Polyhedral Estimates of the Region of Attraction of the Origin of Linear Systems under Aperiodic Sampling and Input Saturation ⋆

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Abstract

This work addresses the stability analysis of linear aperiodic sampled-data systems under saturating inputs. A method to generate an increasing sequence of polyhedral estimates of the region of attraction of the origin of the closed-loop system is proposed. An impulsive system representation, given by a linear flow and a nonlinear jump dynamics due to the saturation term, is employed. From this representation, a convenient partition of the admissible interval for the intersampling time and an appropriate model of the saturation term, the computation of polyhedral contractive sets is carried out considering a convex embedding of the behaviour of the system at the sampling instants. It is then shown that the computed polyhedra are included in the region of attraction of the continuous-time plant driven by the sampled-data control. A numerical example validates the theoretical developments and compares the method presented in this work with other approaches from the literature.

Key words: Sampled-data systems; aperiodic sampling; polyhedral sets; stability analysis; input saturation; convex embeddings.

1 Introduction

Aperiodic sampled-data systems are commonly employed to model the effect of imperfections on the communication channel of networked control systems, like sampling jitters, fluctuations and packet dropouts [15]. In this context, several methods were developed to perform the stability analysis of systems subject to a time-varying sampling interval. In [10,21], for instance, a time-delay systems framework is considered and the resulting method is based on Lyapunov-Krasovskii functionals. Similar ideas are considered in [25,26], where the looped-functional method is presented. In [24,5], the problem is tackled using a hybrid system framework. In [29,17], an uncertain discrete-time model that describes the behavior of the state at the sampling instants is considered. Numerical tractable criteria for this approach can be obtained using polytopic embeddings for the system transition matrix [6,22]. Alternatively, norm-bounded approximations can be employed [11,12,20]. A survey on the subject can be found in [16].

In presence of control input saturation, the situation is more complex. In particular, for exponentially unstable open-loop plants it is not possible to ensure the global asymptotic stability of the origin and hence only local stability properties can be guaranteed [27,23]. It is then useful to characterize and compute estimates of the region of attraction of the origin (RAO) of the closed-loop system. Moreover, even if the open-loop system is exponentially stable, to satisfy performance criteria around the origin, it may be interesting to use a control law that ensures only local stability of the origin and to compute the corresponding RAO [28]. In this case, one can also use a switched control law, starting with a global stabilizing control law and, as the state approaches the origin, switching to a more performing one. The estimate of the RAO will then define the switching region. To compute this estimate in the periodic sampling case, an exact discretization is possible and the problem can be treated in a discrete-time framework (see [28] and the references therein). For aperiodic sampled-data systems, we can cite, for instance, [26,19], where results based on quadratic functions, leading to ellipsoidal estimates of the RAO, are proposed.

On the other hand, the use of methods based on polyhedral

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sets to address the stability analysis of dynamical systems is quite appealing [3,7,4,9]. Because of their flexibility, adopting polyhedra instead of ellipsoids allows a reduction of conservativeness [3, Pg. 188]. In this context, in [18], an approach to generate polyhedral estimates of the RAO of linear systems under sampled-data saturating controls is proposed. The main drawback of the method in [18] concerns the numerical execution of the proposed algorithm, which can become prohibitively complex before its stopping criterion is reached. This is due to the fact that the algorithm computes a decreasing sequence of nested polytopes whose complexity tends to increase at each iteration, and then the execution time related to each iteration also increases. Moreover, the polytopes generated at each intermediate iteration of the proposed algorithm cannot be considered as estimates of the RAO, i.e. regions of safe behavior for the closed-loop system. Thus, in practice, the method in [18] may fail to provide an estimate of the RAO of the system in an acceptable amount of time.

In this work we adopt a different strategy and propose a method which does not have the aforementioned drawbacks. It is based on the computation of an increasing sequence of polytopes, where at each step a new polyhedral estimate of the RAO (larger than the previous one) is obtained. In this case the trade-off between number of iterations and complexity of the resulting polytope can be managed. Differently from [18], to derive our results, an impulsive system representation is employed, with a linear flow and a nonlinear (due to the saturation term) jump dynamics, as shown in Section 2. Some concepts on set invariance theory are recalled in Section 3. Section 4, in turn, introduces the saturated and nonsaturated (SNS) embedding of the saturation function [1], which is used in this work. Next, Section 5 presents the main theoretical results, which lead to the proposition of an algorithm to generate polytopic estimates of the RAO of the closed-loop system. The results are then applied in a numerical example in Section 6, where our approach is compared to other ones from the literature. Some concluding remarks end the paper.

Notation. For $x:\mathbb{R}\to\mathbb{R}^n$, $x(t^-)=\lim_{\tau\mapsto t,\tau<t} x(\tau)$ and similarly for $x(t^+)$. A C-set is a convex and compact set containing the origin in its interior. For $\lambda\in\mathbb{R},\Lambda\in\mathbb{R}^{m\times n},\Omega\subseteq\mathbb{R}^n$, $\lambda\Omega=\{\lambda x: x\in\Omega\}$ and similarly for $\Lambda\Omega$; $\partial\Omega$ is the boundary of Ω and Ω° its interior. For $v,s\in\mathbb{R}^n$, the operators $|v|$ and $v\le s$ must be interpreted component-wise. For $M\in\mathbb{R}^{m\times n}$, $M_{(i)}$ is its i-th row, $M^{(i)}$ its i-th column, $M_{(i,j)}$ its (i,j)-entry and M^T its transpose. For $x\in\mathbb{R}^n$, $\|x\|$ is its Euclidean norm. For $A\in\mathbb{R}^{m\times n}$, $\|A\|=\sigma_{max}(A)$ is its induced 2-norm (largest singular value) and $\sigma_{min}(A)$ is its smallest singular value. For $\Omega,\Theta\subseteq\mathbb{R}^n$, $\Omega+\Theta=\bigcup_{\omega\in\Omega,\theta\in\Theta}\omega+\theta$, $\Omega\setminus\Theta=\{\omega\in\Omega:\ \omega\notin\Theta\}$. $\mathscr{B}_r=\{x\in\mathbb{R}^n:\ \|x\|\le r\}$, $\mathscr{B}=\mathscr{B}_1$, e_k is the k-th canonical base vector of the Euclidean space and $\mathbf{1}=[1\dots1]^T$. $\mathbb{N}_+=\{i\in\mathbb{N}: i\ge1\}$, $\mathbb{N}_m=\{i\in\mathbb{N}:\ 1\le i\le m\}$ and $\mathscr{S}=2^{\mathbb{N}_m}$ is the set of all subsets of $\mathbb{N}_m$. For instance: $2^{\mathbb{N}_2}=\{\emptyset,\{1\},\{2\},\{1,2\}\}$. $\mathscr{P}(H,h)=\{x\in\mathbb{R}^n:\ Hx\le h\},H\in\mathbb{R}^{n_h\times n},h\in\mathbb{R}^{n_h}$, is the H-representation of a polyhedron.

2 Problem formulation

Consider the following continuous-time system:

$$\dot{x}_p(t)=A_px_p(t)+B_pu(t) \tag{1}$$

where $x_p\in\mathbb{R}^{n_p}$ and $u\in\mathbb{R}^m$ are the state and the input of the plant, respectively, and B_p has full column rank. At each sampling instant t_k, $k\in\mathbb{N}$, the state is used to compute the input $u(t)$, which is kept constant until the next sampling time (i.e. on the interval $[t_k,t_{k+1})$) by a zero order hold (ZOH). A linear saturated stabilizing state feedback is considered, i.e.

$$u(t_k)=\mathrm{sat}(K_px_p(t_k^-)),\quad \forall k\in\mathbb{N} \tag{2}$$

where $\mathrm{sat}(\cdot)$ denotes the standard saturation function according to the constraint $u(t)\in\{u\in\mathbb{R}^m:\ |u|\le\mathbf{1}\}$. By convention $t_0=0$ and the difference between two successive sampling instants, given by $\delta_k\triangleq t_{k+1}-t_k$, is considered to be lower and upper bounded by scalars $\tau_m,\tau_M>0$:

$$\delta_k\in\Delta\triangleq[\tau_m,\tau_M],\quad \forall k\in\mathbb{N}, \tag{3}$$

i.e. we assume an aperiodic sampling strategy. Defining the overall system state $x\triangleq[x_p^T\ u^T]^T\in\mathbb{R}^n$, with $n\triangleq n_p+m$, the system can then be represented by the impulsive model:

$$\begin{cases}\dot{x}(t)=A_cx(t), & \forall t\neq t_k,\ t\ge0,\\ x(t_k^+)=A_rx(t_k^-)+B_r\mathrm{sat}(Kx(t_k^-)), & \\ x(0)\triangleq x_0\in\mathbb{R}^n, & \end{cases} \tag{4}$$

as in [8], where $x_0=[x_p^T(0)\ \mathrm{sat}^T(K_px_p(0))]^T$ and $A_c,A_r\in\mathbb{R}^{n\times n}$, $B_r\in\mathbb{R}^{n\times m}$ and $K\in\mathbb{R}^{m\times n}$ are given by

$$A_c=\begin{bmatrix}A_p & B_p\\ 0 & 0\end{bmatrix},\ A_r=\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix},\ B_r=\begin{bmatrix}0\\ I\end{bmatrix},\ K=\begin{bmatrix}K_p & 0\end{bmatrix}. \tag{5}$$

The representation (4) allows to model the sampled-data system between two consecutive sampling instants as the trajectory of a LTI system. This is a key element to derive our results and an important modeling difference w.r.t. [18].

