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GLOBAL WELL-POSEDNESS OF THE KDV EQUATION ON A STAR-SHAPED NETWORK AND STABILIZATION BY SATURATED CONTROLLERS*

HUGO PARADA†, EMMANUELLE CRÉPEAU†, AND CHRISTOPHE PRIEUR‡

Abstract. In this work, we deal with the global well-posedness and stability of the linear and nonlinear Korteweg-de Vries equations on a finite star-shaped network by acting with saturated controls. We obtain the global well-posedness by using the Kato smoothing property for the linear case and then using some estimates and a fixed point argument we deal with the nonlinear system. Finally, we obtain the exponential stability using two different kinds of saturation by proving an observability inequality via a contradiction argument.

Key words. Korteweg-de Vries equation, star-shaped network, stabilization, saturating control

MSC codes. 93C20, 93D15, 35R02, 35A01, 35Q53

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1. Introduction and presentation of our results.

The Korteweg-de Vries (KdV) equation $u_t + u_x + u_{xxx} + uu_x = 0$ was introduced in [13] to model the propagation of long water waves in a channel. The KdV equation has been very well studied in recent years, in particular, the controllability and stabilization properties; see [9, 22] for a complete introduction to these problems. With respect to the KdV equation on networks, we can mention the work [8] where well-posedness of the KdV equation on a star metric graph was studied. In the works [1, 10], stabilization and controllability problems were studied, for the KdV equation on a star-shaped network, and recently the problem of stabilization using internal delay was addressed in [16]. In this work, we are interested in the global well-posedness and stability properties of a KdV equation posed on a star-shaped network using internal saturated feedback terms. Let $K = \{k_n : 1 \leq n \leq N\}$ be the set of the N edges of a network $\mathcal{T}$ described as the intervals $[0, \ell_n]$ with $\ell_n > 0$ for $n = 1, \ldots, N$, the network $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{n=1}^{N} k_n$. Specifically, we are going to consider the next evolution problem for the KdV equation,

(KdV-N)

$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t,x) = 0 & \forall x \in (0,\ell_n),\ t > 0,\ n = 1,\ldots,N, \\ u_n(t,0) = u_{n'}(t,0) & \forall n, n' = 1,\ldots,N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 u_n(t,0) = -\alpha u_1(t,0) - \frac{N}{3} u_1^2(t,0), & t > 0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t > 0,\ n = 1,\ldots,N, \\ u_n(0,x) = u_n^0(x), & x \in (0,\ell_n), \end{cases}$$

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†Université Grenoble Alpes, CNRS, Laboratoire Jean Kuntzmann, F-38000, Grenoble, France (hugo.parada@univ-grenoble-alpes.fr, emmanuelle.crepeau@univ-grenoble-alpes.fr).
‡Université Grenoble Alpes, CNRS, Grenoble-INP, GIPSA- lab, F-38000, Grenoble, France (christophe.prieur@gipsa-lab.fr).

where $\alpha \geq \frac{N}{2}$. The central node conditions are obtained taking account of the following: If we denote by u_n and v_n the dimensionless and scaled variables standing, respectively, for the deflection from rest position and the velocity on the branch n of long water waves, then we get from [25, eq. (13.102)]

$$\begin{cases} \partial_t u_n + \partial_x u_n + \partial_x^3 u_n + u_n \partial_x u_n = 0 & \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N, \\ v_n = u_n - \dfrac{1}{6}u_n^2 + 2\partial_x^2 u_n & \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N. \end{cases}$$

Moreover, at the central node, we can suppose that the elevation of water is the same in all branches and that the sum of the flux is null, which implies

$$\begin{cases} u_n(t,0) = u_{n'}(t,0) & \forall n,n'=1,\dots,N, \\ \displaystyle\sum_{n=1}^N u_n(t,0)v_n(t,0) = 0, & t>0. \end{cases}$$

Then we obtain the following problem:

$$\begin{cases} u_n(t,0) = u_{n'}(t,0) & \forall n,n'=1,\dots,N, \\ \displaystyle\sum_{n=1}^N \partial_x^2 u_n(t,0) = -\frac{N}{2}u_1(t,0) + \frac{N}{6}u_1^2(t,0), & t>0. \end{cases}$$

We adapt the boundary condition at the central node to have a decreasing energy. The hypothesis $\alpha > \frac{N}{2}$ was introduced in [1] and then in [10] the case $\alpha = \frac{N}{2}$ was included. (KdV-N) was studied in [1] by using the following functional setting: Let $H_r^1(0,\ell_n) = \{v \in H^1(0,\ell_n),\ v(\ell_n)=0\}$, where the index r is related to the null right boundary conditions, the space $\mathbb{H}_e^1(\mathcal{T})$ be the Cartesian product of $H_r^1(0,\ell_n)$ including the continuity condition on the central node ($u_n(0) = u_{n'}(0)\ \forall n,n'=1,\dots,N$)

$$\mathbb{H}_e^1(\mathcal{T}) = \left\{ \underline{u} = (u_1,\cdots,u_N)^T \in \prod_{n=1}^N H_r^1(0,\ell_n), u_n(0) = u_{n'}(0)\ \forall n,n'=1,\dots,N \right\},$$

and

$$\|\underline{u}\|^2_{\mathbb{H}_e^1(\mathcal{T})} = \sum_{n=1}^N \|u_n\|^2_{H^1(0,\ell_n)},$$

where the index e is related so that each edge belongs to $H_r^1(0,\ell_n)$. Introduce also the state space

$$\mathbb{L}^2(\mathcal{T}) = \prod_{n=1}^N L^2(0,\ell_n) \quad \text{with} \quad (\underline{u},\underline{v})_{\mathbb{L}^2(\mathcal{T})} = \sum_{n=1}^N \int_0^{\ell_n} u_n v_n dx \quad \forall \underline{u},\underline{v} \in \mathbb{L}^2(\mathcal{T}).$$

We also define the space $\mathbb{B}_T = C([0,T],\mathbb{L}^2(\mathcal{T})) \cap L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))$ with $\|u\|_{\mathbb{B}_T} = \|\underline{u}\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))} + \|\underline{u}\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))}$, and $\mathbb{Y}_T$ be the space of all functions $\underline{v} \in \mathbb{B}_T$ such that $\partial_x^\kappa v_n \in L_x^\infty(0,\ell_n;H^{\frac{1-\kappa}{3}}(0,T))$ for $\kappa = 0,1,2$, with the induced norm

$$\|\underline{v}\|_{\mathbb{Y}_T} = \|\underline{v}\|_{\mathbb{B}_T} + \sum_{\kappa=0}^2 \|\partial_x^\kappa \underline{v}\|_{\prod_{n=1}^N L_x^\infty(0,\ell_n;H^{\frac{1-\kappa}{3}}(0,T))}.$$

In [1, 10] the next well-posedness result was proved for small initial condition and for any time horizon.

