ON THE CONTROL OF A BIMORPH MIRROR

Pierre Le Gall * **Christophe Prieur** ** **Lionel Rosier** *

* *Institut Elie Cartan, Université Henri Poincaré Nancy 1, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France*
** *LAAS-CNRS, 7 avenue du colonel Roche, 31077 Toulouse, France*

Abstract: We consider a large bimorph mirror which is composed of three layers: a purely elastic layer, a layer equipped with a distribution of sensor piezoelectric inclusions, and a layer equipped with a distribution of actuator piezoelectric inclusions. Such a device is modelized by a system of two coupled PDE, the first one involving a second-order operator without time derivative, and the second one being a plate equation. The controllability properties are investigated for both the 1D model and the 2D model, and an output feedback law is proposed for the stabilization of the 1D model.

Keywords: Bimorph mirror, beam equation, plate equation, controllability, output feedback stabilization.

1. INTRODUCTION

The general problem under study in this paper is the control and the stabilization of a bimorph mirror. Such a structure is an active multi-layered flexible plate (see Figure 1):

- a layer is assumed to be a purely flexible plate (this is the mirror);
- another layer is equipped with a distribution of piezoelectric inclusions, which are used as *actuators*;
- the last layer is also equipped with a distribution of piezoelectric inclusions, which are used as *sensors*.

Such a device is used in Adaptive Optics with large ground-based telescopes. Recall that the main goal of Adaptive Optics is to compensate in real time for random wavefront disturbances.

This kind of flexible structure has been investigated in (Lenczner and Prieur, 2006). A PDE-type model has been obtained by making the characteristic dimension of the heterogeneities tend to zero in elastic plates including small inclusions. Two-scale convergence for homogenization as in (Allaire, 1992) was used. The goal of the present paper is to investigate the controllability and the stabilization properties of that model.

There exists a wide literature for the controllability and the stabilization of flexible plates equipped with piezoelectric inclusions. See e.g. (Tucsnak, 1996; Crépeau and Prieur, to appear) where controllability results for the Bernoulli Euler equation are obtained by using the Hilbert Uniqueness Method and Diophantine approximations. Stabilization results for the beam equation are given in (Ammari and Tucsnak, 2000;

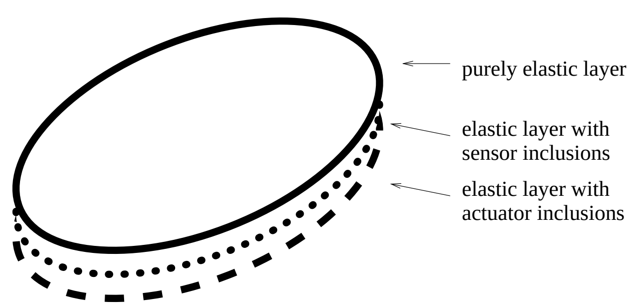

Fig. 1. An active bimorph mirror

Ammari *et al.*, 2001), (Liu and Zheng, 1999) and (Le Gall *et al.*, 2006).

An important feature of the model considered here is that there are two coupled PDE of very different nature: the first one is defined by a second-order operator without any time derivative, and the second one is a Euler-Bernoulli equation.

The model under consideration will be investigated in 1D and in 2D.

1D MODEL (BEAM)

$$\begin{cases} -\partial_x^2 u = -s\partial_x\varphi + a\partial_x^2\varphi, \\ \partial_t^2 w + \partial_x^4 w + w = \partial_x^2\varphi, \\ w(0,t) = w(L,t) = \partial_x w(0,t) = \partial_x w(L,t) = 0, \\ u(0,t) = u(L,t) = 0. \end{cases}$$

In above system, $x \in (0,L)$ is the spatial coordinate and t is time, w stands for the transverse deflection of the beam, u is the scalar longitudinal displacement, and φ is the voltage applied to the inclusions of the actuator layer. We shall assume that $a > 0$ and that s is any real number.

