# Stability Analysis of Output Feedback Control Systems With a Memory-Based Event-Triggering Mechanism

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***Abstract*—This work is concerned with the stability analysis of an output feedback control system possibly influenced by unknown disturbances, where both the plant output and the controller output are subject to event-triggered sampling. We propose a new event-triggering mechanism based on the history of the measured outputs instead of the current outputs only. This novel feature provides a simple link between the parameters of the sampling criterion and the speed of convergence. Accumulation of sampling times is prevented by enforcing a minimum interevent time. The effectiveness of the proposed event-triggered scheme is illustrated by several numerical examples, including nonlinear and linear systems.**

***Index Terms*—Disturbance rejection, event-triggered control, nonlinear system, sampled-data.**

## I. Introduction

Sampled-data control for continuous-time dynamical systems is a very active research topic, in which a continuous-time plant is controlled with a digital device. Traditionally, the control task has been assumed to be executed periodically, which simplifies the implementation of the control system. However, the periodic sampling schemes may produce unnecessary updates of the sampled signals, which will cause high utilization of resources (e.g., computation time, communication bandwidth, etc.). To overcome that limitation, the event-triggering approach was proposed, where the sampling actions are determined by some function of the system state, rather than by progression of time. Several experimental results (see [9], [13], and [19]) have shown the potential of the event-triggered control to reduce the number of samplings.

In the past few years, a multitude of strategies for event-triggered control have been proposed (see [8] and [15]). Some strategies are based on the difference between the current value of the state and the previous sample (see [14] and [21]), assuming in particular input-to-state stability (ISS). Other more recent approaches require less strong assumptions and update the measure of the state only when a Lyapunov function has a sufficiently negative derivative, as the solution approaches to the equilibrium (see [18] and [20]). Other techniques are based on an observer (or a norm-observer) and require the knowledge of the (sampled) output only (see [22] and [23]). Most of the work in the literature assumes that the full plant state is available, which is a strong assumption for many practical applications where only a part of the state can be directly measured. Generalizing event-triggered control techniques to output feedback control is definitively nontrivial; the simple strategy [21] leads to Zeno phenomenon, as shown in [1] and [3]. Different event-triggering mechanisms have been proposed to solve this problem. For instance, [4] and [12] are based on state observers, which lead to more complex event-triggering schemes. In [3] and [17], the authors modified the triggering condition to guarantee an ultimate boundedness property instead of asymptotic stability. Another approach linked to the time regularization technique is presented in [1], where time-triggered control and event-triggered control are combined to rule out Zeno phenomenon, while asymptotic stability and ISS property are preserved.

In this note, we focus on the analysis of the internal stability and the input-to-output stability (IOS), under unknown disturbances, of nonlinear output feedback even-triggered control systems. In addition, we provide a procedure to upper bound the $\mathcal{L}_\infty$-gain of linear time-invariant (LTI) systems. We consider the scenario in which the sensor and the actuator are colocated, and both the plant output and the controller output are sampled synchronously. To provide asymptotic stability and IOS, we propose an event-triggering mechanism, where the sampling times are computed from the difference between the current plant output and controller output and the last sample. A novel feature of the proposed mechanism is that the history of the outputs is used to determine the sampling times. Inspired by the results in [1] and [16], the proposed triggering mechanism enforces a minimum interevent time in order to avoid accumulation of the sampling times. Our stability analysis exploits techniques inspired by the Lyapunov–Razumikhin theorem and Halanay's inequality (see, e.g., [5]). For the particular case of LTI systems, the proposed exponential stability conditions and the procedure for computing the $\mathcal{L}_\infty$-gain upper bound are written in terms of linear matrix inequalities (LMI). In addition, less conservative results in terms of the interevent times are developed by considering piecewise quadratic Lyapunov functions. The main advantage of our approach is the relation of the parameters of the sampling algorithm with the speed of convergence. Moreover, several examples suggest that these parameters are related to the interevent times, leading to a tradeoff between the speed of convergence and the number of needed updates. A preliminary version of this work is [2], where a more restricted scenario is analyzed and no disturbances are considered.

The rest of this paper is organized as follows. First, the problem under consideration and the event-triggered setup are introduced in Section II. Section III contains the stability analysis of nonlinear control systems. The results are particularized for LTI systems in Section IV. The proposed technique is illustrated by numerical examples in Section V.

*Notation:* The sets $\mathbb{S}^n$ and $\mathbb{S}^n_+$ denote the sets of symmetric matrices of dimension $n \times n$ and the set of symmetric positive-definite matrices of dimension $n \times n$, respectively. The notation $P > 0$ for $P \in \mathbb{S}^n$ means that $P$ is positive definite ($P < 0$ means negative definite). For a matrix $A \in \mathbb{R}^{n\times n}$, the notation $He(A)$ refers to $A + A^\top$. For a

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symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_m(A)$ and $\lambda_M(A)$ stand for the minimum and maximum eigenvalue, respectively. The notation $\|x\|$ is the Euclidean norm for $x \in \mathbb{R}^n$, and for a function $f : [a, b] \to \mathbb{R}^n$, a norm is defined as $\|f\| := \sup_{s\in[a,b]} \|f(s)\|$. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing, and $f(0) = 0$. The function $f$ is of class $\mathcal{K}_\infty$, if $f \in \mathcal{K}$ and $\lim_{s\to\infty} f(s) = \infty$. A continuous function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed $s$, the function $r \mapsto f(r, s)$ belongs to class $\mathcal{K}$, and for each fixed $r$, the function $s \mapsto f(r, s)$ is nonincreasing and $\lim_{s\to\infty} f(r, s) = 0$. The space of essentially bounded measurable functions is denoted by $\mathcal{L}_\infty$.

## II. Problem Statement

Consider an output-based control system formed by the feedback interconnection of a plant $\mathcal{P}$ and a controller $\mathcal{C}$. The plant is given by

$$\mathcal{P} : \begin{cases} \dot{x}_p(t) = f_p(x_p(t), u_p(t), w(t)) \\ y_p(t) = g_p(x_p(t)) \end{cases} \tag{1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state of the plant, $u_p \in \mathbb{R}^{n_{up}}$ the control input applied to the plant, $w(t) \in \mathbb{R}^{n_w}$ an unknown disturbance, and $y_p \in \mathbb{R}^{n_{yp}}$ the output of the plant. The controller is given by

$$\mathcal{C} : \begin{cases} \dot{x}_c(t) = f_c(x_c(t), u_c(t)) \\ y_c(t) = g_c(x_c(t)) \end{cases} \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, $u_c \in \mathbb{R}^{n_{uc}}$ the input of the controller, and $y_c \in \mathbb{R}^{n_{yc}}$ the control signal. In addition, let us assume that the feedback interconnection between the plant and the controller is affected by an exogenous signal $e(t) := \left[e_y^\top(t), e_u^\top(t)\right]^\top \in \mathbb{R}^{n_e}$ with $n_e := n_{yp} + n_{yc}$, such that the interconnection is given by $u_p(t) = y_c(t) + e_u(t)$ and $u_c(t) = y_p(t) + e_y(t)$. Considering the state $x := [x_p^\top, x_c^\top]^\top \in \mathbb{R}^n$ with $n := n_p + n_c$, the closed-loop system is described by

$$\begin{cases} \dot{x}(t) = f(x(t), e(t), w(t)) \\ z(t) = g(x(t)) \end{cases} \tag{3}$$

where $z(t) \in \mathbb{R}^{n_z}$ is a performance variable and

$$f(x, e, w) := \begin{bmatrix} f_p(x_p, g_c(x_c) + e_u, w) \\ f_c(x_c, g_p(x_p) + e_y) \end{bmatrix}. \tag{4}$$