THEOREM 1.1 (Theorem 2.7 of [1]). *Let* $(\ell_n)_{n=1,\dots,N} \in (0,\infty)^N$, $\alpha \geq \frac{N}{2}$ *and* $T > 0$. *Then there exist* $\epsilon > 0$ *and* $C > 0$ *such that for all* $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ *with* $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq \epsilon$, *there exists a unique solution of* (KdV-N). *Moreover, it satisfies* $\|\underline{u}\|_{\mathbb{B}_T} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.

The main problem to get a global well-posedness result is the action of the nonlinear boundary condition on the central node. Similar boundary conditions appear for the first time to our knowledge in the work [21] where a wave maker control for a single KdV equation was studied and then in the work [5] where a well-posedness result was given. The system studied in these papers was the next one

$$\begin{cases} \partial_t u(t,x) + \partial_x u(t,x) + u(t,x)\partial_x u(t,x) + \partial_x^3 u(t,x) = 0 & \forall x \in (0,L),\ t > 0, \\ \partial_x^2 u(t,0) = -u(t,0) + \dfrac{1}{6}u^2(t,0) + h(t), & t > 0, \\ u(t,L) = \partial_x u(t,L) = 0, & t > 0, \\ u(0,x) = \phi(x), & x \in (0,L), \end{cases} \tag{1.1}$$

and the following well-posedness result local-in-time for bounded initial data was proven in [5].

THEOREM 1.2 (Theorem 1.1 of [5]). *Let* $T > 0$ *and* $\gamma > 0$ *be given. There exists* $T^* \in (0,T]$ *such that for any* $\phi \in L^2(0,L)$ *and* $h \in H^{-\frac{1}{3}}(0,T)$ *satisfying,* $\|\phi\|_{L^2(0,L)} + \|h\|_{H^{-\frac{1}{3}}(0,T)} \leq \gamma$. *Then the problem* (1.1) *admits a unique solution* $u \in C([0,T^*];L^2(0,L)) \cap L^2(0,T^*;H^1(0,L))$. *Moreover, the corresponding solution map is Lipschitz continuous and the solution possesses the hidden regularities (the sharp Kato smoothing properties)* $\partial_x^\kappa u \in L_x^\infty(0,L;H^{\frac{1-\kappa}{3}}(0,T^*))$, $\kappa = 0,1,2$.

The first main result of our work is the following global-in-time well-posedness theorem.

THEOREM 1.3. *Let* $(\ell_n)_{n=1,\dots,N} \in (0,\infty)^N$, $\alpha \geq \frac{N}{2}$, *and* $T > 0$. *Then, for all* $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, *there exists a unique solution* $\underline{u} \in \mathbb{B}_T$ *of* (KdV-N). *Moreover, there exist* $0 < T^* \leq T$, $C > 0$ *such that* $\underline{u} \in \mathbb{Y}_{T^*}$ *and* $\|\underline{u}\|_{\mathbb{Y}_{T^*}} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.

Note that our result generalized Theorem 1.1 in the sense that the smallness assumption on the initial data is not needed. Our idea is to follow [5] to obtain a similar sharp Kato smoothing regularity presented in Theorem 1.2 for a linear problem of the KdV equation on a star-shaped network. In order to deal with the nonlinear part, we use a fixed point argument to obtain global well-posedness for small time. Finally, we use an energy estimation to obtain a global well-posedness in time. Similar ideas were applied in the case of a single KdV equation in [18]. From the point of view of stabilization, we can refer to the work [26] in which the boundary exponential stabilization problem in the bounded spatial domain $x \in (0,1)$ was studied. It is well known that the length L of the spatial domain plays an important role in the stabilization and controllability properties of the KdV equation. For example, when $L = 2\pi$ it is possible to find a solution of the linearization around 0 of KdV ($u(t,x) = 1 - \cos(x)$) that has constant energy. More generally, if $L \in \mathcal{N}$, where $\mathcal{N}$ is the set of critical lengths defined by

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{N}^* \right\},$$

we can find suitable initial data such that the solution of the linear KdV equation has constant energy. For the case of internal stabilization, it is proved in [18, 17] that for any critical length we achieve local exponential stability for the nonlinear KdV equation by adding a localized damping. In most real-life settings we have to take into account the saturation in the input control due to some (physical, economical, etc.) constraints. With respect to saturated control in infinite-dimensional systems, we can refer to [19] where a wave equation with distributed and boundary saturated feedback law was studied, [14] where the saturated internal stabilization of a single KdV equation was studied and recently [15] where a saturated feedback control law was derived for a linear reaction-diffusion equation. Our idea closely follows works [14] and [16] to prove the stability of the KdV equation in a star-shaped network with saturated internal control. In this work, we consider a saturation map $\mathfrak{sat}$ that could be any of the following cases:

- $\mathfrak{sat} = \mathfrak{sat}_{\mathrm{loc}}$: First consider the following scalar saturation,

$$\mathrm{sat}(f) = \begin{cases} -M & \text{if } f \leq -M, \\ f & \text{if } -M \leq f \leq M, \\ M & \text{if } f \geq M, \end{cases}$$

  where $M > 0$ is given and denotes the saturation level. Then we take the next extension to an infinite-dimensional setting

$$\mathfrak{sat}_{\mathrm{loc}}(f)(x) = \mathrm{sat}(f(x)). \tag{1.2}$$

- $\mathfrak{sat} = \mathfrak{sat}_2$: For $f \in L^2(0,L)$ we define

$$\mathfrak{sat}_2(f)(x) = \begin{cases} f(x) & \text{if } \|f\|_{L^2(0,L)} \leq M, \\ \dfrac{f(x)M}{\|f\|_{L^2(0,L)}} & \text{if } \|f\|_{L^2(0,L)} \geq M. \end{cases} \tag{1.3}$$

In what follows, $\mathfrak{sat}$ corresponds to either $\mathfrak{sat}_{\mathrm{loc}}$ or $\mathfrak{sat}_2$. In order to consider the saturated stabilization problem, we study the next system

