Combinatorial Maps

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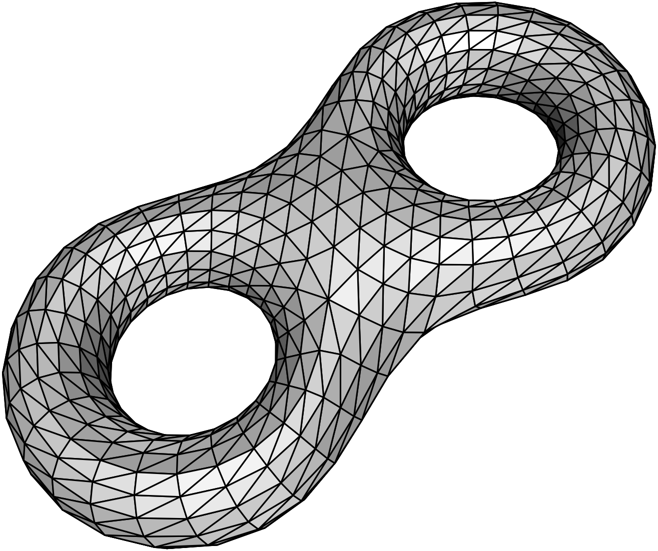
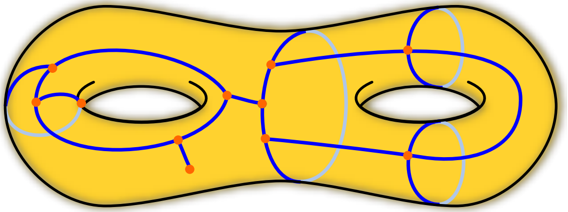


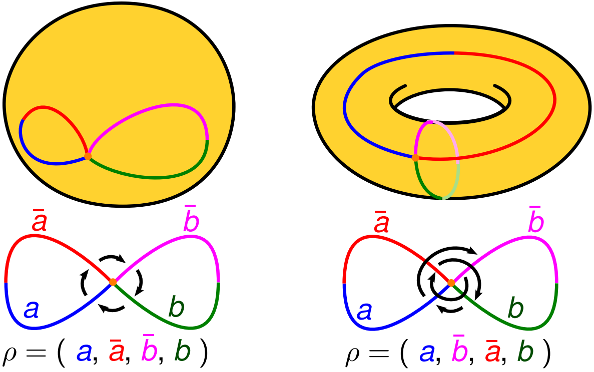
Combinatorial Maps

A *combinatorial map* encodes a graph cellularly embedded in a surface.



It is also called a *combinatorial surface* or a *cellular embedding of a graph*.

Combinatorial (oriented) Maps



Definition

A combinatorial map (G, ρ) is the data of a graph G and a *rotation system* ρ. The rotation system is a permutation on $A(G)$ whose cycles coincide with stars of G.

Combinatorial (oriented) Maps

Definition

Equivalently, a map is a triple $S = (A, \rho, ^{-1})$ where ρ is a permutation of A and $^{-1}$ is a fixed-point free involution on A.

- A vertex of S is a cycle of ρ,
- an edge of S is a cycle of $^{-1}$,
- a face of S is a cycle of $^{-1} \circ \rho$,
- $G(S) = (A/\langle\rho\rangle, A, o, ^{-1})$ is the graph of S, with $o(a) = \langle\rho\rangle a$.

Definition

A map $(A, \rho, ^{-1})$ is connected if the *monodromy group* $\langle \rho, ^{-1} \rangle$ acts transitively on A. Equivalently $G(S)$ is connected.

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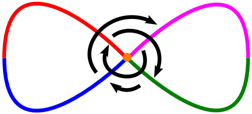
Map Realization

Proposition

To every cellular embedding $p : G \to \mathcal{M}$ we can associate a map $S(p)$. Conversely, every map S can be realized as an embedding p_S of its graph $G(S)$ such that $S(p_S)$ is isomorphic to S.