2D MODEL (PLATE)

$$-(\lambda+\mu)\partial_1 \text{div}\, U - \mu\Delta u_1 = -s_\beta\partial_\beta\varphi + a_{\gamma\delta}\partial^2_{\gamma\delta}\varphi, \quad (1)$$

$$-(\lambda+\mu)\partial_2 \text{div}\, U - \mu\Delta u_2 = -t_\beta\partial_\beta\varphi + b_{\gamma\delta}\partial^2_{\gamma\delta}\varphi, \quad (2)$$

$$\partial_t^2 w + \Delta^2 w + w = g_{\alpha\beta}\partial^2_{\alpha\beta}\varphi, \quad (3)$$

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad (4)$$

$$u_1 = u_2 = 0 \quad \text{on } \Omega. \quad (5)$$

In above system, $x = (x_1, x_2) \in \Omega$, where $\Omega \subset \mathbb{R}^2$ is a Lipschitzian bounded open set, $\partial_1\varphi = \partial\varphi/\partial x_1$, $\partial_2\varphi = \partial\varphi/\partial x_2$, $U = (u_1, u_2)$ stands for the 2D-longitudinal displacement, $\text{div}\, U = \partial_1 u_1 + \partial_2 u_2$, and Einstein's convention of summation for repeated indices has been adopted. As usual, $\lambda > 0$ and $\mu > 0$ stand for the Lame coefficients. We shall assume that the 2×2 matrices $A = (a_{\gamma\delta})$, $B = (b_{\gamma\delta})$, and $G = (g_{\alpha\beta})$ are positive definite.

The paper is organized as follows. In Section 2, we prove the controllability of our model by treating separately the 1D model and the 2D model. In Section 3 we investigate the stabilization of the 1D model by identifying a compatibility condition.

2. CONTROLLABILITY PROPERTIES

2.1 Toy problem

We first investigate the possibility of achieving the exact controllability in finite time. The main feature of the control problem under study is the fact that we have to control the solution u (resp $U = (u_1, u_2)$) of an elliptic equation, in addition of the solution w of the beam (resp. plate) equation. In a certain sense, u or U may be viewed as an "output" depending only on the input φ. The consideration of the following "toy problem" is quite illuminating. Consider three matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{k\times m}$ and the following controlled linear system in finite dimension:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cu. \end{cases} \quad (6)$$

We shall say that (x,y) is *controllable in time T* if for all pairs (x^0, y^0), (x^T, y^T) in $\mathbb{R}^n \times \mathbb{R}^k$ one may find a control input $u \in C([0,T];\mathbb{R}^m)$ and a solution $(x(t), y(t))$ of (6) connecting (x^0, y^0) to (x^T, y^T). Then the following result holds.

Proposition 1. (x,y) is controllable in time T for any $T > 0$ if and only if the pair of matrices (A,B) is controllable and the matrix C is onto.

Proof. The sense $\Rightarrow$ is obvious. Conversely, assume that the pair (A,B) is controllable and that the matrix C is onto (i.e., the map $u \in \mathbb{R}^m \mapsto Cu \in \mathbb{R}^k$ is onto). Let two pairs (x^0, y^0) and (x^T, y^T) be given in $\mathbb{R}^n \times \mathbb{R}^k$. In a first step we construct a trajectory $(\overline{x}, \overline{y})$ associated with a control $\overline{u}$ such that $\overline{y}(0) = y^0$ and $\overline{y}(T) = y^T$. To do this we pick some vectors u^0 and u^T such that $y^0 = Cu^0$ and $y^T = Cu^T$, and we set $\overline{u}(t) := (1 - t/T)u^0 + (t/T)u^T$. Let $\overline{x}$ denote the solution of $\dot{\overline{x}} = A\overline{x} + B\overline{u}$, $\overline{x}(0) = 0$, and let $\overline{y}(t) = C\overline{u}(t)$. Performing the change of variables $\hat{x} = x - \overline{x}$, $\hat{u} = u - \overline{u}$, and $\hat{y} = y - \overline{y}$, we notice that the pair $(\hat{x}, \hat{y})$ has to satisfy the system

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B\hat{u}, \\ \hat{y} = C\hat{u}, \end{cases}$$

the constraints being now $C\hat{u}(0) = C\hat{u}(T) = 0$, $\hat{x}(0) = x_0 - \overline{x}(0)$, and $\hat{x}(T) = x_T - \overline{x}(T)$. The pair (A,B) being controllable, it is well known that we may find a control input $u \in C_0^\infty(0,T)$ steering $\hat{x}$ from $x_0 - \overline{x}(0)$ to $x_T - \overline{x}(T)$. ■

Roughly speaking, to design the control we first construct a "static" control allowing to connect y^0 to y^T, and next we add to it a dynamical correction allowing to connect x^0 to x^T. The point is that this correction may be chosen with the additional constraint that it vanishes at both extremities of $[0,T]$. We shall see in the next section that this method works as well for the control of the bimorph mirror.