(KdV-S)
$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n \partial_x u_n + \partial_x^3 u_n)(t,x) & \\ \qquad + \mathfrak{sat}(a_n(x) u_n(t,x)) = 0, & x \in (0,\ell_n),\ t > 0,\ n = 1,\dots,N, \\ u_n(t,0) = u_{n'}(t,0) & \forall n, n' = 1,\dots,N, \\ \sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha u_1(t,0) - \frac{N}{3} u_1^2(t,0), & t > 0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t > 0,\ n = 1,\dots,N, \\ u_n(0,x) = u_n^0(x), & x \in (0,\ell_n), \end{cases}$$

where the damping terms $(a_n)_{n=1,\dots,N} \in \prod_{n=1}^N L^\infty(0,\ell_n)$ act locally on all branches, formally written as

$$a_n \geq c_n > 0 \ \text{ in an open nonempty set } \omega_n \text{ of } (0,\ell_n), \text{ for all } n = 1,\dots,N. \tag{1.4}$$

In this work, we are going to consider the following energy $E(t)$ of $\underline{u} = (u_1,\dots,u_N)^T \in \mathbb{L}^2(\mathcal{T})$ by

$$E(t) = \frac{1}{2} \|\underline{u}\|^2_{\mathbb{L}^2(\mathcal{T})}. \tag{1.5}$$

The second main result of this paper states the semiglobal exponential stability of (KdV-S).

THEOREM 1.4. *Assume that the damping terms* $(a_n)_{n=1,\ldots,N}$ *satisfy* (1.4). *Let* $(\ell_n)_{n=1}^N \subset (0,\infty)$ *and* $R>0$*, then there exist* $C(R)>0$ *and* $\mu(R)>0$ *such that for all* $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ *with* $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$*, the energy of any solution of* (KdV-S) *defined by* (1.5) *satisfies* $E(t) \leq C(R)E(0)e^{-\mu(R)t}$ *for all* $t>0$.

Then, in order to add damped terms only on the critical lengths as in [1], we neglect the term $u_n\partial_x u_n$ in the KdV equation (KdV-N). Let $I_c = \{n \in \{1,\cdots,N\};\ \ell_n \in \mathcal{N}\}$ be the set of critical lengths and I_c^* be the subset of I_c where we remove one index. We consider now the following problem,

$$\text{(LKdV-S)} \quad \begin{cases} (\partial_t u_n + \partial_x u_n + \partial_x^3 u_n)(t,x) & \\ \quad + \mathfrak{sat}(a_n(x)u_n(t,x)) = 0, & x \in (0,\ell_n),\ t>0,\ n=1,\ldots,N, \\ u_n(t,0) = u_{n'}(t,0) & \forall n,n'=1,\ldots,N, \\ \sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha u_1(t,0), & t>0, \\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t>0,\ n=1,\ldots,N, \\ u_n(0,x) = u_n^0(x), & x\in(0,\ell_n), \end{cases}$$

where the damping $(a_n)_{n=1,\ldots,N} \in \prod_{n=1}^N L^\infty(0,\ell_n)$ satisfy

$$(1.6) \quad \begin{cases} a_n = 0 \quad \text{for} \quad n \in \{1,\ldots,N\}\backslash I_c^*, \\ a_n \geq c_n \quad \text{in an open nonempty set } \omega_n \text{ of } (0,\ell_n), \text{ for all } n\in I_c^*, \\ \text{and } c_n>0 \text{ is a constant.} \end{cases}$$

Then we are able to prove the following global stabilization result, which is the last main result.

THEOREM 1.5. *Assume that the damping terms* $(a_n)_{n=1,\ldots,N}$ *satisfy* (1.6) *and let* $(\ell_n)_{n=1}^N \subset (0,\infty)$*. Then, there exist* $C>0$ *and* $\mu>0$ *such that for all* $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$*, the energy of any solution of* (LKdV-S) *defined by* (1.5) *satisfies* $E(t) \leq CE(0)e^{-\mu t}$ *for all* $t>0$.

Remark 1. Note that for the system (LKdV-S) the stabilization result is global, instead of the one for (KdV-S) which is semiglobal. This difference comes from the action of the term $u_n\partial_x u_n$: The condition $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ is necessary to handle this term. ∘

Remark 2. A global stabilization result for (KdV-S) is, to our knowledge, an open problem. ∘

2. Well-posedness.

This section is devoted to prove well-posedness results for (KdV-N)-(KdV-S) and (LKdV-S); in particular, we focus on Theorem 1.3. Our scheme will be to consider appropriate linear systems to derive regularity properties. Then, using a fixed point result, we obtain the well-posedness for the nonlinear systems.

2.1. Linear problems.

We start by considering the following linear system for the KdV equation on a star-shaped network $\mathcal{T}$:

(LKdV-N)

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n = f_n & \forall x\in(0,\ell_n),\ t>0,\ n=1,\ldots,N, \\ u_n(t,0) = u_{n'}(t,0) & \forall n,n'=1,\ldots,N, \\ \displaystyle\sum_{n=1}^N \partial_x^2 u_n(t,0) = h(t), & t>0, \\ u_n(t,\ell_n)=0,\quad \partial_x u_n(t,\ell_n) = 0, & t>0,\ n=1,\ldots,N, \\ u_n(0,x) = u_n^0(x) & \forall x\in(0,\ell_n), j=1,\ldots,N. \end{cases}$$

The terms f_n and h are internal and boundary functions that are useful for the fixed point approach. First, we deal with the linear system (LKdV-N) with homogeneous initial condition and homogeneous internal source terms ($f_n = 0$):

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n = 0 & \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N, \\ u_n(t,0) = u_{n'}(t,0), & t>0,\ \forall n,n'=1,\dots,N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 u_n(t,0) = h(t), & t>0, \\ u_n(t,\ell_n)=0,\quad \partial_x u_n(t,\ell_n)=0, & t>0,\ n=1,\dots,N, \\ u_n(0,x)=0, & \forall x\in(0,\ell_n), n=1,\dots,N, \end{cases} \tag{2.1}$$

The fact that we work with the linear system $\partial_t u_n + \partial_x^3 u_n = 0$ instead of $\partial_t u_n + \partial_x u_n + \partial_x^3 u_n = 0$ is motivated by [3, 5]. It is well known, that the term $\partial_x u_n$ yields problematic behaviors with respect to regularity and controllability properties, as well noted Rosier in [20] and then in several works [7, 27, 4]. Now, formally we apply the usual Laplace transform with respect to time to the system (2.1) and obtain

$$\begin{cases} s\hat{u}_n + \partial_x^3 \hat{u}_n = 0 & \forall x \in (0,\ell_n),\ n=1,\dots,N, \\ \hat{u}_n(s,0) = \hat{u}_{n'}(s,0) & \forall n,n'=1,\dots,N, \\ \displaystyle\sum_{n=1}^{N} \partial_x^2 \hat{u}_n(s,0) = \hat{h}(s), & \\ \hat{u}_n(s,\ell_n)=0,\quad \partial_x \hat{u}_n(s,\ell_n)=0, & n=1,\dots,N, \\ \hat{u}_n(0,x)=0 & \forall x\in(0,\ell_n), n=1,\dots,N, \end{cases} \tag{2.2}$$

where

$$\hat{u}_n(s,x) = \int_0^\infty e^{-st} u_n(t,x)dt, \quad \hat{h}(s) = \int_0^\infty e^{-st} h(t)dt \quad \forall x \in (0,\ell_n).$$