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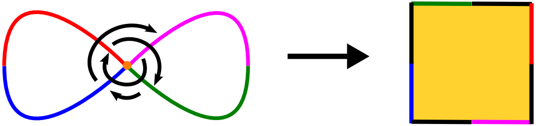

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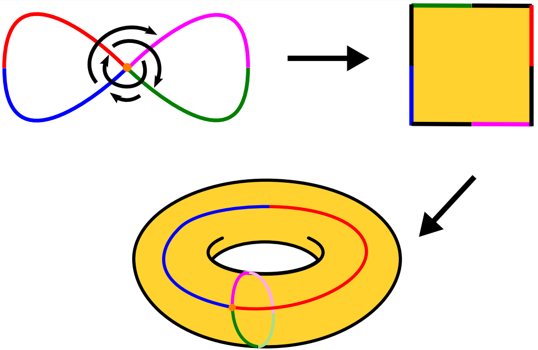

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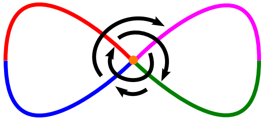

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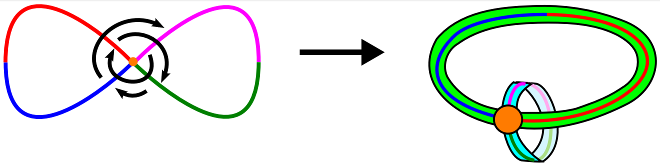

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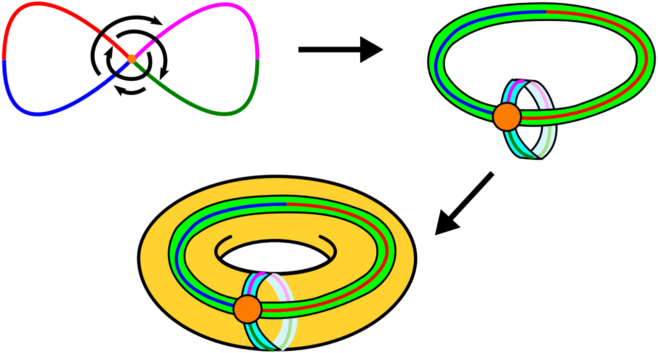


Map Morphisms

Definition

There is an evident notion of map morphism.

Map Morphisms

Definition

A map morphism $(A, \rho, ^{-1}) \to (B, \sigma, ^{-1})$ is a function $f : A \to B$ such that

- $f \circ \rho = \sigma \circ f$ and
- $f \circ ^{-1} = ^{-1} \circ f$.

Map coverings

Definition

A map covering is morphism $p : (A, \rho, ^{-1}) \to (B, \sigma, ^{-1})$ such that

- The restriction of p to each cycle of ρ is one-to-one, and
- The restriction of p to each cycle of $(^{-1} \circ \rho)$ is one-to-one.

i.e., the edges incident to a vertex or to a face are mapped bijectively to the edges incident to the image vertex or face.

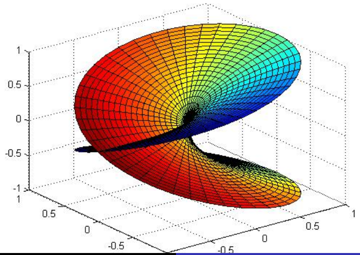
Morphisms and Branched Coverings

A morphism $f : (A, \rho, {}^{-1}) \to (B, \sigma, {}^{-1})$ can be realized as a branched covering. Since f commutes with ρ a cycle of ρ wraps its image k times:

$$f \circ \rho^n(a) = \sigma^n \circ f(a) \implies \exists k \in \mathbb{N}, |\langle \rho \rangle.a| = k\,|\langle \sigma \rangle.f(a)|$$

The integer k is the ramification index of $o(a)$. We often write $e_{o(a)}$ for k.

The same holds for cycles of ${}^{-1} \circ \rho$. The branched covering is ramified at the center of the corresponding faces.



Morphisms and Branched Coverings

Lemma

A morphism $f : (A, \rho, \imath) \to (B, \sigma, \jmath)$ of connected maps is onto.

