\cdot)$ when $n \to \infty$. What does this property imply about the limit configurations? We know from Theorem 4 that $\lim_{t\to\infty} x(t) = x_* \in F \cap \{x : \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \bar{x}_i\}$, and from Theorem 2 that $\lim_{t\to\infty} x^n(t) = x_*^n \in F_n \cap \{x : \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \bar{x}_i^n\}$. By compactness, $x_*^n$ converges to a limit $\tilde{x}_*$ when $n \to \infty$ and

$\tilde{x}_* \in F \cap \{x : \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} \bar{x}_i\}$. Then, it is natural to ask whether $\tilde{x}_*$ and $x_*$ are equal. Simulations (cf. Figure 2) seem to support a positive answer. However, the following example shows that the answer is negative in general, because of the lack of uniform convergence on $(0, \infty)$.

**Example 3** ($t$ and $n$ limits do not commute)**.** In the framework of Example 1, let us consider systems (2) with $N = 2$,

$$s^n(t) = \begin{cases} 1 & \text{if} \quad |t| \le 1 - \epsilon_n \\ 0 & \text{if} \quad |t| \ge 1 \\ -\frac{1}{\epsilon_n}(t-1) & \text{if} \quad 1 - \epsilon_n < t < 1 \\ \frac{1}{\epsilon_n}(t+1) & \text{if} \quad -1 < t < -1 + \epsilon_n \end{cases}$$

initial conditions $\bar{x}^n$ such that $\bar{x}^n \to \bar{x}$ and $\bar{x}_2^n - \bar{x}_1^n = 1 - \delta_n$, with $0 < \delta_n < \epsilon_n$ and $\epsilon_n \to 0$ as $n \to +\infty$.

We denote $z^n = x_2^n - x_1^n$ and we have

$$\dot{z}^n = \begin{cases} 0 & \text{if} \quad z^n \ge 1 \\ \frac{2}{\epsilon_n}(z^n - 1)z^n & \text{if} \quad 1 - \epsilon_n < z^n < 1 \\ -2z^n & \text{if} \quad 0 \le z^n \le 1 - \epsilon_n \end{cases}$$

and $z^n(0) = 1 - \delta_n$. The (classical) solutions to these initial value problems are the functions

$$z^n(t) = \begin{cases} \frac{1-\delta_n}{1-\delta_n+\delta_n e^{\frac{2}{\epsilon_n}t}} & t \in [0, T^n] \\ (1-\epsilon_n)e^{-2(t-T^n)} & t \ge T^n \end{cases}$$

where $T^n = \frac{\epsilon_n}{2} \log \frac{\epsilon_n(1-\delta_n)}{(1-\epsilon_n)\delta_n}$ is such that $z^n(T^n) = 1 - \epsilon_n$. If we choose $\delta_n = e^{-\epsilon_n^{-2}}$, we have that if $n \to +\infty$, then $T^n \to +\infty$ and $z^n(t)$ tends to $z(t) \equiv 1$ pointwise and uniformly on any interval $[0, T]$. On the other hand we remark that $\lim_{t\to+\infty} z^n(t) = 0$ for any $n$. In terms of the coordinates $x_1, x_2$, this means that for each solution $x^n$ of the approximating system, $\lim_{t\to+\infty} x^n(t) = (\frac{\bar{x}_1^n+\bar{x}_2^n}{2}, \frac{\bar{x}_1^n+\bar{x}_2^n}{2})$, while the Carathéodory solution $x(t) \equiv (\bar{x}_1, \bar{x}_2)$, which is the uniform limit of $x^n$ on any interval $[0, T]$, is clearly such that $\lim_{t\to+\infty} x(t) = (\bar{x}_1, \bar{x}_2)$. Of course, as $n \to \infty$ we have that $(\frac{\bar{x}_1^n+\bar{x}_2^n}{2}, \frac{\bar{x}_1^n+\bar{x}_2^n}{2}) \to (\frac{\bar{x}_1+\bar{x}_2}{2}, \frac{\bar{x}_1+\bar{x}_2}{2})$, which is not equal to $(\bar{x}_1, \bar{x}_2)$. □

We conclude that a sequence of solutions to the continuous approximating H-K models may not behave well asymptotically. More precisely a sequence of solutions $x^n(\cdot)$ to (2) may converge to a Carathéodory solution $x(\cdot)$ to (1) but their limits in time may fail to converge to the limit in time of $x(\cdot)$.

# 5 Conclusions

In this paper, we have developed a rigorous and complete analysis of the continuous-time Hegselmann-Krause model and of a class of approximating systems with a continuous right-hand side. The presented results suggest that either approach can be used in modeling opinion dynamics, leading to similar qualitative conclusions. We leave open a few further mathematical questions about the relationship between DHK and its continuous approximations. For instance, Theorem 6 implies that some Carathéodory solutions to DHK are (pointwise) limits of solutions to CHK. This fact asks for a characterization of the Carathéodory (Krasovskii) solutions which can be obtained as limits of solutions to (2). Moreover, one might consider different approximating systems, with $s^n(\cdot)$ converging to $s(\cdot)$ in a weaker sense. May solutions to such systems approximate all Carathéodory (Krasovskii) solutions to (2)?

Besides the interest of the technical problems, in the big picture we intend this paper as a contribution to modeling opinion dynamics. We have indeed compared, in the case study of the Hegselmann-Krause model, two methods for dealing with (opinion) dynamics in which the bounded confidence constraint is modeled by a discontinuity. One method consists of smoothing out the discontinuity and considering a (sequence of) suitable continuous approximation(s), whereas the other consists of undertaking the discontinuity by a Krasovskii differential inclusion. The former method has the advantage of avoiding some mathematical difficulties, and leads to the same qualitative picture as the original discontinuous counterpart, although when it comes to compare sample solutions to the two systems, the relationships are far from being trivial. On the other hand, the choice of the best approximations can be tricky from the point of view of modeling. The latter method, instead, preserves the discontinuous definition, which is appealing and intuitive, but requires more advanced analytical tools and challenges to give an interpretation of multiple and sliding-mode solutions. Moreover, the Krasovskii differential inclusion has the feature of inherently smoothing the discontinuity by taking a convex combination of possible actions: how this can be interpreted in opinion dynamics is a question which we leave to the application-oriented reader.

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