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ON TWO HYBRID ROBUST OPTIMAL STABILIZATION PROBLEMS

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Abstract: We report on some recent results obtained by the authors concerning robust hybrid stabilization of control systems. In (Prieur and Trélat, 2005*a*), we state a result of semi-global minimal time robust stabilization for analytic control systems with controls entering linearly, by means of a hybrid state feedback law, under the main assumption of the absence of minimal time singular trajectories. In (Prieur and Trélat, 2005*c*), we investigate the Martinet case, which is a model case in $\mathbb{R}^3$ where singular minimizers appear, and show that such a stabilization result still holds. Namely, in both cases, we prove that the solutions of the closed-loop system converge to the origin in quasi minimal time (for a given bound on the controller) with a robustness property with respect to small measurement noise, external disturbances and actuator errors.

Keywords: Hybrid modes, Robust stabilizability, Measurement noise, System noise, Disturbance, Actuators, Optimal control, Singular control, Geometrical theory.

1. INTRODUCTION

Let m and n be two positive integers. Consider on $\mathbb{R}^n$ the driftless control-affine system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t)\, f_i(x(t)), \tag{1}$$

where $f_1, \ldots, f_m$ are analytic vector fields on $\mathbb{R}^n$, and where the control function $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ satisfies the constraint

$$\sum_{i=1}^{m} u_i(t)^2 \leq 1. \tag{2}$$

The system (1), together with the constraint (2), is said to be *globally asymptotically stabilizable* at the origin if, for each point $x \in \mathbb{R}^n$, there exists a control law satisfying the constraint (2) such that the solution of (1), associated to this control law and starting from x, tends to 0 as t tends to $+\infty$.

Brockett's condition (Brockett, 1983, Theorem 1, (iii)), implies that, if $m < n$, then there does not exist any continuous stabilizing feedback law for (1). This fact has generated a wide-ranging research on the asymptotic stabilization problem, and there exists a huge literature on this specific problem. Several control laws have been derived for such control systems (see for instance (Kolmanovsky and McClamroch, 1995; Astolfi, 1998) and references therein). The *robust asymptotic stabilization problem* is under current and active research. Many notions of controllers have been introduced for such issues: discontinuous sampling feedbacks (Clarke *et al.*, 2000; Sontag, 1999*a*), time varying control laws (Coron, 1992; M'Closkey and Murray, 1997; Morin and Samson, 2003), patchy feedbacks (as in (Ancona and Bressan, 2002)), ..., enjoying different robustness properties depending on the errors under consideration.

We consider here feedback laws having both discrete and continuous components, generating closed-loop systems with *hybrid* terms (see for instance (Bensoussan and Menaldi, 1997; Tavernini, 1987)). To construct such feedbacks, one has to define a switching strategy between several smooth control laws defined on a partition of the state space. Many results on the stabilization problem of nonlinear systems by means of hybrid controllers have been recently established (see for instance (Branicky, 1998; Goebel and Teel, 2005; Goebel *et al.*, 2004; Liberzon, 2003)). The notion of solution, connected with the robustness problem, is by now well defined in the hybrid context (see (Goebel and Teel, 2005; Prieur and Astolfi, 2003) among others).

Our strategy consists in combining a minimal time controller, which is smooth on a part of the state space, and other controllers defined on the complement of this part. We then define a switching strategy between all control laws, yielding a quasi minimal time hybrid controller, enjoying a robustness property with respect to small measurement noise, actuator errors and external disturbances.