The function $f$ is assumed to be continuous in all its arguments and $f(x, e, w) = 0$ if $x = 0$, $e = 0$, $w = 0$. The functions $g_p$ and $g_c$ are assumed be continuously differentiable, and there exists a function $\xi \in \mathcal{K}$ such that

$$\| \left[g_p^\top(x_p), g_c^\top(x_c)\right]^\top \| \leq \xi(\|x\|). \tag{5}$$

The function $g$ is assumed to be continuous, and in addition, there exists a function $\xi_z \in \mathcal{K}$ such that $\|g(x)\| \leq \xi_z(\|x\|)$. In order to derive the results in this work, the following assumption is considered:

*Assumption 1:* There exist a locally Lipschitz positive-definite function $V : \mathbb{R}^n \to \mathbb{R}_+$, functions $\underline{\alpha}, \overline{\alpha}, \alpha, \beta_{1w} \in \mathcal{K}_\infty$, a locally Lipschitz positive semidefinite function $\beta_e : \mathbb{R}^{n_e} \to \mathbb{R}_+$, a real number $\theta > 0$, a continuous function $H : \mathbb{R}^n \to \mathbb{R}_+$, and a continuous nonnegative function $\delta : \mathbb{R}^{n_{yp}} \to \mathbb{R}_+$ such that, for all $x \in \mathbb{R}^n$,

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \overline{\alpha}(\|x\|) \tag{6}$$

and for all $e \in \mathbb{R}^{n_e}$, and $w \in \mathbb{R}^{n_w}$

$$\begin{aligned} \langle \nabla V(x), f(x, e, w)\rangle \leq &-\alpha(V(x)) - H^2(x) - \delta(y_p) \\ &+ \theta^2\beta_e^2(e) + \beta_{1w}(\|w\|). \end{aligned} \tag{7}$$

*Remark 1:* Assumption 1 is a $\mathcal{L}_2$-gain stability property[1] of (3), which has been used for instance in [1] and [16] with slight changes.

Consider the feedback interconnection of the plant (1) and the controller (2), where both the plant output and the controller output are made through a sampling mechanism. Therefore, the input of the plant and the controller are updated at some instants $t_k$, $k \in \mathbb{N}$, referred to as *sampling* times (or *triggering* times in the context of event-triggered control). In this way, the interconnection is given by

$$u_c(t) = y_p(t_k), \quad u_p(t) = y_c(t_k) \tag{8}$$

for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$.

The sampling times can be generated in several ways. In event-triggered control, the sampling times are governed by event-triggered mechanisms, that continuously monitor the behavior of the plant and the controller, and generate events when some condition is satisfied. This work focuses on the emulation-based approach, where first the controller is designed to get some desired behavior for the continuous loop, and second, an event-triggering scheme is designed to provide a bounded deviation of the event-triggered implementation from the continuous one under Assumptions 1. Therefore, the problem is to design a sampling algorithm, i.e., the computation of the sequence $(t_k)$, $k \in \mathbb{N}$, in order to guarantee stability properties of the system and at the same time to prevent Zeno solutions.

Let us define $\zeta(t) := \left[y_p^\top(t), y_c^\top(t)\right]^\top$, where $y_p(t)$ and $y_c(t)$ are the output of the plant and the controller of the system (3). The dynamics of the event-triggered closed-loop system can be described by (3) and (4), where now the exogenous signal $e : \mathbb{R}_+ \to \mathbb{R}^{n_e}$ represents the sampling-induced error given by

$$\begin{aligned} &e(t) = -\zeta(t), \quad t \in [0, t_1) \\ &e(t) = \zeta(t_k) - \zeta(t), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{N} \end{aligned} \tag{9}$$

and whose evolution is governed between two consecutive sampling instants by $\dot{e}(t) = f_e(e(t), x(t), w(t))$ with

$$f_e(e, x, w) := \begin{bmatrix} -\dfrac{\partial}{\partial x_p} g_p(x_p) f_p(x_p, g_c(x_c) + e_u, w) \\ -\dfrac{\partial}{\partial x_c} g_c(x_c) f_c(x_c, g_p(x_p) + e_y) \end{bmatrix}. \tag{10}$$

In order to develop the main results of this work, we extend the initial condition of the system (3) on the interval $[-h, 0]$ as follows: $x(t) = x(0)$, $t \in [-h, 0]$, where $h > 0$ will be a design parameter of the proposed event-triggered mechanism. The error signal is extended similarly, $e(t) = e(0)$, $t \in [-h, 0]$. In addition, for the sake of simplicity, we define the function $V_t : [-h, 0] \to \mathbb{R}_+$ given by $V_t(s) = V(x(t + s))$, $s \in [-h, 0]$, where $V(x(t))$ is the value of the Lyapunov function in Assumption 1 along the solution to the system for some initial condition $x(0)$ and disturbance $w$.

In order to force a minimum interevent time in the line of [1] and [16], an exponential growth condition on the sampling-induced error $e$ is assumed.

*Assumption 2:* There exist $\beta_{2w} \in \mathcal{K}_\infty$ and real numbers $L_1, L_2 \geq 0$ such that for all $x \in \mathbb{R}^n$, $e \in \mathbb{R}^{n_e}$, and $w \in \mathbb{R}^{n_w}$

$$\langle \nabla\beta_e(e), f_e(e, x, w)\rangle \leq L_1\beta_e(e) + L_2H(x) + L_2\beta_{2w}(\|w\|). \tag{11}$$

*Remark 2:* The technique proposed in this work is also applicable to a control system with a static output controller given by

[1] Function $\beta_{1w}$ in Assumption 1 can be defined as a continuous positive semidefinite function, but no improvement has been found for the purpose of this work.

$y_c(t) = g_c(u_c(t))$, where $u_c \in \mathbb{R}^{n_{uc}}$ and $y_c \in \mathbb{R}^{n_{yc}}$. In this case, for analysis purpose, it can be assumed that the controller is directly connected to the plant. Hence, the event-triggered control system is modeled by (3) with $f(x, e, w) := f_p(x_p, g_c(g_p(x_p) + e), w)$, where the error signal is given by (9) with $\zeta(t) := y_p(t)$ and its evolution between sampling instants is governed by $f_e(e, x, w) := -\frac{\partial}{\partial x_p} g_p(x_p) f_p(x_p, g_c(g_p(x_p) + e), w)$.

## III. Main Results

In this section, we first present the proposed triggering mechanism. Second, asymptotic, exponential, and IOS criteria are provided for nonlinear systems.

### A. Memory-Based Event-Triggered Mechanism

The proposed triggering condition is based on the results in [1] and the idea of memory-based event-triggering proposed in our recent work [2]. The sampling algorithm consists in checking when the sampling-induced error exceeds a bound involving a moving window of the history of the plant output and control signal. In addition, the algorithm prevents from Zeno phenomenon by imposing a minimum interevent time. Consider a continuous positive-definite function $\sigma : \mathbb{R}^{n_e} \to \mathbb{R}_+$, which is assumed to satisfy

$$\sigma(\zeta) \leq \beta_V(V(x)) \tag{12}$$

for all $x \in \mathbb{R}^n$ and some function $\beta_V \in \mathcal{K}$; then, we propose the following sampling algorithm:

$$\begin{aligned} t_{k+1} := \inf \{ & t > t_k + T \text{ such that} \\ & \theta^2 \beta_e^2(e(t)) \geq \delta(y_p(t)) + \max_{s \in [t-h, t]} \sigma(\zeta(s)) \} \end{aligned} \tag{13}$$

with $0 < T \leq \mathcal{T}(\eta, \theta, \lambda, L)$, where $\beta_e$, $\theta$, and $L = [L_1, L_2]$ are from Assumptions 1 and 2, $\eta > 0$, $\lambda \in (0, 1)$, $h > 0$, and

$$\mathcal{T}(\eta, \theta, \lambda, L) := \begin{cases} \dfrac{1}{L_1 r_1} \arctan(r_2), & (1+\eta)\theta L_2 > L_1 \\ \dfrac{1}{L_1} \dfrac{1-\lambda}{1+\lambda}, & (1+\eta)\theta L_2 = L_1 \\ \dfrac{1}{L_1 r_1} \operatorname{arctanh}(r_2), & (1+\eta)\theta L_2 < L_1 \end{cases} \tag{14}$$

with

$$r_1 := \sqrt{\left| \left( \tfrac{(1+\eta)\theta L_2}{L_1} \right)^2 - 1 \right|}, \; r_2 := \frac{r_1 (1-\lambda^2)}{\frac{1}{L_1}\lambda(1+\eta)\theta(L_2^2+1) + 1 + \lambda^2}. \tag{15}$$