In this paper we deal with the stability analysis of the sampled-data system (4), providing polyhedral estimates of its RAO, denoted as $\Gamma_c\subseteq\mathbb{R}^n$, which is defined as the set of initial conditions $x(0)\in\mathbb{R}^n$ such that $x(t)\overset{t\to\infty}{\to}0$ for all sequences $\{t_k\}_{k\in\mathbb{N}}$ satisfying (3). From (4), one gets [8]:

$$\begin{aligned}x(t_{k+1}^+)&=A_rx(t_{k+1}^-)+B_r\mathrm{sat}(Kx(t_{k+1}^-))\\ &\in\{A_re^{A_c\delta}x(t_k^+)+B_r\mathrm{sat}(Ke^{A_c\delta}x(t_k^+)):\ \delta\in\Delta\}.\end{aligned}$$

Then, with $x_k\triangleq x(t_k^+)$, $A(\delta)\triangleq A_re^{A_c\delta}$ and $K(\delta)\triangleq Ke^{A_c\delta}$:

$$\begin{aligned}x_{k+1}&\in\{A(\delta)x_k+B_r\mathrm{sat}(K(\delta)x_k):\ \delta\in\Delta\}\\ &\triangleq\{\mathscr{F}(x_k,\delta):\ \delta\in\Delta\}\triangleq\mathscr{F}(x_k,\Delta).\end{aligned} \tag{6}$$

It also follows that $\|x(t)\| \leq \max_{\delta\in[0,\tau_M]} \|e^{A_c\delta}\|\|x_k\|, \forall t \in [t_k, t_{k+1})$. Hence, the stability of the origin of the closed-loop system (4) is implied by the one of the discrete-time system (6) [8]. Moreover, the RAO of (6), denoted by Γ_d, coincides with the RAO of (4), i.e. $\Gamma_d = \Gamma_c$.

3 Basic concepts on set invariance

Given a C-set $\Omega \subset \mathbb{R}^n$, its Minkowski function $\Psi_\Omega : \mathbb{R}^n \to \mathbb{R}$ is defined as: $\Psi_\Omega(x) \triangleq \min\{\alpha \geq 0 : \ x \in \alpha\Omega\}$. This function satisfies the following properties.

Lemma 1 ([3]) *(Minkowski function properties)*

- *It is positive definite, continuous and convex*
- $\Psi_\Omega(\lambda x) = \lambda\Psi_\Omega(x)$ *for* $\lambda \geq 0$
- $\Psi_\Omega(x_1 + x_2) \leq \Psi_\Omega(x_1) + \Psi_\Omega(x_2)$
- *Its unitary level set corresponds to the set* Ω

Given a C-set $\Omega \subset \mathbb{R}^n$, the Minkowski function of a compact set $S \subset \mathbb{R}^n$ can also be defined as $\Psi_\Omega(S) \triangleq \max_{x\in S} \Psi_\Omega(x)$.

Given a set Ω, its (positive) invariance/contractivity w.r.t. a discrete-time system is defined below.

Definition 1 *Given $\lambda \in [0,1)$, the set $\Omega \subseteq \mathbb{R}^n$ is said to be λ-contractive for a generic difference inclusion $x_{k+1} \in \mathscr{F}(x_k)$ if $\mathscr{F}(x_k) \subseteq \lambda\Omega$ for all $x_k \in \Omega$. If $\lambda = 1$, Ω is an invariant set for $x_{k+1} \in \mathscr{F}(x_k)$.*

For simplicity, in the continuous-time case we will present the concept of invariance directly for the system of interest:

$$\dot{x}(t) = A_c x(t). \tag{7}$$

Definition 2 ([3]) *Given $\beta \in \mathbb{R}$, the C-set $\Omega \subset \mathbb{R}^n$ is said to be β-invariant for (7) if for all $x \in \partial\Omega$:*

$$D^+\Psi_\Omega(x) \triangleq \limsup_{h\to 0^+} \frac{\Psi_\Omega(x + hA_c x) - \Psi_\Omega(x)}{h} \leq \beta \tag{8}$$

Notice that the set is invariant in the usual sense only if $\beta \leq 1$, but it will be useful afterwards to consider also the case $\beta > 1$. Condition (8) is rather technical. The lemma below presents a concrete consequence of it, giving a bound to the potential expansion of Ω along the trajectories of (7). This result will be used in the proof of Theorem 2 in Section 5.

Lemma 2 ([9]) *If the C-set Ω is β-invariant for (7) then $e^{A_c\tau}\Omega \subseteq \alpha\Omega, \forall \tau \in [0, \bar{\tau}]$, where $\alpha \triangleq \max\{1, e^{\beta\bar{\tau}}\}$.*

An important feature of β-invariance is that it can be characterized in terms of a linear programming problem for LTI systems and polyhedral sets, as shown below. The proof of this lemma is analogous to the one of [3, Th. 4.33] without negativity constraint on β.

Lemma 3 ([3]) *Consider the C-set $\Omega = \mathscr{P}(H, \mathbf{1}), H \in \mathbb{R}^{n_h\times n}$. The following statements are equivalent.*

(1) Ω is β-invariant for the system $\dot{x}(t) = A_c x(t)$
(2) There exists $T \in \mathbb{R}^{n_h\times n_h}$ such that

$$HA_c = TH, \quad T\mathbf{1} \leq \beta\mathbf{1}, \quad T_{(i,j)} \geq 0, \ \forall i \neq j \tag{9}$$

Moreover, the following technical result, which will be useful later, holds, where the index set I is in general not finite. For the proof see Appendix A.

Lemma 4 *Given a family of C-sets $\{\Omega_i\}_{i\in I}$ belonging to $\mathbb{R}^n$, where I is the corresponding index set, such that*

$$\mathscr{B}_{r_1} \subseteq \Omega_i \subseteq \mathscr{B}_{r_2}, \quad \forall i \in I \tag{10}$$

where $r_1, r_2 > 0$, there exists $\beta \in \mathbb{R}$ such that Ω_i is β-invariant for (7) for all $i \in I$.

4 SNS contractive sets

We define the one-step set for system (6) as follows.

Definition 3 *Given $\Omega \subseteq \mathbb{R}^n$, the one-step set $P(\Omega)$ w.r.t. (6) is given by $P(\Omega) \triangleq \{x \in \mathbb{R}^n : \ \mathscr{F}(x, \Delta) \subseteq \Omega\}$.*

Notice that $P(\Omega)$ is the set of all $x_k \in \mathbb{R}^n$ such that x_{k+1} will belong to Ω for all possible values of the intersampling time. It is well known that Ω is an invariant set for (6) if and only if $\Omega \subseteq P(\Omega)$ [3]. Given an invariant C-set Ω_0 for (6) that belongs to its RAO, the increasing sequence of nested sets

$$\Omega_{i+1} = P(\Omega_i), \quad i \in \mathbb{N}, \tag{11}$$

gives approximations of increasing accuracy of $\Gamma_d = \Gamma_c$. However, the computation of such sets is in general not possible in practice. In particular, $P(\Omega)$ may be nonconvex even if Ω is a C-set because of the saturation function. That is why we use in this paper the concepts of SNS (saturated and nonsaturated) invariance and SNS RAO according to the definitions presented next [1]. In particular, it is known that the SNS model is less conservative than classical polytopic embeddings of the saturation function [1, Section 5],[28, Chapter 1]. The SNS system that corresponds to (6) has, besides δ, an additional parameter $S \in \mathscr{S} = 2^{\mathbb{N}_m}$ which is related to the saturation function and indicates which components of this function are activated or not. Define $\mathrm{sat}_S : \mathbb{R}^m \to \mathbb{R}^m$ as

$$\mathrm{sat}_S(z) \triangleq \sum_{i\in S^c} e_i z_{(i)} + \sum_{i\in S} e_i \mathrm{sat}(z_{(i)}), \tag{12}$$

where $S \in \mathscr{S}$ and $S^c = \mathbb{N}_m \setminus S$. Define also

$$\mathscr{F}_{SNS}(x_k, \delta, S) \triangleq A(\delta)x_k + B_r \mathrm{sat}_S(K(\delta)x_k)$$

The SNS system related to (6) is then given by

$$x_{k+1} \in \mathscr{F}_{SNS}(x_k,\Delta,\mathscr{S}) \triangleq \{\mathscr{F}_{SNS}(x_k,\delta,S):\ \delta\in\Delta,\ S\in\mathscr{S}\}. \quad (13)$$

Notice that the difference inclusion above takes into account all 2^m possible combinations of saturated/nonsaturated inputs given by $S\in\mathscr{S}=2^{\mathbb{N}_m}$ simultaneously.