The strategy goes in two steps. At first, we consider the minimal time problem for the system (1), (2), of steering a point $x \in \mathbb{R}^n$ to the origin. This problem is solvable as soon as *Hörmander's condition* holds for $(f_1, \ldots, f_m)$, although one is unable in general to compute explicitly the time optimal controllers. Hence, the regularity of optimal feedback laws is in question. For the system (1), the minimal time problem under the constraint (2) is equivalent to the *sub-Riemannian problem* associated to the m-tuple of vector fields $(f_1, \ldots, f_m)$; in these conditions, the minimal time function to the origin is equal to the sub-Riemannian distance to the origin. The analytic regularity of the sub-Riemannian distance appears to be related to the existence of singular minimizing trajectories (see (Agrachev, 1998)). Namely, if there does not exist any nontrivial singular minimizing trajectory starting from the origin, then the sub-Riemannian distance to the origin is subanalytic outside the origin. In particular, this function is analytic outside a stratified submanifold $\mathcal{S}$ of $\mathbb{R}^n$, of codimension greater than or equal to 1 (see (Tamm, 1981)). As a consequence, outside this submanifold, it is possible to provide an analytic time optimal feedback controller for the system (1) with the constraint (2).

Here, the analytic context is used so as to ensure stratification properties, which do not hold a priori if the system is smooth only. These properties are related to the notion of *o-minimal category* (see (van den Dries and Miller, 1996)).

The second step consists in achieving a minimal time robust stabilization procedure, using a hybrid feedback law, by defining a suitable switching strategy (using an hysteresis) between this minimal time feedback controller and other controllers defined on a neighborhood of $\mathcal{S}$.

Note that, in (Prieur and Trélat, 2005*b*), this program was achieved on the so-called Brockett system, for which $n = 3$, $m = 2$, and, denoting $x = (x_1, x_2, x_3)$,

$$f_1 = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \quad f_2 = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}. \tag{3}$$

In this case, there does not exist any nontrivial singular trajectory, and the manifold $\mathcal{S}$ coincides with the axis $(0x_3)$. A simple explicit hybrid strategy was described. In (Prieur and Trélat, 2005*a*), a general result was derived, that requires a countable number of components in the definition of the hysteresis hybrid feedback law.

In (Prieur and Trélat, 2005*c*), we investigate the so-called *Martinet system* in $\mathbb{R}^3$,

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x), \tag{4}$$

where, denoting $x = (x_1, x_2, x_3)$,

$$f_1 = \frac{\partial}{\partial x_1} + \frac{x_2^2}{2} \frac{\partial}{\partial x_3}, \ f_2 = \frac{\partial}{\partial x_2}, \tag{5}$$

and the control function $u = (u_1, u_2)$ satisfies the constraint

$$u_1^2 + u_2^2 \leq 1. \tag{6}$$

This is a well known case in $\mathbb{R}^3$ for which there exist singular minimizing trajectories. However, the previous procedure can be applied, for two main reasons. First, the minimal time function can be proved to belong to the *log-exp* class (see (van den Dries *et al.*, 1994)), which is a o-minimal extension of the subanalytic class, and thus, its singular set $\mathcal{S}$ is a stratified submanifold of codimension greater than or equal to one. This stratification property allows to define a switching strategy near the manifold $\mathcal{S}$. Second, the set of extremities of singular trajectories is small in $\mathcal{S}$, and invariance properties for the optimal flow thus still hold in $\mathbb{R}^3 \setminus \mathcal{S}$. In general, however, this fact is far from being true.

2. PRELIMINARIES

2.1 The minimal time problem

Consider the minimal time problem for the system (1) with the constraint (2). Since *Hörmander's condition* holds for $(f_1, \ldots, f_m)$, any two points of $\mathbb{R}^n$ can be joined by a minimal time trajectory of (1), (2). Denote by $T(x)$ the minimal time needed to steer the system (1) with the constraint (2) from a point $x \in \mathbb{R}^n$ to the origin 0 of $\mathbb{R}^n$.

For $T > 0$, let $\mathcal{U}_T$ denote the (open) subset of $u(\cdot)$ in $L^\infty([0, T], \mathbb{R}^m)$ such that the solution of (1), starting from 0 and associated to a control $u(\cdot) \in \mathcal{U}_T$, is well defined on $[0, T]$. The mapping

$$E_T : \begin{array}{rcl} \mathcal{U}_T & \longrightarrow & \mathbb{R}^n \\ u(\cdot) & \longmapsto & x(T), \end{array}$$

which to a control $u(\cdot)$ associates the end-point $x(T)$ of the corresponding solution $x(\cdot)$ of (1) starting at 0, is called *end-point mapping* at time T; it is a smooth mapping.