The parameters $\eta$, $\lambda$, and $h$ are design parameters of the event-triggering mechanism. The function $\mathcal{T}(\eta, \theta, \lambda, L)$ is based on a combination of the functions proposed in [1] and [16]. The main difference is the constant $L_2$, which allows us to easily encompass the linear case by the nonlinear theory. In addition, note that the event-triggering algorithm proposed in [1] is directly obtained by setting $\sigma(\zeta) = 0$, $L_2 = 1$, and $\lambda = 0$. Moreover, if $L_2 = 1$, then as $\eta \to 0$, $\mathcal{T}(\eta, \theta, \lambda, L)$ approaches the maximum allowable transmission interval given in [16]. In previous results based on the ISS property, the sampling algorithm aims at keeping sufficiently small the sampling-induced error to guarantee that the Lyapunov function is strictly decreasing. However, the proposed algorithm aims at guaranteeing that the maximum of the Lyapunov function in a moving time window is decreasing. This allows local increments of the Lyapunov function while still ensuring the asymptotic convergence to zero.

### B. Stability Analysis of Nonlinear Systems

*Definition 1:* The trivial solution to the event-triggered control system (3) with (9), (13), and $w = 0$ is

1) stable if $\forall \varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(0)\| \leq \delta$ implies $\|x(t)\| \leq \varepsilon$ for all $t \geq 0$;
2) attractive if there exists a $\delta_a > 0$ such that for any $\eta_a > 0$ there exists $T := T(\delta_a, \eta_a)$ such that $\|x(0)\| \leq \delta_a$ implies $\|x(t)\| \leq \eta_a$ for all $t \geq T$;
3) asymptotically stable if it is stable and attractive;
4) exponentially stable with decay rate $\gamma$ if there exist $\delta_e > 0$ and $\eta_e > 0$ such that $\|x(0)\| \leq \delta_e$ implies $\|x(t)\| \leq \eta_e e^{-\gamma t}\|x(0)\|$ for all $t \geq 0$;
5) globally asymptotically (respectively exponentially) stable if $\delta_a$ (respectively $\delta_e$) can be an arbitrarily large, finite number.

*Definition 2:* The event-triggered control system (3) with (9) and (13) is input-to-output stable if there exist functions $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}$ such that

$$\|z(t)\| \leq \max(\beta(\|x(0)\|, t), \kappa(\|w\|_\infty)) \tag{16}$$

for all $t \geq 0$, where $z$ is the performance variable along the solution to the system with initial condition $x(0) \in \mathbb{R}^n$, and disturbance $w \in \mathcal{L}_\infty$.

*Theorem 1:* Under Assumptions 1 and 2, suppose there exist a continuous nondecreasing function $\rho(s) > s$ and a function $\varrho \in \mathcal{K}_\infty$ satisfying $\varrho(s_1 + s_2) \leq \alpha(s_1) + \eta\theta\lambda s_2$ for all $s_1, s_2 \geq 0$, such that the function $\upsilon$ defined by

$$\upsilon : s \mapsto \varrho(s) - \beta_V(\rho(s)) \tag{17}$$

is of $\mathcal{K}$-class; then, the event-triggered control system (3) with (9), (13), and $w = 0$ is globally asymptotically stable. Moreover, if $\varrho$ and $\beta_V$ are Lipschitz continuous functions, then system (3) is input-to-output stable.

*Proof:* The first part of the proof is based on [1] and [16], and some details are omitted. Consider a function $R(q) = V(x) + \max(0, \theta\phi(\tau)\beta_e^2(e))$, where $q = (x, e, \tau)$, $\tau \in [t_k, t_{k+1})$, for all $k \in \mathbb{N}$ is a clock variable introduced to describe the time elapsed since the last sampling instant, and $\phi$ is the solution to $\dot{\phi} = -2L_1\phi - (1+\eta)\theta(L_2^2\phi^2 + 1)$ with $\phi(0) = \lambda^{-1}$. Consider the case $\phi(\tau) \geq 0$; then, Assumptions 1 and 2 and the sampling algorithm (13) imply[2]

$$\begin{aligned} \dot{R}(q) \leq & -\alpha(V(x)) - H^2(x) - \delta(y_p) + \theta^2\beta_e^2(e) \\ & + \beta_{1w}(\|w\|) + \theta\beta_e^2(e)\big(-2L_1\phi - (1+\eta)\theta(L_2^2\phi^2 + 1)\big) \\ & + 2\theta\phi\beta_e(e)\big(L_1\beta_e(e) + L_2 H(x) + L_2\beta_{2w}(\|w\|)\big). \end{aligned} \tag{18}$$

Consider that $\delta(y_p) \geq 0$; then, applying twice the fact that $2ab \leq \frac{1}{\kappa}a^2 + \kappa b^2$ for any $a, b \geq 0$, and $\kappa > 0$, it follows that

$$\dot{R}(q(t)) \leq -\alpha(V(x(t))) - \eta\theta^2\beta_e^2(e(t)) + \beta_w(\|w(t)\|) \tag{19}$$

where $\beta_w(s) = \max(\beta_{w1}(s), \frac{1}{\eta}\beta_{w2}^2(s))$. Considering the function $\varrho$ in (17) and the fact that $\phi(\tau) \leq \lambda^{-1}$ for all $\tau \geq 0$, it is obtained that

$$\dot{R}(q(t)) \leq -\varrho(R(q(t)) + \beta_w(\|w(t)\|). \tag{20}$$

Now, let us consider the case $\phi(\tau) \leq 0$; then, $\tau > T$ with $T$ from (13)–(15). First, we get $R(q) = V(x)$; then, Assumption 1 and $H^2(x) \geq 0$ imply $\dot{R}(q(t)) \leq -\varrho(R(q(t))) - \delta(y_p) + \theta^2\beta_e^2(e(t)) + \beta_w(\|w(t)\|)$. Using the sampling mechanism (13),

[2]The notation $\dot{R}(q)$ should be understood as the generalized directional derivative of Clarke (see [1]).

(12), the notation $V_t$, and the fact that $V(x) \le R(q)$, it is obtained that $\dot{R}(q(t)) \le -\varrho(R(q(t))) + \beta_V(\|R_t\|) + \beta_w(\|w(t)\|)$, where $R_t : [-h, 0] \to \mathbb{R}_+$ is given by $R_t(s) = R(q(t-s))$, $s \in [-h, 0]$. Since $\beta_V(\|R_t\|) \ge 0$, the term $\beta_V(\|R_t\|)$ can be added to the term on the right-hand side of (20), and thus, we get

$$\dot{R}(q(t)) \le -\varrho(R(q(t)) + \beta_V(\|R_t\|) + \beta_w(\|w(t)\|) \tag{21}$$

for all $t \ge 0$.

The rest of the proof is organized in two parts: first, we prove the global asymptotic stability, and, second, the IOS.

1) *Proof of global asymptotic stability:* Let us consider $w = 0$ and deal with stability and attractivity, separately.

