Computing control invariant sets is easy

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***Abstract*—In this paper we consider the problem of computing control invariant sets for linear controlled systems with constraints on the input and on the states. We focus in particular on the complexity of the computation of the N-step operator, given by the Minkowski addition of sets, that is the basis of many of the iterative procedures for obtaining control invariant sets. Set inclusions conditions for control invariance are presented that involve the N-step sets and are posed in form of linear programming problems. Such conditions are employed in algorithms based on LP problems that allow to overcome the complexity limitation inherent to the set addition and can be applied also to high dimensional systems. The efficiency and scalability of the method are illustrated by computing in less than two seconds an approximation of the maximal control invariant set, based on the 15-step operator, for a system whose state and input dimensions are 20 and 10 respectively.**

I. Introduction

Invariance and contractivity of sets are central properties in modern control theory. For a dynamical system, a set is invariant if the trajectories starting within the set remain in it. For controlled systems, if the state can be maintained by an admissible input in the set, then it is referred as control invariant. Although the first important results on invariance date back to the beginning of the seventies [4], this topic gained considerable interest in the recent years, mainly due to its relation with constrained control and popular optimization-based control techniques as Model Predictive Control. The existence of an invariant set to be imposed as terminal constraint is, in fact, an essential ingredient to assure recursive feasibility and constraints satisfaction for many classical MPC control strategies [20] as well as more recent techniques [1].

The study of invariance and set theory methods for control gained interest also thanks to the foundational works by Blanchini and coauthors [6], [8], [9]. In these works, results are provided that proves that the existence of polyhedral Lyapunov functions, and then of contractive polytopes, are necessary and sufficient for stability of parametric uncertain linear systems [21], [7]. Moreover, iterative procedures are given for the computation of control invariant sets that permit their practical implementation. Most of those procedures are substantially based on the one-step backward operator that associates to any set the states that can be steered in it by an admissible input, for every possible realizations of the eventual uncertainty. Different algorithms based on the one-step operator exist for computing control invariants, that substantially differs from the initial set. For instance, if the algorithm are initialized with the state constraints set, [6], [17], [25], the one-step operator generates a sequence of outer approximations of the maximal control invariant that converges to it under compactness assumptions, see [4]. Nevertheless, the finite determination of the algorithm, that is ensured for autonomous systems [18], cannot be assured in general in presence of control input. If, instead, the procedure is initialized with a control invariant set, a non-decreasing sequence of control invariant sets are obtained that converges from the inside to the maximal control invariant set, see the considerations on minimum-time ultimate boundedness problem in [9]. A particular case of the latter approach, that needs no preliminary knowledge of a control invariant set, suggests to initialize the procedure with the set containing the origin only (which is a control invariant in the general framework), obtaining the sequence of *i*-step null-controllable sets, that are control invariant and converges to the maximal control invariant set, see [15], [12].

Thus, although the abstract iterative procedures for obtaining control invariant sets apply also for nonlinear systems, and some constructive results are given [13], [14], the practical computation of the one-step set, that is the basis for them, is often prohibitively complex for their application in high dimension even in the linear context. A common solution to circumvent this major practical issue has been fixing the sets complexity to get conservative but more computationally affordable results. For instance, by considering linear feedback and ellipsoidal control invariant sets, see the monograph [11], or by fixing the polyhedral set complexity [10], [2], [26].

In this paper we address the main problem related to the complexity of the N-step operator, for discrete-time deterministic controlled systems, with polyhedral constraints on the input and on the state. Considering polyhedral sets, such operator can be expressed in terms of Minkowski sum of polyhedra and then as an NP-complete problem [27], hardly manageable in high dimension. An algorithm is presented for determining control invariant sets that is based on a set inclusion condition involving the N-step set of a polyhedron but does not require to explicitly compute the Minkowski sum. Such condition is posed as an LP feasibility problem, then solvable even in high dimension. Once the condition is satisfied, the control invariant set is given by the convex hull of several k-step sets that can be represented through a set of linear equalities and inequality. A second algorithm, based on the previous results on Minkowski sum and convex hull, is also given. The methods, consisting in solving LP problems, are proved to be applicable to high dimensional systems. Examples that show the low conservatism and the high scalability of the approach are provided.

Notations: Denote with $\mathbb{R}_+$ the set of nonnegative real numbers. Given $n \in \mathbb{N}$, define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. The i-th element of a finite set of matrices or vectors is

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denoted as A_i. Using the notation from [24], given a mapping $M:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$, its inverse mapping is denoted $M^{-1}:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$. If M is a single-valued linear mapping, we also denote, with slight abuse of notation, the related matrices $M \in \mathbb{R}^{n\times m}$ and, if M is invertible, $M^{-1} \in \mathbb{R}^{m\times n}$. Given $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ we use the notation $(a,b) = [a^T \ b^T]^T \in \mathbb{R}^{n+m}$. The symbol 0 denotes, besides the zero, also the matrices of appropriate dimensions whose entries are zeros and the origin of a vectorial space, its meaning being determined by the context. The symbol $\mathbf{1}$ denotes the vector of entries 1 and I the identity matrix, their dimension is determined by the context. The subset of $\mathbb{R}^n$ containing the origin only is $\{0\}$. The symbol $\oplus$ denotes the Minkowski set addition, i.e. given $C,D \subseteq \mathbb{R}^n$ then $C \oplus D = \{x+y \in \mathbb{R}^n : \ x \in C, \ y \in D\}$. To simplify the notation, the propositions involving the existential quantifier in the definition of sets are left implicit, e.g. $\{x \in A : f(x,y) \leq 0, \ y \in B\}$ means $\{x \in A : \ \exists y \in B \text{ s.t.} f(x,y) \leq 0\}$. The unit box in $\mathbb{R}^n$ is denoted $\mathscr{B}^n$.

II. Problem formulation and preliminary results

The objective of this paper is to provide a constructive method to compute a control invariant set for controlled linear systems with constraints on the input and on the state. We would like to obtain a polytopic invariant set that could be computed through convex optimization problems. The main aim is to provide a method to obtain admissible control invariant sets for high-dimensional systems, thus no complex computational operations are supposed to be allowed.

The system is given by

$$x^+ = Ax + Bu \tag{1}$$

with constraints

$$x \in X = \{y \in \mathbb{R}^n : \ Fy \leq f\}, \quad u \in U = \{v \in \mathbb{R}^m : \ Gv \leq g\}. \tag{2}$$

Assumption 1. *The sets X and U are closed, convex and contain the origin.*

Note that Assumption 1 implies $f \geq 0$ and $g \geq 0$. Most of the iterative methods for obtaining invariant sets involve the image and the preimage of linear mappings.

Remark 1. *Given a polyhedron $\Omega = \{x \in \mathbb{R}^n : \ Hx \leq h\}$, its preimage through the linear single-valued mapping $A:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$, denoted $A^{-1}\Omega$, is well defined, even if matrix A is singular. Indeed, $A^{-1}\Omega$ is the set of $x \in \mathbb{R}^n$ such that $Ax \in \Omega$ and then it is given by*

$$A^{-1}\Omega = \{x \in \mathbb{R}^n : Ax \in \Omega\} = \{x \in \mathbb{R}^n : \ HAx \leq h\},$$

while the image of Ω through A is

$$A\Omega = \{Ax \in \mathbb{R}^n : x \in \Omega\} = \{Ax \in \mathbb{R}^n : \ Hx \leq h\}.$$

Moreover, for every $\gamma \in \mathbb{R}$ one has

$$\gamma\Omega = \{\gamma x \in \mathbb{R}^n : x \in \Omega\} = \{\gamma x \in \mathbb{R}^n : \ Hx \leq h\}$$

and, defining the mapping $M:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ through the matrix $M = \gamma I$, both the image and the preimage of Ω through M are defined. That is $M\Omega = \gamma\Omega$ and

$$M^{-1}\Omega = \{x \in \mathbb{R}^n : \gamma x \in \Omega\} = \{x \in \mathbb{R}^n : \ \gamma Hx \leq h\}.$$

Note that, also in this case, $M^{-1}\Omega$ is well defined even for $\gamma = 0$: $M^{-1}\Omega = \mathbb{R}^n$ if $0 \in \Omega$ and $M^{-1}\Omega = \emptyset$ if $0 \notin \Omega$.

The one-step backward operator is defined as

$$Q(\Omega) = A^{-1}(\Omega \oplus (-BU)) = \{x \in \mathbb{R}^n : Ax = y - Bu, \ u \in U, \ y \in \Omega\} = \{x \in \mathbb{R}^n : Ax + Bu \in \Omega, \ u \in U\}$$

and provides the set of points in the state space that can be mapped into Ω by an admissible input with dynamics (1).

Considering $X = \mathbb{R}^n$, one way to obtain a control invariant set is by iterating the one-step operator starting from a given initial set Ω, compact, convex set containing the origin in its interior, and then checking whether the union of the sets obtained at iteration k contains Ω. Thus the sketch of the algorithm is:

Algorithm 1 Control invariant

Input: matrices A, B, sets Ω, U

1: $\Omega_0 \leftarrow \Omega$
2: $k \leftarrow 0$
3: **repeat**
4: $\quad \Omega_{k+1} \leftarrow A^{-1}(\Omega_k \oplus (-BU))$
5: $\quad k \leftarrow k+1$
6: **until** $\Omega \subseteq \mathrm{co}\left(\bigcup_{j=1}^{k} \Omega_j\right)$
7: $N \leftarrow k$

Output: $\Omega_\infty \leftarrow \mathrm{co}\left(\bigcup_{k=1}^{N} \Omega_k\right)$

In practice, a bound on the maximal number of iteration should be imposed to avoid an infinite loop. Considering the alternative, direct, definition of Ω_k

$$\begin{aligned} \Omega_k &= A^{-k}\left(\Omega \oplus \bigoplus_{i=0}^{k-1}(-A^iBU)\right) \\ &= \{x \in \mathbb{R}^n : \ A^kx + \sum_{i=0}^{k-1} A^iBu_{i+1} \in \Omega, \ u_i \in U \ \forall i \in \mathbb{N}_k\}, \end{aligned} \tag{3}$$

the algorithm above reduces to search, given Ω, for the minimal N such that

$$\Omega \subseteq \mathrm{co}\left(\bigcup_{k=1}^{N} \Omega_k\right) = \mathrm{co}\left(\bigcup_{k=1}^{N} A^{-k}\left(\Omega \oplus \bigoplus_{i=0}^{k-1}(-A^iBU)\right)\right). \tag{4}$$

As a matter of fact, all the N for which (4) holds, lead to a control invariant set. Moreover, if (4) is satisfied, then it is satisfied for every $K \geq N$, leading to a non-decreasing sequence of nested control invariant sets.