Definition 4 *A SNS invariant (contractive) set for system* (6) *is an invariant (contractive) set for* (13).

Definition 5 *(SNS Region of Attraction of the Origin) The SNS RAO of system* (6)*, denoted by* Γ_{SNS}*, is the RAO of* (13).

The dynamics of (13) encompasses the one of (6), i.e.

$$\begin{aligned}\mathscr{F}(x_k,\Delta) &= \{A(\delta)x_k + B_r\mathrm{sat}(K(\delta)x_k):\ \delta\in\Delta\}\\ &= \{A(\delta)x_k + B_r\mathrm{sat}_S(K(\delta)x_k):\ \delta\in\Delta,\ S=\mathbb{N}_m\}\\ &\subseteq \{A(\delta)x_k + B_r\mathrm{sat}_S(K(\delta)x_k):\ \delta\in\Delta,\ S\in\mathscr{S}\}\\ &= \mathscr{F}_{SNS}(x_k,\Delta,\mathscr{S}).\end{aligned}$$

Therefore, $\Gamma_{SNS}\subseteq\Gamma_d$. This fact will be exploited by the method presented here. Actually, to obtain a numerical tractable procedure, the idea is to compute polyhedral estimates of Γ_d through the computation of estimates of Γ_{SNS}.

As in Definition 3, the one-step set related to (13) is given by $Q(\Omega)\triangleq\{x\in\mathbb{R}^n:\ \mathscr{F}_{SNS}(x,\Delta,\mathscr{S})\subseteq\Omega\}$. Moreover, analogously to (11), we can consider the recursion:

$$\Omega_{i+1} = Q(\Omega_i), \quad i\in\mathbb{N} \quad (14)$$

The next theorem states some properties of the sequence $\{\Omega_i\}_{i\in\mathbb{N}}$ defined above.

Theorem 1 *Consider the recursion* (14) *where* $\Omega_0\subseteq\Gamma_{SNS}$ *is an initial SNS invariant C-set for* (6)*. Then:*

- *Ω_i is a SNS invariant set for (6) for all $i\in\mathbb{N}$;*
- *$\Omega_i\subseteq\Omega_{i+1},\ \forall i\in\mathbb{N}$;*
- *$\Omega_i\subseteq\Gamma_{SNS},\ \forall i\in\mathbb{N}$;*
- *The sequence $\{\Omega_i\}_{i\in\mathbb{N}}$ converges to Γ_{SNS}.*

Proof: The proof is analogous to the one of [1, Th. 2]. The only difference is that, besides S, there is an additional degree of freedom δ in the difference inclusion (13). ■

Unfortunately, as discussed in [9] for the linear case, under aperiodic sampling the one-step set $Q(\Omega)$ is in general not polyhedral even if Ω is a polytope. This is due to the dependence of the discrete-time model (13) on an uncertain matrix exponential term $e^{A_c\delta}$, where the value of δ varies within an interval Δ. A common approach to deal with this term consists in obtaining convex embeddings for it, which can be polytopic [6,22] or norm-bounded [11,12,20]. In this work, in order to obtain numerically tractable conditions for the stability analysis, it will be convenient to adopt a different strategy as in [9,19,18], which consists in considering the following grid of the interval Δ:

$$\Delta_J \triangleq \{\tau_m + (j-1)\tau_J:\ j\in\mathbb{N}_J\}, \quad \tau_J \triangleq \frac{\tau_M-\tau_m}{J}, \quad (15)$$

where $J\in\mathbb{N}_+$ is the number of points of Δ_J. In this case, consider the following approximation of $Q(\Omega)$:

Definition 6 *Given* $\Omega\subseteq\mathbb{R}^n$ *and* $J\in\mathbb{N}_+$,

$$Q_J(\Omega) \triangleq \{x\in\mathbb{R}^n:\ \mathscr{F}_{SNS}(x,\Delta_J,\mathscr{S})\subseteq\Omega\}. \quad (16)$$

Notice that $Q_J(\Omega)$ takes into account only the finite subset Δ_J of possible values for the intersampling time δ_k. So $Q(\Omega)\subseteq Q_J(\Omega)$ but these sets are different in general. In the next section we will see how to deal with this fact. Notice also that, using (15), (13) can be expressed as:

$$x_{k+1} \in \bigcup_{\tau\in[0,\tau_J]} \mathscr{F}_{SNS}(e^{A_c\tau}x_k,\Delta_J,\mathscr{S}). \quad (17)$$

The lemma below, derived straightforwardly from [1, Th. 1], provides a polyhedral characterization of $Q_J(\Omega)$.

Lemma 5 *Given* $\Omega=\mathscr{P}(H,h), H\in\mathbb{R}^{n_h\times n}, h\in\mathbb{R}^{n_h}$,

$$Q_J(\Omega) = \bigcap_{\delta\in\Delta_J}\bigcap_{S\in\mathscr{S}} \mathscr{P}\left(H\Big(A(\delta)+\sum_{i\in S^c} B_r^{(i)}K_{(i)}(\delta)\Big),\, h+\sum_{i\in S}|HB_r^{(i)}|\right) \quad (18)$$

The following two lemmas, which will be used later and whose proofs are in Appendices B and C, respectively, provide some properties of the operators $Q(\cdot)$ and $Q_J(\cdot)$.

Lemma 6 *Assume that* $\Delta=[\tau_m,\tau_M]$ *with* $\tau_m<\tau_M$ *and define* $J^\star$ *as the number of elements of the finite set*

$$\Delta^\star \triangleq \{\tau_m,\tau_M\}\cup(\{\tau_m+2\pi r/\omega_l:\ l\in\mathbb{N}_{n_\omega}, r\in\mathbb{Z}\}\cap\Delta)$$

where $\pm j\omega_l$ *is the* l*-th pair of pure imaginary eigenvalues of* A_p *with* $l=1,\ldots,n_\omega$*. Given a polyhedral C-set* Ω*,*

a) $Q(\Omega)$ and $Q_J(\Omega)$ are both C-sets for all $J\geq J^\star$;
b) For $\gamma>0$ satisfying $\Omega\subseteq\mathscr{B}_\gamma$, $\exists r=r(\gamma)>0$ such that $Q(\Omega)\subseteq Q_J(\Omega)\subseteq\mathscr{B}_r, \forall J\geq J^\star$.

Among all properties that $Q_J(\Omega)$ has to satisfy to be a C-set, the requirement $J\geq J^\star$ in the lemma above is important only to guarantee that $Q_J(\Omega)$ is bounded, avoiding some "pathological" cases where it would be unbounded even if Ω is bounded. In the simplest case, where A_p has no pure imaginary eigenvalues, $\Delta^\star=\{\tau_m,\tau_M\}$ and $J^\star=2$. Otherwise, it is possible to show that $J^\star=2$ if the frequencies $f_{s,k}\triangleq 2\pi/\delta_k\in[2\pi/\tau_M, 2\pi/\tau_m]$ are larger than the natural frequencies ω_l of the open-loop system.

Given two C-sets Θ_1, Θ_2 such that $\Theta_1 \subset \Theta_2^\circ$, it follows straightforwardly that $Q(\Theta_1) \subseteq Q(\Theta_2^\circ)$. However, it is not trivial to see if the set inclusion still holds inverting the order of the operators $Q(\cdot)$ and $(\cdot)^\circ$, i.e. $Q(\Theta_1) \subset Q(\Theta_2)^\circ$. The role of Lemma 7 below is to prove this relation.

Lemma 7 *Given two C-sets Θ_1, Θ_2 such that $\Theta_1 \subset \Theta_2^\circ$, it follows that $Q(\Theta_1) \subset Q(\Theta_2)^\circ$.*

5 Computation of estimates of the RAO

We want to obtain estimates of the RAO of (6) through the computation of polyhedral estimates of the SNS RAO. As explained before, it is not possible to simply replace the operator $Q(\cdot)$ by $Q_J(\cdot)$ in the recursion (14) and use the result of Theorem 1 since $Q_J(\cdot)$ does not take into account all possible values of δ_k, unless in the trivial case where $\tau_m = \tau_M$. Thus, from now on, we will only deal with the nontrivial case $\tau_m < \tau_M$, i.e. δ_k is uncertain. For technical reasons, we will also consider that the number of partitions J of Δ satisfies $J \geq J^\star$, where $J^\star$ is defined in Lemma 6, in which case $Q_J(\cdot)$ is a C-set.

The objective of this section is to show how to construct, in a numerically tractable way, an increasing sequence $\{\Omega_i\}_{i\in\mathbb{N}}$ of SNS λ_i-contractive polyhedral C-sets for (6) using an initial SNS λ_0-contractive polyhedral C-set Ω_0. From the contractivity property, it follows that these sets are included in $\Gamma_{SNS} \subseteq \Gamma_d = \Gamma_c$, being therefore estimates of Γ_c. This statement is a direct consequence of the lemma below.

