Graphs and Coverings

Francis Lazarus

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Definition

Definition A group is a set with a binary operation such that
- the order of successive operations does not matter (in time, \textit{not} in space),
- there is a \textit{unit},
- every element has an inverse.

Example

The permutation groups
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A group **morphism** is a structure preserving map: it *commutes* with the operations.

\[
\exp : (\mathbb{R}, +) \rightarrow (\mathbb{R}^*_+, \times)
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Groups and morphisms constitute a category.
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\( \exp : (\mathbb{R}, +) \to (\mathbb{R}^*_+, \times) \)

Groups and morphisms constitute a category
The operation of a group $G$ induces a group structure on the left cosets $\{gH\}_{g \in G}$ of $H < G$ iff $H$ is a normal subgroup. Then, $p : G \rightarrow G/H$, $g \mapsto gH$ and $\ker p = H$.

Conversely, the kernel of a morphism $f : G \rightarrow J$ is normal and $G/\ker f \cong \text{Im} f$.

**Example**

The (derived) subgroup $[G, G]$ of commutators of $G$. It is the smallest subgroup $D$ such that $G/D$ is commutative. $\forall f : G \rightarrow H$ with $H$ commutative, $\exists! \bar{f} :$ 

\[ 
\begin{array}{ccc}
G & \xrightarrow{f} & H \\
\downarrow \bar{f} & & \downarrow \bar{f} \\
G/[G, G] & & 
\end{array}
\]
Quick refresher on groups

Subgroups

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$\forall f : G \rightarrow H$ with $H$ commutative, $\exists! \bar{f}$:

```
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   \downarrow \bar{f}
G/[G, G] \xrightarrow{\bar{f}} H
```
Eilenberg - Mac Lane, 1945

**Definition**

A *category* consists of

- a class of *objects*,
- for any two objects $a, b$, a set $\text{Hom}(a, b)$ of morphisms with an obvious associative law of composition, such that $\text{Hom}(a, a)$ contains an identity element.

**Example**

- $\text{Grp}$,
- any group $G$ with a single object $a$ and $\text{Hom}(a, a) = G$,
- any preordered set with $|\text{Hom}(a, b)| = 1 \iff a \leq b$,
- any oriented graph with $\text{Hom}(a, b) = \{ \text{oriented } a \to b \text{ paths} \}$. 
### Categories

**Eilenberg - Mac Lane, 1945**

#### Definition

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#### Example

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Functors

Definition

A functor \( F \) between two categories \( C \) and \( D \) consists of:
- A map \( \text{Objects}(C) \rightarrow \text{Objects}(D) \),
- maps \( \text{Hom}(a, b) \rightarrow \text{Hom}(F(a), F(b)) \) that preserve identities and the composition laws.

Example

- by forgetting the group structure, we get: \( \text{Grp} \rightarrow \text{Set} \),
- a group morphism \( f : G \rightarrow H \) induces a functor between the corresponding categories,
- Algebraic topology is mainly concerned with \( \text{Top} \rightarrow \text{Grp} \) and \( \text{Top} \rightarrow \text{Ab} \).
**Functors**

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Definition

The free group $F(S)$ on a set $S$ is defined by the universal property: $\forall f : S \to G$, $\exists! \varphi$:

$$
\begin{align*}
S \xrightarrow{\iota} F(S) \\
\xrightarrow{f} \xrightarrow{\varphi} G
\end{align*}
$$

$F(S)$ can be realized as the set of freely reduced words in $S$:

$$
F(S) = \{ s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \mid s_i \in S, \ s_{i+1}^{\varepsilon_{i+1}} \neq s_i^{-\varepsilon_i} \}.
$$

Example

$F(\{s\}) \cong \mathbb{Z}$
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For a set $S$ and a set $R \subset F(S)$ of relators, the groups with presentation $\langle S ; R \rangle$ is the quotient $F(S)/N$ where $N$ is the normal closure or $R$ in $F(S)$.