Thus, the algorithm computes the preimages of Ω until the stop condition (4) holds. Then all the states in Ω_∞ defined

$$\Omega_\infty = \mathrm{co}\left(\bigcup_{k=1}^{N} \Omega_k\right) \tag{5}$$

can be steered in Ω, thus in Ω_∞ itself, in N steps at most, by means of admissible controls, as proved in the following proposition.

Proposition 1. *Given Ω and Ω_j as defined in (3) if condition (4) holds for $k \in \mathbb{N}$ then the set Ω_∞ defined in (5) is control invariant for the system (1) under the constraint $u \in U$.*

Proof: Given $x \in \Omega_\infty$ we prove that there exists $u \in U$ such that $Ax+Bu \in \Omega_\infty$. From the definition (5) of Ω_∞, $x \in \Omega_\infty$ implies the existence of $x_k \in \Omega_k$ and $\lambda_k \geq 0$, with $k \in \mathbb{N}_N$, such that $x = \sum_{k=1}^N \lambda_k x_k$ and $\sum_{k=1}^N \lambda_k = 1$. Moreover, by definition of Ω_k, for every $y \in \Omega_k$ there exists $u_k(y) \in U$ such that $Ay + Bu_k(y) \in \Omega_{k-1}$, for all $k \in \mathbb{N}_N$ (and with $\Omega_0 = \Omega$). Then denoting $u_k = u_k(x_k)$ and defining $u(x) = \sum_{k=1}^N \lambda_k u_k$, one has that $u(x) \in U$ from convexity of U, and

$$
\begin{aligned}
Ax + Bu(x) &= A\sum_{k=1}^N \lambda_k x_k + B\sum_{k=1}^N \lambda_k u_k \\
&= \sum_{k=1}^N \lambda_k \left(Ax_k + Bu_k\right) \in \mathrm{co}\left(\bigcup_{k=1}^{N-1} \Omega_k \cup \Omega\right) \subseteq \Omega_\infty
\end{aligned}
$$

from condition (4). ■

This means that the set given by (5) is control invariant, in the absence of state constraints, if (4) is satisfied.

To take into account the constraints on the state $x \in X$, recall that, under Assumption 1, if Ω is a control invariant set, then also $\alpha\Omega$ is a control invariant set, in absence of state constraints. Thus a first method would consists, given a control invariant set Ω_∞ in absence of state constraints, in computing the greatest $\alpha \in [0,1]$ such that $\alpha\Omega_\infty \subseteq X$. This method, together with a less conservative one which takes explicitly into account X in the computation of Ω_∞, are illustrated in Section IV-C. Both methods are based on the results valid in absence of state constraints.

Remark 2. *The algorithm sketched above is not the standard one for obtaining a control invariant set. Usually, in fact, one should start with $\Omega = X$ and intersect the preimages with X at every iteration and then check if the inclusion $\Omega_k \subseteq \Omega_{k+1}$ holds, see [9]. This approach provides a sequence of non-increasing nested sets that are outer approximations of the maximal control invariant set and whose intersection converges to it, if X and U are compact, see [4]. Unfortunately, nevertheless, the maximal control invariant set is in general not finitely determined and the sets generated by the iteration are not control invariant. An alternative, related to the approach presented here, is to start with Ω that is already control invariant, which leads to a non-decreasing sequence of nested control invariant sets. The algorithm presented here has the benefit of not requiring the* a priori *knowledge of a control invariant set Ω, but, on the other hand, does not assure that the stop condition is satisfied at some iteration for a given Ω. A scaling procedure will be employed in order to guarantee that the stop condition holds.*

Given the initial set Ω, an alternative condition characterizing an invariant set is the following

$$
\Omega \subseteq A^{-N}\left(\Omega \oplus \bigoplus_{i=0}^{N-1}(-A^i BU)\right), \tag{6}
$$

which is equivalent to the fact that every state in Ω_N can be steered in Ω in exactly N steps.

This means that (6) implies, but is not equivalent to, (4) and the resulting invariant set would be Ω_∞ as in (5). Condition (6), which will be referred to as N-step condition in what follows, is just sufficient for (4) to hold but it does not require the computation of the convex hull of several sets at every iteration. The related algorithm follows, in which the N-step condition and the explicit representation of Ω_k (3) have been used.

Algorithm 2 N-step condition control invariant

Input: matrices A, B, sets Ω, U

$k \leftarrow 0$
repeat
  $k \leftarrow k+1$
until $\Omega \subseteq A^{-k}\left(\Omega \oplus \bigoplus_{i=0}^{k-1}(-A^i BU)\right)$
$N \leftarrow k$

Output: $\Omega_\infty \leftarrow \mathrm{co}\left(\bigcup_{k=1}^{N} \Omega_k\right)$

The main issue which impedes the application of both algorithms in high dimension is the fact that computing the Minkowski set addition is a complex operation, as it is an NP-complete problem, see [27]. Moreover the addition leads to sets whose representation complexity increases. Considering, in fact, two polytopic sets Ω and Δ, their sum has in general more facets and vertices those of Ω and Δ. Thus, the algorithm given above requires the computation of the Minkowski sum, hardly manageable in high dimension, and generates polytope with an increasing number of facets and vertices. Another source of complexity is the convex hull in (4) or (5), as the explicit computation of the convex hull is a non-convex operation whose complexity grows exponentially with the dimension, see [3].

The main objective of this paper is to design a method for testing conditions (4) and (6) by means of convex optimization problems, then applicable also to relatively high dimensional systems, for obtaining a control invariant set.

III. N-step condition for control invariance

As noticed above, a first main issue is related to check whether the sum of several polytopes contains a polytope, see the N-step stop condition (6). Then, also the fact that the convex hull computation could be required, as in condition (4), would introduce additional complexity. We consider first the N-step stop condition used in Algorithm 2 and the computation of the induced control invariant Ω_∞. The stop condition (4) of Algorithm 1 is based on these results and will be illustrated afterward.

A. Minkowski sum and inclusion

Consider the N-step condition (6), characterized by the Minkowski sum of several sets. The explicit definition of the Minkowski sum of sets could be avoided by employing its implicit representation. Indeed, given two polyhedral sets $\Gamma = \{x \in \mathbb{R}^m : Hx \in h\}$ and $\Delta = \{y \in \mathbb{R}^p : Gy \leq g\}$ and $P \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{n\times p}$ we have that $P\Gamma \oplus T\Delta = \{x \in \mathbb{R}^n : x = Py + Tz,\ Hy \leq h,\ Gz \leq g\}$. Thus, the explicit hyperplane or vertex representation of the sum can be replaced by the implicit one, given by the projection of a polyhedron in higher dimension. On the other hand, one might wonder if the stop condition $\Omega \subseteq \Omega_N$ could be checked without the explicit representation of Ω_N.

The first remark to do is that the inclusion condition is testable through a set of LP problems provided the vertices of Ω are available. Such an assumption is not very restrictive, since Ω is a design parameter that could be determined such that both the hyperplane and vertices representation should be available, a box for instance. Nevertheless, and since we are aiming at invariant sets for high dimensional systems, the use of vertices should be avoided if possible. Consider for instance, in fact, a system with $n = 20$. The unit box in $\mathbb{R}^{20}$ is characterized by 40 hyperplanes, but it has $2^{20} \simeq 10^6$ vertices. Then checking if it is contained in a set could require to solve more than a million of LP problems.

We consider then the possibility of testing whether a polyhedron is included in the sum of polyhedra by employing only their hyperplane representations and without the explicit representation of the sum of sets. The following result, based on the Farkas lemma and widely used on set theory and invariant methods for control, is useful for this purpose.

Lemma 1. *Two polyhedral sets $\Gamma = \{x \in \mathbb{R}^n : Hx \leq h\}$, with $F \in \mathbb{R}^{p\times n}$, and $\Delta = \{x \in \mathbb{R}^n : Gx \leq g\}$, with $G \in \mathbb{R}^{q\times n}$, satisfy $\Gamma \subseteq \Delta$ if and only if there exists a non-negative matrix $T \in \mathbb{R}^{q\times p}$ such that*

$$\begin{array}{l} TH = G, \\ Th \leq g. \end{array}$$

Consider now the stop condition (6), which is suitable for applying the Lemma 1, as illustrated below.