PROOF. $f(A) = f(\langle \rho, \imath \rangle.a) = \langle \sigma, \jmath \rangle.f(a) = B$. □

Lemma

All edge fibers have the same size called the degree of f.

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All edge fibers have the same size called the degree of f.

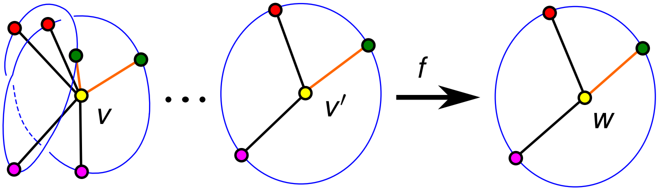
PROOF. Let $b, b' \in B$. $\exists a, a' \in A : b = f(a)$ and $b' = f(a')$.
Let $h \in \langle \rho, \imath \rangle$ with $a' = h(a)$. By commutation
$\exists g \in \langle \sigma, \jmath \rangle : f \circ h = g \circ f$. So,
$g(b) = g(f(a)) = f(h(a)) = f(a') = b'$.
Then $h : f^{-1}(b) \to f^{-1}(b')$ is a bijection since
$f(c) = b \implies f(h(c)) = g \circ f(c) = g(b) = b'$. $\square$

Morphisms and Branched Coverings

Index formula

Let $f : (A, \rho, \imath) \to (B, \sigma, \jmath)$ a morphism of degree n. For any vertex or face w of $(B, \sigma, \jmath)$:

$$\sum_{f(v)=w} e_v = n$$



$e_v = 2$ $\qquad$ $e_{v'} = 1$

PROOF. For an arc a incident to w, partition $f^{-1}(a)$ according to the origin. In each group with origin v, we have e_v arcs of $f^{-1}(a)$. □

Morphisms and Branched Coverings

Riemann-Hurwitz Formula

For a morphism $f : S \to T$ of degree n we have

$$\chi(S) = n.\chi(T) + \sum_{v \in V(S) \cup F(S)} (e_v - 1)$$

PROOF. We now that $|A(S)| = n|A(T)|$ and by Index formula $n = \sum_{f(v)=w} e_v = \sum_{f(v)=w}(e_v - 1) + |f^{-1}(w)|$. So,

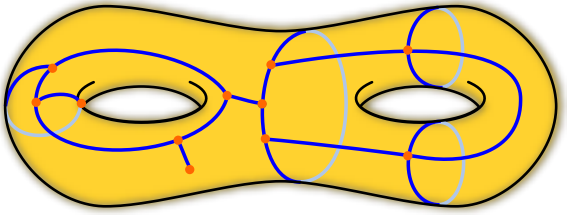
$$\begin{aligned}\chi(S) &= |V(S)| - |A(S)| + |F(S)| \\ &= \sum_{w \in V(T)} |f^{-1}(w)| - n|A(T)| + \sum_{w \in F(T)} |f^{-1}(w)| \\ &= \sum_{w \in V(T) \cup F(T)} \left(n - \sum_{f(v)=w} (e_v - 1) \right) - n|A(T)|\end{aligned}$$

□

Combinatorial Equivalence and Classification

Definition

Combinatorial equivalence is the transitive closure of the relation on maps generated by edge and face splitting.



- Combinatorial equivalence preserves connectivity, orientability and Euler characteristic.

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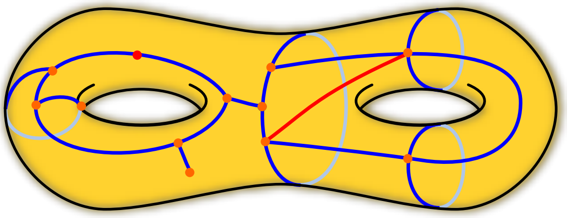