$\langle S ; R \rangle = (S \cup S^{-1})^*/\sim$ with $uv \sim uss^{-1}v \sim urv$, $\forall s \in S \cup S^{-1}, \forall r \in R$.

**Example**

- $F(S) = \langle S ; - \rangle$,
- For any group, $G = \langle G ; xyz^{-1}, z = xy \rangle$
- $\langle \{s\} ; s^n \rangle \simeq \mathbb{Z}/n\mathbb{Z}$,
- $\langle S ; \{[s, t]\}_{s,t \in S} \rangle \simeq \mathbb{Z}^n$. 

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Graphs and Coverings
Free Groups

Group Presentations

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- $F(S) = \langle S ; \rangle$,
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- $\langle S ; \{[s, t]\}_{s,t \in S} \rangle \sim \mathbb{Z}^n$. 
Basic definitions and notation

Formally, a general graph $\Gamma$ consists of three things: a set $V\Gamma$, a set $E\Gamma$, and an incidence relation, that is, a subset of $V\Gamma \times E\Gamma$. An element of $V\Gamma$ is called a vertex, an element of $E\Gamma$ is called an edge, and the incidence relation is required to be such that an edge is incident with either one vertex (in which case it is a loop) or two vertices. If every

Algebraic Graph Theory, Biggs, 1974/1993

1.1 Graphs

A graph $X$ consists of a vertex set $V(X)$ and an edge set $E(X)$, where an edge is an unordered pair of distinct vertices of $X$. We will usually use $xy$ rather than $\{x, y\}$ to denote an edge. If $xy$ is an edge, then we say that $x$ and $y$ are adjacent, and denote this by

Algebraic Graph Theory, Godsil and Royle, 2001
What Really is a Graph

Definition

A graph is a quadruple \((V, A, o, -1)\), with \(o : A \rightarrow V\) and \(-1\) is a fixed-point free involution of \(A\).

*Trees, Serre, 1977 (translated by Stillwell)*
1.4 Homomorphisms

Let $X$ and $Y$ be graphs. A mapping $f$ from $V(X)$ to $V(Y)$ is a homomorphism if $f(x)$ and $f(y)$ are adjacent in $Y$ whenever $x$ and $y$ are adjacent in $X$. (When $X$ and $Y$ have no loops, which is our usual case, this definition implies that if $x \sim y$, then $f(x) \neq f(y)$.)

"There is a evident notion of morphisms for graphs", *Trees*, Serre.

**Definition I**

A morphism $(V, A, o, o^{-1}) \rightarrow (W, B, o, o^{-1})$ is given by $f : V \rightarrow W, g : A \rightarrow B$ with $o \circ g = f \circ o$ and $g \circ o^{-1} = o^{-1} \circ g$.

A non-loop edge contraction is not a morphism for Definition I.

**Definition II**

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Graph Morphisms

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Definition II

A morphism \((V, A, o, \circ^{-1}) \rightarrow (W, B, o, \circ^{-1})\) is given by \(f : V \cup A \rightarrow W \cup B\), with \(f(V) \subset W\) and \(f\) commutes with \(o\) and \(\circ^{-1}\).
Graph Morphisms

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\(f\) commutes with \(o\) and \(-1\).
A loop with basepoint $v$ in $G = (V, A, o, o^{-1})$ is a sequence of arcs $(a_1, \ldots, a_n)$ with $o(a_1) = o(a_n^{-1}) = v$ and $o(a_i) = o(a_{i+1}^{-1})$.

**Definition**

We say that $(a_1, \ldots, a, a^{-1}, \ldots, a_n)$ and $(a_1, \ldots, a_n)$ are elementarily homotopic. **Homotopy** is the transitive closure of elementary homotopies.

**Lemma**

The set of homotopy classes with basepoint $v$ is a group for the concatenation of paths. It is denoted $\pi_1(G, v)$. 