Following [3], we can see that the N component solutions to (2.2) can be written as

$$\hat{u}_n(s,x) = \sum_{j=1}^{3} c^N_{3(n-1)+j}(s) e^{\lambda_j(s)x}, \tag{2.3}$$

where $\lambda_j(s)$, $j=1,2,3$ are the solutions of the characteristic equation $s+\lambda^3=0$ and $c^N = (c_k)^N_{k=1,\dots,3N}$ solves the following linear system

$$\begin{cases} \displaystyle\sum_{n=1}^{N}\sum_{j=1}^{3} c^N_{3(n-1)+j}\lambda_j^2 = \hat{h}, \\ \displaystyle\sum_{j=1}^{3} c_j^N e^{\lambda_j \ell_1} = 0, \\ \displaystyle\sum_{j=1}^{3} c_j^N \lambda_j e^{\lambda_j \ell_1} = 0, \\ \left.\begin{array}{l} \displaystyle\sum_{j=1}^{3} c_j^N = \sum_{j=1}^{3} c^N_{3(n-1)+j}, \\ \displaystyle\sum_{j=1}^{3} c^N_{3(n-1)+j} e^{\lambda_j \ell_n} = 0, \\ \displaystyle\sum_{j=1}^{3} c^N_{3(n-1)+j}\lambda_j e^{\lambda_j \ell_n} = 0 \end{array}\right\} \quad \forall n=2,\dots,N. \end{cases} \tag{2.4}$$

We write this previous system in its matrix form $A_N c^N = \hat{h} e_1$, where e_1 is the first vector of the canonical basis in $\mathbb{R}^{3N}$. We can see easily that $A_N \in M_{3N}$ can be decomposed by induction in blocks as

$$A_1 = \begin{bmatrix} (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ e^{\lambda_1 \ell_1} & e^{\lambda_2 \ell_1} & e^{\lambda_3 \ell_1} \\ \lambda_1 e^{\lambda_1 \ell_1} & \lambda_2 e^{\lambda_2 \ell_1} & \lambda_3 e^{\lambda_3 \ell_1} \end{bmatrix}, \tag{2.5}$$

$$A_N = \left[\begin{array}{c|c} A_{N-1} & \begin{matrix} (\lambda_1)^2 & (\lambda_2)^2 & (\lambda_3)^2 \\ & \mathbf{0}_{3(N-1)-1\times 3} & \end{matrix} \\ \hline \begin{matrix} 1 & 1 & 1 & \\ 0 & 0 & 0 & \mathbf{0}_{3\times 3(N-2)} \\ 0 & 0 & 0 & \end{matrix} & D_N \end{array}\right] = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix} \tag{2.6}$$

for an appropriate choice of B_N, C_N, and

$$D_N = \begin{bmatrix} -1 & -1 & -1 \\ e^{\lambda_1 \ell_N} & e^{\lambda_2 \ell_N} & e^{\lambda_3 \ell_N} \\ \lambda_1 e^{\lambda_1 \ell_N} & \lambda_2 e^{\lambda_2 \ell_N} & \lambda_3 e^{\lambda_3 \ell_N} \end{bmatrix}. \tag{2.7}$$

Formally, taking the inverse of the Laplace transform of $\hat{u}_n$ in (2.3), we get for $t \geq 0$ and $x \in (0, \ell_n)$

$$u_n(t,x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\hat{u}_n(s,x)ds = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st} c^N_{3(n-1)+j}\hat{h}(s)e^{\lambda_j(s)x}ds.$$

If we denote, for $t \geq 0$ and $x \in (0, \ell_n)$,

$$I_n(t,x) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{0}^{i\infty} e^{st} c^N_{3(n-1)+j}\hat{h}(s)e^{\lambda_j(s)x}ds,$$

$$J_n(t,x) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st} c^N_{3(n-1)+j}\hat{h}(s)e^{\lambda_j(s)x}ds,$$

we have

$$u_n(t,x) = I_n(t,x) + J_n(t,x). \tag{2.8}$$

Now we introduce the notation, super index, $^{+\backslash -}$ which corresponds to taking $s = \pm i\rho^3$, $\rho > 0$, in the characteristic equation. Then the roots of the characteristic equation are given by

$$\begin{cases} \lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{1}{2}\rho(\sqrt{3} - i), \quad \lambda_3^+(\rho) = \frac{1}{2}\rho(-\sqrt{3} - i), \\ \lambda_j^-(\rho) = \overline{\lambda_j^+(\rho)}, \; j = 1,2,3. \end{cases}$$

Let $\Delta^{N,+}(\rho)$ be the determinant of $A_N(i\rho^3)$ and $\Delta^{N,+}_{3(n-1)+j}(s)$ be the determinant of the matrix that is obtained by replacing the column $3(n-1)+j$ of the matrix $A_N(i\rho^3)$ by $[1\ 0 \dots\ 0]^T$ and $\hat{h}^+(\rho) = \hat{h}(i\rho^3)$. Assuming that $\Delta^{N,+}(\rho) \neq 0$ (this property will be justified in Proposition 2.1), Cramer's rule implies that $c^{N,+}_{3(n-1)+j}(\rho) = c^{N}_{3(n-1)+j}(i\rho^3)$ is given by

$$c^{N,+}_{3(n-1)+j}(\rho) = \frac{\Delta^{N,+}_{3(n-1)+j}(\rho)}{\Delta^{N,+}(\rho)}\hat{h}^+(\rho). \tag{2.9}$$

Thus, I_n and J_n can be seen as

$$I_n(t,x) = \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{\infty} e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}\frac{\Delta^{N,+}_{3(n-1)+j}(\rho)}{\Delta^{N,+}(\rho)}\hat{h}^+(\rho)3\rho^2 d\rho, \tag{2.10}$$

$$J_n(t,x) = \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{\infty} e^{-i\rho^3 t}e^{\lambda_j^-(\rho)x}\frac{\Delta^{N,-}_{3(n-1)+j}(\rho)}{\Delta^{N,-}(\rho)}\hat{h}^-(\rho)3\rho^2 d\rho, \tag{2.11}$$

where we use the notation $\Delta^{N,-}_k(\rho) = \overline{\Delta^{N,+}_k(\rho)}$, $\Delta^{N,-}(\rho) = \overline{\Delta^{N,+}(\rho)}$, and $\hat{h}^-(\rho) = \overline{\hat{h}^+(\rho)}$. Our idea now is to obtain estimates for u_n; for that we are going to prove some asymptotic properties for $\frac{\Delta^{N,+}_{3(n-1)+j}(\rho)}{\Delta^{N,+}(\rho)}$, the following proposition collects these properties.