Lemma 8 *If the C-set $\Omega \subset \mathbb{R}^n$ is SNS λ-contractive for* (6)*:*

a) $\varepsilon\Omega$ also is SNS λ-contractive for (6) *for all $\varepsilon \in [0,1]$;*
b) All trajectories of (13) *have the following property:*

$$x_k \in \varepsilon\Omega \Rightarrow x_{k+p} \in \lambda^p \varepsilon\Omega, \quad \forall \varepsilon \in [0,1], \forall p \in \mathbb{N}. \quad (19)$$

c) All trajectories of (6) *have property* (19)*.*

Proof: The proof of a) follows the one of [18, Lemma 1] *mutatis mutandis*; b) is implied by the recursive application of a); and c) follows directly from b) and the fact that every trajectory of (6) is also a trajectory of (13). ■

Corollary 1 *If the C-set $\Omega \subset \mathbb{R}^n$ is SNS contractive for* (6)*, then $\Omega \subseteq \Gamma_{SNS} \subseteq \Gamma_d = \Gamma_c$.*

Proof: Follows from item b) of Lemma 8 and the fact that $\lambda \in [0,1)$ and Ω is bounded by definition. ■

In order to construct the sequence $\{\Omega_i\}_{i\in\mathbb{N}}$, the operator $Q(\Omega)$ in (14) will be replaced not by $Q_J(\Omega)$ (as already explained, this strategy would not work) but by the set $\hat{Q}_J(\Omega)$ defined below. The main difference is that $\hat{Q}_J(\Omega)$ corresponds to an inner approximation of $Q(\Omega)$ while $Q_J(\Omega)$ corresponds to an outer approximation of $Q(\Omega)$. The set $\hat{Q}_J(\Omega)$ will be obtained by scaling down $Q_J(\Omega)$.

Definition 7 *Given the polyhedral C-set $\Omega \subset \mathbb{R}^n$, $J \geq J^\star$ and the H-representation*[1] $Q_J(\Omega) = \mathscr{P}(H,\mathbf{1}), H \in \mathbb{R}^{n_h \times n}$,

$$\beta^J(\Omega) \triangleq \inf_{T,\beta} \beta \text{ s.t.} \begin{cases} HA_c = TH \\ T\mathbf{1} \leq \beta\mathbf{1} \\ T_{(i,j)} \geq 0, \qquad \forall i \neq j \end{cases} \quad (20)$$

$$\alpha^J(\Omega) \triangleq \max\{1, e^{\beta^J(\Omega)\tau_J}\} \quad (21)$$

$$\hat{Q}_J(\Omega) \triangleq Q_J(\Omega)/\alpha^J(\Omega) \quad (22)$$

where $T \in \mathbb{R}^{n_h\times n_h}, \beta \in \mathbb{R}$, A_c is defined in (5) *and τ_J in* (15)*.*

Remark 1 *According to Lemma 3, the constraints in* (20) *are feasible for some $\beta \in \mathbb{R}$ if and only if the polyhedral C-set $Q_J(\Omega) = \mathscr{P}(H,\mathbf{1})$ is β-invariant for $\dot{x}(t) = A_c x(t)$. Therefore, $\beta^J(\Omega)$ is the smallest number β such that $Q_J(\Omega)$ (or equivalently $\hat{Q}_J(\Omega)$) is β-invariant for this system.*

The theorem below guarantees that $\hat{Q}_J(\Omega) = Q_J(\Omega)/\alpha^J(\Omega) \subseteq Q(\Omega)$. The motivation for choosing the scale factor $\alpha^J(\Omega)$ will become clear in the proof of this theorem. The number $\alpha^J(\Omega)$ is related to the possible expansion of the set $\hat{Q}_J(\Omega)$ along the trajectories of (7) in a time interval $[0,\tau] \subseteq [0,\tau_J]$.

Theorem 2 *Given a polyhedral C-set $\Omega \subset \mathbb{R}^n$ and $J \geq J^\star$, it follows that*

$$\hat{Q}_J(\Omega) \subseteq Q(\Omega) \quad (23)$$

Proof: Given $x_k \in \hat{Q}_J(\Omega)$, we have to prove that $x_k \in Q(\Omega)$. This is equivalent to show that $x_{k+1} \in \Omega$ for all possible values of $\delta_k \in \Delta$ and $S_k \in \mathscr{S}$, where x_{k+1} is given by (13). Given $\delta_k \in \Delta$ and $S_k \in \mathscr{S}$, we know from (17) that there exists $\tau \in [0,\tau_J]$ such that

$$x_{k+1} \in \mathscr{F}_{SNS}(e^{A_c\tau}x_k, \Delta_J, \mathscr{S}) \quad (24)$$

Consider now that $x_k \in \hat{Q}_J(\Omega) = Q_J(\Omega)/\alpha^J(\Omega)$ where $\hat{Q}_J(\Omega)$ is $\beta^J(\Omega)$-invariant for $\dot{x}(t) = A_c x(t)$. Applying Lemma 2 (with $\Omega = \hat{Q}_J(\Omega)$, $\beta = \beta^J(\Omega)$, $\bar{\tau} = \tau_J$ and $\alpha = \alpha^J(\Omega)$), we conclude that $e^{A_c\tau}x_k \in e^{A_c\tau}\hat{Q}_J(\Omega) \subseteq \alpha^J(\Omega)\hat{Q}_J(\Omega) = Q_J(\Omega)$.

Since $e^{A_c\tau}x_k \in Q_J(\Omega)$ it follows that $\mathscr{F}_{SNS}(e^{A_c\tau}x_k, \Delta_J, \mathscr{S}) \subseteq \Omega$ (see (16)). Combining this set inclusion with (24) we conclude that $x_{k+1} \in \Omega$, proving the result. ■

Considering the result below it is possible to conclude that the set $\hat{Q}_J(\Omega) \subseteq Q(\Omega)$ converges to $Q(\Omega)$ as $J \to \infty$.

Lemma 9 *Given a polyhedral C-set $\Omega \subset \mathbb{R}^n$ and $c \in [0,1)$, there exists $\bar{J} \in \mathbb{N}, \bar{J} \geq J^\star$, such that*

$$cQ(\Omega) \subset \hat{Q}_J(\Omega)^\circ, \quad \forall J \geq \bar{J} \quad (25)$$

[1] $Q_J(\Omega)$ is a polyhedral C-set according to Lemmas 5 and 6 and we can assume without loss of generality that $h = \mathbf{1}$ in the H-representation $Q_J(\Omega) = \mathscr{P}(H,h)$ because $0 \in Q_J(\Omega)^\circ$.

Proof: See Appendix D. ■

The lemma above leads directly to the next theorem.

Theorem 3 *Given a SNS contractive polyhedral C-set* $\Omega \subset \mathbb{R}^n$ *for* (6)*, there exists* $\bar{J} \in \mathbb{N}, \bar{J} \geq J^\star$*, such that*

$$\Omega \subset \hat{Q}_J(\Omega)^\circ, \quad \forall J \geq \bar{J} \tag{26}$$

Proof: Since Ω is contractive for (13), there exists $\lambda \in (0,1)$ such that $\Omega \subseteq Q(\lambda\Omega)$. Moreover, using the relation $\lambda\Omega \subset \Omega^\circ$, Lemma 7 guarantees that $Q(\lambda\Omega) \subset Q(\Omega)^\circ$. Combining these two set inclusions we conclude that $\Omega \subset Q(\Omega)^\circ$, so there exists $c \in (0,1)$ such that $\Omega \subseteq cQ(\Omega)$. Combining this relation with the result from Lemma 9, it follows that $\Omega \subseteq cQ(\Omega) \subset \hat{Q}_J(\Omega)^\circ$ for J sufficiently large. ■

Figure 1 presents a geometrical interpretation of Theorems 2 and 3 and Lemma 9. As the value of J increases, the set $\hat{Q}_J(\Omega)$ converges to $Q(\Omega)$ from the inside, while $Q_J(\Omega)$ converges to $Q(\Omega)$ from the outside. Therefore, for J sufficiently large, $\Omega \subset \hat{Q}_J(\Omega)^\circ$, as illustrated in the right image. The complexity of sets $Q_J(\Omega)$ and $\hat{Q}_J(\Omega)$ will, in principle, increase as $J \to \infty$ (as reasonable from the expression (18)). Theorem 4 combines the results of Theorems 2 and 3 in order to guarantee the contractivity of the set $\hat{Q}_J(\Omega)$.

Theorem 4 *Given a polyhedral C-set* $\Omega \subset \mathbb{R}^n$ *and* $J \geq J^\star$*, if* $\Omega \subset \hat{Q}_J(\Omega)^\circ$ *then* $\hat{Q}_J(\Omega) \subset \mathbb{R}^n$ *is a SNS contractive polyhedral C-set for* (6).