2.2 Controllability of the 1D system.

Let us introduce the spaces $V = H^2(0,L) \cap H^1_0(0,L)$, $\tilde{V} = H^2_0(0,L), H = L^2(0,L)$ and the operator $\mathcal{A}(w,v) = (v, -w^{(4)} - w)$ with domain $\mathcal{D}(\mathcal{A}) = (H^4(0,L) \cap H^2_0(0,L)) \times H^2_0(0,L) \subset \tilde{V} \times H$. Then we have the following result.

Theorem 1. Each pair $((w^0,w^1),u^0)$, $((w^{0,T},w^{1,T}), u^T)$ of triplets of functions in $\mathcal{D}(\mathcal{A}) \times V$ may be connected by a trajectory associated with a control function $\varphi \in C([0,T];H^2(0,L))$.

Proof: As the system is time-reversible, we may assume without loss of generality that $w^{0,T} = w^{1,T} = u^T = 0$. Pick any triplet $((w^0,w^1),u^0)$ in $\mathcal{D}(\mathcal{A}) \times V$. As $a \neq 0$, there exists a unique solution $\varphi^0 \in V$ to the elliptic problem

$$\begin{cases} a\partial_x^2\varphi^0 - s\partial_x\varphi^0 = -\partial_x^2 u^0, \\ \varphi^0(0) = \varphi^0(L) = 0. \end{cases}$$

Let us set $\overline{\varphi}(t) = (1-t/T)\varphi^0$ and let $(\overline{w},\overline{u})$ be the corresponding solution of the system

$$\begin{cases} -\partial_x^2\overline{u} = -s\partial_x\overline{\varphi} + a\partial_x^2\overline{\varphi}, \\ \partial_t^2\overline{w} + \partial_x^4\overline{w} + \overline{w} = \partial_x^2\overline{\varphi}, \\ \overline{w}(0,t) = \overline{w}(L,t) = \partial_x\overline{w}(0,t) = \partial_x\overline{w}(L,t) = 0, \\ \overline{u}(0,t) = \overline{u}(L,t) = 0, \\ \overline{w}(x,0) = \partial_t\overline{w}(x,0) = 0. \end{cases}$$

Notice that $\overline{u}(x,0) = u^0(x)$ and $\overline{u}(x,T) = 0$, for $\overline{\varphi}(0) = \varphi^0$ and $\overline{\varphi}(T) = 0$. As $\partial_x^2\overline{\varphi} \in C^1([0,T];H)$, we infer from a classical result in semigroup theory (see e.g. (Cazenave and Haraux, 1998)) that $(\overline{w},\partial_t\overline{w}) \in C([0,T];\mathcal{D}(\mathcal{A})) \cap C^1([0,T],\tilde{V} \times H)$. Next, we perform a change of unknown functions. We set $\hat{w} = w - \overline{w}$, $\hat{u} = u - \overline{u}$, and $\hat{\varphi} = \varphi - \overline{\varphi}$. Then the pair $(\hat{w},\hat{u})$ solves

$$\begin{cases} -\partial_x^2\hat{u} = -s\partial_x\hat{\varphi} + a\partial_x^2\hat{\varphi}, \\ \partial_t^2\hat{w} + \partial_x^4\hat{w} + \hat{w} = \partial_x^2\hat{\varphi}, \\ \hat{w}(0,t) = \hat{w}(L,t) = \partial_x\hat{w}(0,t) = \partial_x\hat{w}(L,t) = 0, \\ \hat{u}(0,t) = \hat{u}(L,t) = 0, \\ \hat{w}(x,0) = w^0(x),\ \partial_t\hat{w}(x,0) = w^1(x). \end{cases}$$