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A loop with basepoint $v$ in $G = (V, A, o, o^{-1})$ is a sequence of arcs $(a_1, \ldots, a_n)$ with $o(a_1) = o(a_n^{-1}) = v$ and $o(a_i) = o(a_{i+1}^{-1})$.

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Lemma
For a bouquet $B_n$ of $n$ cycles: $\pi_1(B_n) \cong F(n)$

**Proof.** Paths in $B_n$ are words in $A(B_n)$. Two paths are homotopic iff they freely reduce to the same word. So, $\pi_1(B_n) \cong F(A_+ (B_n))$. □
The Homotopy Functor

A morphism \( f : (G, v) \to (H, w) \) induces a group morphism \( f_* : \pi_1(G, v) \to \pi_1(H, w) \) using

\[(a_1, \ldots, a_n) \mapsto (f(a_1), \ldots, f(a_n))\]

Example

A non-loop edge contraction induces a group isomorphism: \( \beta \sim \beta' \) with \( \beta = f(\alpha) \) and \( \beta' = f(\alpha') \) \( \implies \) \( \alpha \sim \alpha' \).
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Let $T$ be a spanning tree of a graph $G$ with basepoint $v$. 
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The contraction $c : G \to B$ induces an isomorphism.

$$\pi_1(G,v) \simeq F(A_+(B))$$

The edges of $B$ are the chords of $T$ in $G$. 
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The edges of $B$ are the chords of $T$ in $G$. 
Theorem

If $G$ is connected,

$$\pi_1(G, v) \simeq \langle A_+(G \setminus T); - \rangle$$

rank $\pi_1(G, v) = |A_+(G \setminus T)| = \frac{|A|}{2} - |V| + 1$
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$$\gamma_a = T[v, o(a)] \cdot a \cdot T(o(a^{-1}), v)$$
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$$\pi(\gamma a) = a \quad (in \ B) \implies \pi_1(G, v) = \langle \{\gamma a\}_{a \in A_+(G \setminus T)} ; - \rangle$$
Graph Coverings

Gao et al., 1998
Let $G$ be a graph. For $v \in V(G)$, let

$$\text{Star}(v) := \{ a \in A(G) \mid o(a) = v \}.$$ 

**Definition**

- A graph (epi)morphism $p : H \to G$ is a **covering** if the restriction $p : \text{Star}(w) \to \text{Star}(p(w))$ is bijective for all $w \in V(H)$.
- $G$ is the **base** of $p$. For $v \in V(G)$, the set $p^{-1}(v)$ is the **fiber** above $v$. 

![Diagram](image-url)
Let \( G \) be a graph. For \( v \in V(G) \), let 
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Definition

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- \( G \) is the base of \( p \). For \( v \in V(G) \), the set \( p^{-1}(v) \) is the fiber above \( v \).
Let $p : H \rightarrow G$ be a covering and let $\gamma$ be a path in $G$.

**Definition**

A path $\delta$ in $H$ with $p(\delta) = \gamma$ is called a **lift** of $\gamma$.

**Lemma**

Let $w \in V(H)$ with $p(w) = o(\gamma)$. There exists a *unique* lift of $\gamma$ with origin $w$. 
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Lift of Homotopies

Let $p : H \to G$ be a covering.

**Lemma**

Let $\alpha \sim \beta$ be two homotopic paths in $G$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be respective lifts with the same origin. Then $\tilde{\alpha} \sim \tilde{\beta}$.

**Proof.** By induction on the number of elementary homotopies separating $\alpha$ and $\beta$. \qed

**Corollary**

$p_*$ is injective.