A trajectory $x(\cdot)$ of (1), with $x(0) = 0$, is said *singular* on $[0, T]$ if its associated control $u(\cdot)$ is a singular point of the end-point mapping E_T (*i.e.*, if the Fréchet derivative of E_T at $u(\cdot)$ is not onto). In that case, the control $u(\cdot)$ is said to be singular.

2.2 Class of controllers and notion of hybrid solution

In this section, we recall the general setting for hybrid systems. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be defined by $f(x, u) = \sum_{i=1}^m u_i f_i(x)$. The system (1) writes

$$\dot{x}(t) = f(x(t), u(t)). \qquad (7)$$

The controllers under consideration depend on the continuous state $x \in \mathbb{R}^n$ and also on a discrete variable $s_d \in \mathcal{N}$, where $\mathcal{N}$ is a nonempty finite subset of $\mathbb{N}$.

A hybrid feedback is a 4-tuple (C, D, k, k_d), where

- C and D are subsets of $\mathbb{R}^n \times \mathcal{N}$;
- $k : \mathbb{R}^n \times \mathcal{N} \to \mathbb{R}^m$ is a function;
- $k_d : \mathbb{R}^n \times \mathcal{N} \to \mathcal{N}$ is a function.

The sets C and D are respectively called the *controlled continuous evolution set* and the *controlled discrete evolution set*.

We next recall the notion of robustness to small noise. Consider two functions e and d satisfying the following *regularity assumptions*:

$$\begin{array}{l} e(\cdot,\cdot), d(\cdot,\cdot) \in L^\infty_{loc}(\mathbb{R}^n \times [0,+\infty); \mathbb{R}^n), \\ e(\cdot,t), d(\cdot,t) \in C^0(\mathbb{R}^n, \mathbb{R}^n), \ \forall t \in [0,+\infty). \end{array} \qquad (8)$$

We introduce these functions as a measurement noise e and an external disturbance d. Below, define the perturbed hybrid system $\mathcal{H}_{(e,d)}$. The notion of solution of such hybrid perturbed systems has been well studied in the literature (see *e.g.* (Bensoussan and Menaldi, 1997; Branicky, 1998; Prieur, 2005; Prieur and Astolfi, 2003; Tavernini, 1987)). Here, we recall the notion of solution given in (Goebel and Teel, 2005; Goebel *et al.*, 2004).

Let $S = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$, where $J \in \mathbb{N} \cup \{+\infty\}$ and $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$. The domain S is said to be a hybrid time domain. A map $(x, s_d) : S \to \mathbb{R}^n \times \mathcal{N}$ is said to be a solution of $\mathcal{H}_{(e,d)}$ with the initial condition (x_0, s_0) if

- the map x is continuous on S;
- for every j, $0 \leq j \leq J-1$, the map $x : t \in (t_j, t_{j+1}) \mapsto x(t, j)$ is absolutely continuous;
- for every j, $0 \leq j \leq J-1$ and almost every $t \geq 0$, $(t, j) \in S$, we have

$$\begin{array}{rcl} (x(t,j) + e(x(t,j),t), s_d(t,j)) \in C, & & \\ \dot{x}(t,j) & = & f(x(t), k(x(t,j) \\ & & +e(x(t,j),t), s_d(t,j))) \\ & & +d(x(t,j),t), \\ \dot{s}_d(t,j) = 0; & & \end{array}$$

- for every $(t, j) \in S$, $(t, j+1) \in S$, we have

$$\begin{array}{rcl} (x(t,j) + e(x(t,j),t), s_d(t,j)) \in D, & & \\ x(t,j+1) & = & x(t,j), \\ s_d(t,j+1) & = & k_d(x(t,j) + e(x(t,j),t), \\ & & s_d(t,j)); \end{array}$$

- $(x(0,0), s_d(0,0)) = (x_0, s_0)$.