Remark 3. *The right-hand side term in (6) cannot be expressed directly as the Minkowski sum of several sets, unless A is nonsingular. In fact, given $A \in \mathbb{R}^{n\times n}$ with $\det(A) \neq 0$ and $\Gamma, \Delta \subseteq \mathbb{R}^n$ then the matrix A^{-1} is defined and thus*

$$A^{-1}\Omega = \{x \in \mathbb{R}^n : \; Ax \in \Omega\} = \{A^{-1}x \in \mathbb{R}^n : x \in \Omega\}, \quad (7)$$

which implies that

$$\begin{aligned} A^{-1}(\Gamma \oplus \Delta) &= \{x \in \mathbb{R}^n : Ax \in \Gamma \oplus \Delta\} = \{x \in \mathbb{R}^n : Ax = \\ & y + z, \; y \in \Gamma, \; z \in \Delta\} = \{x \in \mathbb{R}^n : x = A^{-1}y + A^{-1}z, \\ & y \in \Gamma, \;\; z \in \Delta\} = \{A^{-1}y + A^{-1}z \in \mathbb{R}^n : \; y \in \Gamma, \;\; z \in \Delta\} \\ &= \{y + z \in \mathbb{R}^n : \; Ay \in \Gamma, \;\; Az \in \Delta\} = A^{-1}\Gamma \oplus A^{-1}\Delta, \end{aligned}$$

since the matrix A^{-1} exists. On the contrary, if $\det(A) = 0$ then we have that $A^{-1}(\Gamma \oplus \Delta) \neq A^{-1}\Gamma \oplus A^{-1}\Delta$ in general. Indeed, considering for instance

$$\begin{array}{l} \Gamma = \{x \in \mathbb{R}^2 : \; 1 \leq x_{(1)} \leq 2, \quad -1 \leq x_{(2)} \leq 1\}, \\ \Delta = \{x \in \mathbb{R}^2 : \; -3 \leq x_{(1)} \leq -1, \quad -1 \leq x_{(2)} \leq 1\}, \end{array}$$

and $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, it follows that $A^{-1}\Gamma = A^{-1}\Delta = \emptyset$ but

$$\begin{array}{l} \Gamma \oplus \Delta = \{x \in \mathbb{R}^2 : \; -2 \leq x_{(1)} \leq 1, \quad -2 \leq x_{(2)} \leq 2\}, \\ A^{-1}(\Gamma \oplus \Delta) = \{x \in \mathbb{R}^2 : \; -2 \leq x_{(2)} \leq 2\}. \end{array}$$

The main issue for applying Lemma 1 is the fact that obtaining the explicit hyperplane representation of the set at right-hand side of (6) is numerically hardly affordable, mainly in relatively high dimension. In fact, given two polyhedra $\Gamma \subseteq \mathbb{R}^m$ and $\Delta \subseteq \mathbb{R}^p$, to determine L and l such that $P\Gamma \oplus Q\Delta = \{x \in \mathbb{R}^n : \; Lx \leq l\}$ is an NP-complete problem, see [27]. Nevertheless, a sufficient condition in form of LP feasibility problem is given below for testing if a polyhedral set Ω is contained in $P\Gamma \oplus Q\Delta$.

Proposition 2. *Consider the sets $\Omega = \{x \in \mathbb{R}^n : Hx \leq h\}$, $\Gamma = \{y \in \mathbb{R}^m : Fy \leq f\}$, $\Delta = \{z \in \mathbb{R}^p : Gz \leq g\}$ and with $H \in \mathbb{R}^{n_h \times n}$, $F \in \mathbb{R}^{n_f \times m}, G \in \mathbb{R}^{n_g \times m}$ and the matrices $P \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{n\times p}$. Then $\Omega \subseteq P\Gamma \oplus Q\Delta$ if there exist $T \in \mathbb{R}^{n_{\bar{g}} \times n_{\bar{h}}}$ and $M \in \mathbb{R}^{(n+m+p)\times(n+m+p)}$, with $n_{\bar{g}} = 2n + n_f + n_G$ and $n_{\bar{h}} = n_h + 2m + 2p$ such that*

$$\left\{ \begin{array}{l} T\bar{H} = \bar{G}M \\ T\bar{h} \leq \bar{g} \\ \left[\begin{array}{ccc} I & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc} I & 0 & 0 \end{array}\right] M \end{array} \right. \quad (8)$$

holds with

$$\bar{G} = \begin{bmatrix} I & -P & -Q \\ -I & P & Q \\ 0 & F & 0 \\ 0 & 0 & G \end{bmatrix} \in \mathbb{R}^{n_{\bar{g}}\times(n+m+p)} \quad \bar{g} = \begin{bmatrix} 0 \\ 0 \\ f \\ g \end{bmatrix} \in \mathbb{R}^{n_{\bar{g}}}. \quad (9)$$

and

$$\bar{H} = \begin{bmatrix} H & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & 0 \\ 0 & 0 & I \\ 0 & 0 & -I \end{bmatrix} \in \mathbb{R}^{n_{\bar{h}}\times(n+m+p)} \quad \bar{h} = \begin{bmatrix} h \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n_{\bar{h}}}. \quad (10)$$

Proof: The Minkowski sum of $P\Gamma$ and $Q\Delta$ has an implicit hyperplane representation given by

$$P\Gamma \oplus Q\Delta = \{x \in \mathbb{R}^n : \; x = Py + Qz, \; y \in \Gamma, \; z \in \Delta\} \subseteq \mathbb{R}^n$$

which is equivalent to the projection on $\mathbb{R}^n$ of a polyhedron in $\mathbb{R}^{n+m+p}$, that is

$$P\Gamma \oplus Q\Delta = \mathrm{proj}_x \; \Omega_\oplus \quad (11)$$

where proj_x is the projection on the subspace of x, i.e. $\mathrm{proj}_x \; \Omega_\oplus = [I \; 0 \; 0]\Omega_\oplus$, and

$$\begin{aligned} \Omega_\oplus &= \{(x, y, z) \in \mathbb{R}^{n+m+p} : \; x = Py + Qz, \; Fy \leq f, \; Gz \leq g\} \\ &= \{\bar{x} \in \mathbb{R}^{n+m+p} : \; \bar{G}\bar{x} \leq \bar{g}\} \subseteq \mathbb{R}^{n+m+p}, \end{aligned}$$

with $\bar{x} = (x, y, z) \in \mathbb{R}^{n+m+p}$ and $\bar{G}, \bar{g}$ as in (9). Thus, to prove that $\Omega \subseteq P\Gamma \oplus Q\Delta$ without computing the hyperplane representation of the set $P\Gamma \oplus Q\Delta$ is equivalent to check whether the projection of $\Omega_\oplus$ on $\mathbb{R}^n$ contains $\Omega \subseteq \mathbb{R}^n$. This is equivalent to consider the set

$$\begin{aligned} \bar{\Omega} &= \Omega \times \{0\} \times \{0\} = \{(x, y, z) \in \mathbb{R}^{n+m+p} : \; Hx \leq h, \\ & y = 0, \; z = 0\} = \{\bar{x} \in \mathbb{R}^{n+m+p} : \; \bar{H}\bar{x} \leq \bar{h}\} \subseteq \mathbb{R}^{n+m+p} \end{aligned}$$

with $\bar{x} = (x, y, z)$ and $\bar{H}, \bar{h}$ as in (10), and test if

$$\mathrm{proj}_x \bar{\Omega} \subseteq \mathrm{proj}_x \Omega_\oplus, \quad (12)$$

since $\Omega = \mathrm{proj}_x \bar{\Omega}$ and from (11). Unfortunately, condition (12) is not suitable for using Lemma 1 and then we search for a sufficient condition for (12) to hold such that the lemma can be applied directly.

Consider any linear single-valued mapping $M : \mathbb{R}^{n+m+p} \rightrightarrows \mathbb{R}^{n+m+p}$, characterized by a, possibly non-invertible, matrix $M \in \mathbb{R}^{(n+m+p)\times(n+m+p)}$, such that the value of x through M is preserved, i.e. $\mathrm{proj}_x M((x,y,z)) = x$ for all $(x,y,z) \in \mathbb{R}^{n+m+p}$. Clearly, the value of x is preserved also through the inverse mapping of M, that is $\mathrm{proj}_x M^{-1}((x,y,z)) = x$ for all $(x,y,z) \in \mathbb{R}^{n+m+p}$. This means that $\mathrm{proj}_x \Omega_\oplus = \mathrm{proj}_x M^{-1}\Omega_\oplus$ and then (12) is equivalent to

$$\mathrm{proj}_x \bar{\Omega} \subseteq \mathrm{proj}_x M^{-1}\Omega_\oplus. \tag{13}$$

Then, the existence of M preserving the x and such that

$$\bar{\Omega} \subseteq M^{-1}\Omega_\oplus \tag{14}$$

holds, is a sufficient condition for (13), and thus also for (12), to be satisfied. Notice that necessity of (14) for (13) is not straightforward, since $\mathrm{proj}_x \Gamma \subseteq \mathrm{proj}_x \Delta$ does not imply $\Gamma \subseteq \Delta$, in general.

The condition on the matrix M such that $\mathrm{proj}_x M((x,y,z)) = x$ for all $(x,y,z) \in \mathbb{R}^{n+m+p}$ is

$$\begin{bmatrix} I & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \end{bmatrix} M \tag{15}$$

and then, from Lemma 1 and Remark (1), it follows that conditions (14) and (15) are equivalent to the existence of $T \in \mathbb{R}^{n_{\bar{g}}\times n_{\bar{h}}}$ and $M \in \mathbb{R}^{(n+m+p)\times(n+m+p)}$ satisfying (8). Then (8) is a sufficient condition for $\Omega \subseteq P\Gamma \oplus Q\Delta$. ∎

Thus, the inclusion of a set in the sum of sets can be tested by solving an LP feasibility problem. This results is applied to the stop condition for control invariance.

B. N-step invariance condition as an LP problem

Consider now condition (6) with

$$\Omega = \{x \in \mathbb{R}^n : \ Hx \leq h\}, \quad U = \{u \in \mathbb{R}^m : \ Gu \leq g\} \tag{16}$$

where $H \in \mathbb{R}^{n_h \times n}$ and $G \in \mathbb{R}^{n_g \times m}$. Following the reasonings of the proof of Proposition 2, a tractable condition for the set inclusion (6) to hold is given.