**Proof.**

$p_*[\alpha] = p_*[\beta] \iff p(\alpha) \sim p(\beta) \implies \alpha \sim \beta \implies [\alpha] = [\beta]$. \qed
Graph Coverings

Lift of Homotopies: Application

Lemma

\[ F(\aleph_0) \triangleleft F(n) \triangleleft F(2) \]

Remark: \( \gamma \in p_*\pi_1(Flower_n) \iff |\gamma|_a \equiv 0 \mod n \implies p_*\pi_1(Flower_n) \triangleleft \pi_1(Flower_2) \), i.e., \( F(n) \triangleleft F(2) \). What is \( F(2)/F(n) \)?
**Lemma**

\[ F(\aleph_0) < F(n) < F(2) \]

**Proof.**

Remark: \( \gamma \in p_\ast \pi_1(Flower_n) \iff |\gamma|_a \equiv 0 \mod n \implies p_\ast \pi_1(Flower_n) \triangleleft \pi_1(Flower_2) \), i.e., \( F(n) \triangleleft F(2) \). What is \( F(2)/F(n) \)?
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\[ \text{Remark: } \gamma \in p_* \pi_1(\text{Flower}_n) \iff |\gamma|_a \equiv 0 \pmod{n} \implies p_* \pi_1(\text{Flower}_n) \triangleleft \pi_1(\text{Flower}_2), \text{ i.e., } F(n) \triangleleft F(2). \text{ What is } F(2)/F(n)? \]
Proposition

Let $G$ be a connected graph. For every subgroup $U < \pi_1(G, v)$, there exists a connected covering $p_U : (G_U, w) \to (G, v)$ with $p_U \ast \pi_1(G_U, w) = U$.

Let $T$ be spanning tree of $G$ and $\gamma_a = T[v, o(a)] \cdot a \cdot T(o(a^{-1}), v)$. Define $G_U, p_U$ by

- $V(G_U) = V(G) \times \{Ug\}_{g \in \pi_1(G, x)}$,
- $A(G_U) = A(G) \times \{Ug\}_{g \in \pi_1(G, x)}$
- $o(a, Ug) = o(a)$ and $(a, Ug)^{-1} = (a^{-1}, Ug[\gamma_a])$,
- $p_U$ is the proj. on first component.

$$(o(a), Ug) \overset{(a, Ug)}{\leftrightarrow} (o(a^{-1}), Ug[\gamma_a])$$
Example

Put \( g := [\gamma_a] \), so \( \pi_1(G, v) = \langle g \rangle \). Let \( U = \langle g^2 \rangle \).
\( p_U \) is a covering: \( \text{Star}(x, Ug) = \text{Star}(x) \times \{Ug\} \)

\( G_U \) is connected: For a path \( \alpha = (a_1, \ldots, a_n) \), put 
\( \gamma_\alpha = \gamma_{a_1} \cdots \gamma_{a_n} \). Observe that \( (v, U).[\alpha] = (t(\alpha), U[\gamma_\alpha]) \).

\( p_U \ast \pi_1(G_U, (v, U)) = U: \)
\[ [\lambda] \in \text{Im} p_U \ast \iff (v, U).[\lambda] = (v, U) \iff U[\lambda] = U \iff [\lambda] \in U \]
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\[ p_U \mid_{T[v, x]} \]

\[ \lambda \in \text{Im}p_U \iff (v, U)[\lambda] = (v, U) \iff U[\lambda] = U \iff [\lambda] \in U \]
**Subgroups and Coverings**

- \( p_U \text{ is a covering: } Star(x, Ug) = Star(x) \times \{Ug\} \)
- \( G_U \text{ is connected: } \) For a path \( \alpha = (a_1, \ldots, a_n) \), put
  \[
  \gamma_\alpha = \gamma_{a_1} \cdots \gamma_{a_n}.
  \]
  Observe that \( (v, U).[\alpha] = (t(\alpha), U[\gamma_\alpha]) \).
- \( p_U \ast \pi_1(G_U, (v, U)) = U: \)
  \[
  [\lambda] \in \text{Im}p_U \ast \iff (v, U).[\lambda] = (v, U) \iff U[\lambda] = U \iff [\lambda] \in U
  \]
Definition

When $U = \{1\}$, $G_U$ is the **universal cover**.

