Rend. Sem. Mat. Univ. Pol. Torino - Vol. 64, 1 (2006)
Control Theory and Stabil., II

C. Prieur

ROBUST STABILIZATION OF NONLINEAR CONTROL SYSTEMS BY MEANS OF HYBRID FEEDBACKS

Abstract. The general problem under study in this paper is the robust asymptotic stabilization of nonlinear systems. Different types of systems are considered: the Artstein's circles, the systems which are asymptotically controllable to the origin and also the chained systems. We recall some results on the stabilization by means of hybrid feedback laws (namely controller with a mixed continuous/discrete component) and by means of discontinuous feedbacks. We note that hybrid feedbacks yield a robust stabilization property with respect to measurement noise, actuator errors and exogeneous disturbances, for a larger class of solutions than those obtained with discontinuous state-feedbacks.

1. Introduction

There exists now a very large literature about the hybrid systems (*i.e.* control system with a mixing discrete-continuous state) and the stability properties of these systems see *e.g.* [37, 5, 6, 18]. We can use this notion of systems and their study to construct hybrid feedback laws (*i.e.* state-feedback laws with an additional discrete dynamic state variable) and to stabilize the origin of nonlinear control systems with *a priori* only a continuous state. The state of the closed-loop system we get is hybrid due to the presence of the discrete component of the hybrid feedback law. A generalization of the Carathéodory solutions in the context of the hybrid systems is given in [5]. In [21, 26] it is introduced a generalization of π-solutions and Euler solutions for the hybrid systems.

It appears that this new class of feedback laws has, for some problems in control theory, the following nice properties:

- The hybrid feedback laws could be constructed by making an "hysteresis" between different control laws. To this end, we overlap the sets of definition of different control laws and we define the switching strategy with a suitable dynamical discrete variable. It is done *e.g.* in Section 5 for two control laws and in Section 4 for a countable number of control laws.
- With such a strategy, the hybrid feedback laws could be guaranteed to avoid chattering between the different control laws.
- As this chattering should be avoided to obtain a robust stability property (see [34] for a discussion about this general idea), the hybrid feedback laws can be naturally used for the robust stabilization of control systems.

The aim of this paper is to compare the class of hybrid feedback laws with other class of feedback laws used in control theory. To do this we recall some results on stabilization by means of (dis)-continuous feedback laws and by means of hybrid feedback laws, more precisely the following results:

- [22] The robust stabilization of the Artstein's circles by means of a hybrid feedback law
- [25] The robust stabilization by a hybrid feedback law of any asymptotically controllable system
- [28] The robust stabilization by a hybrid feedback law of the Brockett's integrator and more generally of the chained systems.

Note that in all control problems under study in this paper, we construct (and even we explicit in the cases of the Artstein's circles and the chained systems) a robust hybrid feedback law by using different continuous feedbacks defined on subsets of the state space. These continuous component of our hybrid feedback law are chosen completly independently of the definition of the dynamical discrete variable. We prove the robustness of the hybrid feedback law together with the properties inherited of properties of the different continuous components. More precisely

- For the robust stabilization of asymptotically controllable systems, we use independently properties of the continuous feedback laws components (namely the "patchy feedback" built in [1]) and hybrid strategy to construct an hybrid feedback law which renders the origin a robust asympticaly stable equilibrium. The stabilization property is inherited from the property of the patchy feedback and the robustness is due to the hybrid strategy.
- For the Brockett's integrator and the chained systems, we use independently the speed of the convergence of the stabilization of the feedback of [38] (namely the exponential stabilization) and the hybrid feedback law to get a robust exponential stable equilibrium for the Brockett's integrator and the chained systems.

Thus, given a control system and continuous feedback laws defined on a covering of the state space, our technique should be seen as a new possibility to construct simple and robust feedback laws for nonlinear systems.

The paper is organized as follows. First in Section 2 we recall some well-known results about the stabilization of the origin of a control system by means of *continuous feedback laws*. In particular we introduce the Brockett's integrator.

Then in Section 3 we investigate the class of continuous periodic time-varying feedback laws and we show how, in some cases, *hybrid feedback laws* can be used to stabilize control systems for which there *does not exist any stabilizing continuous periodic time-varying feedback law*.

In Section 4 we study the classical problem of the robust stabilization of the origin of an asymptotically controllable system. We give the history of this problem, in particular the state of the art of the stabilization by means of *discontinuous feedback laws*. We also give a result about the robust stabilization of asymptotically controllable systems by means of *hybrid feedback laws*. We show that the class of solutions considered in our result is larger than those obtained with any discontinuous feedback laws and thus the stabilization property is more rich.