Theorem 1. *Consider Ω and U as in (16), with $H \in \mathbb{R}^{n_h\times n}$ and $G \in \mathbb{R}^{n_g\times m}$, and suppose that $0 \in \Omega$ and $0 \in U$. Then the set Ω_∞ as in (5) is a control invariant set if there exist $T \in \mathbb{R}^{n_{\bar{g}}\times n_{\bar{h}}}$ and $M \in \mathbb{R}^{\bar{n}\times\bar{n}}$, with $n_{\bar{g}} = n_h + Nn_g$, $n_{\bar{h}} = n_h + 2Nm$ and $\bar{n} = n + Nm$, such that*

$$\begin{cases} T\bar{H} = \bar{G}M \\ T\bar{h} \leq \bar{g} \\ \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} M \end{cases} \tag{17}$$

hold with

$$\bar{G} = \begin{bmatrix} HA^N & HB & HAB & \dots & HA^{N-1}B \\ 0 & G & 0 & \dots & 0 \\ 0 & 0 & G & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & G \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} h \\ g \\ g \\ \dots \\ g \end{bmatrix} \tag{18}$$

where $\bar{G} \in \mathbb{R}^{n_{\bar{g}}\times\bar{n}}$ and $\bar{g} \in \mathbb{R}^{n_{\bar{g}}}$, and

$$\bar{H} = \begin{bmatrix} H & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ 0 & -I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ 0 & 0 & -I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & -I \end{bmatrix}, \quad \bar{h} = \begin{bmatrix} h \\ 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

where $\bar{H} \in \mathbb{R}^{n_{\bar{h}}\times\bar{n}}$ and $\bar{h} \in \mathbb{R}^{n_{\bar{h}}}$.

Proof: Consider condition (6), sufficient for Ω_∞ to be a control invariant set. The proof follows the lines of the one of Proposition 2. From Remark 1, the right-hand side term of (6) is given by

$$\begin{aligned} \Omega_N = A^{-N}\Bigg(\Omega \oplus \bigoplus_{i=0}^{N-1}(-A^iBU)\Bigg) &= \{x \in \mathbb{R}^n : A^N x = y - Bu_1 \\ &-ABu_2 - \ldots - A^{N-1}Bu_N, \ Hy \leq h, \ Gu_i \leq g \ \forall i \in \mathbb{N}_N\} \\ &= \{x \in \mathbb{R}^n : HA^N x + HBu_1 + HA^1Bu_2 + \ldots \\ &+HA^{N-1}Bu_N \leq h, \ Gu_i \leq g \ \forall i \in \mathbb{N}_N\} \end{aligned}$$

and then is the projection on $\mathbb{R}^n$ of the set

$$\begin{aligned} \Omega_\oplus = \{(x,u_1,u_2,\ldots,u_N) \in \mathbb{R}^{\bar{n}} : \ HA^N x + HBu_1 + HABu_2 \ldots \\ +HA^{N-1}Bu_N \leq h, \ Gu_i \leq g \ \forall i \in \mathbb{N}_N\} = \{\bar{x} \in \mathbb{R}^{\bar{n}} : \ \bar{G}\bar{x} \leq \bar{g}\}, \end{aligned}$$

with $\bar{x} = (x,u_1,u_2,\ldots,u_N) \in \mathbb{R}^{\bar{n}}$ and $\bar{G}$ and $\bar{g}$ as in (18). The set $\bar{\Omega}$ in this case would result in

$$\begin{aligned} \bar{\Omega} &= \{(x,u_1,u_2,\ldots,u_N) \in \mathbb{R}^{\bar{n}} : \ Hx \leq h, \ u_i = 0 \ \forall i \in \mathbb{N}_N\} \\ &= \{\bar{x} \in \mathbb{R}^{\bar{n}} : \ \bar{H}\bar{x} \leq \bar{h}\} \subseteq \mathbb{R}^{\bar{n}} \end{aligned}$$

with $\bar{H}$ and $\bar{h}$ as in (19). From Proposition 2, the condition (6) is satisfied if there are $T \in \mathbb{R}^{n_{\bar{g}}\times n_{\bar{h}}}$ and $M \in \mathbb{R}^{\bar{n}\times\bar{n}}$ such that (17) holds ∎

Finally, given the set Ω and U, to obtain the greatest multiple of Ω, i.e. $\Omega_\alpha = \alpha\Omega$ such that (6) holds, that is the greatest $\alpha \in \mathbb{R}$ such that

$$\alpha\Omega = \Omega^\alpha \subseteq \Omega_N^\alpha, \tag{20}$$

with

$$\Omega_k^\alpha = A^{-k}\left(\Omega^\alpha \oplus \bigoplus_{i=0}^{k-1}(-A^iBU)\right), \quad \forall k \in \mathbb{N}, \tag{21}$$

is equivalent to compute the smallest nonnegative β, with $\beta = \alpha^{-1}$, such that

$$\Omega \subseteq A^{-N}\left(\Omega \oplus \bigoplus_{i=0}^{N-1}(-A^iB\beta U)\right).$$

This consists in replacing g with βg in (18) and leads to the following LP problem in T, M and β

$$\begin{aligned} \alpha^{-1} = \beta_N = \min_{\beta\in\mathbb{R}_+} \ & \beta \\ \text{s.t.} \ & T\bar{H} = \bar{G}M \\ & T\bar{h} \leq \beta\hat{g} + \tilde{g} \\ & \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \dots & 0 \end{bmatrix} M \end{aligned} \tag{22}$$

with $\hat{g}=(0,\ g,\ g,\ \ldots,\ g)$ and $\tilde{g}=(h,\ 0,\ 0,\ \ldots,\ 0)$, sufficient for the N-step invariant condition

$$\beta_N^{-1}\Omega \subseteq A^{-N}\left(\beta_N^{-1}\Omega \oplus \bigoplus_{i=0}^{N-1}(-A^iBU)\right) \quad \text{with} \quad \alpha=\beta_N^{-1},$$

to hold. Note that using directly α would yield to replacing h by αh in (18) and (19) and then to a nonlinear optimization problem.

IV. Control invariant set and state constraints

If the stop condition (6) is satisfied after appropriately scaling Ω, i.e. with $\Omega=\Omega^\alpha$ satisfying (22), the set Ω_N^α is such that if $x\in\Omega_N^\alpha$ then it can be steered in Ω^α in N steps by a sequence of admissible control inputs $u_i\in U$ with $i\in\mathbb{N}_N$. Recall that, until now, the constraints on the state have not been taken into account, they will in Section IV-C.

Once Ω^α is computed, one possible choice to obtain a control invariant set is considering Ω_∞ as in (3) and (5). This would require to compute the convex hull of the union of several sets, each one given by the Minkowski sum of sets, but the convex hull operation is numerically demanding.

For this, given an arbitrary collection of non-empty convex sets $\Gamma_i\subseteq\mathbb{R}^n$ with $I\in\mathbb{N}$ and $i\in\mathbb{N}_I$, note that

$$\operatorname{co}\left(\bigcup_{i\in\mathbb{N}_I}\Gamma_i\right)=\bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}}\left(\bigoplus_{i\in\mathbb{N}_I}\lambda_i\Gamma_i\right) \quad \text{and} \quad \lambda\bigoplus_{i\in\mathbb{N}_I}\Gamma_i=\bigoplus_{i\in\mathbb{N}_I}\lambda\Gamma_i,$$

see Chapter 3 in [23]. Then, provided condition (20) is satisfied and with definition of Ω and U as in (16), the invariant set is given by

$$\begin{aligned}\Omega_\infty^\alpha &= \operatorname{co}\left(\bigcup_{k=1}^{N}\Omega_k^\alpha\right)=\bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}}\left(\bigoplus_{k=1}^{N}\lambda_k\Omega_k^\alpha\right)\\ &= \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}}\left(\bigoplus_{k=1}^{N}\lambda_kA^{-k}\left(\Omega^\alpha\oplus\bigoplus_{i=0}^{k-1}(-A^iBU)\right)\right).\end{aligned} \tag{23}$$

Before proceeding, it is essential to notice that, given a convex set Ω, the set $\gamma A^{-1}\Omega$ is well defined for all $\gamma\in\mathbb{R}$ and $A\in\mathbb{R}^{n\times n}$, even for $\gamma=0$ and singular matrices A. In fact, it is given by

$$\gamma A^{-1}\Omega=\{\gamma x\in\mathbb{R}^n:\ x\in A^{-1}\Omega\}=\{\gamma x\in\mathbb{R}^n:\ Ax\in\Omega\}.$$

This means, for instance, that, for $n=1$, if $\gamma=0$ and $A=0$ one has $\gamma A^{-1}\Omega=\{0\}$ if $0\in\Omega$, even if $A^{-1}\Omega=\mathbb{R}$.

Lemma 2. *For every $\Omega\subseteq\mathbb{R}^n$, if $\gamma\neq 0$, with $\gamma\in\mathbb{R}$, or $\det(A)\neq 0$ then $\gamma A^{-1}\Omega=A^{-1}\gamma\Omega$.*

Proof: If $\gamma\neq 0$, one has

$$\begin{aligned}&x\in\gamma A^{-1}\Omega \ \Leftrightarrow\ x=\gamma y,\ y\in A^{-1}\Omega \ \Leftrightarrow\ x=\gamma y,\ Ay\in\Omega\\ &\Leftrightarrow\ \gamma^{-1}x=y,\ Ay\in\Omega \ \Leftrightarrow\ A\gamma^{-1}x\in\Omega \ \Leftrightarrow\ \gamma^{-1}Ax\in\Omega\\ &\Leftrightarrow\ Ax=\gamma y,\ y\in\Omega \ \Leftrightarrow\ Ax\in\gamma\Omega \ \Leftrightarrow\ x\in A^{-1}\gamma\Omega,\end{aligned}$$

whereas if $\det(A)\neq 0$ it follows that

$$\begin{aligned}&x\in\gamma A^{-1}\Omega\Leftrightarrow\ x=\gamma y,\ y\in A^{-1}\Omega \ \Leftrightarrow\ x=\gamma y,\ Ay\in\Omega \ \Leftrightarrow\\ &x=\gamma A^{-1}z,\ z\in\Omega \ \Leftrightarrow\ x=A^{-1}\gamma z,\ z\in\Omega \ \Leftrightarrow\ x\in A^{-1}\gamma\Omega.\end{aligned}$$

■

This means, in practice, that the operators γ and A^{-1} actuating on Ω can be switched, if either $\gamma\neq 0$ or $\det(A)\neq 0$. Note that, if $\gamma=0$ and A is singular, then the equality $\gamma A^{-1}\Omega=A^{-1}\gamma\Omega$ does not hold in general, as illustrated in the following example.