We have

\[ \alpha \in p_\ast \pi_1(\mathbb{Z}^2\text{Grid}) \iff |\alpha|_a = |\alpha|_b = 0 \iff \alpha \in [F(2), F(2)]. \]

So, for $G = B_2$, $G_{[G,G]} = \mathbb{Z}^2\text{Grid}$.
Nielsen-Schreier theorem, mid 1920’s

Every subgroup of a free group is free.

\textbf{Proof.} Realize $F(S)$ as the $\pi_1$ of a bouquet of $|S|$ circles. A subgroup of $F(S)$ is the $\pi_1$ of a covering graph, which we know to be free. 

Note: Another proof uses the fact that a group is free iff it acts freely on a tree (Bass-Serre). Any subgroup acts obviously freely on the same tree.
Definition

A morphism $f$ between coverings $p : H \to G$ and $q : K \to G$ sends fibers to fibers. It satisfies:

$$
\begin{align*}
H & \xrightarrow{f} K \\
\downarrow p & & \downarrow q \\
G & & G
\end{align*}
$$
There is a morphism \( f \) between coverings \( p : (H, v) \to (G, u) \) and \( q : (K, w) \to (G, u) \) iff \( p_\ast \pi_1(H, v) < q_\ast \pi_1(K, w) \) in \( \pi_1(G, u) \).

**Proof.** For \( x \in V(H) \), \( \gamma : v \leadsto x \), set \( f(x) = w.p_\ast[\gamma] \).

If \( \lambda : v \leadsto x \) then \( w.p_\ast[\lambda] = w.p_\ast[\lambda \gamma^{-1}]p_\ast[\gamma] = w.p_\ast[\gamma] \)

We have \( q \circ f = p : q(f(x)) = q(w.p_\ast[\gamma]) = t(p(\gamma)) = p(x) \).

We also check that \( f \) commutes with \( o \) and \( -1 \).
Covering Morphisms

**Lemma**

There is a morphism $f$ between coverings $p : (H, v) \rightarrow (G, u)$ and $q : (K, w) \rightarrow (G, u)$ iff $p_*\pi_1(H, v) < q_*\pi_1(K, w)$ in $\pi_1(G, u)$.

**Corollary**

There is an isomorphism $f$ between coverings $p : H \rightarrow G$ and $q : K \rightarrow G$ iff $p_*\pi_1(H, v)$ and $q_*\pi_1(K, w)$ are in the same conjugacy class in $\pi_1(G, u)$ for $p(v) = q(w) = u$.

**Proof.** $\Rightarrow$: By the lemma we must have $p_*\pi_1(H, v) = q_*\pi_1(K, f(v))$ in $\pi_1(G, u)$.

$\Leftarrow$: Suppose $p_*\pi_1(H, v) = [\gamma]^{-1}.q_*\pi_1(K, w).[\gamma]$. But $[\gamma]^{-1}.q_*\pi_1(K, w).[\gamma] = q_*\pi_1(K, w.[\gamma])$ and we can apply the lemma with $f(v) = w.[\gamma]$ $\square$

**Lemma**

A morphism between coverings is a covering. **Proof.** Since restrictions of $p$ and $q$ to stars are one-to-one and since $q \circ f = p$ it must be the case for $f$. 

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Graphs and Coverings
Covering Morphisms

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There is a morphism $f$ between coverings $p : (H, v) \rightarrow (G, u)$ and $q : (K, w) \rightarrow (G, u)$ iff $p_*\pi_1(H, v) < q_*\pi_1(K, w)$ in $\pi_1(G, u)$.

**Corollary**

There is an isomorphism $f$ between coverings $p : H \rightarrow G$ and $q : K \rightarrow G$ iff $p_*\pi_1(H, v)$ and $q_*\pi_1(K, w)$ are in the same conjugacy class in $\pi_1(G, u)$ for $p(v) = q(w) = u$.

**Lemma**

A morphism between coverings is a covering.

