International Journal of Tomography & Statistics
Summer 2007, Vol. 6, No. S07; Int. J. Tomogr. Stat.; 104-109
ISSN 0972-9976;

STABILIZATION OF A CLAMPED-FREE BEAM WITH COLLOCATED PIEZOELECTRIC SENSOR/ACTUATOR

Pierre Le Gall * **Christophe Prieur** ** **Lionel Rosier** *

* *Institut Elie Cartan, Université Henri Poincaré Nancy 1, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France*
** *LAAS-CNRS, 7 avenue du colonel Roche, 31077 Toulouse, France*

Abstract: We consider a Bernoulli-Euler beam, which is clamped at one boundary and free at the other, and to which are attached a piezoelectric actuator and a collocated sensor. We provide an output feedback law and characterize the sensor/actuator location for which the strong stabilization holds.

Keywords: Bernoulli-Euler beam equation, collocated piezoelectric sensor/actuator, strong stabilization.

1. INTRODUCTION

There exists now an important literature devoted to the controllability and stabilizability of flexible structures. We focus in this paper on the use of piezoelectric devices to control flexible structures, as it has been modeled in (Destuynder *et al.*, 1992; Destuynder, 1999) with second order infinite dimensional models.

The controllability of a PDE is often considered as a first step towards the stabilization of the system. Classical controllability results for the beam (or plate) equation may be found in (Komornik, 1994). See also (Rebarber, 1989), where the eigenvalue specification problem is studied for the Bernoulli-Euler beam (with the same boundary condition but with different control than in our paper), and (Fliess *et al.*, 1997), where flatness technics are used for approximate controllability problems.

The main purpose of our paper is to study the strong stabilization of a clamped-free beam, i.e. a beam clamped at one end and free at the other end. There exist technological and industrial motivations to study the stabilization of a beam equipped with piezoelectric actuators/sensors with such boundary conditions (see (Leleu, 2002)).

A very important work related to our study is (Crépeau and Prieur, to appear), where the exact controllability of a clamped-free beam equipped with a piezoelectric actuator is studied in detail. Another important work related to our study is (Ammari and Tucsnak, 2000), where an unbounded feedback is designed for the Bernoulli-Euler beam equation with *different* boundary conditions.

Here, we consider the control problem modelling the vibrations of a Bernoulli-Euler beam that is subject to the action of an attached piezoelectric actuator. If we assume that the beam is clamped at one boundary and free at the other we obtain the system:

$$
\begin{aligned}
&w_{tt}(x,t) + w_{xxxx}(x,t) = h(t)\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)],\\
&w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t),\\
&w(x,0) = w^0(x),\ w_t(x,0) = w^1(x)
\end{aligned}
$$

In the above equations, $x \in (0,\pi)$ is the spatial coordinate, t is time, w stands for the transverse deflection of the beam, $w_t = \partial w/\partial t$, etc., η and ξ stand for the ends of the actuator ($0 \le \eta < \xi \le \pi$), δ_y is the Dirac mass at the point y, and h stands for the control input (see Figure 1).

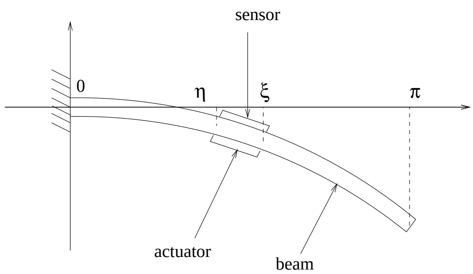

Fig. 1. A beam equipped with two piezoelectric devices.

We are interested in the stabilization of the above system, the control h being expressed as a function of the output $w_x(\eta) - w_x(\xi)$. This corresponds to the situation where the output comes from a piezoelectric sensor located on the same interval (η, ξ) as the actuator. Formal computations on the variations of the energy

$$E(w) = \int_0^\pi (|w_{xx}|^2 + |w_t|^2)\, dx$$

lead to take

$$h := k(w_{xt}(\eta) - w_{xt}(\xi))$$

where k is some number whose range will be specified later. Summing up, we will investigate the stability property of the system

$$w_{tt}(x,t) + w_{xxxx}(x,t) = k(w_{xt}(\eta) - w_{xt}(\xi))\frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)], \quad (1)$$