In Section 5 we study the problem of the robust stabilization of the *n-chained systems* with two controls which is a slight generalization of the Brockett's integrator. We state that there exists a hybrid feedback law (with only two discrete states) such that the origin of the closed-loop system is a globally *exponentially* stable equilibrium with a robustness with respect to measurement noise and external disturbances.

Section 6 summarizes the main contributions of the work and points to some open problems and future research directions.

2. Stabilization by continuous feedback laws

Let us consider the following system:

$$\dot{x} = f(x, u) \tag{1}$$

assuming that the control set $K \subset \mathbb{R}^m$ is a compact subset of $\mathbb{R}^m$ and that the map $f : \mathbb{R}^n \times K \to \mathbb{R}^n$ is locally Lipschitz in x, uniformly with respect to u, and continuous in u. In the following we refer very often to the asymptotically controllability property.

DEFINITION 1. *The system (1) is said to be* globally asymptotically controllable to the origin *if the following properties hold:*

1. *For each x_0 in $\mathbb{R}^n$, there exists an admissible control u_0 (*i.e. *a measurable function* $[0, +\infty) \to K$*) such that the maximal Carathéodory solution X starting from x_0 of*

$$\dot{x} = f(x, u_0) \tag{2}$$

   is defined for all $t \geq 0$ and satisfies $X(t) \to 0$ as $t \to +\infty$.

2. *For each $\varepsilon > 0$ there exists $C > 0$ such that for each x_0 in $B(0, C)$, there is an admissible control u_0 as in 1. such that*

$$X(t) \in B(0, \varepsilon), \ \forall t \geq 0 \, .$$

The general problem under consideration in this paper is the asymptotic stabilization via state-feedback. Let us recall that *asymptotic stabilization* means that the following two properties hold:

- Stability of the origin of the closed-loop system.
- Convergence to the origin of all the solutions.

In this section we restrict ourself to the class of continuous feedback law *i.e.* of continuous mappings $u : \mathbb{R}^n \to K$.

There exists a necessary condition for the existence of a continuous control law which makes the origin locally asymptotically stable as stated by the following result:

THEOREM 1. *[7] It there exist a continuous state feedback law rendering the origin of (1) a locally asymptotically stable equilibrium, then the map* $(x, u) \mapsto f(x, u)$ *is open at zero.*

There exist systems which do not satisfy this necessary condition hence for which there does not exist a continuous stabilizing feedback law (see [35, 7]). Consider *e.g.* the so-called "Brockett's integrator":

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_2 \\ \dot{x}_3 & = & x_2 u_1 - x_1 u_2 \end{array} \right. \tag{3}$$

3. Time-varying feedbacks and the Artstein's circles

We can also consider the class of periodic time-varying feedback laws to make the origin of (1) an asymptotically stable equilibrium. This class of feedback laws is sufficiently large to derive an asymptotic stability property from a controllability property. Indeed it can be proved (see the precise statement in [12], see also [11], for the mono-dimensional case $n = 1$)

THEOREM 2. *[12] If the origin is locally continuously reachable in small time, if the strong algebra rank condition is satisfied and if* $n \geq 4$*, then there exists a periodic time-varying feedback law such that the origin of (1) in closed-loop is an asymptotically stable equilibrium.*

However let us make the following remark concerning the Artstein's circles (see [2] for an introduction of this system). This system is well-known and studied in several papers as [34, 22, 8, 4]. This is the following two dimensional system:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & u(-x_1^2 + x_2^2) \\ \dot{x}_2 & = & -2u x_1 x_2 \end{array} \right. \tag{4}$$

with u in $\mathbb{R}$. In (4), all motions are allowed along the integral curves of g *i.e.*:

- the origin,
- all circles centered on the x_2-axis and tangent to the x_1-axis,
- the x_1-axis.

With $u > 0$ the circle is followed clockwise if $x_2 > 0$ and, anticlockwise if $x_2 < 0$. Let θ in $(-\pi, \pi]$ denote the polar angle of a point $x \neq 0$.

It can easily proved (see *e.g.* [34]) that the origin of this system cannot be asymptotically stabilizable by a continuous feedback law.

It is proved in [4, page 98] that the origin of the Artstein's circles cannot be stabilized by a continuous time-varying feedback law. However it can be stabilized (even in presence of small measurement noises) by means of a hybrid feedback law. See [22] for a precise statement of this result.

Therefore, in some cases, *hybrid feedback laws* can be used to stabilize control systems for which there *does not exist any stabilizing continuous periodic time-varying feedback law*.

4. Stabilization of asymptotically controllable systems

Let us assume that the system (1) is globally asymptotically controllable to the origin. We have seen in Section 2 that in many cases, we can not restrict ourself to the class of continuous feedback laws to render the origin a locally asymptotically stable equilibrium. The first result concerning the use of discontinuous feedback laws is [36] but the author assumes that the system is analytic and completely controllable. The following property is proved in [10] (see also a construction in [31] using semiconcave control-Lyapunov function of [30])

($\mathcal{P}$) Any asymptotically controllable systems can be asymptotically stabilized by a discontinuous controller.