and $\hat{\varphi}$ has to be designed in such a way that $\hat{u}(.,0) = 0$ and $(\hat{w}(.,T),\partial_t\hat{w}(.,T),\hat{u}(.,T)) = (-\overline{w}(.,T),-\partial_t\overline{w}(.,T),0)$. In particular, the condition $\hat{\varphi}(.,0) = \hat{\varphi}(.,T) = 0$ is required. A classical result (Lions, 1988) on the controllability of the plate equation gives the existence of some control input $\hat{\varphi}$, which is compactly supported in time, and such that the corresponding trajectory fulfills all the above conditions. Alternatively, using the fact that the control is applied on the *whole* domain, we may design an *explicit* control input by specifying some trajectory $(\hat{w},\hat{w}_t)$ in the class $C([0,H];\mathcal{D}(\mathcal{A})) \cap C^1([0,T];\tilde{V} \times H)$ and fulfilling the above conditions. Next, $\hat{\varphi}(t)$ may be defined for each $t \in [0,T]$ as the solution of the elliptic problem

$$\begin{cases} \partial_x^2\hat{\varphi} = \partial_t^2\hat{w} + \partial_x^4\hat{w} + \hat{w}, \\ \hat{\varphi}(0,t) = \hat{\varphi}(L,t) = 0. \end{cases}$$

To complete the proof of Theorem 1, we need the following result:

Proposition 2. Let v^0, v^1, v^2 be three functions in $H^4(0,L) \cap H^2_0(0,L)$, $H^2_0(0,L)$ and $L^2(0,L)$, respectively. There there exists a function $v \in C([0,T];H^4(0,L)) \cap C^1([0,T];H^2_0(0,L)) \cap C^2([0,T];$ $L^2(0,L))$ fulfilling $v(0) = v^0$, $v'(0) = v^1$, $v''(0) = v^2$ and $v(T) = v'(T) = v''(T) = 0$.

Proof of Proposition 2. Let $(\psi_k)_{k\geq 1}$ denote an orthonormal basis of $L^2(0,L)$ constituted of eigenfunctions for the operator $Aw = w^{(4)}$ with the boundary conditions $w(0) = w(L) = w'(0) = w'(L) = 0$. The eigenvalue associated with the function ψ_k is denoted by λ_k. Pick any function $h \in C^4(\mathbb{R}^+)$ such that $h(0) = h^{(3)}(0) = 1$, $h'(0) = h''(0) = h^{(4)}(0) = 0$, and any function $g \in C^4(\mathbb{R}^+)$ such that $g(t) = 1$ for $t \leq T/4$ and $g(t) = 0$ for $t \geq T/2$. If the functions v^0, v^1 and v^2 are decomposed along the ψ_k's as

$$v^0 = \sum_{k\geq 1} a_k\psi_k,\ v^1 = \sum_{k\geq 1} b_k\psi_k,\ v^2 = \sum_{k\geq 1} c_k\psi_k$$

then $\sum_{k\geq 1}\left(k^8|a_k|^2 + k^4|b_k|^2 + |c_k|^2\right) < \infty$. The function v is then defined as

$$v(x,t) := g(t)\sum_{k\geq 1}\Big(a_k h(\sqrt{\lambda_k}t) + \frac{b_k}{\sqrt{\lambda_k}}h''(\sqrt{\lambda_k}t) + \frac{c_k}{\lambda_k}h'(\sqrt{\lambda_k}t)\Big)\psi_k(x).$$

As $\lambda_k \sim Ck^4$ as $k \to \infty$, one readily obtains that $v \in C([0,T];H^4(0,L)) \cap C^1([0,T];H^2_0(0,L)) \cap C^2([0,T];L^2(0,L))$. The properties $v(0) = v^0$, $v'(0) = v^1$, $v''(0) = v^2$ and $v(T) = v'(T) = v''(T) = 0$ are obvious. ■

The proof of Theorem 1 is completed by applying Proposition 2 to $v^0 = w^0$, $v^1 = w^1$, and $v^2 = -\partial_x^4 w^0 - w^0$ on the interval $[0,T/2]$, and next to $v^0 = -\overline{w}(T)$, $v^1 = \overline{w}_t(T)$ and $v^2 = -\partial_x^4\overline{w}(T) - \overline{w}(T)$ on the interval $[T/2,T]$ by reversing the time. ■

Remark 1. To simplify the exposition, we have imposed Dirichlet boundary conditions ($\varphi(0,t) = \varphi(L,t) = 0$) to the control input φ, but Neumann boundary conditions (i.e. $\partial_x\varphi(0,t) = \partial_x\varphi(L,t) = 0$ together with e.g. $\int_0^L \varphi(x,t)\,dx = 0$) may be taken instead.