Example 1. *Consider $\lambda=0$, $A=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ and $\Omega=\mathscr{B}^n$. Then $A^{-1}\gamma\Omega=\{x\in\mathbb{R}^n: Ax\in\{0\}\}=\{x\in\mathbb{R}^n: x_{(2)}=0\}$ and $\gamma A^{-1}\Omega=\{\lambda x\in\mathbb{R}^n: -1\leq x_{(2)}\leq 1\}=\{0\}$.*

The cases of nonsingular and singular matrix A are considered individually.

A. Invariant for nonsingular A

If A is nonsingular the invariant set is the polyhedron give below.

Proposition 3. *Let Ω and U as in (16). If $\det(A)\neq 0$ then Ω_∞^α defined in (23) is equal to $\check{\Omega}_\infty^\alpha$ where*

$$\begin{aligned}\check{\Omega}_\infty^\alpha=\{x\in\mathbb{R}^n: x=\sum_{k=1}^{N}z_k,\ HA^kz_k+\sum_{i=0}^{k-1}HA^iBv_{i+1,k}\leq\lambda_k\alpha_Nh,\\ Gv_{i,k}\leq\lambda_kg\ \forall i\in\mathbb{N}_k\ \forall k\in\mathbb{N}_N,\ \lambda\geq 0,\ \sum_{k=1}^{N}\lambda_k=1\}.\end{aligned} \tag{24}$$

Proof: From (23) and Lemma 2 it follows

$$\begin{aligned}&\Omega_\infty^\alpha=\{x\in\mathbb{R}^n:\ x=\sum_{k=1}^{N}z_k,\ z_k\in\lambda_k\Omega_k^\alpha\ \forall k\in\mathbb{N}_N,\ \lambda\geq 0,\\ &\sum_{k=1}^{N}\lambda_k=1\}=\{x\in\mathbb{R}^n:\ x=\sum_{k=1}^{N}z_k,\ z_k\in A^{-k}\lambda_k\Big(\Omega^\alpha\\ &\oplus\bigoplus_{i=0}^{k-1}(-A^iBU)\Big)\forall k\in\mathbb{N}_N,\ \lambda\geq 0,\ \sum_{k=1}^{N}\lambda_k=1\}=\{x\in\mathbb{R}^n:\\ &x=\sum_{k=1}^{N}z_k,\ \ A^kz_k\in\left(\lambda_k\Omega^\alpha\oplus\bigoplus_{i=0}^{k-1}(-A^iB\lambda_kU)\right)\forall k\in\mathbb{N}_N,\\ &\lambda\geq 0,\ \sum_{k=1}^{N}\lambda_k=1\}=\{x\in\mathbb{R}^n:\ x=\sum_{k=1}^{N}z_k,\ A^kz_k=y_k\\ &-\sum_{i=0}^{k-1}A^iBv_{i+1,k},\ y_k\in\lambda_k\Omega^\alpha,\ v_{i,k}\in\lambda_kU\ \forall i\in\mathbb{N}_k\ \forall k\in\mathbb{N}_N,\\ &\lambda\geq 0,\ \sum_{k=1}^{N}\lambda_k=1\}=\{x\in\mathbb{R}^n:\ x=\sum_{k=1}^{N}z_k,\ A^kz_k=y_k-\\ &\sum_{i=0}^{k-1}A^iBv_{i+1,k},\ Hy_k\leq\lambda_k\alpha_Nh,\ Gv_{i,k}\leq\lambda_kg\ \forall i\in\mathbb{N}_k\\ &\forall k\in\mathbb{N}_N,\ \lambda\geq 0,\ \sum_{k=1}^{N}\lambda_k=1\}=\check{\Omega}_\infty^\alpha\end{aligned}$$

where the second equality holds since A is nonsingular and then λ_k and A^{-k} can be switched. ■

Note that the invariant set Ω_∞^α is then characterized by linear equalities and inequalities, that is by a polytope in higher dimension. This means that testing if a state is in Ω_∞^α reduces to solve a feasibility problem with linear constraints. Also the problem of enforcing state constraints, see Section IV-C below, can be solved through convex optimization by using the representation (24). Moreover, such a representation is particularly suitable to be used in optimization-based control, as model predictive control for instance, since it reduces to enforcing the linear constraints characterizing Ω_∞^α.

B. Invariant for singular A

In the other case, namely if A is singular, the polyhedral form of the invariant set is less straightforward.

Proposition 4. *Let Ω and U as in (16). If $\det(A)=0$ then Ω_∞^α defined in (23) is equal to $\hat{\Omega}_\infty^\alpha$ where*

$$\hat{\Omega}_\infty^\alpha = \{x\in\mathbb{R}^n : x=\sum_{k=1}^N \lambda_k w_k,\, HA^k w_k + \sum_{i=0}^{k-1} HA^i B v_{i+1,k} \leq \alpha_N h, \\ G v_{i,k} \leq g \ \forall i\in\mathbb{N}_k \forall k\in\mathbb{N}_N,\ \lambda\geq 0,\ \sum_{k=1}^N \lambda_k = 1\}. \tag{25}$$

Proof: The set Ω_∞^α is given by

$$\Omega_\infty^\alpha = \{x\in\mathbb{R}^n:\ x=\sum_{k=1}^N z_k,\ z_k\in\lambda_k\Omega_k^\alpha\ \forall k\in\mathbb{N}_N, \lambda\geq 0, \\ \sum_{k=1}^N \lambda_k = 1\} = \{x\in\mathbb{R}^n:\ x=\sum_{k=1}^N z_k,\ z_k\in\lambda_k A^{-k}\Bigg(\Omega^\alpha \\ \oplus\bigoplus_{i=0}^{k-1}(-A^iBU)\Bigg)\ \forall k\in\mathbb{N}_N,\ \lambda\geq 0, \sum_{k=1}^N\lambda_k=1\} = \{x\in\mathbb{R}^n: \\ x=\sum_{k=1}^N z_k,\ z_k=\lambda_k w_k,\ w_k\in A^{-k}\left(\Omega^\alpha\oplus\bigoplus_{i=0}^{k-1}(-A^iBU)\right) \\ \forall k\in\mathbb{N}_N, \lambda\geq 0,\ \sum_{k=1}^N\lambda_k=1\} = \{x\in\mathbb{R}^n:\ x=\sum_{k=1}^N\lambda_k w_k, \\ A^k w_k = y_k - \sum_{i=0}^{k-1} A^iBv_{i+1,k},\ y_k\in\Omega^\alpha,\ v_{i,k}\in U\ \forall i\in\mathbb{N}_k \\ \forall k\in\mathbb{N}_N, \lambda\geq 0, \sum_{k=1}^N\lambda_k=1\} = \{x\in\mathbb{R}^n:\ x=\sum_{k=1}^N\lambda_k w_k, \\ A^k w_k = y_k - \sum_{i=0}^{k-1} A^iBv_{i+1,k},\ Hy_k\leq\alpha_N h,\ Gv_{i,k}\leq g\ \forall i\in\mathbb{N}_k \\ \forall k\in\mathbb{N}_N, \lambda\geq 0, \sum_{k=1}^N\lambda_k=1\} = \hat{\Omega}_\infty^\alpha.$$

■

Unfortunately, this representation of Ω_∞^α is not suitable to be directly tested through an LP feasibility problem, due to the nonlinearities $\lambda_k w_k$. This means that checking whether a state is contained in $\hat{\Omega}_\infty^\alpha$ could not be solved through LP problems, as for nonsingular A. Neither the problem of computing the biggest copy of $\hat{\Omega}_\infty^\alpha$ satisfying the state constraints (treated in Section IV-C, see (34)) could be addressed by convex optimization problems.

What we are going to prove is that the expression of Ω_∞^α as in (24) holds also when A is a singular matrix, that is $\Omega_\infty^\alpha = \hat{\Omega}_\infty^\alpha = \check{\Omega}_\infty^\alpha$. For notational simplicity we define

$$\bar{\Omega}_k = \Omega^\alpha \oplus \bigoplus_{i=0}^{k-1}(-A^iBU), \tag{26}$$

so that $\Omega_k^\alpha = A^{-k}\bar{\Omega}_k$, for all $k\in\mathbb{N}_N$. Then the sets $\check{\Omega}_\infty^\alpha$ and $\hat{\Omega}_\infty^\alpha$ defined in (24) and (25) become

$$\begin{aligned}\check{\Omega}_\infty^\alpha &= \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}}\left(\bigoplus_{k=1}^N A^{-k}\lambda_k\bar{\Omega}_k\right),\\ \hat{\Omega}_\infty^\alpha &= \mathrm{co}\left(\bigcup_{k=1}^N A^{-k}\bar{\Omega}_k\right) = \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}}\left(\bigoplus_{k=1}^N \lambda_k A^{-k}\bar{\Omega}_k\right),\end{aligned} \tag{27}$$

which are equal if A is nonsingular. We prove that they are equal also for singular A. For this aim, some preliminary results are to be recalled or introduced here.