   1) *Stability:* First, note that (5) implies that $\|(x(0), e(0))\| \le \omega(\|x(0)\|)$ with $\omega \in \mathcal{K}$ given by $\omega(s) := \sqrt{s^2 + \xi(s)^2}$. In addition, the fact that $\beta_e$ is continuous positive semi-definite and the inequality (6) imply that there exists a function $\overline{\alpha}_R \in \mathcal{K}_\infty$ such that $R(q) \le \overline{\alpha}_R(\|(x,e)\|)$. Now, for any given $\varepsilon > 0$, pick $\delta > 0$ such that $0 < \delta \le \omega^{-1}(\overline{\alpha}_R^{-1}(\underline{\alpha}(\varepsilon)))$. Function $\upsilon$ satisfies the conditions in Proposition 3 (given in the Appendix), and thus, Proposition 3 can be applied to function $R(q(t))$ with $\mu < R(q(0))$, which leads to $R(q(t)) \le R(q(0)) \le \overline{\alpha}_R(\omega(\delta)) \le \underline{\alpha}(\varepsilon)$ for all $t \ge 0$. Therefore, it is obtained $\underline{\alpha}(\|x(t)\|) \le V(x(t)) \le R(q(t)) \le \underline{\alpha}(\varepsilon)$ for all $t \ge 0$, and the proof of stability is complete.
   2) *Attractivity:* For any given $\delta_a$, $\eta_a > 0$, let $\mu = \underline{\alpha}(\eta_a)$ and $\vartheta = \overline{\alpha}_R(\omega(\delta_a))$; then, $\|x(0)\| \le \delta_a$ implies $R(q(0)) \le \vartheta$. The application of Proposition 3 guarantees that there exists $T(\vartheta, \mu)$ such that $R(q(t)) \le \mu$ for all $t \ge T(\vartheta, \mu)$. Therefore, it follows $V(x(t)) \le R(q(t)) \le \mu = \underline{\alpha}(\eta_a)$ for all $t \ge T(\overline{\alpha}_R(\omega(\delta_a)), \underline{\alpha}(\eta_a))$, which completes the proof of attractivity.

   The stability and the attractivity imply the asymptotic stability of the system. Since $\delta_a$ can be chosen arbitrarily large, the global asymptotic stability is proved.

2) *Proof of IOS:* Let define the function $\chi(s) := \varepsilon\upsilon(s)$, where $\upsilon$ is as in (17), for some $\varepsilon$ satisfying $0 < \varepsilon < 1$. For a given initial condition $x(0)$ and a disturbance $w$, let $\hat{t} := \inf\{t > 0 : R(q(t)) \le \chi^{-1}(\beta_w(\|w\|_\infty))\}$ with $\hat{t} = \infty$ if $R(q(t)) > \chi^{-1}(\beta_w(\|w\|_\infty))$ for all $t > 0$. Then, (21) leads to $\dot{R}(q(t)) \le -\varrho(R(q(t))) + \beta_V(\|R_t\|) + \chi(R(q(t)))$ for all $t \in [0, \hat{t})$. Lemma 2 with $\mu = \chi^{-1}(\beta_w(\|w\|_\infty))$ (note that $\varrho(s) - \beta_V(\rho(s)) - \chi(s) = (1-\varepsilon)\upsilon(s) \in \mathcal{K}$) guarantees that there exists a function $\beta \in \mathcal{KL}$ such that $R(q(t)) \le \max(\beta(R(q(0), t), \chi^{-1}(\beta_w(\|w\|_\infty)))$ for all $t \ge 0$. The bound of $R$ and the facts that $V(x) \le R(q)$ and $\|z(t)\| \le \xi_z(\|x(t)\|) \le \xi_z(\underline{\alpha}^{-1}(V(x(t))))$ lead to $\|z(t)\| \le \xi_z(\underline{\alpha}^{-1}(\max(\beta(\overline{\alpha}_R(\omega(\|x(0)\|, t), \kappa(\|w\|_\infty))))$ with $\kappa(s) = \chi^{-1}(\beta_w(s))$, and that completes the proof. ■

*Theorem 2:* Under Assumptions 1 and 2, for a given $\delta > 0$, assume that there exist positive scalars $\underline{k}$, $\overline{k}$, $k_e$, $k_\xi$, and $\lambda_1 > \lambda_2$ such that

$$\begin{aligned} &\underline{k}s^2 \le \underline{\alpha}(s), \quad \overline{k}s^2 \ge \overline{\alpha}(s), \quad \beta_e(e) \le k_e\|e\| \\ &\xi(s) \le k_\xi s, \quad \lambda_1 s \le \alpha(s), \quad \lambda_2 s \ge \beta_V(s) \end{aligned} \tag{22}$$

for all $0 \le s \le \delta$ and $e \in \mathbb{R}^{n_e}$ with $\|e\| \le \delta$, where $\beta_V$ is as in (12); then, the event-triggered control system (3) with (9), (13), $0 < T \le \mathcal{T}(\eta, \theta, \lambda, L)$ with $\eta\theta\lambda \ge \lambda_1$ and $w = 0$ is locally exponentially stable with decay rate $\gamma > 0$ given as the unique solution of

$$2\gamma = \lambda_1 - \lambda_2 e^{2\gamma h}. \tag{23}$$

In addition, if (22) holds for all $s \in [0, \infty)$ and all $e \in \mathbb{R}^{n_e}$, then the system is globally exponentially stable.

*Proof:* Let $\delta_e^2 = \frac{2}{\overline{k} + k_e^2 k_\xi^2} \min(\underline{k}\delta^2, \delta)$ and take $\varrho(s) = \lambda_1 s$. Considering inequality (21) with $w = 0$ and the bounds (22), it follows $\dot{R}(q(t)) \le -\lambda_1 R(q(t)) + \lambda_2\|R_t\|$ for all $t \ge 0$ and $\|x(0)\| \le \delta_e$. Since $\lambda_1 > \lambda_2$, then Lemma 1 (Halanay's inequality) can be applied to the above inequality. Hence, there exists $\gamma > 0$ being the unique solution to (23) such that $R(q(t)) \le R(q(0))e^{-2\gamma t}$. The bounds (22) lead to $\underline{k}\|x(t)\|^2 \le V(x(t)) \le R(q(t)) \le (\overline{k} + \theta\lambda^{-1}k_e^2 k_\xi^2)\|x(0)\|^2 e^{-2\gamma t}$. Therefore, the system (3) with (9) and (13) is exponentially stable with decay rate $\gamma$. ■

*Remark 3:* Theorem 2 provides a relation between the parameters of the sampling algorithm and the decay rate. First, note that from (23), the decay rate decreases when the parameter $h$ increases. In addition, $0.5(\lambda_1 - \lambda_2)$ is the supremum of $\gamma$, that is, $\lim_{h\to 0}\gamma(h) = 0.5(\lambda_1 - \lambda_2)$, where $\gamma(h)$ is the solution of (23) as a function of $h$. In addition, the function $\sigma$ is related to $\gamma$ through the functions $\beta_V$ and $\lambda_2$. On the other hand, small values of $\eta$ and $\lambda$ may be desirable to increase the minimum of the interevent times; however, this leads to small values of $\lambda_1$ and, thus, small values of the decay rate through (23).

## IV. Application to LTI Systems

In this section, we focus on a closed-loop system formed by an LTI plant given by

$$\mathcal{P} : \begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u_p(t) + B_{pw} w(t) \\ y_p(t) = C_p x_p(t) \end{cases} \tag{24}$$

and an LTI controller described as follows:

$$\mathcal{C} : \begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \\ y_c(t) = C_c x_c(t) \end{cases} \tag{25}$$

where $A_p$, $B_p$, $C_p$, $B_{pw}$, $A_c$, $B_c$, and $C_c$ are matrices of appropriate dimensions. The dynamics of the event-triggered closed-loop system can be described as

$$\begin{cases} \dot{x}(t) = (A + BC)x(t) + Be(t) + B_w w(t) \\ z(t) = C_z x(t) \end{cases} \tag{26}$$

where $C_z$ is some nonzero matrix of appropriate dimensions that defines the performance variable, $e(t)$ is given by (9), and

$$\begin{aligned} A &:= \begin{bmatrix} A_p & 0 \\ 0 & A_c \end{bmatrix}, \; B := \begin{bmatrix} 0 & B_p \\ B_c & 0 \end{bmatrix} \\ C &:= \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}, \; B_w := \begin{bmatrix} B_{pw} \\ 0 \end{bmatrix}. \end{aligned} \tag{27}$$

For the analysis of the above system, the general Assumptions 1 and 2 are replaced by an asymptotic stability assumption on the LTI system.