$$w(0,t) = w_x(0,t) = w_{xx}(\pi,t) = w_{xxx}(\pi,t) = 0, \quad (2)$$

$$w(x,0) = w^0(x),\ w_t(x,0) = w^1(x). \quad (3)$$

2. WELL-POSEDNESS OF (1)-(3)

The well-posedness of (1)-(3) in the standard energy space is obtained in this section as a direct application of the classical semi-group theory. To specify the operator, we have to see for which function w the r.h.s. of (1) belongs to $L^2(0,\pi)$. Let us introduce some notations. If w is any function in $H^1(0,\eta) \cap H^1(\eta,\xi) \cap H^1(\xi,\pi)$, we define $\{w_x\} \in L^2(0,\pi)$ by

$$\{w_x\}(x) := \begin{cases} w_x^{\mathcal{D}'(0,\eta)}(x) & \text{if } x \in (0,\eta), \\ w_x^{\mathcal{D}'(\eta,\xi)}(x) & \text{if } x \in (\eta,\xi), \\ w_x^{\mathcal{D}'(\xi,\pi)}(x) & \text{if } x \in (\xi,\pi). \end{cases}$$

We set also $[w]_\eta := w(\eta^+) - w(\eta^-)$, and $[w]_\xi := w(\xi^+) - w(\xi^-)$. Then it follows that

$$w_x = \{w_x\} + [w]_\eta \delta_\eta + [w]_\xi \delta_\xi \quad \text{in } \mathcal{D}'(0,\pi).$$

Assume now that $w \in H^2(0,\pi)$ and that $v \in H^2(0,\pi)$, and define $u \in \mathcal{D}'(0,\pi)$ by $u := -w_{xxxx} + k(v_x(\eta) - v_x(\xi))\frac{d}{dx}(\delta_\eta - \delta_\xi)$. If $u \in L^2(0,\pi)$, then the restriction of u to each of the intervals $(0,\eta)$, (η,ξ) and (ξ,π) has also to be a square integrable function. The same conclusion holds for w_{xxxx}, hence $w \in H^4(0,\eta) \cap H^4(\eta,\xi) \cap H^4(\xi,\pi)$. We may then compute w_{xxx}, w_{xxxx} and u. We obtain

$$w_{xxx} = \{w_{xxx}\} + [w_{xx}]_\eta \delta_\eta + [w_{xx}]_\xi \delta_\xi,$$

$$w_{xxxx} = \{w_{xxxx}\} + [w_{xxx}]_\eta \delta_\eta + [w_{xxx}]_\xi \delta_\xi + [w_{xx}]_\eta \frac{d}{dx}\delta_\eta + [w_{xx}]_\xi \frac{d}{dx}\delta_\xi,$$

and

$$\begin{aligned} u = -\{w_{xxxx}\} & -[w_{xxx}]_\eta \delta_\eta - [w_{xxx}]_\xi \delta_\xi \\ & -[w_{xx}]_\eta \frac{d}{dx}\delta_\eta - [w_{xx}]_\xi \frac{d}{dx}\delta_\xi \\ & +k(v_x(\eta) - v_x(\xi))\frac{d}{dx}(\delta_\eta - \delta_\xi). \end{aligned}$$

Then $u \in L^2(0,\pi)$ provided that all the coefficients in front of the Dirac masses vanish, i.e.

$$[w_{xx}]_\eta = k(v_x(\eta) - v_x(\xi)) = -[w_{xx}]_\xi$$

and

$$[w_{xxx}]_\eta = [w_{xxx}]_\xi = 0.$$

We are now in a position to define the operator associated with (1)-(3).

Let $V = \{w \in H^2(0,\pi) \,|\, w(0) = w'(0) = 0\}$, $(w_1, w_2)_V = \int_0^\pi w_1'' \overline{w_2''} dx$, and $H = L^2(0,\pi)$, $(v_1, v_2)_H = \int_0^\pi v_1 \overline{v_2} dx$. Then $\mathcal{H} = V \times H$ endowed with the usual product norm is a (complex) Hilbert space. If we introduce $v := w_t$ and define the operator A with domain

