CR13: Computational Topology
Exercises #4
Due October 19th

1. Let $G$ be a connected graph that is not a tree. Recall that the orientable (resp. non-orientable) genus $g(G)$ (resp. $\tilde{g}(G)$) of a graph is the smallest genus of an orientable (resp. non-orientable) surface on which it embeds. Show that they satisfy:

$$\tilde{g}(G) \leq 2g(G) + 1.$$

Let $G_n$ be the family of graphs in Figure 1. The $A_i$ are edges that wrap around and identify opposite points.

2. Show that for every $n$, $G_n$ embeds in the projective plane.

3. Show that for every $n$, $G_n$ embeds on the orientable surface of genus $n$.

   Henceforth, we assume that $G_n$ is embedded on an orientable surface $S_n$ of genus $g$. The subgraph $K_n$ is defined in Figure 2, and inherits an embedding on $S_n$ from the embedding of $G_n$.

4. Show that if $C_1$ bounds a face that is a disk, then $S$ has genus at least $n$. *Hint: Compute the faces of $K_n$.*

5. Show that if $C_1$ bounds a disk $D$ (but not necessarily a face of the embedding), then at most one of the radial arcs $A_i$ is contained in that disk.

6. Deduce from the previous question that in the embedding of $G_n$ on $S_n$, if $C_1$ bounds a disk then this disk is a face.

7. Show that $S_2$, and thus $G_2$, have genus at least 2. *Hint: If $C_1$ bounds a disk, use the previous questions. Otherwise, prove that $G_2 \setminus C_1$ is not planar, for example by finding a forbidden minor.*

8. Show that $S_n$, and thus $G_n$, have genus at least $n$.

   The family of graphs $G_n$ shows that one cannot obtain the inequality from question 1 in the other direction, i.e., bound the orientable genus by the non-orientable one.
Figure 1: The family of graphs $G_n$.

Figure 2: The family of graphs $K_n$. 