Definition 1. *[5] Given a nonempty convex set C, the vector d is a direction of recession of C if $x+\alpha d\in C$ for all $x\in C$ and $\alpha\geq 0$. The set of all directions of recession is a cone containing the origin, called the recession cone of C. The lineality space of C, denoted L_C, is the set of directions of recession d whose opposite, $-d$, are also directions of recession. Given a subspace S, $S^\perp$ is its orthogonal complement.*

Theorem 2. *[23] A subset of $\mathbb{R}^n$ is a convex cone if and only if it is closed under addition and positive scalar multiplication.*

Theorem 3. *[23] If K_1 and K_2 are convex cones containing the origin then $K_1\oplus K_2 = co(K_1\cup K_2)$.*

Lemma 3. *Given the subspaces $S_1, S_2\subseteq\mathbb{R}^n$, we have $S_1 = S_1\oplus S_1$ and $S_1\oplus S_2 = co(S_1\cup S_2)$.*

Proof: It follows from Theorems 2 and 3 and the fact that every subspace is a convex cone containing the origin. ■

Proposition 5. *(Decomposition of a Convex Set [5]) Let C be a nonempty convex subset of $\mathbb{R}^n$. Then, for every subspace S that is contained in the lineality space L_C, we have $C = (C\cap S^\perp)\oplus S$.*

Lemma 4. *Given the nonempty convex set $C\subseteq\mathbb{R}^n$, for every subspace $S\subseteq C$, we have $C\oplus S = C$.*

Proof: From Proposition 5 and Lemma 3, and since $S\subseteq L_C$, it follows that $C\oplus S = (C\cap S^\perp)\oplus S\oplus S = (C\cap S^\perp)\oplus S = C$. ■

Finally, given $K\subseteq\mathbb{N}_N$ and defined $\bar{K} = \mathbb{N}_n/K$ and

$$\Lambda(K) = \{\lambda\in\mathbb{R}^n:\ \lambda_k > 0\ \forall k\in K,\quad \lambda_k = 0\ \forall k\in\bar{K}\}$$

(note that λ_k is strictly positive if and only if $k\in K$) one has

$$\begin{aligned}&\{\lambda\in\mathbb{R}^n:\ \lambda_k\geq 0\ \forall k\in\mathbb{N}_n,\ \mathbf{1}^T\lambda = 1\}\\ &\quad = \bigcup_{K\subseteq\mathbb{N}_N}\{\lambda\in\Lambda(K) : \mathbf{1}^T\lambda = 1\}\end{aligned} \tag{28}$$

where K denotes the set of indices such that λ_k is not zero, in practice. In fact, for every λ in the l.h.s. of (28), there exists a K, that is the set of indices for which $\lambda_k > 0$, such that $\lambda\in\Lambda(K)$. Analogously, every λ in the r.h.s. of (28), also satisfies $\lambda \geq 0$ and then it is contained in the l.h.s. set. Equality (28) implies that

$$\begin{aligned}
\check{\Omega}_\infty^\alpha = & \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N A^{-k}\lambda_k\bar{\Omega}_k\right) = \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} A^{-k}\lambda_k\bar{\Omega}_k \right.\\
& \left. \oplus \bigoplus_{k\in\bar{K}} A^{-k}\lambda_0\bar{\Omega}_k\right),\\
\hat{\Omega}_\infty^\alpha = & \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N \lambda_k A^{-k}\bar{\Omega}_k\right) = \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} \lambda_k A^{-k}\bar{\Omega}_k \right.\\
& \left. \oplus \bigoplus_{k\in\bar{K}} \lambda_0 A^{-k}\bar{\Omega}_k\right),
\end{aligned} \tag{29}$$

with $\lambda_0 = 0$.

Lemma 5. *Given the sets $C, D, E \subseteq \mathbb{R}^n$, one has $(C \cup D) \oplus E = (C \oplus E) \cup (D \oplus E)$.*

Proof: From the definition of Minkowski sum, it follows

$$\begin{aligned}
&(C\cup D)\oplus E = \{x\in\mathbb{R}^n:\ x\in C \text{ or } x\in D\}\oplus E = \{x+y\in\mathbb{R}^n:\\
&(x\in C,\ y\in E) \text{ or } (x\in D,\ y\in E)\} = \{x+y\in\mathbb{R}^n:\ x\in C,\\
&y\in E\}\cup\{x+y\in\mathbb{R}^n: x\in D,\ y\in E)\} = (C\oplus E)\cup(D\oplus E).
\end{aligned}$$

■

Now we are in the position of proving that $\Omega_\infty^\alpha = \hat{\Omega}_\infty^\alpha$, even for singular A.

Theorem 4. *Let Ω and U as in (16) be non-empty and such that $0 \in \Omega$ and $0 \in U$. Then the sets $\check{\Omega}_\infty^\alpha$ and $\hat{\Omega}_\infty^\alpha$, defined in (27), are equal.*

Proof: If A is nonsingular, the equality follows directly from Lemma 2. Consider now the case of A singular. The sets $\lambda_k A^{-k}\bar{\Omega}_k$ and $A^{-k}\lambda_k\bar{\Omega}_k$, involved in (27), are equal for every $k \in \mathbb{N}_N$ provided $\lambda_k > 0$, from Lemma 2. On the other hand, this is no more true if $\lambda_k = \lambda_0 = 0$, in fact

$$\begin{aligned}
\lambda_0 A^{-k}\bar{\Omega}_k &= \{\lambda_0 x\in\mathbb{R}^n:\ x\in A^{-k}\bar{\Omega}_k\}\\
&= \{\lambda_0 x\in\mathbb{R}^n:\ A^k x\in\bar{\Omega}_k\} = \{0\},\\
A^{-k}\lambda_0\bar{\Omega}_k &= \{x\in\mathbb{R}^n:\ A^k x\in\lambda_0\bar{\Omega}_k\}\\
&= \Big\{x\in\mathbb{R}^n:\ A^k x\in\{0\}\Big\} = \ker(A^k),
\end{aligned} \tag{30}$$

as the set of x such that $A^k x \in \bar{\Omega}_k$ is non-empty from $0 \in \Omega$ and $0 \in U$. Moreover, for every $k \in \mathbb{N}_N$ one has

$$\begin{aligned}
\ker(A^k) &= \{x\in\mathbb{R}^n:\ A^k x\in\{0\}\}\\
&\subseteq \{x\in\mathbb{R}^n:\ A^k x\in\lambda_k\bar{\Omega}_k\} = A^{-k}\lambda_k\bar{\Omega}_k,
\end{aligned} \tag{31}$$

since $0 \in \lambda_k\bar{\Omega}_k$ for every $\lambda_k \geq 0$, from $0 \in \Omega$ and $0 \in U$. Inclusion (31) with $\lambda_k = 1$ implies $\bigcup_{k=1}^N \ker(A^k) \subseteq \bigcup_{k=1}^N A^{-k}\bar{\Omega}_k$ and then

$$\bigoplus_{k=1}^N \ker(A^k) = \mathrm{co}\left(\bigcup_{k=1}^N \ker(A^k)\right) \subseteq \mathrm{co}\left(\bigcup_{k=1}^N A^{-k}\bar{\Omega}_k\right) = \hat{\Omega}_\infty^\alpha, \tag{32}$$

where the first equality follows from Lemma 3 and the fact that $\ker(A^k)$ are subspaces. Moreover, (31) yields

$$\bigoplus_{k=1}^N \ker(A^k) = \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N \ker(A^k)\right) \subseteq \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N A^{-k}\lambda_k\bar{\Omega}_k\right) = \Omega_\infty^\alpha. \tag{33}$$

Then, denoting the value $\lambda_0 = 0$, one has

$$\begin{aligned}
\hat{\Omega}_\infty^\alpha &= \mathrm{co}\left(\bigcup_{k=1}^N A^{-k}\bar{\Omega}_k\right) = \mathrm{co}\left(\bigcup_{k=1}^N A^{-k}\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N \lambda_k A^{-k}\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} \lambda_k A^{-k}\bar{\Omega}_k \oplus \bigoplus_{k\in\bar{K}} \lambda_0 A^{-k}\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} \lambda_k A^{-k}\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} A^{-k}\lambda_k\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} A^{-k}\lambda_k\bar{\Omega}_k \oplus \bigoplus_{k=1}^N \ker(A^k)\right)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} A^{-k}\lambda_k\bar{\Omega}_k \oplus \bigoplus_{k=1}^N \ker(A^k) \oplus \bigoplus_{k\in\bar{K}} \ker(A^k)\right)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\left(\bigoplus_{k\in K} A^{-k}\lambda_k\bar{\Omega}_k \oplus \bigoplus_{k\in\bar{K}} \ker(A^k)\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\right)\\
&= \bigcup_{K\subseteq\mathbb{N}_N} \bigcup_{\substack{\lambda\in\Lambda(K)\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k\in K} A^{-k}\lambda_k\bar{\Omega}_k \oplus \bigoplus_{k\in\bar{K}} A^{-k}\lambda_0\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N A^{-k}\lambda_k\bar{\Omega}_k\right) \oplus \bigoplus_{k=1}^N \ker(A^k)\\
&= \bigcup_{\substack{\lambda\geq 0\\ \mathbf{1}^T\lambda=1}} \left(\bigoplus_{k=1}^N A^{-k}\lambda_k\bar{\Omega}_k\right) = \check{\Omega}_\infty^\alpha
\end{aligned}$$

where: the second equality holds from Lemma 4 and (32); the forth from (29); the fifth from (30); the sixth from Lemma 2; the seventh from Lemma 5; the eighth from Lemma 4; the tenth from (30); the eleventh from (29); the twelfth and the last one from Lemma 4 and (33). ■

Theorem 4 implies that checking if $x \in \Omega_\infty^\alpha$ resorts to solve an LP feasibility problem in the variables $x, z_k, v_{i,k}, \lambda_k$ for all $i \in \mathbb{N}_k$ and $k \in \mathbb{N}_N$, then in a space of dimension $n + Nn + 0.5N(N+1)m + N$.