Proof: Relation $\Omega \subset \hat{Q}_J(\Omega)^\circ$ guarantees the existence of $\hat{\lambda} \in (0,1)$ such that $\Omega \subseteq \hat{\lambda}\hat{Q}_J(\Omega)$. Then, from (23), $\hat{Q}_J(\Omega) \subseteq Q(\Omega) \subseteq Q(\hat{\lambda}\hat{Q}_J(\Omega))$ and we conclude from the definition of contractivity that $\hat{Q}_J(\Omega)$ is SNS $\hat{\lambda}$-contractive for (6). ■

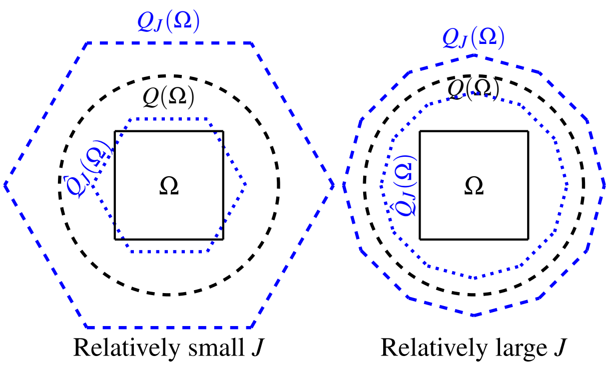

Fig. 1. Interpretation of Theorems 2 and 3 and Lemma 9.

The properties above are used in Algorithm 1, which applies them recursively to provide an increasing sequence of estimates of the (SNS) RAO of the system. The MPT toolbox [14], which has functions to manipulate polyhedral sets, can be used. Moreover, an initial SNS contractive polyhedral C-set $\Omega_0 \subset \mathbb{R}^n$ for (6) is required. It is suggested to obtain this set in the region of linearity $\mathscr{L} \triangleq \{x \in \mathbb{R}^n : K(\delta)x \in \mathscr{U}, \forall \delta \in \Delta\}$ of (6), i.e. the region where the control input does not saturate, using the method proposed in [9], which is able to compute a contractive polyhedral C-set for the linear difference inclusion $x_{k+1} \in \{(A(\delta) + B_r K(\delta))x_k : \delta \in \Delta\}$ provided that it is exponentially stable.

Algorithm 1 Increasing sequence of estimates of $\Gamma_{SNS} \subseteq \Gamma_c$

Input: Initial SNS contractive polyhedral C-set $\Omega_0 \subset \mathbb{R}^n$ for (6), $\bar{i} \in \mathbb{N}$, $J_0 \geq J^\star$

$i \leftarrow 0$, $J \leftarrow J_0$

while $i < \bar{i}$ **do**

Compute $\hat{Q}_J(\Omega_i)$ according to Definition 7

if

$$\Omega_i \subset \hat{Q}_J(\Omega_i)^\circ \tag{27}$$

then

$$\Omega_{i+1} \triangleq \hat{Q}_J(\Omega_i), \quad J_{i+1} \triangleq J \tag{28}$$

$i \leftarrow i+1$

end if

Increment J

end while

Output: Estimate of the RAO: $\Omega_{\bar{i}}$

Remark 2 *Algorithm 1 generates a sequence* $\{\Omega_i\}_{i=0}^{\bar{i}}$ *of polyhedral C-sets and a corresponding strictly increasing sequence of integers* $\{J_i\}_{i=1}^{\bar{i}}$ *that satisfy for all i:*

$$\Omega_{i+1} \subseteq Q(\Omega_i) \quad \textit{(from (28) and Theorem 2)} \tag{29a}$$

$$\Omega_i \subset \Omega_{i+1}^\circ \quad \textit{(from (27) and (28))} \tag{29b}$$

$$\Omega_{i+1} \textit{ is SNS } \lambda_{i+1}\textit{-contractive for (6)} \quad \textit{(from (27),(28) and Theorem 4)} \tag{29c}$$

$$\Omega_{i+1} \subseteq \Gamma_{SNS} \subseteq \Gamma_c \quad \textit{(from (29c) and Corollary 1)} \tag{29d}$$

$$\Omega_{i+1} = \hat{Q}_{J_{i+1}}(\Omega_i) \quad \textit{(from (28))} \tag{29e}$$

where in (29c) *the sets* Ω_i *have not necessarily the same contraction factor* λ_i *since* $\lambda \in (0,1)$ *in Theorem 4 depends on* Ω *and* J.

Theorem 3 guarantees that the test (27) (the *if* statement) will eventually be true since J is always incremented, thus the algorithm has a finite execution time. Moreover, the estimate $\Omega_{\bar{i}}$ of the RAO is related to $x_0 = [x_p^T(0)\ u^T(0)]^T$ but $x_p(0)$ and $u(0)$ are actually coupled by the relation $u(0) = \text{sat}(K_p x_p(0))$. Hence, considering the H-representation $\Omega_{\bar{i}} = \mathscr{P}(H,h) = \{x \in \mathbb{R}^n : Hx \leq h\}$, the "safe" set of plant initial states corresponds to the union of 3^m polytopes [13]:

$$\Omega_{\bar{i},x_p} \triangleq \left\{ x_p \in \mathbb{R}^{n_p} : H \begin{bmatrix} x_p \\ \text{sat}(K_p x_p) \end{bmatrix} \leq h \right\}. \tag{30}$$

Notice that the bigger $\bar{i}$ is, the bigger will be the estimate $\Omega_{\bar{i},x_p}$ of the RAO of the system. However, the polytopes Ω_i

computed by Algorithm 1 tend to become more complex at each iteration. Then, the execution time of the computer code also grows at each iteration and, in practice, the maximum number of iterations $\bar{i}$ cannot be arbitrarily large. The method of [18] has a similar problem. The main difference between these two approaches is that the method of [18] will only provide a valid estimate of the RAO upon termination of the algorithm presented in [18], while the method proposed in this work provides, at each iteration of Algorithm 1, a new estimate Ω_{i,x_p} of the RAO which encompasses the preceding one Ω_{i-1,x_p}. The numerical example of Section 6 presents a case where the execution of the algorithm in [18] becomes prohibitively complex before its stopping criterion is reached. That is, the algorithm fails to provide an estimate of the RAO. On the other hand, the application of Algorithm 1 is successful, even if $\bar{i}$ is relatively small.

5.1 Convergence properties

Consider the non-truncated version of the sequence $\{\Omega_i\}$ generated by Algorithm 1 (i.e. with $\bar{i} \equiv \infty$), which satisfies (29). We will show that $\{\Omega_i\}_{i\in\mathbb{N}}$ converges to the SNS RAO Γ_{SNS} of (6) under the following assumption.

Assumption 1 Γ_{SNS} *is bounded, i.e. there exists* $\gamma > 0$ *such that* $\Gamma_{SNS} \subseteq \mathscr{B}_\gamma$.

The convergence will be proved starting with the lemma below, whose proof is in Appendix E.

Lemma 10 *Given the sequence* $\{\Omega_i\}_{i\in\mathbb{N}}$ *generated by Algorithm 1 and* $i_1 \in \mathbb{N}$, $\exists i_2 \geq i_1$ *such that* $Q(\Omega_{i_1}) \subseteq \Omega_{i_2}$.

It is worth saying that, according to the authors' experience, the result above may also be true even if Assumption 1 does not hold. Using this lemma we obtain the following theorem.

Theorem 5 *The sequence* $\{\Omega_i\}_{i\in\mathbb{N}}$ *generated by Algorithm 1 converges to* Γ_{SNS}.

Proof: Denote the sequence generated by (14) as $\{\Theta_i\}_{i\in\mathbb{N}}$ to avoid confusion with $\{\Omega_i\}_{i\in\mathbb{N}}$, generated by Algorithm 1. The initial SNS contractive polyhedral C-set $\Omega_0 \subset \mathbb{R}^n$ for (6) satisfies the hypothesis of Theorem 1. Thus, Theorem 1 guarantees the convergence of $\{\Theta_i\}_{i\in\mathbb{N}}$ with $\Theta_0 = \Omega_0$ to Γ_{SNS}. Therefore, since $\{\Theta_i\}_{i\in\mathbb{N}}$ and $\{\Omega_i\}_{i\in\mathbb{N}}$ are both increasing sequences of nested sets, it suffices to show that for each $i_A \geq 0$ there exists $i_B \geq 0$ such that $\Theta_{i_A} \subseteq \Omega_{i_B}$. Let us prove it by induction. The case $i_A = 0$ is clearly true since $\Theta_0 = \Omega_0$. Let us assume that $\Theta_{i_A} \subseteq \Omega_{i_B}$ for $i_A, i_B \in \mathbb{N}$ and show that there exists $i_C \in \mathbb{N}$ such that $\Theta_{i_A+1} \subseteq \Omega_{i_C}$. From Lemma 10, $\exists i_C \in \mathbb{N}$ satisfying $Q(\Omega_{i_B}) \subseteq \Omega_{i_C}$. Thus, from (14), $\Theta_{i_A+1} = Q(\Theta_{i_A}) \subseteq Q(\Omega_{i_B}) \subseteq \Omega_{i_C}$, that proves the result. ■