The notion of solutions used by the authors is the notion of π-solutions (*i.e.* solutions with a feedback law computed with an arbitrary small sampling schedule). Recently in [1], the authors prove the property ($\mathcal{P}$) for all Carathéodory solutions by exhibiting a "patchy feedback".

The controllers in [10, 1] are robust with respect to actuator errors and external disturbances (*i.e.* all systems perturbed by small actuator errors and small external disturbances are asymptotically stable) but are not robust with respect to arbitrary small measurement noise. One way to robustly stabilize the system (1) is to enlarge the class of controllers as in [17] where the authors introduce the notion of "dynamic hybrid controller" which is computed with an "external model". This controller compares, at suitable sampling times, the predicted state with the measured state. Due to the measurement noise these can differ substantially, therefore, as remarked in [33], it requires a resetting of the controller which may be difficult to construct. Moreover, with this controller, the origin is a robustly globally asymptotically stable equilibrium for the π-solutions only. Here we state the existence of a hybrid feedback law which renders the origin a robustly globally asymptotically stable equilibrium for a *larger* class of solutions and, moreover, our feedback does not need a *resetting*.

In [33, 9], the authors prove the existence, for all asymptotically controllable systems, of a controller which is robust with respect to measurement noise and makes the origin of the system (1) be a semiglobal *practical* stable equilibrium (*i.e.* driving all states in a given compact set of initial conditions into a specified neighborhood of the origin). It is proved in [34, sec. 5.4] that one can in fact get a more general result: one can prove the existence of a sampling feedback making the origin be a robust global asymptotic equilibrium for all π-solutions with a sampling rate *sufficiently slow*. We can exhibit a hybrid feedback law rendering the origin a robust global asymptotically stable equilibrium for the π-solutions with *any fast* enough sampling schedule, so for a larger class of solutions.

To make more clear this fact let us define the notion of π-solutions of (1) in closed-loop with a hybrid feedback law $u : \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}} \to K$, $k_d : \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}} \to \{1, 2\}^{\mathbb{N}}$, where $\{1, 2\}^{\mathbb{N}}$ denotes the set of the sequences with value in $\{1, 2\}$.

Let ξ, ζ: $\mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ and ψ: $\mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^m$ be an unkwon measurement noise, an external disturbance and an actuator error respectively, satisfying our *standing regularity assumptions*, *i.e.*

$$\begin{gathered}\xi \text{ and } \zeta \text{ are in } \mathcal{L}^{\infty}_{loc}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n) \text{ and are continuous in } x \text{ in } \mathbb{R}^n \text{ for each } t \geq 0,\\ \psi \text{ is in } \mathcal{L}^{\infty}_{loc}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^m) \text{ and is continuous in } x \text{ in } \mathbb{R}^n \text{ for each } t \geq 0,\end{gathered}$$

The control system (1) in closed-loop with a hybrid feedback law $u : \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}} \to K$, $k_d : \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}} \to \{1, 2\}^{\mathbb{N}}$ with the perturbations ξ, ζ and ψ is

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & f(x(t), u(x(t) + \xi(x, t), s_d(t)) + \psi(x, t)) + \zeta(x, t)\\ s_d(t) & = & k_d(x(t) + \xi(x, t), s_d^-(t))\end{array}\right. \tag{5}$$

where $s_d^-(t)$ denotes formally (see Definition 2 below for a precise meaning):

$$s_d^-(t) = \lim_{s \to t,\ s < t} s_d(s)\ .$$

Let π be a sampling schedule of $\mathbb{R}$, *i.e.* a sequence $(t_n)_{n \in \mathbb{N}}$ such that, for all n in $\mathbb{N}$, we have $t_n < t_{n+1}$ and $\lim_{n \to +\infty} t_n = +\infty$. Note that the upper and lower diameters of the sampling schedule are defined by (see [33])

$$\overline{d}(\pi) = \sup_{i \in \mathbb{Z}} (t_{i+1} - t_i) \quad , \qquad \underline{d}(\pi) = \inf_{i \in \mathbb{Z}} (t_{i+1} - t_i)\ .$$

We rewrite the notion of solution given in [5] in the context of the π-solutions as follows

DEFINITION 2. *Let π be a sampling schedule of $\mathbb{R}$, t_0 in π, $T > t_0$, and $(x_0, s_0) \in \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}}$. We say that (X, S_d): $[t_0, T) \to \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}}$ is a π-*solution *of (5) on $[t_0, T)$ with initial condition (x_0, s_0) if*