**Proof.** Since restrictions of $p$ and $q$ to stars are one-to-one and since $q \circ f = p$ it must be the case for $f$. □
The set of coverings of a graph $G$, up to isomorphism, corresponds to the set of conjugacy classes of subgroups of $\pi_1(G)$ with the preorder relation $H \geq K$ if $\exists g \in \pi_1(G)$ with $g^{-1}Hg \subset K$.

The universal covering is the maximal element.
Jenn3D, F. Obermeyer
Definition

Let $\Gamma < Aut(G)$ acts without (arc) inversion. The quotient graph $G/\Gamma$ is given by

- $V(G/\Gamma) = \{\Gamma \cdot v\}_{v \in V(G)}$,
- $A(G/\Gamma) = \{\Gamma \cdot a\}_{a \in A(G)}$,
- $o(\Gamma \cdot a) = \Gamma \cdot o(a)$ and $(\Gamma \cdot a)^{-1} = \Gamma \cdot a^{-1}$

Note: $\Gamma$ acts without inversion $\iff (\Gamma \cdot a)^{-1} \neq \Gamma \cdot a$
Free Actions

Definition

\( \Gamma < Aut(G) \) acts \textbf{freely} if it acts without inversion and \( g \in \Gamma \setminus \{1\} \) does not fix any vertex.

Proposition

If \( \Gamma \) acts without inversion then \( p_\Gamma : G \rightarrow G/\Gamma \) is an epimorphism. It is a covering iff \( \Gamma \) acts freely on \( G \).

\textbf{Proof.} \( p_\Gamma \) restricted to Stars must be injective. Let \( g \neq Id \) fix a vertex. Then \( \exists a \in A(G) : a \neq g(a) \) and \( o(a) = o(g(a)) \).
**Free Actions**

**Definition**

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Lemma I

If $\Gamma$ acts freely on $G$ then $(p_\Gamma)_* \pi_1(G, v) \triangleleft \pi_1(G/\Gamma, \Gamma \cdot v)$

**Proof.** $p_\Gamma(v.\beta) = p_\Gamma(v) \implies \exists g \in \Gamma : g(v) = v.\beta$. So, $v.(\beta p_\Gamma(\alpha)(\beta^{-1})) = g(v).(p_\Gamma(\alpha)(\beta^{-1})) = (g(v).p_\Gamma(\alpha))(\beta^{-1}) = g(v).\beta^{-1} = v$. □
Lemma I

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**Definition**

A $p$-(auto)morphism of a covering $p : H \to G$ satisfies:

$$
\begin{array}{ccc}
H & \xrightarrow{f} & H \\
p & \downarrow & \downarrow \\
\downarrow & & \downarrow \\
G & \xleftarrow{p} & G \\
\end{array}
$$

$Aut(p) :=$ set of $p$-automorphisms.

**Lemma**

$Aut(p)$ acts freely on $H$.

Let $f \in Aut(p)$. $f(a) = a^{-1} \implies p(a) = p(a^{-1}) = p(a)^{-1}$, contradiction.

$f(v) = v \implies \forall \alpha : v \leadsto x, f(x) = f(v).p(\alpha) = v.p(\alpha) = x.$
Lemma II
If $\Gamma$ acts freely on $G$ then $\text{Aut}(p_\Gamma) = \Gamma$.

**Proof.** Obviously, $\Gamma \subset \text{Aut}(p_\Gamma)$ and $\Gamma$ acts transitively on the fibers of $p_\Gamma$. Since $\text{Aut}(p_\Gamma)$ acts freely, $\text{Aut}(p_\Gamma) \subset \Gamma$. \qed

Lemma III
If $p : (H, v) \to (G, u)$ is a covering with $p_*\pi_1(H, v) \triangleleft \pi_1(G, u)$ then $\text{Aut}(p)$ acts transitively on fibers.