$$\begin{aligned} \mathcal{D}(A) = \{\, & z = (w,v) \,|\, (w,v) \in H^2(0,\pi)^2, \\ & w \in H^4(0,\eta) \cap H^4(\eta,\xi) \cap H^4(\xi,\pi), \\ & w(0) = w_x(0) = w_{xx}(\pi) = w_{xxx}(\pi) = 0, \\ & v(0) = v_x(0) = 0, \\ & [w_{xx}]_\eta = k(v_x(\eta) - v_x(\xi)) = -[w_{xx}]_\xi, \\ & [w_{xxx}]_\eta = [w_{xxx}]_\xi = 0 \Big\} \end{aligned}$$

and defined by

$$\begin{aligned} Az &= \left(v, -w_{xxxx} + k(v_x(\eta) - v_x(\xi))\frac{d}{dx}(\delta_\eta - \delta_\xi)\right) \\ &= (v, -\{w_{xxxx}\}), \end{aligned}$$

then the closed-loop system (1)-(3) may be seen as the initial value problem for the abstract first-order evolution equation in $\mathcal{H}$

$$\begin{cases} \dfrac{dz}{dt} = Az,\ t > 0 \\ z(0) = (w^0, w^1). \end{cases}$$

The first main result in this paper is the following one.

Theorem 1. If $k \geq 0$, then $\mathcal{A}$ generates a C_0-semigroup $(S(t))_{t \geq 0}$ of contractions on $\mathcal{H}$.

Proof: Obviously, $\mathcal{D}(A)$ is dense in $\mathcal{H}$. According to a classical result (see e.g. (Liu and Zheng, 1999, Thm 1.2.4)), the theorem is proved if we show that A is dissipative and that $0 \in \rho(A)$, the resolvent set of A. Let us begin with the dissipativity property.

Lemma 1. For any $z = (w, v) \in \mathcal{D}(A)$ we have that

$$(Az, z)_{\mathcal{H}} = 2i\ \mathrm{Im}(\int_0^\pi v_{xx}\overline{w_{xx}}\,dx) - k|v_x(\eta) - v_x(\xi)|^2.$$

In particular, Re $(Az,z)_{\mathcal{H}} = -k|v_x(\eta) - v_x(\xi)|^2 \leq 0$, i.e. A is dissipative.

Proof of Lemma 1: Pick any pair of functions $(w, v) \in \mathcal{H}$. Then

$$(A(w,v),(w,v))_{\mathcal{H}} = \int_0^\pi v_{xx}\overline{w_{xx}}\,dx - \int_0^\pi \{w_{xxxx}\}\overline{v}\,dx.$$

After some integrations by parts on the intervals $(0,\eta)$, (η,ξ), (ξ,π), we obtain that

$$\begin{aligned}
-\int_0^\pi \{w_{xxxx}\}\overline{v} &= -\int_0^\eta \{w_{xxxx}\}\overline{v} - \int_\eta^\xi \{w_{xxxx}\}\overline{v} - \int_\xi^\pi \{w_{xxxx}\}\overline{v} \\
&= -\int_0^\pi w_{xx}\overline{v_{xx}}\,dx + [w_{xx}\overline{v_x}]_{x=0}^{\eta} + [w_{xx}\overline{v_x}]_{x=\eta}^{\xi} \\
&\quad + [w_{xx}\overline{v_x}]_{x=\xi}^{\pi} \\
&= -\int_0^\pi w_{xx}\overline{v_{xx}}\,dx - [w_{xx}]_\eta\overline{v_x(\eta)} - [w_{xx}]_\xi\overline{v_x(\xi)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(A(w,v),(w,v))_{\mathcal{H}} &= \int_0^\pi (v_{xx}\overline{w_{xx}} - w_{xx}\overline{v_{xx}})\,dx \\
&\quad -[w_{xx}]_\eta\overline{v_x(\eta)} - [w_{xx}]_\xi\overline{v_x(\xi)} \\
&= 2i\ \mathrm{Im}(\int_0^\pi v_{xx}\overline{w_{xx}}\,dx) - k|v_x(\eta) - v_x(\xi)|^2.
\end{aligned}$$

This completes the proof of Lemma 1. ■

We now proceed to the study of A^{-1}. The following result holds true.

Proposition 1. $0 \in \rho(A)$.