2.3 Controllability of the 2D system.

Notice first that for any trajectory $(w(t),U(t))$ of (1)-(5) associated with the control input $\varphi(t)$, $t \in [0,T]$, the control φ has to be at each instant t a solution of the following system of elliptic PDE

$$-s_\beta \partial_\beta \varphi + a_{\gamma\delta}\partial^2_{\gamma\delta}\varphi = f_1, \qquad (7)$$

$$-t_\beta \partial_\beta \varphi + b_{\gamma\delta}\partial^2_{\gamma\delta}\varphi = f_2, \qquad (8)$$

where $f_i := -(\lambda+\mu)\partial_i \mathrm{div}\, U - \mu\Delta u_i$ for $i = 1,2$. Obviously, for φ to exist the functions f_1 and f_2 have to satisfy a *compatibility condition*, namely

$$(-t_\beta\partial_\beta + b_{\gamma\delta}\partial^2_{\gamma\delta})\, f_1 = (-s_\beta\partial_\beta + a_{\gamma\delta}\partial^2_{\gamma\delta})\, f_2. \qquad (9)$$

If, moreover, $-t_\beta\partial_\beta + b_{\gamma\delta}\partial^2_{\gamma\delta} = \lambda(-s_\beta\partial_\beta + a_{\gamma\delta}\partial^2_{\gamma\delta})$, then $f_2 = \lambda f_1$.

It is not clear, however, that these conditions are sufficient to guarantee the existence of a solution of (7)-(8). We shall adopt the following

Definition 1. A quadruplet $(w^0, w^1, u_1, u_2) \in (H^4(\Omega)\cap H^2_0(\Omega)) \times H^2_0(\Omega) \times (H^2(\Omega)\cap H^1_0(\Omega))^2$ will be said to be *compatible* if the system (7)-(8) possesses a solution $\varphi \in H^2(\Omega)$, the functions f_1, f_2 being defined as $f_i := -(\lambda+\mu)\partial_i \mathrm{div}\,(u_1,u_2) - \mu\Delta u_i$ for $i = 1,2$.

Then the following result holds true.

Theorem 2. Each pair (w^0, w^1, u_1, u_2), $(w^{0,T}, w^{1,T}, u_1^T, u_2^T)$ of compatible quadruplets may be connected by a trajectory associated with a control function $\varphi \in C([0,T]; H^2(\Omega))$.

Proof. The proof follows the same pattern as for Theorem 1. Once again, we may assume that the terminal quadruplet is $(0,0,0,0)$.

<u>Step 1: Control of the static equations.</u>

As the quadruplet (w^0, w^1, u_1, u_2) is assumed to be compatible, there exists a function $\varphi^0 \in H^2(\Omega)$ solving (7)-(8). We set $\overline{\varphi}(t) := (1-t/T)\varphi^0$. Next, $\overline{w}$ is defined as the solution of the plate equation (3) (with $\overline{\varphi}$ substituted to φ) with the boundary conditions (4) and issuing from $(0,0)$, and $\overline{U} = (\overline{u_1}, \overline{u_2})$ is the solution of the elliptic problem (1), (2) and (5).

<u>Step 2: Control of the plate equation.</u>

We perform the change of unknown functions $\hat{w} = w - \overline{w}$, $\hat{u}_i = u_i - \overline{u_i}$ ($i = 1,2$), and $\hat{\varphi} = \varphi - \overline{\varphi}$. Then $\hat{u}_1, \hat{u}_2, \hat{w}$ and $\hat{\varphi}$ have to fulfill (1)-(5). The constraints at time 0 and T are respectively $(\hat{w}(0), \hat{w}_t(0), \hat{u}_1(0), \hat{u}_2(0)) = (w^0, w^1, 0, 0)$ and
$(\hat{w}(T), \hat{w}_t(T), \hat{u}_1(T), \hat{u}_2(T)) = (-\overline{w}(T), -\overline{w}_t(T), 0, 0)$. To conclude, we apply the following result whose proof is virtually the same as for Proposition 2.