6 Numerical example

Consider the system taken from [26], where $\Delta = [0.5,2]$ and

$$A_p = \begin{bmatrix} 1.1 & -0.6 \\ 0.5 & -1 \end{bmatrix},\ B_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ K_p = \begin{bmatrix} -1.7491 \\ 0.5417 \end{bmatrix}^T. \quad (31)$$

The initial set Ω_0 required by Algorithm 1 was obtained using the method in [9] and, at each iteration, the value of J is incremented using the rule $J \leftarrow \lceil 1.05J \rceil$, where $\lceil c \rceil$ is the smallest integer greater than or equal to c. Considering $J_0 = 20$ and $\bar{i} = 9$, Figure 2 shows the increasing sequence $\{\Omega_{i,x_p}\}_{i=0}^{\bar{i}}$ of estimates of the RAO, computed from $\{\Omega_i\}_{i=0}^{\bar{i}}$ according to (30). For ease of viewing we performed in the plot the transformation of coordinates $z_p \triangleq Tx_p$, where matrix T corresponds to a contraction in the direction of $[\cos(72^\circ)\ \sin(72^\circ)]^T$ by a factor of 20. Notice that Ω_{6,x_p} is considerably close to the last 3 sets of the sequence. However, its complexity is significantly smaller, since the H-representation of Ω_6 has 154 hyperplanes, while the one of Ω_9 has 302 hyperplanes, which shows the trade-off between number of iterations and complexity.

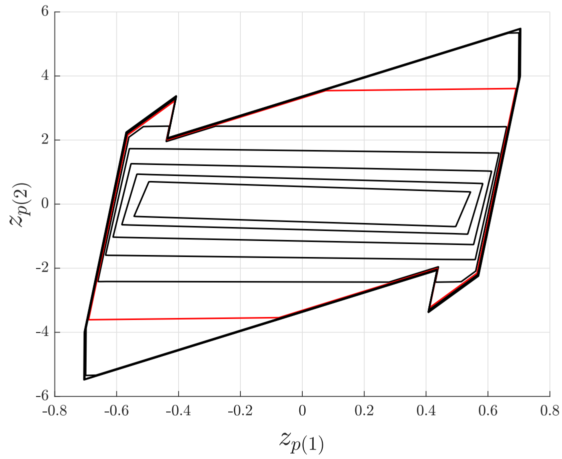

Fig. 2. Sequence $\{\Omega_{i,x_p}\}_{i=0}^{9}$, where Ω_{5,x_p} is in red.

In Figure 3, the estimate $\Omega_{\bar{i},x_p}$ of the RAO is compared to other ones from the literature. We plotted the piecewise quadratic estimate obtained with the method from [8] and the ellipsoidal estimate obtained with the one from [19]. The approach presented here resulted in an estimate of the RAO that encompasses these other two. On the other hand, it was not possible to obtain a valid estimate using the method in [26] since the corresponding matrix inequalities are not feasible for this example. Moreover, the stopping criterion of the algorithm presented in [18] was not satisfied after nearly 3 days of execution on a computer with a `Intel® Core™ i7` processor, i.e. it was not possible to obtain a valid estimate of the RAO using the method in [18] either.

An approximation of the RAO is shown in Figure 3 through black circles, where, for each point of a grid of the state

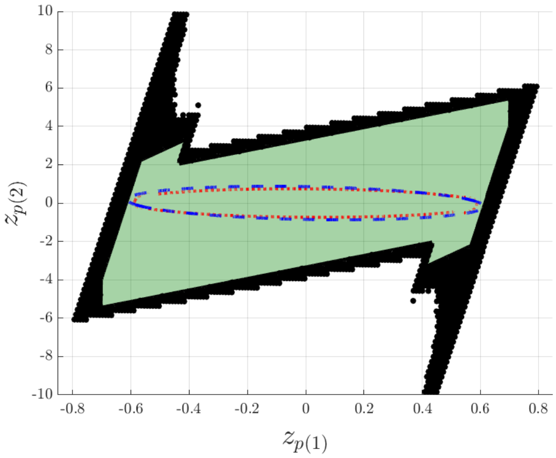

Fig. 3. Estimates of the RAO of (31) obtained with the proposed approach (filled in green) and with the methods of [19] (blue–dashed line) and [8] (red-dotted line). A numerically evaluated approximation of the RAO is depicted by black circles.

space, 2000 trajectories of the closed-loop system starting at it were simulated, considering $\{\delta_k\}_{k\in\mathbb{N}}$ to be a sequence of independent, identically distributed (i.i.d.) random variables with uniform distribution on the interval Δ. As it can be seen, the proposed approach provided a considerably accurate estimate of the RAO (specially if compared to the methods of [19,8]).

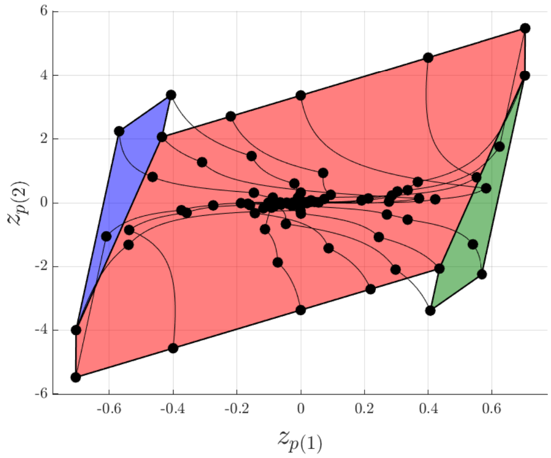

Fig. 4. Trajectories starting at the boundary of $\Omega_{\bar{i},x_p}$.

Figure 4 shows several continuous-time trajectories of system (31) with $x_p(0) \in \partial\Omega_{\bar{i},x_p}$ and δ_k randomly chosen in the interval Δ. It should be noticed that the set $\Omega_{\bar{i},x_p}$ is not invariant for the continuous-time system. It is only invariant with respect to the discrete-time trajectory $\{x_p(t_k)\}_{k\in\mathbb{N}}$ that models the behavior of $x_p(t)$ at the sampling instants t_k, represented in the figure by black circles. Nevertheless, it is ensured that for all initial conditions in $\Omega_{\bar{i},x_p}$ the corresponding continuous-time trajectories converge to the origin. Figure 4 also shows the division of $\Omega_{\bar{i},x_p}$ in $3^m = 3$ polytopes. Notice that, even if $\Omega_{\bar{i}}$ is convex, $\Omega_{\bar{i},x_p}$ is not convex in general.

7 Conclusions

A new method to obtain estimates of the RAO of linear aperiodic sampled-data systems subject to input saturation was developed. It relies on the use of a convex embedding of the difference inclusion that models the behavior of the system state between consecutive sampling instants, leading to a computational algorithm based on linear programming only. As shown by the numerical example, the application of our method outperformed the ones in [26,8,19,18]. In particular, compared to the method in [18], the main advantage of the one presented in this work is related to the numerical applicability of the corresponding algorithm, where at each iteration a valid estimate of the RAO is generated. Hence, it is possible to manage the trade-off between number of iterations and complexity of the resulting polytope.

Since the proposed method does not rely on the special structure of the matrices in (4) (with the exception of Lemma 6, which would possibly have to be adapted depending on the case), it is in principle possible to extend the approach to cope with other types of sampling, as in the case of impulsive inputs. Moreover, since the method computes invariant polyhedral sets for the discrete-time model that describes the trajectory of the system's state at the sampling instants, our method could also be useful to guarantee state constraints satisfaction along the continuous-time trajectories. This fact has not been exploited in this work but it could be the focus of future research related to model predictive control, for instance.