Proof: We have to prove that the operator $A : \mathcal{D}(A) \to \mathcal{H}$ is one-to-one and onto, and that its inverse $A^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous. Let a pair $(f, g) \in \mathcal{H}$ be given, and let us investigate the equation $A(w, v) = (f, g)$, where (w, v) has to be found in $\mathcal{D}(A)$. We have to solve the system

$$\begin{cases} v = f \\ -\{w_{xxxx}\} = g \end{cases}$$

supplemented by adequate boundary conditions. To eliminate g, we introduce the (unique) solution $\tilde{w} \in H^4(0,\pi)$ of the following elliptic problem

$$\begin{cases} -\tilde{w}_{xxxx} = g \\ \tilde{w}(0) = \tilde{w}_x(0) = \tilde{w}_{xx}(\pi) = \tilde{w}_{xxx}(\pi) = 0. \end{cases}$$

Setting $w = \tilde{w} + \hat{w}$ and $K := k(v_x(\eta) - v_x(\xi)) = k(f_x(\eta) - f_x(\xi))$, we have to solve

$$\{\hat{w}_{xxxx}\} = 0 \qquad (4)$$

$$\hat{w}(0) = \hat{w}_x(0) = \hat{w}_{xx}(\pi) = \hat{w}_{xxx}(\pi) = 0 \qquad (5)$$

$$[\hat{w}_{xx}]_\eta = K = -[\hat{w}_{xx}]_\xi \qquad (6)$$

$$[\hat{w}_{xxx}]_\eta = [\hat{w}_{xxx}]_\xi = 0 \qquad (7)$$

where $\hat{w}$ is to be found in $H^2(0,\pi) \cap H^4(0,\xi) \cap H^4(\xi,\eta) \cap H^4(\eta,\pi)$. In particular,

$$[\hat{w}]_\eta = [\hat{w}]_\xi = [\hat{w}_x]_\eta = [\hat{w}_x]_\xi = 0. \qquad (8)$$

We infer from (4) and (5) that there exist some constants $a_2, a_3, b_0, b_1, b_2, b_3, c_0$ and c_1 such that

$$\hat{w}(x) = \begin{cases} a_2x^2 + a_3x^3 & \text{if } 0 < x < \eta, \\ b_0 + b_1x + b_2x^2 + b_3x^3 & \text{if } \eta < x < \xi, \\ c_0 + c_1(x - \pi) & \text{if } \xi < x < \pi. \end{cases}$$

It follows from (7) that $a_3 = b_3 = 0$. (6) gives $2b_2 - 2a_2 = K = 2b_2$, hence $a_2 = 0$ and $b_2 = K/2$. Finally, it is easily seen that b_0, b_1, c_0 and c_1 are uniquely determined by the following system of linear equations (coming from (8))

$$\begin{cases} b_0 + b_1\eta + \dfrac{K}{2}\eta^2 = 0 \\ b_1 + K\eta = 0 \\ b_0 + b_1\xi + \dfrac{K}{2}\xi^2 = c_0 + c_1(\xi - \pi) \\ b_1 + K\xi = c_1 \end{cases}$$

This proves the existence and uniqueness of $\hat{w}$, and the existence and uniqueness of a pair $(w, v) \in \mathcal{D}(A)$ such that $A(w, v) = (f, g)$. To see that $0 \in \rho(A)$, it remains to prove that the map $A^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous. Let $(w, v) = A^{-1}(f, g)$. Then $||v||_H = ||f||_H$, $||\tilde{w}||_{H^4(0,\pi)} \leq Const||g||_H$, $||\hat{w}||_V \leq Const|K| \leq Const||f||_V$. Therefore

$$||(w,v)||_{\mathcal{H}} \leq Const||(f,g)||_{\mathcal{H}}.$$

This completes the proof of Proposition 1 and of Theorem 1. ■

Proposition 2. The operator $A^{-1} : \mathcal{H} \to \mathcal{H}$ is compact.