PROPOSITION 2.1. *For all $\rho > 0$, $\Delta^{N,+}(\rho) \neq 0$. Moreover, the following asymptotic properties hold, for $\rho \to \infty$,*

$$\begin{aligned} &\frac{\Delta^{N,+}_{3(n-1)+1}}{\Delta^{N,+}} \sim -\delta_N\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, \quad \frac{\Delta^{N,+}_{3(n-1)+2}}{\Delta^{N,+}} \sim \delta_N\rho^{-2}e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}},\\ &\frac{\Delta^{N,+}_{3(n-1)+3}}{\Delta^{N,+}} \sim \delta_N\rho^{-2}e^{-i\frac{\pi}{3}}, \quad \sum_{j=1}^{3}\frac{\Delta^{N,+}_{3(n-1)+j}}{\Delta^{N,+}} \sim \delta_N\rho^{-2}e^{-i\frac{\pi}{3}}, \quad n = 1,\dots,N, \end{aligned} \tag{2.12}$$

where $\delta_N > 0$ only depends on N and satisfies $\delta_N = \dfrac{\delta_{N-1}}{\delta_{N-1}+1}$.

Proof. The main problem in this proof is to deal with the determinant of the matrix without making explicit computations. Recall that, in the case of N branches, the matrix A_N has size $3N \times 3N$. Our proof is based on an induction argument over the number N of branches of the network.

- $N = 1$: in this case, system (2.4) is exactly the system studied in [5] for $\ell_1 = 1$. By Appendix B, it holds that $\Delta^{1,+}(\rho) \neq 0$ for all $\rho > 0$. Moreover, following the explicit calculations given in [5] we can deduce

$$\begin{aligned} &\frac{\Delta^{1,+}_1}{\Delta^{1,+}} \sim -\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_1 - i\frac{3}{2}\rho\ell_1}, \quad \frac{\Delta^{1,+}_2}{\Delta^{1,+}} \sim \rho^{-2}e^{-\rho\sqrt{3}\ell_1 + i\frac{\pi}{3}}, \quad \frac{\Delta^{1,+}_3}{\Delta^{1,+}} \sim \rho^{-2}e^{-i\frac{\pi}{3}},\\ &\sum_{j=1}^{3}\frac{\Delta^{1,+}_j}{\Delta^{1,+}} \sim \rho^{-2}e^{-i\frac{\pi}{3}}. \end{aligned}$$

  That gives (2.12) in the case $N = 1$.
- Suppose now that $\Delta^{N-1,+}(\rho) \neq 0$ for all $\rho > 0$ and that the asymptotic property (2.12) is true for any network of $N-1$ branches. Let us prove that

$\Delta^{N,+}(\rho) \neq 0$ for all $\rho > 0$ and that the asymptotic property (2.12) holds for a network of N branches. As

$$A_N = \begin{bmatrix} A_{N-1} & B_N \\ C_N & D_N \end{bmatrix},$$

and we have $\det(A_{N-1}) = \Delta^{N-1,+} \neq 0$ by hypothesis, we can write

$$A_N = \begin{bmatrix} I_{3(N-1)} & \mathbf{0}_{3(N-1)} \\ C_N A_{N-1}^{-1} & I_{3(N-1)} \end{bmatrix} \begin{bmatrix} A_{N-1} & \mathbf{0}_{3(N-1)} \\ \mathbf{0}_{3(N-1)} & D_N - C_N A_{N-1}^{-1} B_N \end{bmatrix} \times \begin{bmatrix} I_{3(N-1)} & A_{N-1}^{-1} B_N \\ \mathbf{0}_{3(N-1)} & I_{3(N-1)} \end{bmatrix},$$

which implies directly that

$$\Delta^{N,+} = \det(A_N) = \det(A_{N-1}) \det(D_N - C_N A_{N-1}^{-1} B_N). \tag{2.13}$$

The difficulty of the last expression is the role of the matrix A_{N-1}^{-1}. In fact, to calculate this inverse explicitly is quite complicated. Note now that if

$$A_{N-1}^{-1} = \begin{bmatrix} x_1 & \vdots & \dots & \vdots \\ x_2 & \vdots & \dots & \vdots \\ x_3 & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \end{bmatrix},$$

then, we have

$$C_N A_{N-1}^{-1} B_N = \begin{bmatrix} (\lambda_1^+)^2(x_1+x_2+x_3) & (\lambda_2^+)^2(x_1+x_2+x_3) & (\lambda_3^+)^2(x_1+x_2+x_3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \tag{2.14}$$

from here we can see that it is not necessary to calculate all the entries of the matrix A_{N-1}^{-1}. Indeed, we only need the 3 first entries of the first column. Straightforward calculations show that

$$x_1 = \frac{\Delta_1^{N-1,+}}{\Delta^{N-1,+}}, \quad x_2 = \frac{\Delta_2^{N-1,+}}{\Delta^{N-1,+}}, \quad x_3 = \frac{\Delta_3^{N-1,+}}{\Delta^{N-1,+}}. \tag{2.15}$$

Using (2.14) and (2.15) we get

$$C_N A_{N-1}^{-1} B_N = \begin{bmatrix} (\lambda_1^+)^2 \sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & (\lambda_2^+)^2 \sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & (\lambda_3^+)^2 \sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.16}$$

Then with (2.7)

(2.17)

$$
\begin{aligned}
&D_N - C_N A_{N-1}^{-1} B_N \\
&= \begin{bmatrix} -1-(\lambda_1^+)^2 \sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & -1-(\lambda_2^+)^2 \sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} & -1-(\lambda_3^+)^2 \sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \\ e^{\lambda_1^+ \ell_N} & e^{\lambda_2^+ \ell_N} & e^{\lambda_3^+ \ell_N} \\ \lambda_1^+ e^{\lambda_1^+ \ell_N} & \lambda_2^+ e^{\lambda_2^+ \ell_N} & \lambda_3^+ e^{\lambda_3^+ \ell_N} \end{bmatrix}
\end{aligned}
$$

and using the multilinearity of the determinant

$$\det(D_N - C_N A_{N-1}^{-1} B_N) = -\sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N),$$

where

$$F_N = \begin{bmatrix} (\lambda_1^+)^2 & (\lambda_2^+)^2 & (\lambda_3^+)^2 \\ e^{\lambda_1^+ \ell_N} & e^{\lambda_2^+ \ell_N} & e^{\lambda_3^+ \ell_N} \\ \lambda_1^+ e^{\lambda_1^+ \ell_N} & \lambda_2^+ e^{\lambda_2^+ \ell_N} & \lambda_3^+ e^{\lambda_3^+ \ell_N} \end{bmatrix}. \tag{2.18}$$