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A Proof of Lemma 4

First notice that a C-set Ω is always γ-invariant for (7) with $\gamma = \Psi_\Omega(A_c\Omega)$, since for $x \in \partial\Omega$ one has (using Lemma 1):

$$\begin{aligned} D^+\Psi_\Omega(x) =& \limsup_{h\to 0^+} \frac{\Psi_\Omega(x+hA_cx)-\Psi_\Omega(x)}{h} \\ \leq& \limsup_{h\to 0^+} \frac{\Psi_\Omega(x)+h\Psi_\Omega(A_cx)-\Psi_\Omega(x)}{h} \\ =& \Psi_\Omega(A_cx) \leq \Psi_\Omega(A_c\Omega) = \gamma. \end{aligned}$$

The result of the lemma follows directly from the fact that (10) implies, for $\beta = \dfrac{\|A_c\|r_2}{r_1}$, that:

$$\Psi_{\Omega_i}(A_c\Omega_i) \leq \beta, \quad \forall i \in I. \tag{A.1}$$

To see this, consider $i \in I$ arbitrary. For all $x \in \Omega_i \subseteq \mathscr{B}_{r_2}$, $\|A_cx\| \leq \|A_c\|\|x\| \leq \|A_c\|r_2$. Thus,

$$A_cx \in \|A_c\|r_2\mathscr{B} = \frac{\|A_c\|r_2}{r_1}\mathscr{B}_{r_1} \subseteq \frac{\|A_c\|r_2}{r_1}\Omega_i = \beta\Omega_i.$$

B Proof of Lemma 6

We will show the result only for $Q_J(\Omega)$, $\forall J \geq J^\star$. The proof for $Q(\Omega)$ is analogous (it suffices to replace Δ_J by Δ). From (18) it follows that $Q_J(\Omega)$ is an intersection of closed and convex sets. Hence it is also closed and convex. Let us show that the origin is in the interior of $Q_J(\Omega)$. Notice that the set-valued map $\mathbb{R}^n \ni x \mapsto \mathscr{F}_{SNS}(x,\Delta_J,\mathscr{S}) \subset \mathbb{R}^n$ is continuous (according to Definition 1.4.3 of [2]). In particular, it is upper semicontinuous [2, Def. 1.4.1] at the origin. Thus, since Ω is a neighborhood of $\mathscr{F}_{SNS}(0,\Delta_J,\mathscr{S}) = \{0\}$ (recall that $0 \in \Omega^\circ$), there exists by definition of upper semicontinuity $c > 0$ such that $\mathscr{F}_{SNS}(x,\Delta_J,\mathscr{S}) \subseteq \Omega$ for all $x \in \mathscr{B}_c$. That is, $\mathscr{B}_c \subseteq Q_J(\Omega)$ and the conclusion follows.

To guarantee that $Q_J(\Omega)$ is a C-set, we still have to show it is bounded. So let us prove the existence of $r = r(\gamma) > 0$ satisfying the statement of the lemma. Using the definitions of $A(\delta)$ and $K(\delta)$, we deduce from (13) that

$$\begin{aligned} \mathscr{F}_{SNS}(x_k,\delta_k,S_k) = x_{k+1}(x_k,\delta_k,S_k) &= \begin{bmatrix} x_{p,k+1}(x_k,\delta_k) \\ u_{k+1}(x_k,\delta_k,S_k) \end{bmatrix} \\ &= \begin{bmatrix} x_{p,k+1}(x_k,\delta_k) \\ \mathrm{sat}_{S_k}\left(K_p x_{p,k+1}(x_k,\delta_k)\right) \end{bmatrix} \\ x_{p,k+1}(x_k,\delta_k) &= \begin{bmatrix} e^{A_p\delta_k} & \int_0^{\delta_k} e^{A_ps}\mathrm{d}sB_p \end{bmatrix} x_k \end{aligned}$$

Given $\delta_A, \delta_B \in \Delta_J$, notice that

$$\begin{bmatrix} x_{p,k+1}(x_k,\delta_A) \\ x_{p,k+1}(x_k,\delta_B) \end{bmatrix} = \begin{bmatrix} e^{A_p\delta_A} & \int_0^{\delta_A} e^{A_p s}\mathrm{d}sB_p \\ e^{A_p\delta_B} & \int_0^{\delta_B} e^{A_p s}\mathrm{d}sB_p \end{bmatrix} x_k \triangleq \Lambda(\delta_A,\delta_B)x_k$$

Then

$$\max\{\|x_{p,k+1}(x_k,\delta_A)\|,\|x_{p,k+1}(x_k,\delta_B)\|\} \geq \frac{1}{\sqrt{2}}\|\Lambda(\delta_A,\delta_B)x_k\| \geq \frac{1}{\sqrt{2}}\sigma_{min}\left(\Lambda(\delta_A,\delta_B)\right)\|x_k\|. \quad \text{(B.1)}$$

Claim 1 *$\sigma_{min}\left(\Lambda(\tau_m,\delta)\right) > 0$ if $\delta \in \Delta \setminus \Delta^\star$*

Proof: This is equivalent to the full column rank property of $\Lambda(\tau_m,\delta)$ because $n_p \geq m$. Since

$$\Lambda(\tau_m,\delta) = \underbrace{\begin{bmatrix} I_{n_p} & \int_0^{\tau_m} e^{A_p s}ds \\ I_{n_p} & \int_0^{\delta} e^{A_p s}ds \end{bmatrix}}_{\triangleq \Lambda_0(\tau_m,\delta)} \begin{bmatrix} I_{n_p} & 0 \\ A_p & I_{n_p} \end{bmatrix} \begin{bmatrix} I_{n_p} & 0 \\ 0 & B_p \end{bmatrix}$$

(recall that $\int_0^\delta Ae^{As}ds = \int_0^\delta \frac{d}{ds}[e^{As}]ds = [e^{As}]|_0^\delta$ for a matrix A) and B_p has full column rank by assumption, it suffices to prove that $\Lambda_0(\tau_m,\delta)$ is nonsingular. Assume by contradiction that $v = [v_1^T \; v_2^T]^T, v \neq 0$, with $v_1, v_2 \in \mathbb{R}^{n_p}$, satisfies $\Lambda_0(\tau_m,\delta)v = 0$. Then $v_2 \neq 0$ and $\int_{\tau_m}^{\delta} e^{A_p s}dsv_2 = 0$, i.e. $\int_{\tau_m}^{\delta} e^{A_p s}ds$ is singular. But the eigenvalues of this matrix are given by $\bar{\lambda} = \int_{\tau_m}^{\delta} e^{\lambda s}ds$, where λ is the corresponding eigenvalue of A_p, and $\delta \in \Delta \setminus \Delta^\star$, i.e. we are excluding the case $\delta = \tau_m + 2\pi z/\omega_l, z \in \mathbb{Z}$, which would lead to $\bar{\lambda} = 0$. △

Consider now an ordering of the elements of $\Delta^\star$, i.e. $\tau_m = p_1 < p_2 < \ldots < p_{J^\star-1} < p_{J^\star} = \tau_M$, and take $i^\star \in \arg\max_{i \in \mathbb{N}_{J^\star-1}} (p_{i+1} - p_i)$. Notice in particular that

$$p_{i^\star+1} - p_{i^\star} \geq \frac{\tau_M - \tau_m}{J^\star - 1} > \frac{\tau_M - \tau_m}{J^\star} = \tau_{J^\star}. \quad \text{(B.2)}$$

From (B.2) it follows that

$$\bar{\Delta} \triangleq [\frac{p_{i^\star+1} + p_{i^\star} - \tau_{J^\star}}{2}, \frac{p_{i^\star+1} + p_{i^\star} + \tau_{J^\star}}{2}] \subset (p_{i^\star}, p_{i^\star+1}) \subseteq \Delta \setminus \Delta^\star \quad \text{(B.3)}$$

Moreover, since the length of $\bar{\Delta}$ is equal to $\tau_{J^\star}$ and the elements of Δ_J are equally spaced by the distance $\tau_J \leq \tau_{J^\star}, \forall J \geq J^\star$, one has

$$\Delta_J \cap \bar{\Delta} \neq \emptyset, \; \forall J \geq J^\star. \quad \text{(B.4)}$$

Define $\bar{\sigma} \triangleq \inf_{\delta \in \bar{\Delta}} \sigma_{min}\left(\Lambda(\tau_m,\delta)\right)$. It follows from the continuous dependence of $\delta \mapsto \sigma_{min}\left(\Lambda(\tau_m,\delta)\right)$, the compactness of $\bar{\Delta}$ and Claim 1 (which can indeed be applied since (B.3) holds) that $\bar{\sigma} > 0$. Define now $r = r(\gamma) \triangleq \frac{\sqrt{2}}{\bar{\sigma}}\gamma$. Given arbitrary $J \geq J^\star$ and $x_k \notin \mathscr{B}_r$, let us show that $x_k \notin Q_J(\Omega)$, that is, $Q_J(\Omega) \subseteq \mathscr{B}_r, \forall J \geq J^\star$, as stated by the lemma.

In (B.1) choose $\delta_A = \tau_m \in \Delta_J$ and $\delta_B = \bar{\delta}$ for some $\bar{\delta} \in \Delta_J \cap \bar{\Delta}$ ($\Delta_J \cap \bar{\Delta} \neq \emptyset$ from (B.4)). Then, since $\|x_{k+1}\| \geq \|x_{p,k+1}\|$ and $x_k \notin \mathscr{B}_r$, it follows that

$$\max\{\|x_{k+1}(x_k,\tau_m,S_k)\|,\|x_{k+1}(x_k,\bar{\delta},S_k)\|\} \geq \frac{1}{\sqrt{2}}\sigma_{min}\left(\Lambda(\tau_m,\bar{\delta})\right)\|x_k\| \geq \frac{1}{\sqrt{2}}\bar{\sigma}\|x_k\| > \frac{1}{\sqrt{2}}\bar{\sigma}r = \gamma$$

Recalling that $\Omega \subseteq \mathscr{B}_\gamma$, this inequality means that $\mathscr{F}_{SNS}(x_k,\delta_k,S_k) = x_{k+1}(x_k,\delta_k,S_k) \notin \Omega$ at least for $\delta_k = \tau_m$ or $\delta_k = \bar{\delta}$ (independently of S_k). Since $\tau_m, \bar{\delta} \in \Delta_J$, it follows from Definition 6 that $x_k \notin Q_J(\Omega)$, as we wanted to show.