Proof: Since

$$||A^{-1}(f,g)||_{\mathcal{D}(A)} = ||A^{-1}(f,g)||_{\mathcal{H}} + ||(f,g)||_{\mathcal{H}},$$

we see that A^{-1} is continuous from $\mathcal{H}$ into $\mathcal{D}(A)$, hence it is sufficient to prove that the embedding $\mathcal{D}(A) \to \mathcal{H}$ is compact. Let $\left((w^n, v^n)\right)_{n \geq 0}$ be any bounded sequence in $\mathcal{D}(A)$. We have to prove that a subsequence $\left((w^{n'}, v^{n'})\right)$ converges (strongly) in $\mathcal{H}$. As $||v^n||_{H^2(0,\pi)} \leq Const$, there exists a subsequence $(v^{n'})$ and some function $v \in H = L^2(0,\pi)$ such that

$v^{n'} \rightarrow v$ in H. On the other hand, $||w^n||_{H^2(0,\pi)} + ||\{w^n_{xxxx}\}||_{L^2(0,\pi)} \leq Const$, so

$$||w^n||_{H^4(0,\eta)} + ||w^n||_{H^4(\eta,\xi)} + ||w^n||_{H^4(\xi,\pi)} \leq Const.$$

Extracting a subsequence once again, we infer that there exists a function $w \in V$ such that $w^{n'} \rightharpoonup w$ in V, and $w^{n'} \rightharpoonup w$ in $H^4(0,\eta)$, in $H^4(\eta,\xi)$ and in $H^4(\xi,\pi)$. It follows that $w^{n'} \rightarrow w$ in $H^2(0,\eta)$, in $H^2(\eta,\xi)$ and in $H^2(\xi,\pi)$. We conclude that $w^{n'} \rightarrow w$ in V. ■

3. STRONG STABILITY

3.1 Free evolution

In this section we recall some useful facts about the *free* evolution of (1)-(3) (i.e., when $k = 0$). Thus we consider the homogeneous Cauchy problem

$$\phi_{tt} + \phi_{xxxx} = 0, \quad (9)$$

$$\phi(0,t) = \phi_x(0,t) = \phi_{xx}(\pi,t) = \phi_{xxx}(\pi,t) = 0, \quad (10)$$

$$\phi(.,0) = \phi^0,\ \phi_t(.,0) = \phi^1. \quad (11)$$

Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow L^2(0,\pi)$ be the operator with domain

$$\mathcal{D}(\mathcal{A}) := \{\phi \in H^4(0,\pi);\ \phi(0) = \phi_x(0) = = \phi_{xx}(\pi) = \phi_{xxx}(\pi) = 0\}$$

and defined by $\mathcal{A}\phi = \phi_{xxxx}$. Obviously, $\mathcal{A}^{-1}$ is a compact symmetric operator on $\mathcal{H}$, hence there exists a countable orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $\mathcal{A}^{-1}$. The following result (Crépeau and Prieur, to appear, Lemma 2.1) provides useful results about the eigenvectors of $\mathcal{A}$.

Proposition 3. The $L^2(0,\pi)$-normalized eigenfunctions of $\mathcal{A}$ are the functions $(\psi_k)_{k\geq 1}$, defined by

$$\psi_k(x) = \gamma_k\big(\cos(\alpha_k x) - \cosh(\alpha_k x) + \mu_k(\sinh(\alpha_k x) - \sin(\alpha_k x))\big), \quad (12)$$

where α_k is the k-th positive root of

$$1 + \cos(\alpha_k\pi)\cosh(\alpha_k\pi) = 0, \quad (13)$$

and

$$\mu_k = \frac{\cos(\alpha_k\pi) + \cosh(\alpha_k\pi)}{\sin(\alpha_k\pi) + \sinh(\alpha_k\pi)},$$

$$\gamma_k = \frac{1}{\sqrt{\pi}},$$

and the eigenvalue associated with ψ_k is $\lambda_k = \alpha_k^4$. Moreover, we have as $k \rightarrow +\infty$

$$\alpha_k = k - \frac{1}{2} + (-1)^{k+1}\frac{2e^{\frac{\pi}{2}}}{\pi}e^{-\pi k} + o(e^{-\pi k}), \quad (14)$$

$$\mu_k = 1 + 2(-1)^k e^{-\alpha_k\pi} + o(e^{-\alpha_k\pi}), \quad (15)$$

and

$$-\sinh(\alpha_k\rho) + \mu_k\cosh(\alpha_k\rho) = \begin{cases} e^{-\alpha_k\rho} + o(e^{-\alpha_k\rho}) & \text{if } 0 < \rho < \frac{\pi}{2}, \\ (-1)^k e^{-\alpha_k(\pi-\rho)} + o(e^{-\alpha_k(\pi-\rho)}) & \text{if } \frac{\pi}{2} < \rho < \pi. \end{cases} \quad (16)$$