Then, it holds that

$$\Delta^{N,+} = \Delta^{N-1,+} \left[-\sum_{j=1}^{3} \frac{\Delta_j^{N-1,+}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N) \right]. \tag{2.19}$$

Using (2.7) and (2.18) we can derive

$$
\begin{aligned}
\det(D_N) = \rho\sqrt{3} e^{-i\rho\ell_N} &+ \left(-\frac{\rho\sqrt{3}}{2} - \frac{3}{2} i\rho \right) e^{\left(-\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2}\right)\ell_N} \\
&+ \left(-\frac{\rho\sqrt{3}}{2} + \frac{3}{2} i\rho \right) e^{\left(\frac{\rho\sqrt{3}}{2} + i\frac{\rho}{2}\right)\ell_N},
\end{aligned} \tag{2.20}
$$

$$\det(F_N) = \sqrt{3}\rho^3 e^{-i\rho\ell_N} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(\sqrt{3}-i)\ell_N} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(-\sqrt{3}-i)\ell_N}. \tag{2.21}$$

Now, to compute $\Delta_{3(n-1)+j}^{N,+}$, let $A_{N,j}^n$ the matrix obtained by replacing the column $3(n-1)+j$ of A_N by $[1\,0 \cdots 0]^T$, for $j=1,2,3$ and $n=1,\dots,N-1$, that is

(2.22)

$$
A_{N,j}^n = \left[\begin{array}{c|c} \overset{(j+3(n-1)-th)}{\overbrace{1}} & \\ 0 & B_N \\ \vdots & \\ \hline \vdots & \\ \vdots & D_N \\ 0 & \end{array} \right]
= \begin{cases}
\left[\begin{array}{c|c} A_{N-1,j}^n & B_N \\ \hline \begin{matrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \mathbf{0} & D_N \end{array} \right] & \text{if } j=1,\ n=1 \\
\left[\begin{array}{c|c} A_{N-1,j}^n & B_N \\ \hline \begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \mathbf{0} & D_N \end{array} \right] & \text{if } j=2,\ n=1 \\
\left[\begin{array}{c|c} A_{N-1,j}^n & B_N \\ \hline \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \quad \mathbf{0} & D_N \end{array} \right] & \text{if } j=3,\ n=1 \\
\left[\begin{array}{c|c} A_{N-1,j}^n & B_N \\ \hline C_N & D_N \end{array} \right] & \text{if } j=1,2,3,\ n=2,\cdots,N-1.
\end{cases}
$$

We claim the following property of $\Delta_{3(n-1)+j}^{N,+}$.

Lemma 2.2.

$$\Delta^{N,+}_{3(n-1)+j} = \Delta^{N-1,+}_{3(n-1)+j} \det(D_N), \quad n = 1, \dots, N-1, \quad j = 1, 2, 3. \tag{2.23}$$

Proof. Using the decomposition given by (2.22), we get

$$A^n_{N,j} = \begin{bmatrix} A^n_{N-1,j} & B_N \\ C^n_{N,j} & D_N \end{bmatrix}$$

for an appropriate choice of $C^n_{N,j}$. Thus, with the same idea as (2.13) it holds that

$$\Delta^{N,+}_{3(n-1)+j} = \det(A^n_{N,j}) = \det(A^n_{N-1,j}) \det(D_N - C^n_{N,j}(A^n_{N-1,j})^{-1} B_N). \tag{2.24}$$

Similarly, as before, we need to study the product $C^n_{N,j}(A^n_{N-1,j})^{-1}B_N$, in particular, the first column of the matrix $(A^n_{N-1,k})^{-1}$. To do that, note that

$$A^n_{N-1,j} v = \left[\begin{array}{c|c} \overbrace{1}^{(j+3(n-1)-th)} & \\ 0 & B_{N-1} \\ \vdots & \\ \hline \vdots & \\ \vdots & D_N \\ 0 & \end{array}\right] v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

by simple inspection; the solution of this problem is $v = [0 \ \cdots \ \overbrace{1}^{j+3(n-1)} \ \cdots \ 0]^T$ which we know coincides with the first column of $(A^n_{N-1,j})^{-1}$ and, therefore, $C^n_{N,j}(A^n_{N-1,j})^{-1}B_N = \mathbf{0}_{3\times 3}$; therefore, with (2.24)

$$\Delta^{N,+}_{3(n-1)+j} = \Delta^{N-1,+}_{3(n-1)+j} \det(D_N), \quad n = 1, \dots, N-1, \quad j = 1, 2, 3,$$

which finishes the proof of Lemma 2.2. ∎

In order to show that $\Delta^{N,+} \neq 0$, note that by (2.19) we get

$$\Delta^{N,+} = -\sum_{j=1}^{3} \Delta^{N-1,+}_j \det(F_N) + \Delta^{N-1,+} \det(D_N), \quad j = 1, 2, 3.$$

Using (2.23) recursively, we get

$$\Delta^{N-1,+}_j = \Delta^{1,+}_j \prod_{\ell=2}^{N-1} \det(D_\ell).$$

Noticing that $\Delta^{1,+} = \det(F_1)$, $-\sum_{j=1}^{3} \Delta^{1,+}_j = \det(D_1)$ and invoking inductively (2.19), we deduce

$$\Delta^{N,+} = \sum_{j=1}^{N} \det(F_j) \prod_{\ell=1,\ \ell\neq j}^{N} \det(D_\ell).$$