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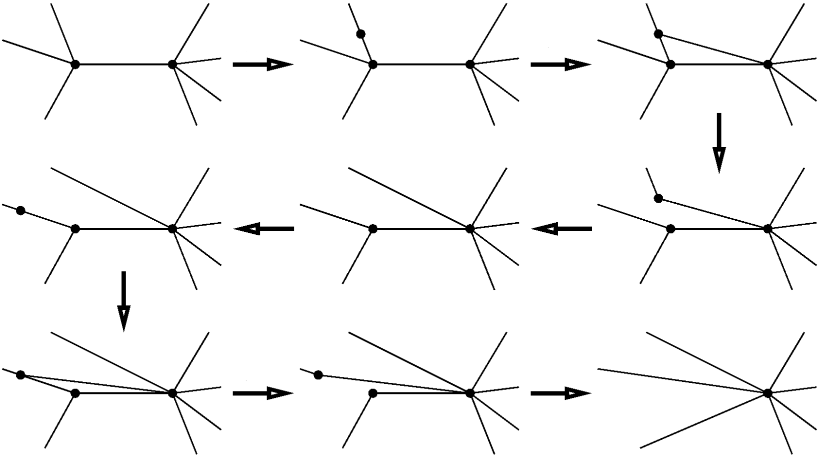
Edge Contraction

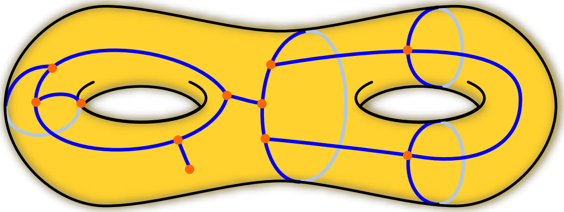
non-loop edge contraction is a combinatorial equivalence.



Reduced Maps

Lemma

Every (connected) map is combinatorially equivalent to a reduced map with one vertex and one face.



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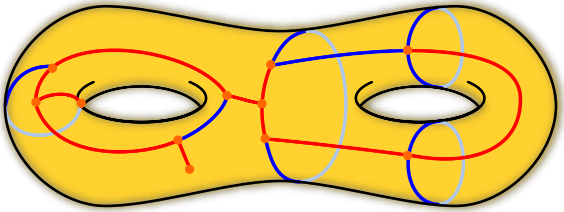

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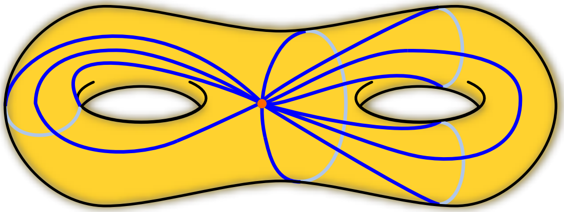

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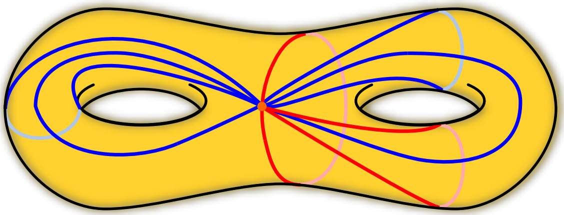

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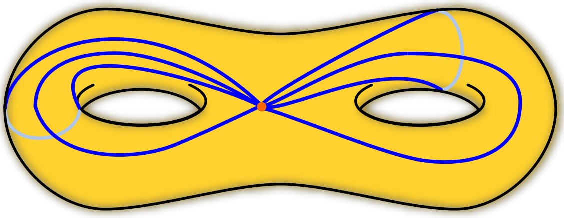

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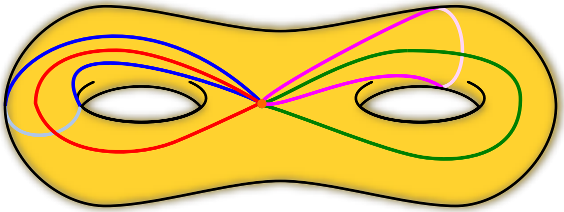