1. *The map X is absolutely continuous on $[t_0, T)$.*
2. *We have, for all t in $[t_0, \min(t_1, T))$,*

   $$S_d(t) = S_d(t_0)\ , \tag{6}$$

   for all i in $\mathbb{N}_{>0}$ and for all t in $[\min(t_i, T), \min(t_{i+1}, T))$,

   $$S_d(t) = k_d(X(t_i) + \xi(X(t_i), t_i), S_d(t_{i-1}))\ . \tag{7}$$

3. *We have, for all i in $\mathbb{N}$ and for almost all t in $[\min(t_i, T), \min(t_{i+1}, T))$,*

   $$\dot{X}(t) = f\,(X(t), u(X(t_i) + \xi(X(t_i), t_i), S_d(t_i)) + \psi(X(t_i), t_i)) + \zeta(X(t), t).$$

4. *We have*

$$X(t_0) = x_0 \quad , \qquad S_d(t_0) = k_d(x_0 + \xi(x_0, t_0), s_0) \; . \tag{8}$$

By invoking Zorn's Lemma exactly as in the proof of [32, Proposition 1], one can prove that every π-solution can be extended to a maximal solution:

DEFINITION 3. *Let t_0 in $\mathbb{R}$, $T > t_0$, (x_0, s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathbb{N}}$ and $d_0 > 0$. We say that $(X, S_d) : [t_0, T) \to \mathbb{R}^n \times \{1,2\}^{\mathbb{N}}$ is a d_0*-maximal solution *starting from (x_0, s_0) of (5) on $[t_0, T)$, if the following properties hold*

- *for all $T' < T$, there exists a sampling schedule π of $[0, +\infty)$ such that*

$$\overline{d}(\pi) \le d_0 \; , \tag{9}$$

  and such that (X, S_d) is a π-solution starting from x_0 of (5) on $[t_0, T')$,
- *for all $T' > T$ and for all sampling schedule π of $[0, +\infty)$ such that (9) holds, there does not exist any π-solution (X', S'_d) starting from (x_0, s_0) and defined on $[t_0, T')$ such that the restriction of (X', S'_d) to $[t_0, T)$ is (X, S_d).*

Any globally asymptotically controllable system to the origin is robustly asymptotically stabilizable by means of a hybrid feedback law, indeed we have

THEOREM 3. *[25] Let (1) be a globally asymptotically controllable system to the origin. Then there exists a hybrid feedback law, $u : \mathbb{R}^n \times \{1,2\}^{\mathbb{N}} \to K$, $k_d : \mathbb{R}^n \times \{1,2\}^{\mathbb{N}} \to \{1,2\}^{\mathbb{N}}$ such that the origin is a robustly globally asymptotically stable equilibrium for the system in closed-loop with (u, k_d), for any fast π-solutions. More precisely there exists (u, k_d) such that, the following properties hold:*

1. Existence of solutions: *For all $C > 0$, there exists $\chi_0 = \chi_0(C) > 0$ such that for all ξ, ζ, ψ satisfying our standing regularity assumptions and such that*

$$\begin{aligned} &\sup_{x \in \mathbb{R}^n,\, t \ge 0} |\xi(x,t)| \le \chi_0 \; , \\ &\operatorname{esssup}_{x \in \mathbb{R}^n,\, t \ge 0} |\zeta(x,t)| \le \chi_0 \; , \\ &\operatorname{esssup}_{x \in \mathbb{R}^n,\, t \ge 0} |\psi(x,t)| \le \chi_0 \; , \end{aligned} \tag{10}$$

   for all (x_0, s_0) in $B(0, C) \times \{1,2\}^{\mathbb{N}}$, and for all sampling schedules π of $\mathbb{R}$, there exists a π-solution of (5) starting from (x_0, s_0) at $t_0 = 0$.
2. Completeness: *Moreover, there exists $d_0 = d_0(C)$ such that, all the d_0-maximal solutions of (5) are defined on $[0, +\infty)$.*
3. Global stability: *There exists δ of class $\mathbb{K}_\infty$ such that, for all $\varepsilon > 0$, there exist $\chi_0 = \chi_0(\varepsilon) > 0$ and $d_0 = d_0(\varepsilon) > 0$ such that, for all ξ, ζ, ψ satisfying our standing regularity assumptions and (10), for all (x_0, s_0) in $B(0, \delta(\varepsilon)) \times \{1,2\}^{\mathbb{N}}$, and for every d_0-maximal solution (X, S_d) of (5) starting from (x_0, s_0) at $t_0 = 0$, one has*