Remark 4. *From Theorem 4, also the stop condition (4), employed in Algorithm 1, can be posed as an LP problem, once α_N is fixed. In fact, by reasonings analogous to those of Theorem 1, the inclusion $\alpha_N\Omega \subseteq \Omega_\infty^\alpha$ can be posed in form of* *the LP problem (17), by appropriately defining the matrices $\bar{G}$, $\bar{g}$, $\bar{H}$, $\bar{h}$ from (24). Such an LP problem could be also used to approximate the optimal α_N, by griding it for instance. This would also have the benefit of leading to smaller values of N, since the the stop condition (4) holds if the N-step one (6) is satisfied, but the inverse is not true in general. On the other hand, the dimension of such an LP problem might be much bigger than for the N-step condition, in fact: $\bar{n}$ would be $(1+N)n+0.5N(N+1)m+N$ instead of $n+Nm$ defined for the N-step; $n_{\bar{g}} = 2n+Nn_h+0.5N(N+1)n_g+N+1$ instead of n_h+Nn_g and $n_{\bar{h}} = n_h+2nN+N(N+1)m+2N$ instead of n_h+2Nm.*

C. State constraints

Concerning the state constraints, recall first that every smaller multiple of Ω_∞^α, i.e. every $\sigma\Omega_\infty^\alpha$ with $\sigma \in [0,1]$, is still control invariant, under Assumption 1 and if $0 \in \Omega$. Then the greatest invariant multiple of Ω_∞^α contained in X is given by $\Omega_\infty^{\alpha,\sigma} = \sigma\Omega_\infty^\alpha$ with

$$\begin{aligned}\sigma = \ & \max_{\delta\in[0,1]} \delta \\ & \text{s.t.} \quad \delta\Omega_\infty^\alpha \subseteq X,\end{aligned}$$

which is equivalent, for X as in (2), to

$$\sigma = \min\left\{1, \frac{f_1}{\delta_1}, \ldots, \frac{f_{n_f}}{\delta_{n_f}}\right\} \quad \text{with} \quad \begin{aligned}\delta_i = \ & \max_x F_i x \\ & \text{s.t.} \quad x \in \Omega_\infty^\alpha,\end{aligned} \tag{34}$$

for every $i \in \mathbb{N}_{n_f}$. Then the scaling factor σ can be obtained by computing δ_i solving (34), which are n_f LP problems, in both cases of A singular and nonsingular, from Theorem 4. In fact, the constraint $x \in \Omega_\infty^\alpha$ is a set of linear constraints in a space of dimension $(1+N)n+0.5N(N+1)m+N$, see (24) and Remark 4.

The method presented above does not explicitly take into account the shape of X in computing $\Omega_\infty^{\alpha,\sigma}$, and then could lead to some conservatism. Alternatively, the state constraints could be considered by defining the analogous of the preimage set Ω_k^α as in (21). Given the constraint sets $X \subseteq \mathbb{R}^n$ and $U \in \mathbb{R}^m$ and $\sigma \in \mathbb{R}_+$, the set defined by

$$\begin{aligned}&\Omega_k^\sigma(X,U) = \{x \in \mathbb{R}^n: \ A^k x + \sum_{i=1}^k A^{i-1}Bu_i \in \sigma\Omega, \\ &\quad A^{k-1}x + A^{k-2}Bu_k + \ldots + Bu_2 \in X, \ \ldots \ Ax + Bu_k \in X, \\ &\quad x \in X, \ u_i \in U \ \forall i \in \mathbb{N}_k\}\end{aligned} \tag{35}$$

is the set of initial states $x \in X$ for which a sequence of length k of input $u_i \in U$, with $i \in \mathbb{N}_k$, exists such that the generated trajectory is maintained in X and ends in $\sigma\Omega$ at time k. Hence, the set given by

$$\Omega_\infty^\sigma(X,U) = \mathrm{co}\left(\bigcup_{k=1}^N \Omega_k^\sigma(X,U)\right)$$

is the set of states that can be steered in $\sigma\Omega$ in N steps at most through an admissible trajectory that does not violate the state constraints X, if $\sigma\Omega \subseteq \Omega_\infty^\sigma(X,U)$. Hence, solving the problem

$$\begin{aligned}\hat{\sigma} = \ & \max_{\sigma\in\mathbb{R}_+} \sigma \\ & \text{s.t.} \quad \sigma\Omega \subseteq \Omega_\infty^\sigma(X,U),\end{aligned} \tag{36}$$

leads to the control invariant set $\Omega_\infty^{\hat{\sigma}}(X,U)$ contained in X. The problem (36) does not yield to a convex problem, as for the sufficient condition (20) and the considerations that follow it. Nevertheless, the following problem

$$\begin{aligned}\mu_N = \ & \min_{\mu\in\mathbb{R}_+} \mu \\ & \text{s.t.} \quad \Omega \subseteq \Omega_\infty^1(\mu X, \mu U),\end{aligned} \tag{37}$$

leads to an LP analogous to (22) and equivalent to (36), with $\mu = \sigma^{-1}$, since it can be proved that $\sigma^{-1}\Omega_k^\sigma(X,U) = \Omega_k^1(\mu X, \mu U)$ for all $k \in \mathbb{N}$.

Finally, note that $\Omega_k^1(\mu X, \mu U)$ as in (35) is the projection on $\mathbb{R}^n$ of a polytope on a space of dimension $n+km+1$. Then the constraint $\Omega \subseteq \Omega_\infty^1(\mu X, \mu U)$ would lead to linear constraints analogous to those of Theorem 1 but with $\bar{n} = (1+N)n + 0.5N(N+1)m+N$, $n_{\bar{g}} = 2n+Nn_h+0.5N(N+1)n_g+N+1$ and $n_{\bar{h}} = n_h+2nN+N(N+1)m+2N$, see also Remark 4. The solution of the lower dimensional optimization problem

$$\begin{aligned}\mu_N = \ & \min_{\mu\in\mathbb{R}_+} \mu \\ & \text{s.t.} \quad \Omega \subseteq \Omega_N^1(\mu X, \mu U),\end{aligned} \tag{38}$$

leads to a more conservative control invariant set contained in X. Note that (38) yields to an LP problem analogous of the N-step condition in absence of state constraints (22).

V. Numerical examples

The different results presented in this paper are illustrated through numerical examples.

A. Example 1

The first example concerns a system with $n = 2$ and $m = 1$. The main interest relies in fact that both the Minkowski sum and the convex hull can be computed efficiently in this low dimensional system, using for instance the MPT toolbox for managing polytopes, [16]. This would allow us to explicitly compute outer approximations of the maximal invariant set and the sets Ω_N^α and Ω_∞^α and then to give a graphical illustration of our results in terms of conservatism.

We consider the systems (1) with

$$A = \begin{bmatrix} 1.2 & 1 \\ 0 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} \tag{39}$$

and constraints on the input $U = \{u \in \mathbb{R}: \|u\| \leq 2\}$. We consider first no constraints in the state. The initial set Ω has been chosen to be the unitary box, i.e. $\Omega = \mathscr{B}^2$. Then the maximal value of α such that sets Ω^α and Ω_N^α satisfy (20) is obtained for different values of N, by solving (22). Given such α, the set Ω_∞^α, defined in (23), is a control invariant set. To give an intuition of the method results and of the conservatism with respect to the maximal control invariant, a sequence of non-increasing nested outer approximations of the maximal control invariant set is computed, by starting with Σ_0 big enough (i.e. containing the maximal control invariant set) and iterating $\Sigma_k = \Sigma_0 \cap A^{-1}(\Sigma_k \oplus (-BU))$, [9]. The sets Σ_0 and Σ_{10k} for $k \in \mathbb{N}_5$ are depicted in Figure 1 in thin lines while Σ_{60} is the white polytope with thick borders. The set Ω^α is the dark-gray box and both Ω_N^α and Ω_∞^α are represented in light gray, for $N=5$. As can be noticed, the sets Ω_N^α and Ω_∞^α are very close, where clearly $\Omega_N^\alpha \subseteq \Omega_\infty^\alpha$.

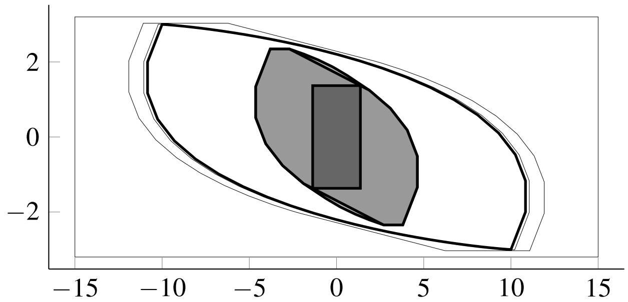

Fig. 1. Sets Σ_0 and Σ_{10k} for $k \in \mathbb{N}_5$ in thin lines; Σ_{60} in white with thick lines; Ω^α in dark gray; Ω_N^α and Ω_∞^α in light gray, for $N=5$.

The sets Σ_{60} and Ω_∞^α for $N=5,10,15,20$ are drawn in Figure 2, in white the former and gray the latter.

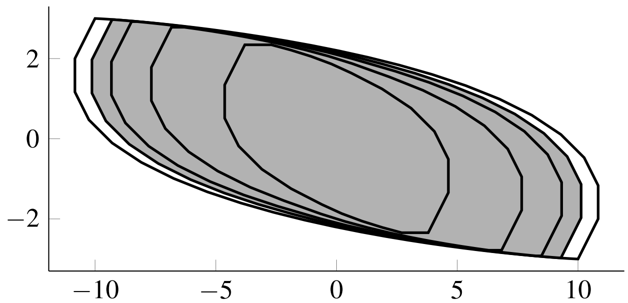

Fig. 2. Set Σ_{60} in white with thick lines and Ω_∞^α in light gray, for $N=5$ (inner), $N=10,15$ and $N=20$ (outer).