*Assumption 3:* The controller $\mathcal{C}$ renders the system (26) with $e(t) = 0$ and $w(t) = 0$ for all $t \ge 0$ asymptotically stable, and thus, for every matrix $Q \in \mathbb{S}_+^n$, there exists a matrix $P \in \mathbb{S}_+^n$ such that

$$-Q = (A + BC)^\top P + P(A + BC). \tag{28}$$

*Definition 3:* The $\mathcal{L}_\infty$-gain of the event-triggered control system (26) with (9) and (13) is defined as

$$\kappa := \inf\{\hat{\kappa} \in \mathbb{R}_+ : \exists\varphi \in \mathcal{K} \text{ s.t.} \|z\|_\infty \le \hat{\kappa}\|w\|_\infty + \varphi(\|x(0)\|), \quad \forall x(0) \in \mathbb{R}^n, \forall w \in \mathcal{L}_\infty\} \tag{29}$$

where $z$ is the performance variable of the solution to (26) with initial condition $x(0) \in \mathbb{R}^n$, and disturbance $w \in \mathcal{L}_\infty$.

Henceforth, the functions $\sigma$ and $\beta_e$ for the sampling algorithm (13) will be given by $\sigma(\zeta) := \sigma_c\zeta^\top U_\sigma\zeta$ and $\beta_e^2(e) := e^\top U_e e$ with $\sigma_c > 0$ and $U_\sigma, U_e \in \mathbb{S}_+^{n_e}$.

*Proposition 1:* For given scalars $\gamma$, $h$, $\sigma_c$, $\eta > 0$, $\lambda \in (0,1)$, with $\eta\lambda \geq 2\gamma + \sigma_c e^{2\gamma h}$ and matrices $U_\sigma, U_e \in \mathbb{S}_+^{n_e}$, assume that there exist matrices $P \in \mathbb{S}_+^n$, and $U_w \in \mathbb{S}_+^{n_w}$, and real numbers $\varsigma_1$, $\varsigma_2 > 0$ such that

$$C^\top U_\sigma C - P \leq 0 \tag{30}$$

$$(2\gamma + \sigma_c(e^{2\gamma h} - 1))P - \varsigma_1 C_z^\top C_z \geq 0 \tag{31}$$

$$\Psi + \mathrm{diag}((2\gamma + \sigma_c e^{2\gamma h})P + \varsigma_2 \Pi, 0, 0) < 0 \tag{32}$$

hold, where $\Pi = (A+BC)^\top C^\top C(A+BC)$, $Q$ is given by (28), and

$$\Psi := \begin{bmatrix} -Q & PB & PB_w \\ \star & -U_e & 0 \\ \star & \star & -U_w \end{bmatrix}. \tag{33}$$

Then, for all positive real value $0 < T \leq \mathcal{T}(\eta, 1, \lambda, L)$ and

$$L_1 = \frac{\sqrt{\|U_e\|\|CB\|}}{\sqrt{\lambda_m(U_e)}},\ L_2 = \max\left(\sqrt{\frac{\|U_e\|}{\varsigma_2}}, \frac{\sqrt{\|U_e\|\|CB_w\|}}{\sqrt{\eta\|U_w\|}}\right) \tag{34}$$

the event-triggered control system given by (9), (13), and (26) with $w = 0$ is globally exponentially stable with decay rate $\gamma$. Moreover, the $\mathcal{L}_\infty$-gain of the system is smaller than or equal to $\sqrt{\frac{1}{\varsigma_1}\|U_w\|}$.

*Proof:* Consider the Lyapunov function $V(x) = x^\top Px$, with time derivative along the solutions to (26) given by $\dot{V}(x(t)) = -x^\top(t)Qx(t) + He\left(x^\top(t)P(Be(t) + B_w w(t))\right)$. Adding $e^\top(t)U_e e(t) - e^\top(t)U_e e(t)$ and $w^\top(t)U_w w(t) - w^\top(t)U_w w(t)$ with $U_e \in \mathbb{S}_+^{n_e}$, $U_w \in \mathbb{S}_+^{n_w}$, it is obtained $\dot{V}(x(t)) \leq \varphi^\top\Psi\varphi + e^\top U_e e + \|U_w\|\|w\|^2$, where $\varphi := \left[x^\top\ e^\top\ w^\top\right]^\top$. In addition, condition (32) implies

$$\dot{V}(x(t)) \leq -\lambda_1 V(x) - \varsigma_2 x^\top x + e^\top U_e e + \|U_w\|\|w(t)\|^2 \tag{35}$$

with $\lambda_1 = (2\gamma + \sigma_c e^{2\gamma h})$. Note that condition (30) leads to $\sigma(\zeta) \leq \beta_V(V(x))$ with $\beta_V(s) = \sigma_c s$. Moreover, Assumptions 1 and 2 are satisfied by considering the following functions and real numbers: $\alpha(s) = \lambda_1 s$, $H(x) = \sqrt{\varsigma_2}\|C(A+BC)x\|$, $\theta = 1$, $\beta_{1w}(s) = \|U_w\|s^2$, $\beta_{2w}^2(s) = \eta\|U_w\|s^2$, $\delta(y_p) = 0$. Therefore, the exponential stability with decay rate $\gamma > 0$ is concluded by applying Theorem 2.

In order to obtain an upper bound of the $\mathcal{L}_\infty$-gain, let us consider (21) from the proof of Theorem 1, with the above functions and real numbers:

$$\dot{R}(q(t)) \leq -(2\gamma + \sigma_c e^{2\gamma h})R(q(t)) + \sigma_c\|R_t\| + \|U_w\|\|w(t)\|^2. \tag{36}$$

Consider some $\varepsilon$ such that $e^{2\gamma h} > \varepsilon > 1$, then (21) leads to $\dot{R}(q(t)) \leq -\varepsilon\sigma_c R(q(t)) + \sigma_c\|R_t\|$, whenever $R(q(t)) \geq \frac{\|U_w\|}{2\gamma + \sigma_c(e^{2\gamma h} - \varepsilon)}\|w\|_\infty^2$. Applying Proposition 3 with $\upsilon_1(s) = \varepsilon\sigma_c s$, $\upsilon_2(s) = \sigma_c s$, and $\rho(s) = \frac{1+\varepsilon}{2}s$, it is obtained

$$R(q(t)) \leq \max\left(\frac{\|U_w\|}{2\gamma + \sigma_c(e^{2\gamma h} - \varepsilon)}\|w\|_\infty^2, R(q(0))\right). \tag{37}$$

Choosing $\varepsilon > 1$ sufficiently close to 1, it follows that (31) implies $(2\gamma + \sigma_c(e^{2\gamma h} - \varepsilon))P - \varsigma C_z^\top C_z \geq 0$, and in addition, $\|z(t)\|^2 \leq \frac{1}{\varsigma_1}\left(2\gamma + \sigma_c(e^{2\gamma h} - \varepsilon)\right)R(q(t))$ for all $t \geq 0$, which yields to the upper bound of the $\mathcal{L}_\infty$-gain, $\kappa \leq \sqrt{\frac{1}{\varsigma_1}\|U_w\|}$, since $2\gamma + \sigma_c(e^{2\gamma h} - 1) > 2\gamma + \sigma_c(e^{2\gamma h} - \varepsilon)$. ■