$$X(t) \in B(0, \varepsilon), \;\; \forall t \ge 0 \; . \tag{11}$$

4. Global attractivity: *For all* $\varepsilon > 0$ *and for all* $C > 0$*, there exist* $T = T(\varepsilon, C) > 0$*,* $\chi_0 = \chi_0(\varepsilon, C) > 0$ *and* $d_0 = d_0(\varepsilon, C) > 0$ *such that, for all* ξ*,* ζ*,* ψ *satisfying our standing regularity assumptions and (10), for each* (x_0, s_0) *in* $B(0, C) \times \{1, 2\}^{\mathbb{N}}$*, and for* d_0*-every maximal solution* (X, S_d) *of (5) starting from* (x_0, s_0) *at* $t_0 = 0$*, one has*

$$X(t) \in B(0, \varepsilon),\ \forall t \geq T\ . \tag{12}$$

REMARK 1. Some observations are in order.

- Note that in Theorem 3 we have the robust global asymptotic stability for the π-solutions for *any fast* enough sampling rate since the only constraint on the sampling schedule under consideration in this result is (9).

  In [33, 9], only for the π-solutions with a sampling rate *sufficiently slow* are considered since, in these papers, it is assumed moreover that the lower diameters of the sampling schedules have a strictly positive lower bound. See in particular the assumption in [33, Theorem 1]

$$|\xi(t)| \leq \underline{d}(\pi),\ \forall t \geq 0\ . \tag{13}$$

  Thus the class of solutions under consideration in Theorem 3 is *larger* than those considered in [33, 9].

  Let us compare (10) and the inequality (13). Given a sampling schedule whose lower diameter is close to zero, this restriction forces the measurement noise to be close to zero. In our context the measurement noise and the lower diameter are completely independent.

  Note that the controller given by [33] is *not robust* with respect to noise which does not satisfy (13) (consider the example of the Artstein's circles given in Section 3). See also the discussion given in [34, Section 4].

- We can reformulate Theorem 3 in terms of generalized solution as defined in [13, 14, 23] or in terms of Euler solutions as defined in *e.g.* [34].

- To prove this result, we use some techniques of [1] to deduce from the asymptotic controllability a "family of nested patchy vector fields", and we introduce hysteresis between an infinite number of controllers as it is done in [22] for two controllers. This allows us to define a *hybrid patchy feedback*.

- Note that Theorem 3 is false if in (10) the supremum *sup* is relaxed by *esssup*. See [23, Theorem 4.2], where it is proved, in an analogous situation, that there exists a noise ξ such that *esssup* $|\xi| = 0$, *sup* $|\xi| \neq 0$ and such that the origin of the perturbed closed-loop system is not an attractive equilibrium.

5. Stabilization of the chained systems

In this section we focus on n-dimensional chained systems with two controls, *i.e.* systems described by equations of the form

$$(14)\qquad \begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 \end{cases}$$

Note that the 3-chained systems is equivalent to the Brockett's integrator introduced in Section 2. (Indeed making the change of variables $z_1 = x_1$, $z_2 = x_2$ and $z_3 = 2x_3 - x_1 x_2$ brings the 3-chained systems into the Brockett's integrator).

The stabilization problem of the chained systems has been widely studied in the literature. See [20, 16] and references therein, where several control laws, yielding diverse asymptotic properties, have been proposed.

On the contrary, the robust stabilization problem for nonholonomic systems in the presence of measurement errors and exogenous disturbances is not yet completely solved. Several attempts have been made to study the robustness properties of existing control laws or to robustify given controllers, see *e.g.* [21] where the problem of local robust stabilization by means of time-varying control laws have been studied. A similar problem has been addressed in [15] using the class of discontinuous control laws introduced in *e.g.* [3]. See also [19], where a discontinuous control law, possessing a Lyapunov stability property, has been constructed.

In this section we recall a recent result about the robust exponential stabilization of the n-chained systems, which is the following

THEOREM 4. *[28] There exists a hybrid feedback law (u, k_d), $u : \{1, 2\} \times \mathbb{R}^n \to \mathbb{R}^2$ and $k_d : \mathbb{R}^n \times \{1, 2\} \to \{1, 2\}$, such that the origin of (14) in closed-loop with (u, k_d) is a robustly globally exponentially stable equilibrium.*

REMARK 2. Note that chained systems are asymptotically controllable to the origin, thus, by the general result Theorem 3, there exists a hybrid feedback rendering the origin a robustly globally asymptotically stable equilibrium. In the present theorem such a feedback is *explicitly constructed* and it is shown that only *two* discrete variables are needed in the construction (instead of a countable number of discrete variables as in Theorem 3). Moreover we have *exponential* stabilization.

Let us summarize the construction of this hybrid feedback law and explicit the result claimed in Theorem 4.