Proposition 3. Let v^0, v^1, v^2 be three functions in $H^4(\Omega)\cap H^2_0(\Omega)$, $H^2_0(\Omega)$, and $L^2(\Omega)$, respectively. Then there exists a function $v \in C([0,T]; H^4(\Omega)) \cap C^1([0,T]; H^2_0(\Omega)) \cap C^2([0,T]; L^2(\Omega))$ fulfilling
$v(0) = v^0,\ v'(0) = v^1,\ v''(0) = v^2$ and $v(T) = v'(T) = v''(T) = 0$.

The proof of Theorem 2 is complete. ■

3. OUTPUT STABILIZATION OF AN ADAPTIVE MIRROR

We consider the system

$$-\partial^2_x u = -s\partial_x\varphi + a\partial^2_x\varphi \qquad (10)$$

$$\partial^2_t w + \partial^4_x w + w = \partial^2_x \varphi \qquad (11)$$

$$I = \partial_t(\partial^2_x w + c\partial_x u) \qquad (12)$$

I is the distributed current field measured through the layer equipped with sensor piezoelectric patches.

We are interested in the output stabilization of above system, the control φ being expressed as a function of the output I. In Adaptive Optics, this corresponds to the problem of the stabilization at the rest position of a large mirror equipped with piezoelectric sensors and actuators. This problem should be seen as a first step towards the tracking problem, for which a control input is designed so that the state of the mirror converges to a given trajectory.

We consider the following initial conditions:

$$w(x,0) = w^0(x),\ \partial_t w(x,0) = w^1(x). \qquad (13)$$

We will prescribe the boundary conditions later.

To stabilize (10)-(12), it is natural to try to impose the following additional feedback condition

$$\partial^2_x\varphi = k\partial_t w + k'\partial_t\partial^2_x w, \qquad (14)$$

k and k' being two real numbers whose range will be specified later.

The feedback condition (14) is consistent with the output feedback condition (namely $\varphi = \Lambda(I)$ for some operator Λ) provided that some “compatibility condition” is fulfilled by any solution of (10)-(12) and (14).

3.1 The compatibility condition

To state this compatibility condition in a simple way, we consider first periodic boundary conditions on u, w and φ and we assume that $L = \pi$. Then we may rewrite (10)-(12) and (14) in using Fourier series. Indeed, we have

$$u(x,t) = \sum_{n\in\mathbb{Z}} \alpha_n(t)e^{inx},$$
$$w(x,t) = \sum_{n\in\mathbb{Z}} \beta_n(t)e^{inx},$$
$$I(x,t) = \sum_{n\in\mathbb{Z}} \gamma_n(t)e^{inx},$$

where (α_n), (β_n) and (γ_n) are three sequences of functions of time which are of class C^2. To express that φ depends only on I, we introduce also a sequence (λ_n) of complex numbers such that

$$\varphi(x,t) = \sum_{n\in\mathbb{Z}} \lambda_n\gamma_n(t)e^{inx}.$$

Let us rewrite (10)-(12) and (14) in terms of these sequences. Easily computations yields

$$\alpha_n = -\frac{is\lambda_n\gamma_n}{n} - a\lambda_n\gamma_n \quad (15)$$
$$\ddot{\beta}_n + \beta_n(n^4+1) = (k-n^2k')\dot{\beta}_n \quad (16)$$
$$\gamma_n = -n^2\dot{\beta}_n + inc\dot{\alpha}_n \quad (17)$$
$$-n^2\lambda_n\gamma_n = (k-n^2k')\dot{\beta}_n \quad (18)$$

The expression of $\beta_n(t)$ may be deduced from (16). Equations (15) and (18) allow us to compute α_n. The condition (17) can be restated as follows (with (15)):

$$c\dot{\gamma}_n\lambda_n(s-ian) - n^2\dot{\beta}_n - \gamma_n = 0. \quad (19)$$

Note that (18) implies that

$$\frac{\dot{\beta}_n}{\gamma_n}(t) \equiv \frac{\dot{\beta}_n}{\gamma_n}(t=0). \quad (20)$$

This is a compatibility condition which has to be fulfilled for (14) to hold.