It is easily seen that if $\phi^0 = \sum_{k\geq 1}\phi_k^0\psi_k$, and $\phi^1 = \sum_{k\geq 1}\phi_k^1\psi_k$, then the solution $\phi = \phi(x,t)$ of (9)-(11) reads

$$\phi(x,t) = \sum_{k=1}^{+\infty}\left(\phi_k^0\cos(\alpha_k^2 t) + \frac{\phi_k^1}{\alpha_k^2}\sin(\alpha_k^2 t)\right)\psi_k(x). \quad (17)$$

Let $L := \xi - \eta$ denote the length of the actuator/sensor. For any $k \geq 1$ and any $L \in (0,\pi]$, let

$$\mathcal{S}_k(L) := \{\eta \in [0,\pi - L] \mid \psi_k'(\eta) - \psi_k'(\eta + L) = 0\}, \quad (18)$$

and set

$$\mathcal{S}(L) := \cup_{k\geq 1}\mathcal{S}_k(L).$$

Definition 1. We say that (1)-(2) is *strongly stable* in $\mathcal{H}$, if for any $(w^0,w^1) \in \mathcal{H}$ we have that

$$E(w(t),w_t(t)) = ||(w(t),w_t(t))||^2_{\mathcal{H}} \rightarrow 0$$

as $t \rightarrow +\infty$.

The following theorem provides a characterization of the location of the actuator/sensor for which the strong stability holds. Its proof rests on LaSalle principle (see also (Ammari and Tucsnak, 2000) for strong stability results obtained this way).

Theorem 2. The system (1)-(2) is strongly stable in $\mathcal{H}$ if and only if $k > 0$ and $\eta \notin \mathcal{S}(L)$.

Proof: First of all, it follows from Lemma 1 that for any $(w^0,w^1) \in \mathcal{D}(A)$ and for any $t \geq 0$,

$$E(w(T),v(T)) - E(w^0,w^1) = -k\int_0^T |v_x(\eta,t) - v_x(\xi,t)|^2\,dt. \quad (19)$$

Thus, the condition $k > 0$ is needed for the energy to decrease. Moreover, the relation (19) shows that the trace $v_x(\eta,.) - v_x(\xi,.)$ exists in $L^2(0,T)$ for any $T > 0$ and any $(w^0,w^1) \in \mathcal{H}$. Furthermore, a density argument shows that (19) still holds true for $(w^0,w^1) \in \mathcal{H}$. Since the embedding $\mathcal{D}(A) \subset \mathcal{H}$ is compact, we obtain that the set $\text{orb}(w^0,w^1) := \{(w(t),v(t))|\ t \geq 0\}\}$ is precompact in $\mathcal{H}$ for any $(w^0,w^1) \in \mathcal{D}(A)$. Therefore, the $\omega-$limit set of (w^0,w^1), defined as

$$\omega(w^0,w^1) = \{z \in \mathcal{H}| \ \exists(t_n) \rightarrow \infty,\ \lim_{n\rightarrow\infty} S(t_n)(w^0,w^1) = z\},$$

is nonempty. On the other hand, according to LaSalle's invariance principle (see (Cazenave and Haraux, 1998)), for any $(\phi^0,\phi^1) \in \omega(w^0,w^1)$, we have that $S(t)(\phi^0,\phi^1) = (\phi(t),\phi_t(t)) \in \omega(w^0,w^1)$ and $E(\phi(t),\phi_t(t)) = E(\phi^0,\phi^1)$. The above relation and (19) imply that ϕ is a solution of (9)-(11) and fulfills

$$\phi_{xt}(\eta,t) - \phi_{xt}(\xi,t) = 0 \qquad \forall t \geq 0.$$

Derivating w.r.t. x and t in (17), we obtain

$$0 \equiv \phi_{xt}(\eta,t) - \phi_{xt}(\xi,t) = -\sum_{k=1}^{+\infty} \big(\alpha_k^2\phi_k^0 \sin(\alpha_k^2 t) - \phi_k^1 \cos(\alpha_k^2 t)\big)\left(\psi_k'(\eta) - \psi_k'(\xi)\right).$$