Let us define the following "local" feedback, u_l. This terminology has been chosen since we can prove that the solutions of the chained systems, in closed-loop with the hybrid feedback law defined below, starting with the *global* feedback law, "switch" at finite time to the *local* feedback law and converges to the origin as t tends to $+\infty$. See [28, Lemma 5.5] for a proof (see also the analogous result in [23]). This local controller u_l: $\mathbb{R}^n \to \mathbb{R}^2$ is defined by, for all $x \in \mathbb{R}^n$,

$$u_l(x_1, x_2, \ldots, x_n) = \begin{pmatrix} -x_1 \\ p_2 x_2 + p_3 \dfrac{x_3}{x_1} + p_4 \dfrac{x_4}{x_1^2} + \cdots + p_n \dfrac{x_n}{x_1^{n-2}}, \end{pmatrix}, \tag{15}$$

with the p_i such that the matrix

$$A = \begin{bmatrix} p_2+1 & p_3 & p_4 & \cdots & p_{n-1} & p_n \\ -1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & n-1 \end{bmatrix}$$

is Hurwitz. Let $P = P^T > 0$ be such that $A^T P + PA < 0$, and let z be a variable in $\mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} z = z(x) &= Y^T P Y \quad \text{if } x_1 \neq 0\,, \\ &= +\infty \quad \text{if } x_1 = 0\,, \end{aligned} \tag{16}$$

for all x in $\mathbb{R}^n$, with $Y \in \mathbb{R}^{n-1}$ defined by, $\forall x \in \mathbb{R}^n$, $x_1 \neq 0$,

$$Y = Y(x) = \begin{bmatrix} \dfrac{x_2}{x_1} & \dfrac{x_3}{x_1^2} & \cdots & \dfrac{x_n}{x_1^{n-1}} \end{bmatrix}^T .$$

Let $\mu > 0$ and consider the "global" feedback $u_g : \mathbb{R}^n \to \mathbb{R}^2$, defined, for all $x \in \mathbb{R}^n$, by

$$u_g(x_1, x_2, \ldots, x_n) = \begin{pmatrix} 1 \\ -\mu x_2 \end{pmatrix}. \tag{17}$$

To this end, for any strictly positive real number M, we define the subset Γ_M of $\mathbb{R}^n$ as

$$\Gamma_M = \{x, x_1 \neq 0,\ z < M\}\,, \tag{18}$$

where z is defined by (16). In the three dimensional case, these sets are cones with axis x_1 and are symmetric with respect to $\{x_1 = 0\}$. See Figure 1 for a sketch of this cone.

Let $M_2 > M_1 > 0$. For simplicity, in what follows, for all $i \in \{1, 2\}$, we define $\Gamma_i := \Gamma_{M_i}$. The hybrid controller (u, k_d) is defined making a hysteresis between u_l

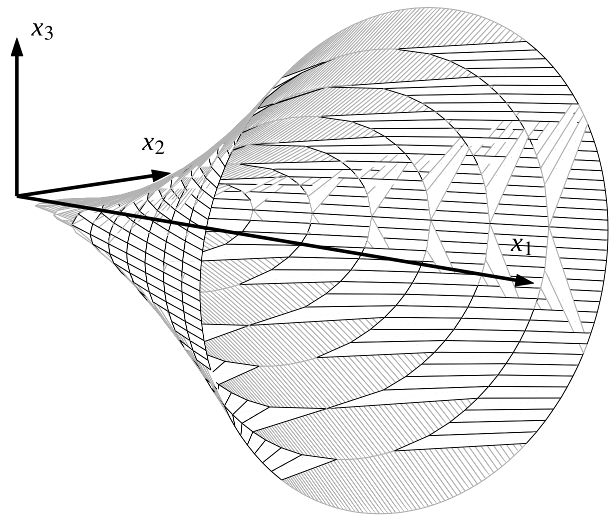

Figure 1: Sketch of one of the sets Γ_i, used to define the hybrid controller, in $\mathbb{R}^3$. Only the intersection with $\{x_1 \geq 0\}$ is shown.

and u_g on Γ_2 and Γ_1, *i.e.*

$$\begin{array}{rcll} u : \{1,2\} \times \mathbb{R}^n & \to & \mathbb{R}^2 & \\ (s_d, x) & \mapsto & u_l(x) & \text{if} \quad s_d = 1,\ x_1 \neq 0\,, \\ & & 0 & \text{if} \quad s_d = 1,\ x_1 = 0\,, \\ & & u_g(x) & \text{if} \quad s_d = 2\,, \end{array} \tag{19}$$

$$\begin{array}{rcll} k_d : \mathbb{R}^n \times \{1,2\} & \to & \{1,2\} & \\ (x, s_d) & \mapsto & 1 & \text{if} \quad x \in \Gamma_1 \cup \{0\}, \\ & & s_d & \text{if} \quad x \in \Gamma_2 \setminus \Gamma_1, \\ & & 2 & \text{if} \quad x \notin \Gamma_2 \cup \{0\}. \end{array} \tag{20}$$