*Remark 4:* Note that for any system (26) satisfying Assumption 3, we can always find $\gamma$, $\sigma_c$, $h$, $U_\sigma$, and $U_e$, such the LMIs (30)–(32) are feasible. For instance, let us set $U_\sigma = \frac{\lambda_m(P)}{\|C^\top C\|}$; then, the LMI (30) is feasible. In addition, LMI (31) is feasible for $\varsigma_1 \leq (2\gamma + \sigma_c(e^{2\gamma h} - 1)\frac{\lambda_m(P)}{\|C_z^\top C_z\|}$. Finally, by taking $U_e = \alpha_e I$ and $U_w = \alpha_w I$ with $\alpha_e, \alpha_w > 0$, and applying the Schur complement twice on (32), it follows that the LMI (32) is feasible if

$$\begin{aligned} Q - (2\gamma + \sigma_c e^{2\gamma h})P - \varsigma_2\Pi - \frac{1}{\alpha_e}PBB^\top P > 0 \\ Q - (2\gamma + \sigma_c e^{2\gamma h})P - \varsigma_2\Pi - \frac{1}{\alpha_e}PBB^\top P - \frac{1}{\alpha_w}PB_w B_w^\top P > 0 \end{aligned} \tag{38}$$

with $\Pi = He(C(A+BC))$, which hold for sufficiently large $\alpha_e$ and $\alpha_w$, and sufficiently small $\gamma$, $\sigma_c$, and $\varsigma_2$.

The conditions in Proposition 1 are obtained by using a quadratic Lyapunov function, which leads to a conservative stability criterion, especially for unstable open-loop systems. A simple relaxation of the quadratic Lyapunov functions consists in dividing the state space in different regions and considering a quadratic Lyapunov function for each region, leading to a piecewise quadratic Lyapunov function (see [10]). In order to divide the state space, we consider a uniform partition of the $\mathbb{R}^2$ subspace, which leads to a partition of the $\mathbb{R}^n$ space through an orthogonal projection defined by a matrix $\Upsilon \in \mathbb{R}^{2\times n}$ (the results may depend on the election of $\Upsilon$). For the sake of the simplicity, the following result only deals with the exponential stability, although the estimation of the $\mathcal{L}_\infty$-gain can be tackled with the same approach.

*Proposition 2:* For given scalars $\gamma$, $h$, $\sigma_c$, $\eta > 0$, $\lambda \in (0,1)$, with $\eta\lambda \geq 2\gamma + \sigma_c e^{2\gamma h}$, matrices $U_\sigma$, $U_e \in \mathbb{S}_+^{n_e}$, and $\Upsilon \in \mathbb{R}^{2\times n}$, assume that there exist matrices $P_i \in \mathbb{S}_+^n$, scalar $\varsigma > 0$, and scalars $\varrho_{1_i}$, $\varrho_{2_i} \geq 0$, $i = 1, \ldots, N$ such that

$$C^\top U_\sigma C - P_i + \varrho_{1_i} S_i < 0 \tag{39}$$

$$\begin{bmatrix} -Q_i + (2\gamma + \sigma_c e^{2\gamma h})P_i + \varsigma\Pi + \varrho_{2_i} S_i & P_i B \\ \star & -U_e \end{bmatrix} < 0 \tag{40}$$

for $i = 1, \ldots, N$, $\Gamma_1^\top(P_N - P_1)\Gamma_1 = 0$, and

$$\Gamma_i^\top(P_i - P_{i-1})\Gamma_i = 0,\quad i = 2, \ldots, N \tag{41}$$

hold, where $\Pi = (A+BC)^\top C^\top C(A+BC)$, $Q_i$ is given by (28) for each $P_i$, $\Gamma_i$ is a basis of the null space of $\Theta_i\Upsilon$ with $\Theta_i := \left[\sin\left(\frac{i\pi}{N}\right)\ -\cos\left(\frac{i\pi}{N}\right)\right]$ and $S_i := -\Upsilon^\top(\Theta_i^\top\Theta_{i-1} + \Theta_{i-1}^\top\Theta_i)\Upsilon$; then, for all positive real value $0 < T \leq \mathcal{T}(\eta, 1, \lambda, L)$ with $L_1 = \frac{\sqrt{\|U_e\|\|CB\|}}{\sqrt{\lambda_m(U_e)}}$, $L_2 = \sqrt{\frac{\|U_e\|}{\varsigma}}$, the event-triggered control system given by (9), (13), and (26) with $w = 0$ is globally exponentially stable with decay rate $\gamma$.

*Proof:* The proof follows as the proof of Proposition 1 by considering the Lyapunov function

$$V(x) = x^\top P_i x,\ \text{if}\ x^\top S_i x \geq 0,\ i = 1, \ldots, N \tag{42}$$

and by adapting [10]. ■

## V. Examples

### A. Nonlinear Example

In this example, we consider the controlled Lorenz equations (see [24]) given by the functions $f_p(x_p) = [-ax_1 + ax_2, bx_1 - x_2 - x_1x_3 + u_p, x_1x_2 - cx_3]^\top$ and $g_p(x) = x_1$ with $a$, $b$, $c > 0$ and $x_p = [x_1, x_2, x_3]^\top$. The plant is controlled by a static output feedback controller given by $g_c(u_c) = -(\frac{p_1}{p_2}a + b)u_c$ (see Remark 2), where $p_1$, $p_2 > 0$. Consider the ISS Lyapunov function $V(x) = p_1x_1^2 + p_2x_2^2 + p_2x_3^2$. By taking suitable values of $p_1$ and $p_2$, Assumptions 1 and 2 are satisfied with functions $\underline{\alpha}(s) = \min(p_1, p_2)s$, $\overline{\alpha}(s) = \max(p_1, p_2)s$, $\alpha(s) = \lambda_1 s$, $\lambda_1 := \min(2c, 2a(1 - \frac{2p_1}{5p_2}) - 18, 2 - \frac{4p_1a}{5p_2} - \frac{25a}{6p_1p_2})$,

TABLE I
EFFECT OF $\sigma_c$ AND $h$ ON THE INTEREVENT TIMES AND THE DECAY RATE (LOWER BOUND AND NUMERICAL ESTIMATION OF THE DECAY RATE ARE IN BRACKETS)

| $\sigma_c$ \ $h$ | 0.5 | 1 | 2 |
|---|---|---|---|
| 0.5 | 0.0095 (0.26, 2.69) | 0.0187 (0.24, 2.68) | 0.0287 (0.19, 1.77) |
| 1 | 0.0112 (0.17, 2.68) | 0.0193 (0.15, 2.66) | 0.0315 (0.11, 1.64) |
| 2 | 0.0151 (0.03, 2.68) | 0.0213 (0.02, 2.53) | 0.0317 (0.01, 1.47) |

$\beta_e^2(e) = \frac{2p_1}{5a}|e|^2$, $\quad H(x) = \sqrt{\frac{2p_1 a}{5}}(|x_1| + |x_2|)$, $\quad \delta(y_p) = 18y_p^2$, $\theta^2 = 0.6(p_1 a + p_2 b)^2$, $L_1 = 0$, $L_2 = 1$. Due to the static controller, it is sufficient to consider the error signal $e(t) = y_p(t_k) - y_p(t)$. The function $\sigma$ of the sampling algorithm (13) is given by $\sigma(\zeta) = \sigma_c \zeta^2$ with $0 < \sigma_c < \lambda_1 \min(p_1, p_2)$, and we select $\beta_V(s) = \frac{\sigma_c}{\min(p_1,p_2)} s$. Therefore, the global exponential stability of the event-triggered implementation of the control system follows from Theorem 2. The decay rate $\gamma$ is obtained by solving the equation $2\gamma = \lambda_1 - \frac{\sigma_c}{\min(p_1,p_2)} e^{2\gamma h}$ for a given $h > 0$. Let consider the parameter values $a = 10$, $b = 28$, $c = 8/3$ used in [24]; then, we set $p_1 = 3$, $p_2 = 3a$, $\eta = 0.03$, $\lambda = 0.04$, and it is obtained that $\mathcal{T}(\eta, \theta, \lambda, L) = 0.0021$. Table I provides the average of all the interevent times of 100 executions of the system with random initial conditions[3] $\|x(0)\| \le 10$, a simulation time of 10 s, $T = 0.002$, and several values of the design parameters. In addition, the lower bound of the decay rate is also shown in Table I. It can be observed that an increment on both $\sigma_c$ and $h$ leads to an increment of the interevent times at the expense of reducing the speed of convergence. It should be pointed out that the event-triggering condition proposed in [1] can be directly recovered by setting $\sigma_c = 0$. In this case, the decay rate is lower bounded by $\lambda_1 = 0.73$, and the average of all the interevent times of the 100 executions of the system is 0.0078. As Table I illustrates, the main advantage of the proposed algorithm is that it allows us to obtain greater interevent times in average at the expense of reducing the decay rate.