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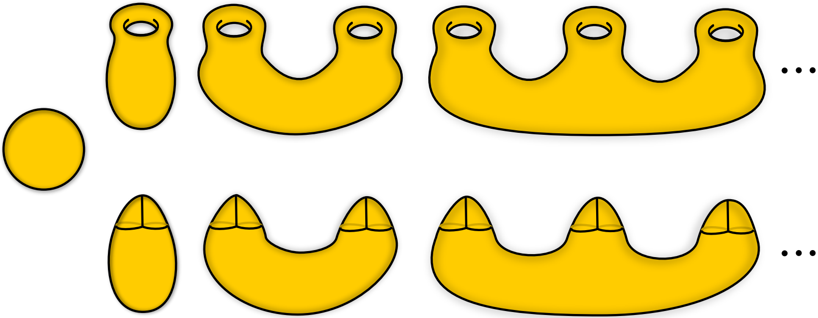
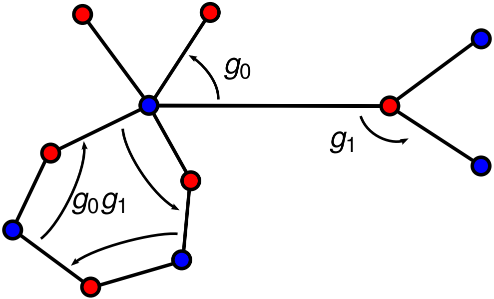


Bipartite Maps

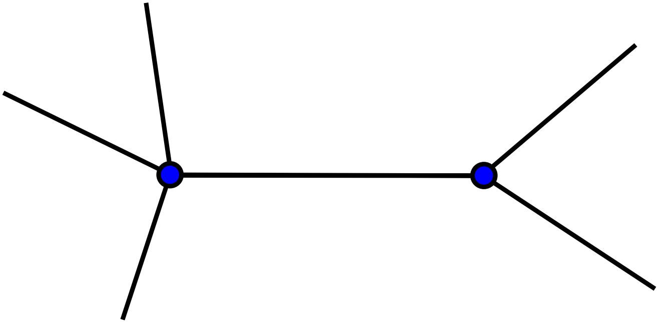
Definition

- A bipartite map is a (monodromy) group $\langle g_0, g_1 \rangle < \mathcal{S}^E$ acting transitively on a set of edges E. The blue (red) vertices are the orbits of g_0 (g_1), the faces are the orbits of $g_0 g_1$.
- a morphism is a map between edges that "commutes" with the g_i's.



Bipartite Maps vs. Maps

Every map *is* a bipartite map.



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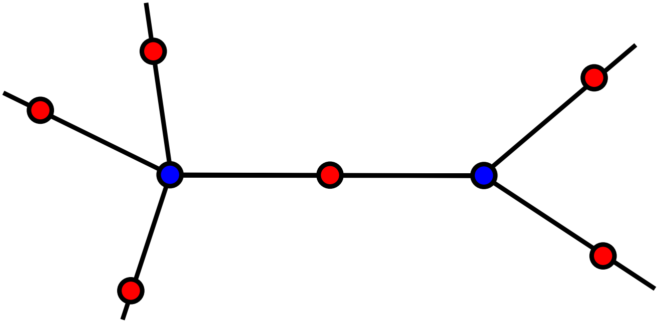
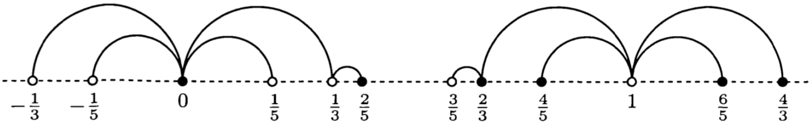


The Universal Map

Let $F(2) = \langle a_0, b_0 \rangle$.

Definition

The universal map U is the left action of $F(2)$ on itself. Vertices and faces have infinite degree.



G. Jones, 1997

The rank 2 free subgroup $\langle \frac{z}{-2z+1}, \frac{z-2}{2z-3} \rangle$ of the modular group $PSL_2(\mathbb{Z})$ of isometries of the upper halfplane acts freely and transitively on U.

