Throughout the homework, when we use homology, we mean homology on surfaces with coefficients in $\mathbb{Z}$. Note that this differs from the course on surfaces where we use $\mathbb{Z}_2$ coefficients, but the lecture notes on homology introduces the (very similar) formalism.

We denote by $S_g$ an orientable surface of genus $g$, which, as usual, is endowed with a cellulary embedded graph $G$. By a closed curve on $S_g$, we mean a map $S^1 \to S_g$. A closed curve is simple if that map is injective. A simple closed curve $\gamma$ is nonseparating if $S_g \setminus \gamma$ has a single connected component. A closed curve $\gamma$ induces a homology class $[\gamma]$ by "pushing" it into a closed walk on the graph $G$, where, as a multiset of oriented edges, it corresponds to a homology class. This "pushing" can be done in multiple ways, but since the resulting walks differ by boundaries of faces, they correspond to the same homology class. A homology class $c$ is represented by a closed curve $\gamma$ if $c = [\gamma]$. Two closed curves are homologous if they induce the same homology class.

Recall from Exercise sheet #5 that an element $(p, q) \in \pi_1(T) = H_1(T) = \mathbb{Z}^2$ can be represented by a simple closed curve if and only if $p$ and $q$ are relatively prime. In this homework, you can use this fact without reproving it. The goal of this homework is to obtain a similar result for other surfaces.

An element $v$ in $H_1(S_g)$ is primitive if $v \neq nw$ for any $w \in H_1(S_g)$ and any integer $n \geq 2$. We denote the greatest common denominator of two integers $a$ and $b$ by $\gcd(a, b)$.

1. Prove that if a nonzero element of $H_1(S_g)$ can be represented by a simple closed curve, then it is primitive.

2. Prove that if $g = 1$, if an element is primitive then it can be represented with a simple closed curve.

3. Prove that $S_g$ admits a basis $B$ for $H_1(S_g)$ represented by simple closed curves $a_1, b_1, \ldots, a_g, b_g$ such that $a_i$ intersects $b_i$ exactly once, and there are no other intersections between these curves. We denote by $N_i$ a small neighborhood of the pair of curves $a_i$ and $b_i$.

4. Let $v$ be an element of $H_1(S_g)$, of which the decomposition on the basis $B$ is denoted $(v_1, w_1, \ldots, v_g, w_g)$. Show that for each $i$, there is a nonseparating simple closed curve $\gamma_i$ in $N_i$ so that

$$\gcd(v_i, w_i)[\gamma_i] = v_i[a_i] + w_i[b_i].$$

5. Show that if there exist two nonseparating disjoint and non-homologous simple closed curves $\alpha$ and $\beta$ on $S_g$ such that $[\alpha] + [\beta] \neq 0$, then there also exists a simple nonseparating closed curve representing $[\alpha] + [\beta]$.

6. Show that there exists a simple closed curve $\gamma_{1,2}$ such that $\gcd(v_1, w_1, v_2, w_2)[\gamma_{1,2}] = v_1[a_1] + w_1[b_1] + v_2[a_2] + w_2[b_2]$. Hint: Use the previous question and the Euclidean algorithm.

7. By induction, show that if an element of $H_1(S_g)$ is primitive, then it can be represented with a simple closed curve.

8. Given a closed curve $\gamma$ on a surface, provide an algorithm to determine whether $\gamma$ is homologous to a simple closed curve $\gamma'$. What is the complexity of your algorithm?