Since $\alpha_{k+1}^2 - \alpha_k^2 \to \infty$, we infer from a generalization of Ingham's inequality that for any $T > 0$

$$0 = \int_0^T |\phi_{xt}(\eta,t) - \phi_{xt}(\xi,t)|^2 dt \geq C_T \sum_{k=1}^{+\infty} \left(|\alpha_k^2\phi_k^0|^2 + |\phi_k^1|^2\right)|\psi_k'(\eta) - \psi_k'(\xi)|^2.$$

Therefore, if $\eta \notin \mathcal{S}(L)$, then $\phi_k^0 = \phi_k^1 = 0$ for all $k \geq 1$ and $S(t)(w^0,w^1) \to (\phi^0,\phi^1) = (0,0)$ in $\mathcal{H}$. Conversely, if $\eta \in \mathcal{S}_k(L)$ for some $k \geq 1$, then any state of the form $(\phi^0,\phi^1) = (\phi_k^0\psi_k, \phi_k^1\psi_k)$ gives rise to a solution of (1)-(3) (or (9)-(11)) whose energy does not tend to 0. The proof of Theorem 2 is achieved. ■

Remark 1. (1) In sharp contrast to (Tucsnak, 1996) and (Crépeau and Prieur, to appear), the length of the actuator/sensor may take here *any* real value. Furthermore, the position of the left endpoint for which the strong stability holds is given explicitly, and the theory of Diophantine approximation is not involved here.

(2) When $\eta \in \mathcal{S}(L)$, then there is also a strong stability in the quotient space $\mathcal{H}/\text{Span}\{(\psi_k,0), (0,\psi_k)\}$ if $\mathcal{S}_{k'}(L) \cap \mathcal{S}_k(L) = \emptyset$ for any $k' \neq k$.

As the set $\mathcal{S}(L)$ plays a crucial role in the stability results, we collect some of its properties in the following proposition.

Proposition 4. For any $L \in (0,\pi)$ the set $\mathcal{S}(L)$ of critical values of η is countable and dense in $[0,\pi - L]$.

To prove that $\mathcal{S}(L)$ is countable, it is sufficient to prove that each set $\mathcal{S}_k(L)$ is finite. But $\mathcal{S}_k(L) = f_k^{-1}(0) \cap [0,\pi - L]$, where $f_k(\eta) := \psi_k'(\eta) - \psi_k'(\eta + L)$. As the function f_k is analytic, we infer that it has only a finite number of zeros in the interval $[0,\pi - L]$ (i.e., $\mathcal{S}_k(L)$ is finite) if it is not identically null. To check the last property, we need first to establish the following

CLAIM 1. $\quad \mu_k \neq 1 \quad \forall k \geq 1.$

Argue by contradiction. If the claim is false, then there exists some $k \geq 1$ with

$$\mu_k = \frac{\cos(\alpha_k\pi) + \cosh(\alpha_k\pi)}{\sin(\alpha_k\pi) + \sinh(\alpha_k\pi)} = 1. \tag{20}$$

Let $x = \cos(\alpha_k\pi)$. Using (13) and (20), we arrive to the equation

$$x - x^{-1} = \pm\sqrt{1 - x^2} + \sqrt{x^{-2} - 1}$$

whose solutions are easily found to be ± 1. Now, the equation (13) has no solution if $\cos(\alpha_k\pi) = \pm 1$. The claim is proved. □

Derivating in (12) we obtain

$$\begin{aligned}\psi_k'(x) &= \gamma_k\big(-\alpha_k\sin(\alpha_k x) - \alpha_k\sinh(\alpha_k x) + \mu_k(\alpha_k\cosh(\alpha_k x) - \alpha_k\cos(\alpha_k x))\big)\\ &\sim \gamma_k\alpha_k(\mu_k\cosh(\alpha_k x) - \sinh(\alpha_k x))\\ &\sim \gamma_k\alpha_k(\mu_k - 1)e^{\alpha_k x}/2\end{aligned}$$

as $x \to +\infty$. Therefore, $f_k(\eta) \sim -\gamma_k\alpha_k(\mu_k - 1)e^{\alpha_k(\eta+L)}/2$ as $\eta \to +\infty$, which shows that $f_k \not\equiv 0$. The proof that $\mathcal{S}(L)$ is countable is achieved.