The Universal (Bipartite) Map

Let $F(2) = \langle a_0, b_0 \rangle$. Let (E, g_0, g_1) be a map. Fix $e \in E$.
We have a group morphism $F(2) \xrightarrow{\theta} \langle g_0, g_1 \rangle$, $a_i \mapsto g_i$.

The map (E, g_0, g_1) is a quotient of the universal map given by the morphism:

$$\begin{array}{rcl} F(2) & \xrightarrow{f} & E \\ b & \longmapsto & \theta(b)(e) \end{array}$$

Its automorphism group is $\theta^{-1}(S_e)$ where S_e is the stabilizer of e in $\langle g_0, g_1 \rangle$.

f indeed “commutes” with a_i: For $b \in F(2)$
$f(a_i(b)) = \theta(a_i(b))(e) = \theta(a_i b)(e) = g_i(\theta(b)(e)) = g_i(f(b))$

Every map has a “canonical” geometric realization.

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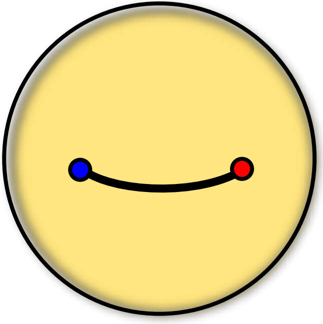
The Trivial (Bipartite) Map

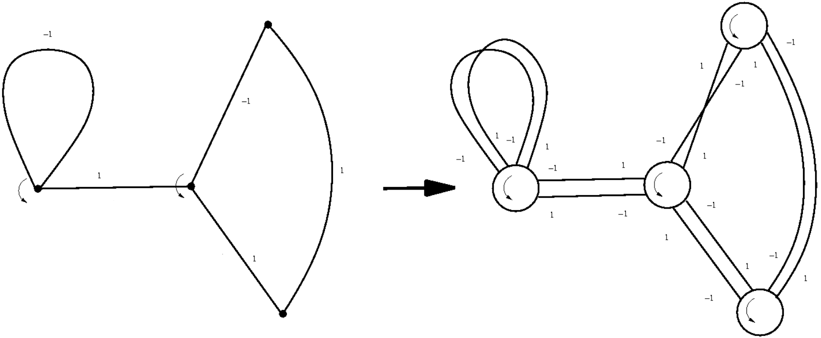
The trivial map $(\{e\}, 1, 1)$ is spherical.



For every map (E, g_0, g_1) there is a (trivial) morphism onto $(\{e\}, 1, 1)$. It defines a branched covering of the sphere whose monodromy group is $\langle g_0, g_1 \rangle$.

Non Orientable Maps

We add a signature $\sigma : A \rightarrow \{-1, 1\}$ to get a map $(A, \rho, ^{-1}, \sigma)$



Consider the set of flags $A \times \{-1, 1\}$, the facial permutation

$$\varphi(a, \epsilon) = (\rho^{\epsilon.\sigma(a)}(a^{-1}), \epsilon.\sigma(a))$$

and the involution

$$\alpha_0(a, \epsilon) = (a^{-1}, -\epsilon.\sigma(a))$$

Non Orientable Maps

We can also describe a non orientable map by a group $\langle \alpha_0, \alpha_1, \alpha_2 \rangle < \mathcal{S}^{\mathcal{D}}$ acting on a set D of darts with $\alpha_i^2 = (\alpha_0\alpha_2)^2 = 1$.

- $V = D/\langle \alpha_1, \alpha_2 \rangle$
- $E = D/\langle \alpha_0, \alpha_2 \rangle$
- $F = D/\langle \alpha_0, \alpha_1 \rangle$

Let $G = \langle a_0, a_1, a_2 \,;\, a_0^2 = a_1^2 = a_2^2 = (a_0a_2)^2 = 1 \rangle$.
G acts on the left onto itself to define a universal map. We obtain a universal (m, n) type map if we take $G_{m,n} = \langle a_0, a_1, a_2 \,;\, a_0^2 = a_1^2 = a_2^2 = (a_0a_2)^2 = (a_0a_1)^m = (a_1a_2)^n = 1 \rangle$.

THANK YOU!