Then, from Appendix C, it holds, for all $j = 1, \dots, N$, $\det(D_j) \neq 0$, thus

$$\Delta^{N,+} = \left(\prod_{\ell=1}^{N} \det(D_\ell)\right) \sum_{j=1}^{N} \frac{\det(F_j)}{\det(D_j)},$$

and from Appendix D, $\sum_{j=1}^{N} \frac{\det(F_j)}{\det(D_j)} \neq 0$, thus $\Delta^{N,+} \neq 0$. Now as $\Delta^{N,+} \neq 0$, we can obtain using (2.19) and (2.23) that

$$\frac{\Delta^{N,+}_{3(n-1)+j}}{\Delta^{N,+}} = \frac{\Delta^{N-1,+}_{3(n-1)+j}}{\Delta^{N-1,+}} \frac{\det(D_N)}{-\displaystyle\sum_{l=1}^{3} \frac{\Delta^{N-1,+}_{l}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N)} \tag{2.25}$$

for $j = 1, 2, 3,\ n = 1, \dots, N-1$. Then, using (2.21) we get $\det(F_N) \sim \sqrt{3}\rho^3 e^{\frac{\rho}{2}\sqrt{3}\ell_N + i\frac{\rho}{2}\ell_N}$ and by the induction assumption $\sum_{l=1}^{3} \frac{\Delta^{N-1,+}_{l}}{\Delta^{N-1,+}} \sim \delta_{N-1}\rho^{-2}e^{-i\frac{\pi}{3}}$. Thus $\sum_{l=1}^{3} \frac{\Delta^{N-1,+}_{l}}{\Delta^{N-1,+}} \det(F_N) \sim \delta_{N-1}\sqrt{3}\rho e^{\frac{\rho}{2}\sqrt{3}\ell_N + i\frac{\rho}{2}\ell_N - i\frac{\pi}{3}}$ and then for $\rho \to \infty$

$$\frac{\det(D_N)}{-\sum_{l=1}^{3} \frac{\Delta^{N-1,+}_{l}}{\Delta^{N-1,+}} \det(F_N) + \det(D_N)} \sim \frac{1}{\delta_{N-1} + 1}. \tag{2.26}$$

Now by the induction assumption

$$\frac{\Delta^{N-1,+}_{3(n-1)+1}}{\Delta^{N-1,+}} \sim -\delta_{N-1}\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, \quad \frac{\Delta^{N-1,+}_{3(n-1)+2}}{\Delta^{N-1,+}} \sim \delta_{N-1}\rho^{-2}e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}},$$

$$\frac{\Delta^{N-1,+}_{3(n-1)+3}}{\Delta^{N-1,+}} \sim \delta_{N-1}\rho^{-2}e^{-i\frac{\pi}{3}},$$

and (2.25)–(2.26) we have

$$\frac{\Delta^{N,+}_{3(n-1)+1}}{\Delta^{N,+}} \sim -\delta_{N}\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_n - i\frac{3}{2}\rho\ell_n}, \quad \frac{\Delta^{N,+}_{3(n-1)+2}}{\Delta^{N,+}} \sim \delta_{N}\rho^{-2}e^{-\rho\sqrt{3}\ell_n + i\frac{\pi}{3}},$$
$$\frac{\Delta^{N,+}_{3(n-1)+3}}{\Delta^{N,+}} \sim \delta_{N}\rho^{-2}e^{-i\frac{\pi}{3}}, \quad \sum_{j=1}^{3} \frac{\Delta^{N,+}_{3(n-1)+j}}{\Delta^{N,+}} \sim \delta_{N}\rho^{-2}e^{-i\frac{\pi}{3}}, \quad n = 1, \dots, N-1, \tag{2.27}$$

where $\delta_N = \frac{\delta_{N-1}}{\delta_{N-1}+1}$. It just remains to study the case $n = N$. Note that using the block decomposition of A_N we get

$$C_N \begin{bmatrix} \frac{\Delta^{N,+}_{1}}{\Delta^{N,+}} \\ \frac{\Delta^{N,+}_{2}}{\Delta^{N,+}} \\ \frac{\Delta^{N,+}_{3}}{\Delta^{N,+}} \\ \vdots \\ \frac{\Delta^{N,+}_{3N-5}}{\Delta^{N,+}} \\ \frac{\Delta^{N,+}_{3N-4}}{\Delta^{N,+}} \\ \frac{\Delta^{N,+}_{3N-3}}{\Delta^{N,+}} \end{bmatrix} + D_N \begin{bmatrix} \frac{\Delta^{N,+}_{3N-2}}{\Delta^{N,+}} \\ \frac{\Delta^{N,+}_{3N-1}}{\Delta^{N,+}} \\ \frac{\Delta^{N,+}_{3N}}{\Delta^{N,+}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and recalling (2.6) and (2.7) explicit calculations show that

$$\begin{bmatrix} \frac{\Delta_{3N-2}^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_{3N-1}^{N,+}}{\Delta^{N,+}} \\ \frac{\Delta_{3N}^{N,+}}{\Delta^{N,+}} \end{bmatrix} = \frac{\left(-\sum_{j=1}^{3}\frac{\Delta_j^{N,+}}{\Delta^{N,+}}\right)}{\det(D_N)} \begin{bmatrix} -\rho\sqrt{3}e^{-i\rho\ell_N} \\ \left(\frac{\rho\sqrt{3}}{2}+\frac{3}{2}i\rho\right)e^{\left(-\frac{\rho\sqrt{3}}{2}+\frac{i\rho}{2}\right)\ell_N} \\ \left(\frac{\rho\sqrt{3}}{2}-\frac{3}{2}i\rho\right)e^{\left(\frac{\rho\sqrt{3}}{2}+\frac{i\rho}{2}\right)\ell_N} \end{bmatrix}, \tag{2.28}$$

and using (2.27) we can conclude from (2.28)

$$\frac{\Delta_{3N-2}^{N,+}}{\Delta^{N,+}} \sim -\delta_N\rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}\ell_N-i\frac{3}{2}\rho\ell_N}, \quad \frac{\Delta_{3N-1}^{N,+}}{\Delta^{N,+}} \sim \delta_N\rho^{-2}e^{-\rho\sqrt{3}\ell_N+i\frac{\pi}{3}},$$
$$\frac{\Delta_{3N}^{N,+}}{\Delta^{N,+}} \sim \delta_N\rho^{-2}e^{-i\frac{\pi}{3}}, \quad \sum_{j=1}^{3}\frac{\Delta_{3(N-1)+j}^{N,+}}{\Delta^{N,+}} \sim \delta_N\rho^{-2}e^{-i\frac{\pi}{3}},$$

which gives the induction and concludes the proof of Proposition 2.1. ∎

Remark 3. Recently, in [11], the problem of small-time local controllability of the nonlinear single KdV equation was addressed. To reach the obstruction to small-time controllability in [11] new regularity results in the spirit of [2] were established. Those results have some connections with the analysis developed in this work. Here, the analysis of the linear problem (2.4) is based on the estimate of the terms I_n and J_n ((2.10) and (2.11)). These involve two integrals of ρ from 0 to infinity, and Proposition 2.1 shows the integrands are well-defined ($\Delta^{N,+} \neq 0$) and deal with their behavior at infinity. However, in [11] the behavior of the integrands might be infinite for finite ρ. This is the case where $L \in \mathcal{N}$, with $2k + l \notin 3\mathbb{N}^*$ [11, Lemma B1]. The main difference between these two different behaviors is because in [11] they worked with the linear system including the term, u_x which is necessary to study controllability issues. ○

Now we are going to state the next regularity result for the solution (2.1) using the Laplace representation obtained in (2.8) and Proposition 2.1.