## B. LTI Example With Stable Plant

In this example, we consider a control system studied in [3], where the matrices are given by

$$A_p = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B_p = B_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$A_c = \begin{bmatrix} -2 & 1 \\ -13 & -2 \end{bmatrix}, \quad B_c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}, \quad C_c = \begin{bmatrix} 5 & 2 \end{bmatrix}. \quad (43)$$

The system is affected by a disturbance $w$, and in order to measure its impact on the system, the performance variable $z$ is defined by the matrix $C_z = [1\ 0\ 0\ 0]$.

Let set the matrix $U_s = \begin{bmatrix} 1-u & 0 \\ 0 & u \end{bmatrix}$ with $u \in (0,1)$; then, it can be expected that for a given $\sigma_c$, higher interevent times can be obtained by minimizing $\|U_e\|$. Hence, let us set the following minimization problem with decision variables $P$, $U_e$, and $U_w$:

$$\begin{cases} \min \alpha\|U_w\| + (1-\alpha)\|U_e\| \\ \text{subject to (30)–(32)} \end{cases} \quad (44)$$

where $\varsigma_1 = 1$ and $\varsigma_2 = 1 \times 10^{-5}$ in (31) and (32). The parameter $\alpha$ allows us to indirectly balance between the interevent times and the $\mathcal{L}_\infty$-gain. Considering the value $\alpha = 0.9$, $\gamma = 0.402$, $\sigma_c = 0.132$, $h = 1$, and $u = 0.01$, the optimization problem is solved with $U_e = 1.56I$ and $U_w = 0.21$. Using the results from the optimization problem, and taking $\eta = 10$ and $\lambda = 0.11$, Proposition 1 applies. Therefore, the event-triggered control system is globally exponentially stable and the $\mathcal{L}_\infty$-gain is smaller or equal to 0.46. The minimum interevent time is given by $\mathcal{T}(\eta, 1, \lambda, L) = 5.19 \times 10^{-6}$. For comparison purpose, we consider the results in [3], which provides the same upper bound of the $\mathcal{L}_\infty$-gain. In order to compare both triggering mechanism, let us consider the disturbance $w(t) = \|w\|_\infty \sin(\frac{\pi}{2}t)$, zero initial condition, and a simulation time of 30 s. Table II provides the obtained average interevent times and the number of triggering events, respectively, denoted as $\tau_{\text{avg}}$ and $N_\tau$, for several values of $\|w\|_\infty$. The number of the triggering events is considered as the sum of the sampling of $y_p$ and $y_c$. We notice that the number of triggering events significantly increases with the increment of $\|w\|_\infty$ for the sampling mechanism in [3], while it remains constant for the proposed mechanism.

[3]The initial conditions are taking inside a ball for the sake of the reproducibility of the results.

TABLE II
SIMULATION COMPARISON WITH [3]

| $\|w\|_\infty$ | 0.5 | 1 | 5 | 10 |
|---|---|---|---|---|
| [3], $\tau_{\text{avg}}$ of $y_p$ | 0.488 | 0.200 | 0.042 | 0.025 |
| [3], $\tau_{\text{avg}}$ of $y_c$ | 0.169 | 0.093 | 0.025 | 0.017 |
| [3], $N_\tau$ | 237 | 429 | 1898 | 2884 |
| Proposed, $\tau_{\text{avg}}$ | 0.12 | | | |
| Proposed, $N_\tau$ | 496 | | | |

TABLE III
EFFECT OF THE NUMBER OF REGIONS $N$ ON THE INTEREVENT TIMES

| N | 2 | 10 | 20 | 40 |
|---|---|---|---|---|
| max $J$ | $7 \times 10^{-4}$ | $35 \times 10^{-4}$ | $40 \times 10^{-4}$ | $41 \times 10^{-4}$ |
| $\mathcal{T}$ | $1.1 \times 10^{-7}$ | $6.33 \times 10^{-7}$ | $6.58 \times 10^{-7}$ | $6.66 \times 10^{-7}$ |
| $\tau_{\text{avg}}$ | 0.0269 | 0.0472 | 0.0489 | 0.0494 |

## C. LTI Example With Unstable Plant

Let now consider a control system (see [3, Example 2]) composed of a plant and a controller with matrices

$$A_p = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \ B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_p = \begin{bmatrix} -1 & 4 \end{bmatrix}$$
$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}, \ B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_c = \begin{bmatrix} 1 & -4 \end{bmatrix}. \quad (45)$$

Let us set the matrix $U_s = \left[\begin{smallmatrix} 0.01 & 0 \\ 0 & 0.99 \end{smallmatrix}\right]$. As aforementioned, we can expect that the maximization of $\sigma_c$ and minimization of $\|U_e\|$ lead to greater interevent times, and thus, it is of interest to maximize $J := \frac{\sigma_c}{\|U_e\|}$. Table III provides the maximum $J$ and $\mathcal{T}(10, 1, 0.024, L)$ as a function of $N$ obtained by Proposition 2 with $\Upsilon = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right]$. In addition, it is shown the average of the interevent times from 100 executions of the system with random initial conditions and a simulation time of 40 s, where the parameters of the proposed algorithm $h = 2$, $U_e$ is set to provide the maximum $J$, and $\sigma_c$ is set with the greatest value that provides a decay rate $\gamma = 0.005$. It can be seen that greater values of $J$ are obtained by increasing the number of regions, which indirectly entails an improvement of the interevent times.

## VI. CONCLUSION

This work proposed an event-triggering mechanism that guarantees the asymptotic/exponential stability and the IOS of event-triggered control systems, where both the plant output and the control output are subject to sampling. The proposed sampling criterion mixes a condition based on the history of the outputs and a dwell-time constraint. Both nonlinear and linear systems are analyzed. For LTI systems, the conditions for exponential stability are given in the form of LMI, and in addition, we provided a procedure to obtain an upper bound of the $\mathcal{L}_\infty$-gain. Several numerical examples showed how the interevent times can be increased by a suitable design of the parameters, but at the price of reducing the convergence rate of the trajectories. For the future work, it could be interesting to consider asynchronous sampling and to apply the proposed mechanism to multiagent systems.

## APPENDIX

*Lemma 1. (Halanay's inequality [7]):* Let $\psi : [-h, \infty) \to \mathbb{R}_+$ be bounded on $[-h, 0]$, $h > 0$, and continuous on $[0, \infty)$. Assume that for some positive constants $\lambda_2 < \lambda_1$, the following inequality holds:

$$\dot{\psi} \leq -\lambda_1 \psi(t) + \lambda_2 \max_{s \in [t-h,t]} \psi(s),\ t \geq 0. \tag{46}$$

Then, $\psi(t) \leq e^{-\gamma t} \max_{s \in [-h,0]} \psi(s), t \geq 0$, where $\gamma > 0$ is the unique positive solution of the equation $\gamma = \lambda_1 - \lambda_2 e^{\gamma h}$.