**Proof.** $p(w) = p(v) \implies p_*\pi_1(H, w) = p_*\pi_1(H, v)$.

We construct $f \in \text{Aut}(p)$ such that $f(v) = w$: If $\alpha : v \leadsto x$ set $f(x) = w.[p(\alpha)]$. If $\beta : v \leadsto x$ then $w.[p(\beta)] = w.[p(\beta \alpha^{-1})][p(\alpha)] = w.[p(\alpha)]$. \qed
**Lemma II**
If $\Gamma$ acts freely on $G$ then $\text{Aut}(p_\Gamma) = \Gamma$.

**Lemma III**
If $p : (H, v) \to (G, u)$ is a covering with $p_*\pi_1(H, v) \lhd \pi_1(G, u)$ then $\text{Aut}(p)$ acts transitively on fibers.

**Proof.**
$p(w) = p(v) \implies p_*\pi_1(H, w) = p_*\pi_1(H, v)$.
We construct $f \in \text{Aut}(p)$ such that $f(v) = w$: If $\alpha : v \rightsquigarrow x$ set $f(x) = w.[p(\alpha)]$. If $\beta : v \rightsquigarrow x$ then $w.[p(\beta)] = w.[p(\beta\alpha^{-1})][p(\alpha)] = w.[p(\alpha)]$. 

\[
\begin{array}{ccc}
  f(a) & & a \\
  f(x) & & x \\
  w & & v \\
  H & & G \\
  \text{p} & & u
\end{array}
\]
Proposition

Let \( p : H \to G \). If \( \Gamma < Aut(H) \) then \( H/\Gamma \cong G \) iff

\[
\begin{array}{c}
p \uparrow \\
\uparrow \\
\uparrow \\
p \downarrow \\
H/\Gamma \cong G
\end{array}
\]

\[ p_\Gamma \]

1. \( \Gamma = Aut(p) \)
2. \( p_\ast \pi_1(H, v) \triangleleft \pi_1(G, p(v)) \)

**Proof.** \( \implies : \)

1. \( \Gamma \) acts freely \( \implies \Gamma = Aut(p_\Gamma) = Aut(p) \) by lemma II.
2. By lemma I, we also have \( p_\Gamma \ast \pi_1(H, v) \triangleleft \pi_1(H/\Gamma, p_\Gamma(v)) \) whence \( p_\ast \pi_1(H, v) \triangleleft \pi_1(G, p(v)) \).

\( \Leftarrow : \) In that case \( Aut(p) \) acts transitively by Lemma III, so \( H/Aut(p) \cong G \). \( \square \)
A covering as above is said **Galois** or **regular** or **normal**.

**Theorem**

If $p : H \to G$ is a covering then

$\text{Aut}(p) \simeq N(p_*\pi_1(H, v))/p_*\pi_1(H, v)$.

If $p$ is Galois $\text{Aut}(p) \simeq \pi_1(G, p(v))/p_*\pi_1(H, v)$.
**Definition**

A covering as above is said Galois or regular or normal.

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If $p : H \to G$ is a covering then

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If $p$ is Galois

\[ \text{Aut}(p) \cong \pi_1(G, p(v)) / p_*\pi_1(H, v) \]

**Example**

Francis Lazarus  
Graphs and Coverings
**Definition**

A covering as above is said **Galois** or **regular** or **normal**.

**Theorem**

If \( p : H \to G \) is a covering then
\[
\text{Aut}(p) \cong N(p_\ast \pi_1(H, v)) / p_\ast \pi_1(H, v).
\]

If \( p \) is Galois \( \text{Aut}(p) \cong \pi_1(G, p(v)) / p_\ast \pi_1(H, v) \).

**Proof.** \( \text{Aut}(p) \overset{F}{\to} p^{-1}(v), f \mapsto f(v) \). Put \( f_w := F^{-1}(w) \), i.e. \( f_w(v) = w \). Let \( \pi_1(G, p(v)) \overset{M}{\to} \text{Aut}(p), \alpha \mapsto f_{v.\alpha} \).