C Proof of Lemma 7

Since the set-valued map $\mathbb{R}^n \ni x \mapsto \mathscr{F}_{SNS}(x,\Delta,\mathscr{S}) \subset \mathbb{R}^n$ is continuous, it follows that the one-step set $Q(\Omega)$ of any open set $\Omega \subseteq \mathbb{R}^n$ is also open [2, Proposition 1.4.4] [2]. In particular, $Q(\Theta_2^\circ) = Q(\Theta_2^\circ)^\circ$ and we conclude that

$$Q(\Theta_1) \underbrace{\subset}_{\Theta_1 \subset \Theta_2^\circ} Q(\Theta_2^\circ) = Q(\Theta_2^\circ)^\circ \subseteq Q(\Theta_2)^\circ$$

where the first inclusion is strict being $Q(\Theta_1)$ a C-set (Lemma 6).

D Proof of Lemma 9

Let us consider the nontrivial case $c \neq 0$. Since by definition $Q_J(\Omega) \supseteq Q(\Omega), \forall J \in \mathbb{N}_+$, and $0 \in Q(\Omega)^\circ$, there exists $r_1 > 0$ such that

$$\mathscr{B}_{r_1} \subseteq Q_J(\Omega), \quad \forall J \in \mathbb{N}_+. \quad \text{(D.1)}$$

Moreover, from Lemma 6, there exists $r_2 > 0$ such that

$$Q_J(\Omega) \subseteq \mathscr{B}_{r_2}, \quad \forall J \geq J^\star. \quad \text{(D.2)}$$

Combining (D.1) and (D.2), Lemma 4 guarantees the existence of $\beta \in \mathbb{R}$ such that $Q_J(\Omega)$ is β-invariant for $\dot{x}(t) = A_c x(t)$ for all $J \geq J^\star$. It then follows from Remark 1 that $\beta^J(\Omega) \leq \beta, \forall J \geq J^\star$. Thus, since $\tau_J \to 0$ as $J \to \infty$,

$$1 \leq \alpha^J(\Omega) = \max\{1, e^{\beta^J(\Omega)\tau_J}\} \leq \max\{1, e^{\beta\tau_J}\} \overset{J\to\infty}{\to} 1. \quad \text{(D.3)}$$

That is, given $c \in (0,1)$, there exists $\bar{J} \geq J^\star$ such that $\alpha^J(\Omega) < 1/c, \forall J \geq \bar{J}$. Thus, $cQ(\Omega) \subset Q(\Omega)^\circ/\alpha^J(\Omega) \subseteq Q_J(\Omega)^\circ/\alpha^J(\Omega) = \hat{Q}_J(\Omega)^\circ$, $\forall J \geq \bar{J}$, where the first set inclusion holds since $Q(\Omega)$ is a C-set (Lemma 6) and the equality follows from the definition of $\hat{Q}_J(\Omega)$.

[2] $Q(\Omega)$ is called the core of Ω by $F_{SNS}(\cdot,\Delta,\mathscr{S})$ in [2].

E Proof of Lemma 10

Consider first the claim below.

Claim 2

$$\{\alpha^J(\Omega_i)\}_{J=J^\star}^{\infty} \to 1 \text{ uniformly in } i \in \mathbb{N} \tag{E.1}$$

Proof: The proof follows the same reasoning of Appendix D:

- *Since* $Q(\Omega_0) \subseteq Q_J(\Omega_0) \subseteq Q_J(\Omega_i), \forall J \in \mathbb{N}_+, \forall i \in \mathbb{N}$, *there exists* $r_1 > 0$ *such that* $\mathscr{B}_{r_1} \subseteq Q_J(\Omega_i), \forall J \in \mathbb{N}_+, \forall i \in \mathbb{N}$. *Moreover, from Assumption 1,* $\Omega_i \subseteq \Gamma_{SNS} \subseteq \mathscr{B}_\gamma, \forall i \in \mathbb{N}$. *Thus, using Lemma 6, it follows that there exists* $r_2 = r_2(\gamma) > 0$ *such that* $Q_J(\Omega_i) \subseteq \mathscr{B}_{r_2}, \forall J \geq J^\star, \forall i \in \mathbb{N}$. *Then, from Lemma 4, there exists* $\beta \in \mathbb{R}$ *such that* $Q_J(\Omega_i)$ *is* β*-invariant for* $\dot{x}(t) = A_c x(t)$, $\forall J \geq J^\star, \forall i \in \mathbb{N}$.
- *It follows that* $\beta^J(\Omega_i) \leq \beta, \forall J \geq J^\star, \forall i \in \mathbb{N}$ *(see Remark 1).*
- *As shown in* (D.3), *the sequence* $\{\alpha^J(\Omega_i)\}_{J \geq J^\star}$ *is lower and upper bounded for all* $i \in \mathbb{N}$ *by sequences that do not depend on i and converge to* 1. △

Since, from (29b), $\Omega_{i_1} \subset \Omega_{i_1+1}^\circ$, it follows that $Q(\Omega_{i_1}) \subset Q(\Omega_{i_1+1})^\circ$ (Lemma 7). Thus, recalling that $Q(\Omega_{i_1}), Q(\Omega_{i_1+1})$ are C-sets (Lemma 6), there exists $c_1 \in (0,1)$ such that

$$Q(\Omega_{i_1}) \subseteq c_1 Q(\Omega_{i_1+1}). \tag{E.2}$$

Consider now $c_2 \in (c_1, 1)$. Applying Lemma 9, there exists $\bar{J} \geq J^\star$ such that

$$c_2 Q(\Omega_{i_1+1}) \subseteq \hat{Q}_J(\Omega_{i_1+1}),\ \forall J \geq \bar{J}. \tag{E.3}$$

We can split the rest of the proof into two cases depending on the value J_{i_1+2} of J used by the algorithm to generate the set Ω_{i_1+2}, according to (29e).

a) $J_{i_1+2} \geq \bar{J}$: the proof end here with $i_2 = i_1 + 2$ because

$$Q(\Omega_{i_1}) \underbrace{\subset}_{\text{(E.2)},c_2>c_1} c_2 Q(\Omega_{i_1+1}) \underbrace{\subseteq}_{\text{(E.3)}} \hat{Q}_{J_{i_1+2}}(\Omega_{i_1+1}) \underbrace{=}_{\text{(29e)}} \Omega_{i_1+2} = \Omega_{i_2}$$

b) $J_{i_1+2} < \bar{J}$: from (E.1), there exists $\hat{J} \geq J^\star$ such that

$$\alpha^J(\Omega_i) < c_2/c_1,\ \forall i \in \mathbb{N},\ \forall J \geq \hat{J}. \tag{E.4}$$

Notice now that the sequence $\{J_i\}_{i \in \mathbb{N}_+}$ generated by the algorithm is strictly increasing (see Remark 2), so sooner or later the value of J_i will reach $\max\{\bar{J}, \hat{J}\}$, i.e. there exists $i_2 > i_1 + 2$ such that $J_{i_2} \geq \max\{\bar{J}, \hat{J}\}$. Then

$$\begin{aligned} c_2 Q(\Omega_{i_1+1}) &\underbrace{\subseteq}_{\text{(E.3)}} \hat{Q}_{J_{i_2}}(\Omega_{i_1+1}) \underbrace{\subseteq}_{\alpha^{J_{i_2}}(\Omega_{i_1+1}) \geq 1} Q_{J_{i_2}}(\Omega_{i_1+1}) \\ &\underbrace{\subseteq}_{\Omega_{i_1+1} \subseteq \Omega_{i_2-1} \text{ since } i_1+1<i_2-1} Q_{J_{i_2}}(\Omega_{i_2-1}) \end{aligned} \tag{E.5}$$

Multiplying (E.5) by c_1/c_2 and combining it to (E.2) we conclude that

$$\begin{aligned} Q(\Omega_{i_1}) &\underbrace{\subseteq}_{\text{(E.2)}} c_1 Q(\Omega_{i_1+1}) \underbrace{\subseteq}_{\text{(E.5)}} \frac{c_1}{c_2} Q_{J_{i_2}}(\Omega_{i_2-1}) \\ &\underbrace{\subset}_{J_{i_2} \geq \hat{J},\text{(E.4)}} Q_{J_{i_2}}(\Omega_{i_2-1}) / \alpha^{J_{i_2}}(\Omega_{i_2-1}) = \hat{Q}_{J_{i_2}}(\Omega_{i_2-1}) \underbrace{=}_{\text{(29e)}} \Omega_{i_2}. \end{aligned}$$