This hybrid feedback law makes the origin of (14) in closed-loop a robustly globally asymptotically stable equilibrium as claimed by Theorem 4. The controller u_l has been computed to make any Carathéodory solution of

$$\dot{x} = f(x, u_l(x))$$

with an initial condition in Γ_2, converge exponentially to the origin as t tends to $+\infty$, whereas the controller u_g has been computed such that any Carathéodory solution of

$$\dot{x} = f(x, u_g(x))$$

with an initial condition in $\mathbb{R}^n \setminus \Gamma_1$, enter in finite time in Γ_1. The discrete variable makes an hysteresis of u_l and u_g on Γ_2 and Γ_1. This is the general idea of the proof of Theorem 4.

Some numerical simulations illustrating this hybrid feedback law can been found in [27].

6. Conclusion

In this paper, we study different stabilization problems of some nonlinear control systems (the Artstein's circles, the systems which are asymptotically controllable to the origin and also the chained systems) and we compare the robustness property of the closed-loop systems with a discontinuous state-feedback and the closed-loop systems with a hybrid controller. We note that hybrid feedbacks yield a robust stabilization property with respect to measurement noise, actuator errors and exogeneous disturbances, for a larger class of solutions than those obtained with discontinuous state-feedbacks.

The technique used in the proof of these results should be seen as a new strategy to construct robust controllers for nonlinear control systems.

Theorical studies with hybrid feedback laws are under actual investigation. Let us cite [29] where the global robust stabilization of the Brockett integrator in minimal time is addressed and solved by means of a hybrid state feedback law. This is a work at the intersection of the optimal control theory and the robust control theory which could be probably be generalized.

Applications of this class of hybrid feedback laws is also under investigation. Nonholomic systems have been already studied in Section 5. Other class of mechanical systems are under actual study.

References

[1] ANCONA F. AND BRESSAN A., *Patchy vectors fi elds and asymptotic stabilization*, ESAIM: Control Optim. Calc. Var. **4** (1999), 445–471.

[2] ARTSTEIN Z., *Stabilization with relaxed controls*, Nonlinear Anal. TMA **7** (11) (1983), 1163–1173.

[3] ASTOLFI A., *Discontinuous control of nonholonomic systems*, Systems Control Lett. **27** (1996), 37–45.

[4] BACCIOTTI A. AND ROSIER L., *Liapunov functions and stability in control theory*, Lect. Notes Cont. Inf. Sc. **267**, Springer-Verlag, London 2001.

[5] BENSOUSSAN A. AND MENALDI J.L., *Hybrid control and dynamic programming*, Dyn. Contin. Discrete Impulsive Syst. **3** (4) (1997), 395–442.

[6] BRANICKY M.S., *Multiple Lyapunov functions and other analysis tools for switched and hybrid systems*, IEEE Trans. Autom. Control **43** (4) (1998), 475–482.

[7] BROCKETT R.W., *Asymptotic Stability and Feedback Stabilization*, in: "Differential Geometric Control Theory", (Eds. Brockett R.W., Millman R.S. and Sussmann H.), Birkhäuser, Boston 1983, 181–191.

[8] CERAGIOLI F., *Some remarks on stabilization by means of discontinuous feedbacks*, Systems and Control Letters **45** (2002) 271–281.

[9] CLARKE F.H., LEDYAEV YU.S., RIFFORD L. AND STERN R.J., *Feedback stabilization and Lyapunov functions*, SIAM J. Contr. Opt. **39** (1) (2000), 25–48.

[10] CLARKE F.H., LEDYAEV YU.S., SONTAG E.D. AND SUBBOTIN A.I., *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Autom. Control **42** (1997), 1394–1407.

[11] CORON J.-M., *Stabilizing time-varying feedback*, IFAC Nonlinear Control Systems Design, Tahoe, USA, 1995.

[12] CORON J.-M., *Stabilization in finite time of locally controllable systems by means of continuous time-varying feedbacks laws*, SIAM J. Contr. Opt. **33** (1995), 804–833.

[13] HÁJEK O., *Discontinuous differential equations, part I*, J. Differential Equations **32** (1979), 149–170.

[14] HERMES H., *Discontinuous vector fields and feedback control*, in: "Differential equations and dynamic systems" (Eds. Hale J. K. and La-Salle J.P.), Academic Press, New York, London 1967.

[15] JIANG Z.P., *Robust exponential regulation of nonholonomic systems with uncertainties*, Automatica J. IFAC **36** (2000), 189–200.