\( M \) is a morphism: \( M(\alpha \beta)(v) = f_{v.\alpha \beta}(v) = v.\alpha \beta = (v.\alpha).\beta = f_{v.\alpha}(v).\beta = f_{v.\alpha}(v.\beta) = f_{v.\alpha} \circ f_{v.\beta}(v) \). So,
\[
M(\alpha \beta) = f_{v.\alpha} \circ f_{v.\beta} = M(\alpha) \circ M(\beta).
\]

\( \ker M = \{ \alpha \mid v.\alpha = v \} = p_\ast \pi_1(H, v) \). \( \square \)
Gross and Tucker, 1987
Definition

A voltage on a graph $G$ with values in group $B$ is a map $\kappa : A(G) \to B$ with

$$\kappa(a^{-1}) = \kappa(a)^{-1}, \quad \forall a \in A(G)$$

If $B$ acts on the right on the set $S$, the voltage induces a covering $p_\kappa : G_\kappa \to G$ where

- $V(G_\kappa) = V(G) \times S$ and $A(G_\kappa) = A(G) \times S$, and
- $o(a, s) := (o(a), s)$ and $(a, s)^{-1} := (a^{-1}, s.\kappa(a))$
Lemma

Every covering \( p : H \to G \) is \((\simeq \text{ to})\) the covering induced by a voltage on \( G \).

**Proof.** Let \( T \) be a spanning tree of \((G, \nu)\). Define \( \kappa : A(G) \to \pi_1(G, \nu) \) by \( \kappa(a) = \gamma_a = T[\nu, o(a)].a.T[o(a^{-1}), \nu] \) with \( \pi_1(G, \nu) \) acting on the fiber \( p^{-1}(\nu) \).

Check that \( G_{\kappa} \xrightarrow{\simeq} H \) □
Proposition

A voltage $\kappa : A(G) \to B$ with $B$ acting on itself and $B = \langle \text{Im}\kappa \rangle$ induces a Galois covering.

**Proof.** Note that for $\alpha \in \pi_1(G, v)$, $(v, 1_B)\cdot\alpha = (v, \kappa(\alpha))$, so that for the induced morphism $\kappa : \pi_1(G, v) \to B$ we have $\ker\kappa = p_{\kappa\ast}\pi_1(G_\kappa, (v, 1_B))$. $\square$

Proposition

Conversely, a Galois covering $p : H \to G$ is induced by a voltage $\kappa : A(G) \to B$ with $B$ acting on itself.

**Proof.** Let $T$ be a spanning tree of $G$. For every $a \in A(G)$ there is a unique $f_a \in \text{Aut}(p)$ with $f_a(v) = v \cdot \gamma_a$. We let $B = \text{Aut}(p)$ and $\kappa(a) = f_a$.

Check that $G \xrightarrow{\kappa} H$. 

Francis Lazarus

Graphs and Coverings
Proposition

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Proposition

Conversely, a Galois covering $p : H \to G$ is induced by a voltage $\kappa : A(G) \to B$ with $B$ acting on itself.

**Proof.** Let $T$ be a spanning tree of $G$. For every $a \in A(G)$ there is a unique $f_a \in \text{Aut}(p)$ with $f_a(v) = v \cdot \gamma_a$. We let $B = \text{Aut}(p)$ and $\kappa(a) = f_a$.

Check that $G_\kappa \xrightarrow{\sim} H \quad \square$

\[ G_\kappa \xrightarrow{p_\kappa} H \xrightarrow{p} G \]
Summary

Quotient graphs

Non-free actions

Galois coverings

Auto-acting voltages

Free actions

Non-Galois coverings

Coverings or voltage graphs
Gross and Tucker, 1987
Topics in Topological Graph Theory, 2009
Trees, J.P. Serre, 1977 (Translation J. Stillwell)
Algebraic Topology: An Introduction, W.S. Massey, 1977
(Translation J. Stillwell)
THANK YOU!