Finally, the sets Σ_0 and Σ_{10k} for $k \in \mathbb{N}_6$ are depicted in Figure 1, in white, together with Ω^α, in dark gray, and Ω_∞^α, in light gray, for $N=40$. The set Ω_∞^α is very close to the outer approximation of the maximal control invariant set Σ_{60}.

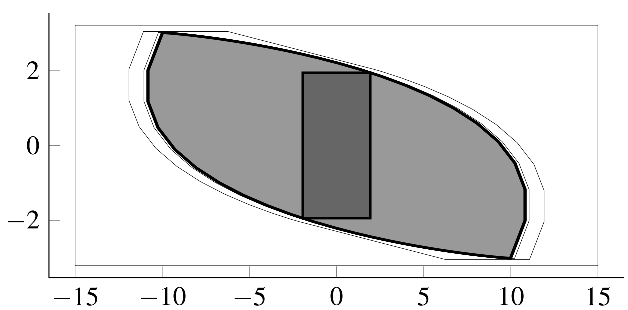

Fig. 3. Sets Σ_0 and Σ_{10k} for $k \in \mathbb{N}_5$ in thin lines; Σ_{60} in white with thick lines; Ω^α in dark gray; Ω_N^α and Ω_∞^α in light gray, for $N=40$.

B. Example 2: singular matrix

Consider the systems (1) with singular transition matrix

$$A = \begin{bmatrix} 1.2 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix} \tag{40}$$

and input constraints sets and initial set as for Example 1, $U = \{u \in \mathbb{R} : \|u\| \leq 2\}$ and $\Omega = \mathscr{B}^n$. The sets Σ_i for $i \in \mathbb{N}_{60}$ have been computed starting with $\Sigma_0 = 1000\mathscr{B}^2$. Figure 4 shows the outer approximations of the maximal control invariant set Σ_i with $i=40,50,60$, the control invariant sets Ω_∞^α for different values of N, in particular $N \in \mathbb{N}_{10}$, and Ω^α related to $N=10$.

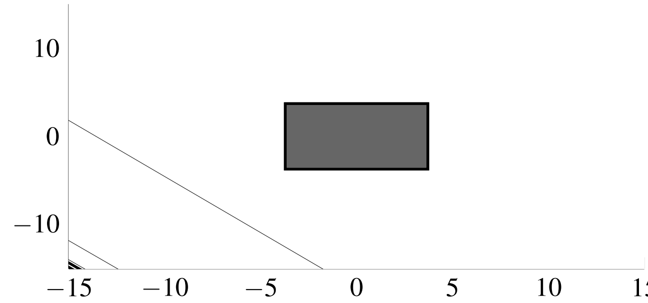

Fig. 4. Sets Σ_{40} and Σ_{50} in thin lines; Σ_{60} in white with thick lines; Ω_∞^α in light gray, for $N \in \mathbb{N}_{10}$, and Ω^α for $N=10$ in dark gray.

The inner and outer approximation of the maximal invariant appear to be rather close for $N=10$.

C. Example 3: state constraints

In this example we consider the same dynamics and same sets Ω and U of Example 1 and the state constraint set given by $X = \{x \in \mathbb{R}^2 : \ -10 \leq x_1 \leq 5, \ -1 \leq x_2 \leq 2\}$. Both method for taking into account the state constraints illustrated in Section IV-C are applied using $N=15$. Figure 5 shows the set $\Omega_\infty^{\alpha,\sigma}$ obtained by solving (34) in middle shade gray and also Ω_∞^σ induced by the solution to (38) in light gray (besides the sets X, Σ_k and $\sigma\Omega$).

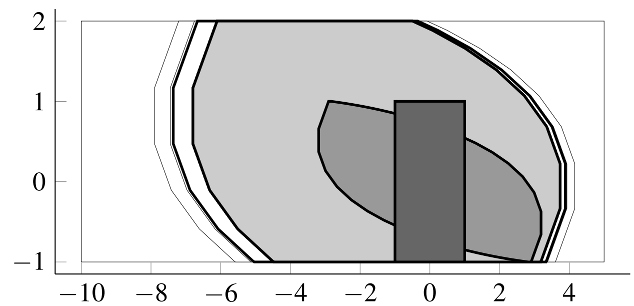

Fig. 5. Sets $\Sigma_0 = X$ and Σ_{10k} with $k \in \mathbb{N}_5$ in thin lines; Σ_{60} in white with thick lines; Ω_∞^σ in light gray; $\Omega_\infty^{\alpha,\sigma}$ in middle shade gray, and $\sigma\Omega$ in dark gray, for $N=15$.

Note how the conservatism with respect to the scaling procedure (34) is reduced by taking into account the shape of X as in the method based on (38). The latter, in fact, provides a good approximation of the maximal control invariant set for $N=15$.

D. Example 4: high dimensional system

We apply now the proposed method to an high dimensional system, in particular with $n=20$ and $m=10$. To provide some hints on the conservatism of the control invariant obtained with respect to the maximal control invariant set, we build a system for which the latter can be computed, or, at least, approximated. Indeed the classical algorithms for computing or approximating the maximal control invariant set are too computationally demanding to be applied to high dimensional systems in general. Then, a particular structure has to be imposed to the system dynamics to apply them and obtain an estimation of the maximal control invariant set to be compared with our results. In particular, we consider system (1) with

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{10} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_{10} \end{bmatrix}$$

where $A_i \in \mathbb{R}^{2\times 2}$ and $B_i \in \mathbb{R}^2$, for $i \in \mathbb{N}_{10}$, are matrices whose entries are randomly generated such that all A_i have instable poles and the pairs (A_i, B_i) are controllable. This means that the whole system is controllable and it is, in practice, composed by 10 decoupled two-dimensional subsystems with one control input each. Hence, the maximal control invariant set for the overall system, Σ_∞, is given by the Cartesian product of the maximal control invariant sets of the 10 subsystems, that is $\Sigma_\infty = \prod_{i=1}^{10} \Sigma_{i,\infty}$ where $\Sigma_{i,\infty}$ are the maximal control invariant set (or an outer approximation of it) for the i-th subsystem. Then Σ_∞ can be computed by computing $\Sigma_{i,\infty}$, being (A_i, B_i) a two-dimensional controllable system, for all $i \in \mathbb{N}_{10}$.

The linear problem (22) has been posed with $N = 3, 5, 9, 15$ and solved with YALMIP interface [19] and Mosek optimizer [22]. In Table I, the dimensions and solution times for the LP problems are reported.

	$N=3$	$N=5$	$N=9$	$N=15$
LP dimension	10002	19602	48402	115602
Solution time	$0.9s$	$0.99s$	$1.25s$	$1.71s$

TABLE I

To quantify the difference between the outer approximation of the maximal control invariant set Σ_∞ and the set Ω_∞^α, 100 vectors $v \in \mathbb{R}^n$ are generated randomly. Then, (a lower approximation of) the maximal values of r_Σ and r_Ω are computed such that $r_\Sigma v \in \Sigma_\infty$ and $r_\Omega v \in \Omega_\infty^\alpha$, through dichotomy method. In practice, we search for (approximations of) the intersections between the ray $v_r = \{rv \in \mathbb{R}^n : \ r \geq 0\}$ and the boundaries of the sets Σ_∞ and Ω_∞^α. The ratio between r_Ω / r_Σ is an indicator of the mismatch between the outer approximation of the maximal control invariant set Σ_∞ and Ω_∞^α, the closer to one, the closer are the intersections between the ray v_r and the two sets.

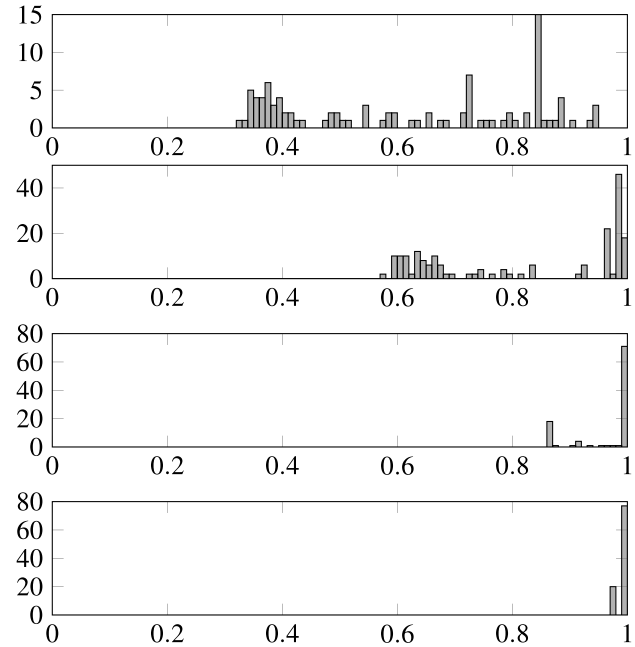

Fig. 6. Histograms of the values r_Ω / r_Σ for $N = 3, 5, 9, 15$ (from top to bottom).

Figure 6 shows the histograms of the ratio r_Ω / r_Σ for the different values of N. As expected, the higher is the horizon N, the closer are the sets Σ_∞ and Ω_∞^α.

VI. Conclusions

In this paper we addressed the problem of computing control invariant sets for linear systems with state and input polyhedral constraints. In particular we considered the computational complexity inherent to the explicit determination of polyhedral one-step sets, that are the basis of many iterative procedures for obtaining control invariant sets. Invariance conditions are given, that are set inclusions involving the N-step sets, which are posed in form of LP optimization problems, instead of Minkowski sum of polyhedra. Then the procedures based on those conditions are applicable even for high dimensional systems.

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