In this context, we next recall the concept of stabilization of (7) by a minimal time hybrid feedback law sharing a robustness property with respect to measurement noise and external disturbances (see (Prieur and Trélat, 2005*b*)). The usual Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$. Recall that a function of class $\mathcal{K}_\infty$ is a function δ: $[0, +\infty) \to [0, +\infty)$ which is continuous, increasing, satisfying $\delta(0) = 0$ and $\lim_{R \to +\infty} \delta(R) = +\infty$.

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

$$\rho(x) > 0, \ \forall x \neq 0. \qquad (9)$$

We say that the *completeness assumption for* ρ holds if, for all (e, d) satisfying the regularity assumptions (8), and so that,

$$\begin{array}{l} \sup_{[0,+\infty)} |e(x,\cdot)| \leq \rho(x), \\ \operatorname{esssup}_{[0,+\infty)} |d(x,\cdot)| \leq \rho(x), \ \forall x \in \mathbb{R}^n, \end{array} \qquad (10)$$

for every $(x_0, s_0) \in \mathbb{R}^n \times \mathcal{N}$, there exists a maximal solution on $[0, +\infty)$ of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0).

We say that the *uniform finite time convergence property* holds if there exists a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying (9), such that the completeness assumption for ρ holds, and if there exists a function $\delta : [0, +\infty) \to [0, +\infty)$ of class $\mathcal{K}_\infty$ such that, for every $R > 0$, there exists $\tau = \tau(\mathrm{diam}(R)) > 0$, for all functions e, d satisfying the regularity assumptions (8) and inequalities (10) for this function ρ, for every $x_0 \in B(0, R)$, and every $s_0 \in \mathcal{N}$, the maximal solution (x, s_d) of $\mathcal{H}_{(e,d)}$ starting from (x_0, s_0) satisfies

$$\begin{array}{c} |x(t,j)| \leq \delta(R), \ \forall t \geq 0, \ (t,j) \in S, \\ x(t,j) = 0, \ \forall t \geq \tau, \ (t,j) \in S. \end{array}$$

The point 0 is said to be a *semi-global minimal time hybrid robust stabilizable equilibrium* for the system (7) if, for every $\varepsilon > 0$ and every compact subset $K \subset$

$\mathbb{R}^n$, there exists a hybrid feedback law (C,D,k,k_d) satisfying the constraint

$$||k(x,s_d)|| \leq 1, \tag{11}$$

where $||\cdot||$ stands for the Euclidian norm in $\mathbb{R}^m$, such that:

- the uniform finite time convergence property holds;
- there exists a continuous function $\rho_{\varepsilon,K} : \mathbb{R}^n \to \mathbb{R}$ satisfying (9) for $\rho = \rho_{\varepsilon,K}$, such that, for every $(x_0,s_0) \in K \times \mathcal{N}$, all functions e,d satisfying the regularity assumptions (8) and inequalities (10) for $\rho = \rho_{\varepsilon,K}$, the maximal solution of $\mathcal{H}_{(e,d)}$ starting from (x_0,s_0) reaches 0 within time $T(x_0)+\varepsilon$, where $T(x_0)$ denotes the minimal time to steer the system (7) from x_0 to 0, under the constraint $||u|| \leq 1$.

2.3 *The subanalytic class and the log-exp class*

In this section, we recall the definition of a subanalytic function, then the one of a log-exp function (see (van den Dries *et al.*, 1994)), and some properties that are used in a crucial way in the proof of the following main results.

Let M be a real analytic finite dimensional manifold. A subset A of M is said to be *semi-analytic* if and only if, for every $x \in M$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} ($1 \leq i \leq p$ and $1 \leq j \leq q$), such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \mid \ g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0,\ j = 1 \ldots q\}.$$

Let SEM(M) denote the set of semi-analytic subsets of M. The image of a semi-analytic subset by a proper analytic mapping is not in general semi-analytic, and thus this class has to be enlarged.