Let us now check that $\mathcal{S}(L)$ is dense in $[0,\pi - L]$. The idea is that $f_k(\eta)$ oscillates like the function $\sin(\alpha_k(\eta + \frac{L}{2}) - \frac{\pi}{4})$, so that the set $\mathcal{S}_k(L)$ is constituted of $O(k)$ points "almost equidistributed" in $[0,\pi - L]$. In a less formal way, using (20) and (16), we have that

$$\begin{aligned}f_k(\eta) &= \psi_k'(\eta) - \psi_k'(\eta + L)\\ &= \gamma_k\alpha_k\big(\sin(\alpha_k(\eta + L)) - \sin(\alpha_k\eta) + \mu_k(\cos(\alpha_k(\eta + L)) - \cos(\alpha_k\eta))\\ &\quad - \sinh(\alpha_k\eta) + \mu_k\cosh(\alpha_k\eta) + \sinh(\alpha_k(\eta + L)) - \mu_k\cosh(\alpha_k(\eta + L))\big)\\ &= \gamma_k\alpha_k\big(2\cos(\alpha_k(\eta + \frac{L}{2}))\sin(\alpha_k\frac{L}{2}) - 2\mu_k\sin(\alpha_k(\eta + \frac{L}{2}))\sin(\alpha_k\frac{L}{2}) + o(e^{-\delta\alpha_k})\big)\\ &= 2\sqrt{2}\gamma_k\alpha_k\big(\sin(\frac{\pi}{4} - \alpha_k(\eta + \frac{L}{2}))\sin(\alpha_k\frac{L}{2}) + o(e^{-\delta\alpha_k})\big)\end{aligned}$$

for some positive constant $\delta \in (0,\pi/2)$, provided that η and $\eta + L$ are both different from $\frac{\pi}{2}$. The following claim is needed.

CLAIM 2. $\forall L \in (0,\pi]$, $\forall k_0 \geq 1$, $\exists k \geq k_0$ such that $|\sin(\alpha_k\frac{L}{2})| \geq e^{-\delta\alpha_k}$.

If the claim is false, then there exists an integer $k_0 \geq 1$ such that

$$|\sin(\alpha_k\frac{L}{2})e^{\delta\alpha_k}| < 1 \quad \forall k \geq k_0.$$

As $|\sin(\alpha_k\frac{L}{2}) - \sin((k - \frac{1}{2})\frac{L}{2})| \leq Ce^{-\pi k}$ by (14), we infer that

$$|\sin((k - \frac{1}{2})\frac{L}{2})e^{\delta\alpha_k}| \leq C' \quad \forall k \geq k_0, \tag{21}$$

where C and C' denote some positive constants. If $L \in \pi\mathbb{Q}$, then the sequence $\left(\sin((k - \frac{1}{2})\frac{L}{2})\right)_{k\geq 0}$ is periodic (not constant). If $L \notin \pi\mathbb{Q}$, then the same sequence is dense in $[-1,1]$. In both cases, (21) cannot hold. Claim 2 is proved. □

With Claim 2 we conclude that for a sequence $k' \to \infty$, $f_{k'}$ vanishes between any pair of successive extrema of $\sin(\frac{\pi}{4} - \alpha_{k'}(\eta + \frac{L}{2}))$. The density of $\mathcal{S}(L)$ follows at once. ■

The sets $\mathcal{S}_k(L)$ have been numerically computed for $L = \frac{\pi}{2}$ and $k \leq 6$. This choice corresponds to the case of an actuator/sensor which covers half of the length of the beam. The repartition is sketched out in Table 1.

Table 1. $S_k(L)$ for small values of k and for $L = \frac{\pi}{2}$

S_1	0.15				
S_2	0.48				
S_3	0.07	0.62	1.28		
S_4	0.21	0.67	1.1		
S_5	0.04	0.35	0.7	1.05	1.41
S_6	0.13	0.43	0.71	1	1.28

4. CONCLUSION

The paper was devoted to the output stabilization of a clamped-free beam with collocated piezoelectric sensor/actuator. It was proved that for any length of the actuator, the strong stability holds provided that the left endpoint η of the actuator does not belong to a dense countable set $\mathcal{S}(L)$. The determination of the decay rate will be the purpose of further research in a near future.

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