[16] KOLMANOVSKY I. AND MCCLAMROCH N. H., *Developments in nonholonomic control problems*, IEEE Control Systems **15** (1995), 20–36.

[17] LEDYAEV Y.S. AND SONTAG E.D., *A remark on robust stabilization of general asymptotically controllable systems*, in: "Proc. Conf. on Information Sciences and Systems (CISS 97)", Baltimore, MD 1997 246–251.

[18] LITSYN E., NEPOMNYASHCHIKH Y.V. AND PONOSOV A., *Stabilization of linear differential systems via hybrid feedback controls*, SIAM J. Contr. Opt. **38** (5) (2000), 1468–1480.

[19] MARCHAND N. AND ALAMIR M., *Discontinuous exponential stabilization of chained form systems*, Automatica **39** (2) (2003), 343–348.

[20] MORIN P., POMET J.-B. AND SAMSON C., *Developments in time-varying feedback stabilization of nonlinear systems*, in: "IFAC Symp. on Nonlinear Control System Design", Enschede, The Netherlands 1998.

[21] MORIN P. AND SAMSON C., *Robust stabilization of driftless systems with hybrid open-loop/feedback control*, American Control Conference, Chicago 2000.

[22] PRIEUR C., *A robust globally asymptotically stabilizing feedback: the example of the Artstein's circles*, in: "Nonlinear Control in the Year 2000" (Eds. Isidori A., Lamnabhi-Lagarrigue F. and Respondek W.), Lect. Notes Cont. Inf. Sc. **258**, Springer Verlag, London 2000, 279–300.

[23] PRIEUR C., *Uniting local and global controllers with robustness to vanishing noise*, Math. Control Signals Systems **14** (2001), 143–172.

[24] PRIEUR C., *Diverses méthodes pour des problèmes de stabilisation*, Ph.D. Thesis, University Paris-Sud, Orsay, France 2001.

[25] PRIEUR C., *Asymptotic controllability and robust asymptotic stabilizability*, SIAM J. Contr. Opt., **43** (5) (2005), 1888–1912.

[26] PRIEUR C., *Hybrid systems with small perturbations, applications in control theory*, in: "Fourth Nonlinear Control Network (NCN) Workshop, Nonlinear and Adaptive Control", (Eds. Zinober A. and Owens D.), Lect. Notes Cont. Inf. Sc. **281**, Springer, Berlin 2002, 285–294.

[27] PRIEUR C. AND ASTOLFI A., *Robust stabilization of chained systems via hybrid control*, in: "Proc. of Conf. on Decision and Control", Las-Vegas, NV, 2002.

[28] PRIEUR C. AND ASTOLFI A., *Robust stabilization of chained systems via hybrid control*, IEEE Trans. Aut. Control **48** (10) (2003), 1768–1772.

[29] PRIEUR C. AND TRÉLAT E., *Robust optimal stabilization of the Brockett integrator via a hybrid feedback*, Math. Control Signals Systems, **13** (3) (2005), 201–216.

[30] RIFFORD L., *Existence of Lipschitz and semiconcave control-Lyapunov functions*, SIAM J. Contr. Opt. **39** (4) (2000), 1043–1064.

[31] RIFFORD L., *Semiconcave control-Lyapunov functions and stabilizing feedbacks*, SIAM J. Contr. Opt. **41** (3) (2002), 659–681.

[32] RYAN E.P., *Discontinuous feedback and universal adaptive stabilization*, in: "Control of Uncertain Systems" (Eds. Hinrichsen D. and Martensson B.), Birkhäuser 1990, 245–258.

[33] SONTAG E.D., *Clocks and insensitivity to small measurement errors*, ESAIM: Control Optim. Calc. Var. **4** (1999), 537–557.

[34] SONTAG E.D., *Stability and stabilization: discontinuities and the effect of disturbances*, in: "Nonlinear Analysis, Differential Equations, and Control", (Eds. Clarke F.H. and Stern R.J.), Kluwer 1999, 551–598.

[35] SONTAG E.D. AND SUSSMANN H., *Remarks on continuous feedback*, IEEE Conf. Decision and Control, Albuquerque 1980, 916–921.

[36] SUSSMANN H.J., *Subanalytic sets and feedback control*, J. Differential Equations **31** (1979), 31–52.

[37] TAVERNINI L., *Differential automata and their discrete simulators*, Nonlin. An., Th., Meth., App. **11** (6) (1987), 665–683.

[38] VALTOLINA E. AND ASTOLFI A., *Local robust regulation of chained systems*, Systems Control Letters **49** (3) (2003), 231–238.

AMS Subject Classification: 93B52, 93D15.

Christophe PRIEUR, LAAS-CNRS, 7, avenue du Colonel Roche, 31077 Toulouse Cedex 4, France
e-mail: Christophe.PRIEUR@laas.fr