A subset A of M is said to be *subanalytic* if and only if, for every $x \in M$, there exist a neighborhood U of x in M and $2p$ couples $(\Phi_i^\delta, A_i^\delta)$ ($1 \leq i \leq p$ and $\delta = 1,2$), where $A_i^\delta \in SEM(M_i^\delta)$, and where the mappings $\Phi_i^\delta : M_i^\delta \to M$ are proper analytic, for real analytic manifolds M_i^δ, such that

$$A \cap U = \bigcup_{i=1}^{p} (\Phi_i^1(A_i^1) \backslash \Phi_i^2(A_i^2)).$$

Let SUB(M) denote the set of subanalytic subsets of M.

The subanalytic class is closed by union, intersection, complementary, inverse image by an analytic mapping, image by a proper analytic mapping. In brief, the subanalytic class is *o-minimal* (see (van den Dries and Miller, 1996)). Moreover subanalytic sets are *stratifiable* in the following sense. A *stratum* of a differentiable manifold M is a locally closed sub-manifold of M. A locally finite partition $\mathcal{S}$ of M is a *stratification* of M if any $S \in \mathcal{S}$ is a stratum such that, for every $T \in \mathcal{S}$,

$$T \cap \partial S \neq \emptyset \Rightarrow T \subset \partial S \text{ and } \dim T < \dim S.$$

Finally, a mapping between two analytic manifolds M and N is said to be *subanalytic* if its graph is a subanalytic subset of $M \times N$.

The *log-exp class*, defined in (van den Dries *et al.*, 1994), is an extension of the subanalytic class with functions log and exp, that shares the same properties than the one of subanalytic sets (namely, it is a o-minimal class). More precisely, a log-exp function is defined by a finite composition of subanalytic functions, of exponentials and logarithms; if $g_1,\ldots,g_m$, are log-exp functions in $\mathbb{R}^n$, and if F is a log-exp function in $\mathbb{R}^m$, then the composition $F \circ (g_1,\ldots,g_m)$ is a log-exp function in $\mathbb{R}^n$. A log-exp set is defined by a finite number of equalities and inequalities using log-exp functions.

Let M be an analytic manifold, and F be a subanalytic (resp., log-exp) function on M. The *analytic singular support* of F is defined as the complement of the set of points x in M such that the restriction of F to some neighborhood of x is analytic. The following property is of great interest.

Proposition 2.1. (van den Dries *et al.*, 1994; Tamm, 1981) The analytic singular support of F is subanalytic (resp., log-exp); thus, in particular, it is stratifiable. If F is moreover locally bounded on M, then it is moreover of codimension greater than or equal to one.

3. THE MAIN RESULTS

3.1 *In the absence of singular minimizers*

The following theorem is the main result of (Prieur and Trélat, 2005*a*).

Theorem 3.1. Assume that Hörmander's condition holds. If there exists no nontrivial minimal time singular trajectory of (1), (2), starting from 0, then the origin is a semi-global minimal time hybrid robust stabilizable equilibrium for the system (1), under the constraint (2).

3.2 *In the presence of singular minimizers*

In (Prieur and Trélat, 2005*c*), we prove the following result.

Theorem 3.2. The origin is a semi-globally minimal time robustly stabilizable equilibrium for the system (4) with the constraint (6).

3.3 The idea of the proof

3.3.1. Regularity of the minimal time function First, to derive Theorem 3.1, the following crucial remark is due. Under the assumption of the absence of nontrivial singular minimizing trajectory, the minimal time function $T(x)$ to steer the system (1) from x to 0, under the constraint (2), is subanalytic. The corresponding minimal time feedback controller is continuous (even analytic) on $\mathbb{R}^n \setminus \mathcal{S}$, where $\mathcal{S}$ is the set of points of $\mathbb{R}^n$ at which T is not analytic. Since T is subanalytic, $\mathcal{S}$ is a stratified submanifold of $\mathbb{R}^n$, of codimension greater than or equal to one.