However, we note that when $c=0$, (19) implies

$$\gamma_n = -n^2\dot{\beta}_n.$$

In this case, the compatibility condition (20) is always satisfied. Thus we will assume thereafter that $c=0$, so that the compatibility condition holds.

3.2 *Stability when $c=0$*

Assuming $c=0$, (19) implies

$$\gamma_n = -n^2\dot{\beta}_n$$

and (18) yields

$$\lambda_n = \frac{k-n^2k'}{n^4},$$

which corresponds to the following output feedback law

$$\partial_x^2\varphi = k\partial_t w + k'\partial_t\partial_x^2 w$$
$$= k\Lambda(I) + k'I,$$

where $\Lambda = (\partial_x^2)^{-1}$ with periodic boundary conditions.

Let us go back to the framework of the clamped beam, and let Λ denote now the operator $(\partial_x^2)^{-1}$ with Dirichlet boundary conditions. The control input φ, which has to fulfill (14), is defined by

$$\varphi = \Lambda(k\Lambda(I)+k'I) = \Lambda(k\partial_t w + k'\partial_t\partial_x^2 w).$$

Then the model under study becomes

$$-\partial_x^2 u = -s\partial_x\varphi + a\partial_x^2\varphi \quad (21)$$
$$\partial_t^2 w + \partial_x^4 w + w = k\partial_t w + k'\partial_t\partial_x^2 w, \quad (22)$$
$$I = \partial_t\partial_x^2 w \quad (23)$$
$$\varphi = \Lambda(k\Lambda(I)+k'I), \quad (24)$$

with the initial conditions

$$w(x,0) = w^0(x),\ \partial_t w(x,0) = w^1(x), \quad (25)$$

and the boundary conditions

$$w(0,t) = \partial_x w(0,t) = w(L,t) = \partial_x w(L,t) = 0, \quad (26)$$
$$u(0,t) = u(L,t) = 0. \quad (27)$$

These boundary conditions mean physically that the beam is clamped at both extremities.

Let us introduce the following energy functional

$$E(w) = \frac{1}{2}\int_0^L (|\partial_x^2 w|^2 + |\partial_t w|^2 + |w|^2)dx.$$

Formal computations along the solutions of (22) yield

$$\frac{d}{dt}E(w) = \int_0^L k|\partial_t w|^2 - k'|\partial_t\partial_x w|^2 dx.$$

Let the spaces V and H be as in Section 2.2. Using Fourier series in the $sin(k\pi x/L)$'s, we may extend Λ as a continuous operator from $H^{-2}(0,L)$ into H. (21) and (27) may then be written as

$$u = s\Lambda(\partial_x\varphi) - a\varphi.$$

The following result is the last main result of the paper.

Theorem 3. Assume that $k<0$ and $k'>0$. Then for any $(w^0,w^1)\in V\times H$, there exists a unique solution (u,w) of (21)-(27) such that $u\in C^0(\mathbb{R}^+;H)$ and $w\in C^0(\mathbb{R}^+;V)\cap C^1(\mathbb{R}^+;H)$.

Moreover we have that

$$||u||_{L^2(0,L)} + E(w) \to 0$$

as $t\to+\infty$.

Proof. Let us introduce the operator $\tilde{A}(w,v) = (v, -w^{(4)} - w + kv + k'v'')$ with domain $\mathcal{D}(\tilde{A}) = (H^4(0,L)\cap H_0^2(0,L))\times H_0^2(0,L) \subset V\times H$. The space $V\times H$ is endowed with the following scalar product

$$\langle \begin{pmatrix} w \\ v \end{pmatrix}, \begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} \rangle = \int_0^L (w''\tilde{w}'' + w\tilde{w})\,dx + \int_0^L v\tilde{v}\,dx. \quad (28)$$

We have, for all $(w,v)\in\mathcal{D}(\tilde{A})$,

$$\langle \tilde{A}\begin{pmatrix} w \\ v \end{pmatrix}, \begin{pmatrix} w \\ v \end{pmatrix} \rangle = k\int_0^L v^2\,dx - k'\int_0^L (v')^2\,dx \quad (29)$$

and thus $\tilde{A}$ is a dissipative operator as soon as $k < 0$ and $k' > 0$. It may be seen that $\tilde{A}$ generates a continuous semigroup $(\tilde{S}(t))_{t\geq 0}$ of contractions in $V \times H$.