*Proposition 3:* Consider a continuous and differentiable almost everywhere function $\psi : [-h, \infty) \to \mathbb{R}_+$ satisfying

$$\dot{\psi}(t) \leq -\upsilon_1(\psi(t)) + \upsilon_2(\|\psi_t\|) \tag{47}$$

whenever $\psi(t) \geq \mu$ and $t \geq 0$, for some $\mu > 0$ and functions $\upsilon_1$ and $\upsilon_2$. In addition, assume that there exists a continuous nondecreasing function $\rho(s) > s$ such that $\upsilon(s) := \upsilon_1(s) - \upsilon_2(\rho(s))$ is nondecreasing and $\upsilon(s) > 0$ for all $s > 0$; then

1) $\psi(t) \leq \max(\mu, \psi(0), \rho^{-1}(\|\psi_0\|)), t \geq 0$;
2) there exists $T = T(\vartheta, \mu)$ such that if $\|\psi_0\| \leq \vartheta$ then $\psi(t) \leq \mu$ for all $t \geq T$ and for all $\vartheta > 0$.

*Proof:* First, we prove Statement 1. To do so, let us consider the following three cases for some $\check{t} \geq 0$:

*Case 1:* Suppose that $\psi(\check{t}) < \mu$; then, there exists $\hat{t} > \check{t}$ such that $\psi(t) \leq \mu$ for all $t \in [\check{t}, \hat{t}]$.

*Case 2:* Suppose that $\psi(\check{t}) < \rho^{-1}(\|\psi_{\check{t}}\|)$; then, there exists $\hat{t} > \check{t}$ such that $\psi(t) < \rho^{-1}(\|\psi_{\check{t}}\|)$ for all $t \in [\check{t}, \hat{t}]$.

*Case 3:* Suppose that $\rho^{-1}(\|\psi_{\check{t}}\|) \leq \psi(\check{t})$ and $\mu \leq \psi(\check{t})$; then, (47) leads to $\dot{\psi}(\check{t}) \leq -\upsilon(\psi(\check{t})) \leq 0$. Hence, it is impossible for $\psi(t)$ to exceed $\psi(\check{t})$, implying $\psi(t) \leq \psi(\check{t})$ for all $t \geq \check{t}$.

The rest of the proof follows from the combination of the three cases for all $t \geq 0$.

Now, let us focus on Statement 2. For given $\vartheta$, $\mu > 0$, suppose that $\mu < \vartheta$; otherwise, Statement 1 implies Statement 2 with $T(\vartheta, \mu) = 0$. The continuity of $\rho$ implies that there exists an $a > 0$, such that $a < s - \rho^{-1}(s)$ for $\mu \leq s \leq \vartheta$. In addition, let $\nu := \min_{\mu \leq s \leq \vartheta} \upsilon(s)$. Now, consider the time $\tau_0 \in [-h, 0]$ such that $\psi(\tau_0) = \|\psi_0\|$; then, there are two cases:

*Case 1:* If $\rho^{-1}(\psi(\tau_0)) > \psi(0)$, then Statement 1 implies $\psi(t) \leq \max(\rho^{-1}(\psi(\tau_0)), \mu) \leq \max(\psi(\tau_0) - a, \mu)$, $\forall t \geq 0$.

*Case 2:* suppose that $\rho^{-1}(\psi(\tau_0)) \leq \psi(0)$. If $\psi(0) > \mu$, then there exists a scalar $d$ with $0 < d \leq \frac{a}{\nu}$ such that $\psi(t) \leq \rho(\psi(t))$ for all $\psi \in [0, d]$, and thus, (47) leads to $\dot{\psi}(t) \leq -\upsilon(\psi(t)) \leq -\nu, t \in [0, d]$. From Statement 1, it follows $\psi(t) \leq \max(\psi(\tau_0) - a, \mu)$, $\forall t \geq d$. Hence, both cases lead to $\psi(t) \leq \max(\psi(\tau_0) - a, \mu), \forall t \geq \frac{a}{\nu}$. If $\psi(\tau_0) - a \leq \mu$, then Statement 2 is proved; otherwise, let us pick a time $\tau_1 \in [\tau_0 + \frac{a}{\nu}, \tau_0 + \frac{a}{\nu} + h]$ such that $\psi(\tau_1) = \|\psi_{\tau_0 + \frac{a}{\nu} + h}\|$. Note that $\tau_1 \geq \tau_0 + \frac{a}{\nu}$. Following the reasoning of $\tau_0$, we obtain $\psi(t) \leq \max(\psi(\tau_1) - a, \mu), \forall t \geq \tau_0 + \frac{a}{\nu} + h$. The process can be repeated for a sequence of times $\tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k$ as long as $\psi(\tau_k) \geq \mu$. Therefore, for a large enough $k$, there exists time $T \geq \tau_k$ such that $\psi(T) \leq \mu$ and Statement 1 implies $\psi(t) \leq \mu$ for all $t \geq T$. ■

*Lemma 2:* For each pair of Lipschitz continuous functions $\upsilon_1$ and $\upsilon_2$ such that there exists a continuous nondecreasing function $\rho(s) > s$ satisfying that $\upsilon(s) := \upsilon_1(s) - \upsilon_2(\rho(s))$ is a nondecreasing function and $\upsilon(s) > 0$ for all $s > 0$, there exists a function $\beta \in \mathcal{KL}$ with the following property: if $\psi : [-h, \infty) \to \mathbb{R}_+$ is a continuous and differentiable almost everywhere function that satisfies (47) for some given $\mu > 0$, then it holds that $\psi(t) \leq \max(\beta(\|\psi_0\|, t), \mu)$ for all $t \geq 0$.

*Proof:* The proof is divided in five steps.

1) Existence of a global solution to $\dot{y}(t) = -\upsilon_1(y(t)) + \upsilon_2(\|y_t\|)$. The proof follows from the Lipschitz continuity of $\upsilon_1$ and $\upsilon_2$ and the results in [6].
2) Existence of $T(\vartheta, \mu)$ such that if $\|y_0\| \leq \vartheta$ then $y(t) \leq \mu$, $\forall t \geq T(\vartheta, \mu)$. It follows by Proposition 3.
3) Comparison principle: $y_0 = \psi_0$ implies $\psi(t) \leq y(t)$, $\forall t \geq 0$. In order to prove it, let us define the continuous function $d(t) := \psi(t) - y(t)$. Assume that $y_0 = \psi_0$; then, $d(t) = 0$, $\forall t \in [-h, 0]$. By way of contradiction, suppose that $d(t) > 0$ for all $t \in (0, \varepsilon)$ for some $\varepsilon > 0$. Considering $\varepsilon$ small enough, it follows that $\dot{d}(t) > 0$ for all $t \in (0, \varepsilon)$. Let us now define $\bar{t} := \sup\{t \in [-h, 0] : y(t) = y_0\}$. If $\bar{t} < 0$ then $\|y_t\| = \|\psi_t\|$ for all $t \in [0, \varepsilon)$. On the contrary, if $\bar{t} = 0$, then Proposition 3 implies that $\dot{y}(t) < 0$ and $\dot{\psi}(t) < 0$ for all $t \in [0, \varepsilon)$, and thus, $\|y_t\| = \|\psi_t\| = \|y_0\|$ for all $t \in [0, \varepsilon)$. Therefore, from (47), it is obtain that $\dot{d}(t) \leq -\upsilon_1(\psi(t)) + \upsilon_1(y(t)) < 0$, which is a contradiction. Therefore, $d(t) \leq 0$ for all $t \in [0, \varepsilon)$. Repeating the same procedure, it can be proved that $d(t) \leq 0$ for all $t \in [0, \infty)$.
4) Steps 2 and 3 lead to $\psi(t) \leq y(t) \leq \mu$ for all $t \geq T(\vartheta, \mu)$ whenever $\|\psi_0\| = \|y_0\| \leq \vartheta$.
5) Construction of the function $\beta \in \mathcal{KL}$ from $T(\vartheta, \mu)$. The proof follows as the proof in [11, Appendix C.6]. ■

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