To derive Theorem 3.2, we note the following fact. It has been proved in (Bonnard *et al.*, 1999) (see also (Agrachev *et al.*, 1997; Bonnard and Trélat, 2001)) that the minimal time function $T(x)$ to steer the system (4) from x to 0, under the constraint (6), belongs to the *log-exp* class, and thus, is stratifiable. Note that it has been proved in (Agrachev *et al.*, 1997) that T is not subanalytic.

3.3.2. The optimal controller Outside the singular set $\mathcal{S}$, the function $T(\cdot)$ is analytic, and the minimal time controllers steering a point $x \in \mathbb{R}^n \setminus \mathcal{S}$ to 0 are given by the closed-loop formula
$u_i(x) = -\frac{1}{2}\langle \nabla(T(x)^2), f_i(x)\rangle,\ i = 1, \ldots, m.$

In the Martinet case, we get the more explicit expression

$$u_1(x) = -\frac{1}{2}\left(\frac{\partial T}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial T}{\partial x_3}\right),$$
$$u_2(x) = -\frac{1}{2}\frac{\partial T}{\partial x_2}.$$

The smoothness of this optimal controller outside the submanifold $\mathcal{S}$ ensures a robustness property of the stability outside $\mathcal{S}$. In the first case, T is subanalytic, and in the second, T is log-exp. In both cases, the singular set $\mathcal{S}$ is a stratified submanifold, of codimension greater than or equal to one. In a neighborhood of $\mathcal{S}$, it is therefore necessary to use other controllers, and to define an adequate switching strategy. Notice that this neighborhood can be chosen arbitrarily thin, and thus, the time ε needed for its traversing is arbitrarily small. Therefore, starting from an initial point x_0, the time needed to join 0, using this hybrid strategy, is equal to $T(x_0) + \varepsilon$. The switching strategy is achieved by adding a dynamical discrete variable s_d and using a hybrid feedback law.

3.3.3. The hybrid strategy The main fact consists in constructing neighborhoods of $\mathcal{S}$ whose complements share invariance properties for the optimal flow (see (Prieur and Trélat, 2005*a*) for a general result on the cut locus, and see (Prieur and Trélat, 2005*c*) for the specific Martinet case).

The second component of the hysteresis then consists of a set of controllers, defined in a neighborhood Ω of $\mathcal{S}$, such that every solution starting from Ω leaves Ω in small time. This is possible using Hörmander's condition (see (Prieur and Trélat, 2005*a*) for details).

Then, a hybrid feedback law is constructed, using an hysteresis. This is the most technical part.

3.3.4. Further comments The crucial fact used in the proof relies on stratification properties of the minimal time function. This holds whenever the minimal time function belongs to the subanalytic class, or to the log-exp class. More generally, this holds in a o-minimal class. For general analytic control systems of the form (1), (2), in the absence of singular minimizing trajectory, the minimal time function to a point can be proved to be subanalytic outside this point. In the Martinet case, the minimal time function is not subanalytic, due to the presence of a singular minimizing trajectory, however, it belongs to the log-exp class, which is also o-minimal, and hence, is still stratifiable.

This situation extends to the so-called Martinet integrable case (see (Bonnard *et al.*, 1999)). In a neighborhood of 0, a model of this latter case is given by the two vector fields

$$f_1 = g_1(x_2)\left(\frac{\partial}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial}{\partial x_3}\right),\ f_2 = g_2(x_2)\frac{\partial}{\partial x_2},$$

where g_1 and g_2 are germs of analytic functions at 0 such that $g_i(0) = 1$. It is proved in (Bonnard *et al.*, 1999) that the minimal time function still belongs to the log-exp class in this case. This is however no longer true whenever the functions g_1, g_2 also depend on x_1 and x_3. In this case, it is conjectured in (Bonnard and Trélat, 2001) that the minimal time function does not belong to the log-exp class. In this latter case, a larger class is due for describing the regularity of the minimal time function, but it is not clear if it is possible to find an adapted o-minimal class.

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