Let us now turn to the strong stability. To prove the strong stability, it is clearly sufficient to show that

$$\lim_{t\to\infty} \tilde{S}(t) \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = 0 \quad \forall \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in \mathcal{D}(\tilde{A}).$$

Since the imbedding $\mathcal{D}(\tilde{A}) \subset V \times H$ is compact, the set

$$\text{orb} \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \bigcup_{t\geq 0} \tilde{S}(t) \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$$

is precompact in $V \times H$ for any $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ in $\mathcal{D}(\tilde{A})$. In this case the ω-limit set of $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ defined by

$$\omega \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \Big\{ W \in V \times H,\ \exists (t_n),\ t_n \to +\infty, \\ S(t_n) \begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \to W \text{ as } n \to \infty \Big\}$$

is nonempty for any $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$ in $\mathcal{D}(\tilde{A})$. Moreover, according to LaSalle's invariance principle, if

$$W \in \omega \begin{pmatrix} w^0 \\ w^1 \end{pmatrix}$$

then, for all $t \geq 0$,

$$||W|| = ||S(t)W||$$

where $||\cdot||$ is the norm associated with (28). Pick any $W = (\overline{w}^0, \overline{w}^1)^T \in \omega(w^0, w^1)^T$, and set $(\overline{w}(t), \overline{v}(t))^T = S(t)(\overline{w}^0, \overline{w}^1)^T$. Using (29), we obtain that $\overline{v}(t) = 0$ for all $t \geq 0$. The function $\overline{w}$ then solves $\partial_x^4 \overline{w} + \overline{w} = 0$ together with the boundary conditions (26), hence $\overline{w} \equiv 0$. Therefore $\overline{w}^0 = \overline{w}^1 = 0$. Thus, when $t \to \infty$, $(w, \partial_t w) \to (0,0)$ in $V \times H$, hence $I \to 0$ in $H^{-2}(0,L)$ and $\varphi \to 0$ and $u \to 0$ in $L^2(0,L)$. ■

Remark 2. When $c = 0$, $k < 0$ and $k' > 0$, the result in Theorem 3 still holds for the 2D model with $V = H_0^2(\Omega)$ and $H = L^2(\Omega)$.

REFERENCES

Allaire, G. (1992). Homogenization and two-scale convergence. *SIAM J. Math. Anal.* **23**(6), 1482–1518.

Ammari, Kais and Marius Tucsnak (2000). Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force. *SIAM J. Control Optim.* **39**(4), 1160–1181 (electronic).

Ammari, Kais, Antoine Henrot and Marius Tucsnak (2001). Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string. *Asymptot. Anal.* **28**(3-4), 215–240.

Cazenave, Thierry and Alain Haraux (1998). *An introduction to semilinear evolution equations*. Vol. 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press. New York. Translated from the 1990 French original by Yvan Martel and revised by the authors.

Crépeau, Emmanuelle and Christophe Prieur (to appear). Control of a clamped-free beam by a piezoelectric actuator. *ESAIM Control Optim. Calc. Var.*

Le Gall, P., C. Prieur and L. Rosier (2006). Stabilization of a clamped-free beam with collocated piezoelectric sensor/actuator. In: *IFAC Control Applications of Optimisation (CAO'06)*. Cachan, France.

Lenczner, M. and C. Prieur (2006). Asymptotic model of an active mirror. In: *IFAC Control Applications of Optimisation (CAO'06)*. Cachan, France.

Lions, J.-L. (1988). *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*. Vol. 8 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson. Paris. Contrôlabilité exacte. [Exact controllability], With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

Liu, Zhuangyi and Songmu Zheng (1999). *Semigroups associated with dissipative systems.* Vol. 398 of *Research Notes in Mathematics*. Chapman & Hall/CRC.

Tucsnak, Marius (1996). Regularity and exact controllability for a beam with piezoelectric actuator. *SIAM J. Control Optim.* **34**(3), 922–930.