PROPOSITION 2.3. *Let $T > 0$ and $h \in H^{-\frac{1}{3}}(0,T)$, then we have a unique solution $\underline{u} \in \mathbb{Y}_T$ of* (2.1)*. Moreover, there exists $C > 0$ such that for all $h \in H^{-\frac{1}{3}}(0,T)$, $\|\underline{u}\|_{\mathbb{Y}_T} \le C\|h\|_{H^{-\frac{1}{3}}(0,T)}$.*

Proof. This proof uses Proposition 2.1 and follows closely [5, Proposition 2.2] and [3], thus it is omitted here. ∎

Note that Proposition 2.3 justifies the formal computations given in (2.8). Let $\underline{W}$ the operator that corresponds to the integral representation obtained in Proposition 2.3, i.e., given $T > 0$ and $h \in H^{-\frac{1}{3}}(0,T)$, the unique solution $\underline{u}$ of (2.1) is given by

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \underline{W}h \in \mathbb{B}_T.$$

Our next step is to consider the linear problem including nonhomogeneous initial data and source terms, as follows:

$$\begin{cases} \partial_t v_n(t,x) + \partial_x^3 v_n(t,x) = f_n(t,x) & \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N, \\ v_n(t,0) = v_{n'}(t,0) & \forall n,n' = 1,\dots,N, \\ \sum_{n=1}^{N}\partial_x^2 v_n(t,0) = h(t), & t>0, \\ v_n(t,\ell_n) = 0, \quad \partial_x v_n(t,\ell_n) = 0, & t>0,\ n=1,\dots,N, \\ v_n(0,x) = v_n^0, & x \in (0,\ell_n). \end{cases} \tag{2.29}$$

We know from [1] that in the case, $h = 0$ the solution of (2.29) can be written as

$$\underline{v}(t,x) = \underline{W}_0(t)\underline{v}^0 + \int_0^t \underline{W}_0(t-\tau)\underline{f}(\tau)d\tau,$$

for any $\underline{v}^0 \in \mathbb{L}^2(\mathcal{T})$ and $\underline{f} \in L^1(0,T;\mathbb{L}^2(\mathcal{T}))$, where $\{\underline{W}_0(t)\}_{t\geq 0}$ is the C_0-semigroup in the space $\mathbb{L}^2(\mathcal{T})$ generated by the operator $\mathcal{A}\underline{v} = -\partial_x^3\underline{v}$, with domain

$$D(\mathcal{A}) = \left\{ \underline{v} \in \left(\prod_{n=1}^N H^3(0,\ell_n)\right) \cap \mathbb{H}_e^2(\mathcal{T}), \sum_{n=1}^N \frac{\mathrm{d}^2 v_n}{\mathrm{d}x^2}(0) = 0, \right\}$$

where $H_r^2(0,\ell_n) = \{v \in H^2(0,\ell_n),\ \left(\frac{d}{dx}\right)^{i-1} v(\ell_n) = 0,\ 1 \leq i \leq 2\}$ and the space $\mathbb{H}_e^2(\mathcal{T})$ is the Cartesian product of $H_r^2(0,\ell_n)$ including the continuity condition on the central node ($u_n(0) = u_{n'}(0)\ \forall n, n' = 1,\dots,N$). Using semigroup theory it is possible to show that $\underline{v} \in C([0,T];\mathbb{L}^2(\mathcal{T}))$ and also using multipliers we can obtain the classical Kato smoothing result $\underline{v} \in L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))$, but it is difficult (if not impossible) to derive the sharp Kato smoothing property established in Proposition 2.3 using energy methods. Now we use the following result obtained in [3] for a single KdV equation posed on a bounded domain.

$$\text{(2.30)} \qquad \begin{cases} \partial_t\psi + \partial_x^3\psi = f, & x \in (0,L), \quad t \geq 0, \\ \psi(t,0) = \psi(t,L) = \partial_x\psi(t,L) = 0, & t \geq 0, \\ \psi(0,x) = \psi^0(x), & x \in (0,L), \end{cases}$$

Proposition 2.4 (Lemma 3.3 of [3]). *Let $T > 0$ and $L > 0$ be given. For any $\psi^0 \in L^2(0,L)$ and $f \in L^1(0,T;L^2(0,L))$, the problem* (2.30) *admits a unique solution $\psi \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$, with $\partial_x^\kappa\psi \in L_x^\infty(0,L;H^{\frac{1-\kappa}{3}}(0,T))$, $\kappa = 0,1,2$. Moreover, there exists $C > 0$ depending only on T and L such that*

$$\begin{aligned} &\|\psi\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))} + \sum_{\kappa=0}^2 \|\partial_x^\kappa\psi\|_{L_x^\infty(0,L;H^{\frac{1-\kappa}{3}}(0,T))} \\ &\leq C\left(\|\psi^0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;H^1(0,L))}\right). \end{aligned}$$

Now for any $v_n^0 \in L^2(0,\ell_n)$ and $f_n \in L^1(0,T;L^2(0,\ell_n))$, consider

$$\psi_n = \psi_n(t,\cdot) = W_1^n(t)v_n^0(\cdot) + \int_0^t W_1^n(t-\tau)f_n(\tau,\cdot)d\tau,$$

where $W_1^n(t)$ is the C_0-semigroup associated with the boundary-value problem (2.30) on $(0,\ell_n)$. Let $\mathfrak{h}(t) = \sum_{n=1}^N \partial_x^2\psi_n(t,0) \in H^{-\frac{1}{3}}(0,T)$ by Proposition 2.4. Now take $h \in H^{-\frac{1}{3}}(0,T)$, then by Proposition 2.3 the function $\underline{w} = \underline{W}(t)(h-\mathfrak{h})$ is well-defined and is the solution of (2.1) with boundary data $h - \mathfrak{h}$. Finally, the solution $\underline{v}$ of (2.29) can be expressed as

$$\underline{v}(t,\cdot) = \underline{W_1}(t)\underline{v}^0(\cdot) + \int_0^t \underline{W_1}(t-\tau)\underline{f}(\tau,\cdot)d\tau + \underline{W}(t)(h-\mathfrak{h})(t).$$

The next result encapsulates these ideas.

Proposition 2.5. *Let $T > 0$ be given, then, for any $\underline{v}^0 \in \mathbb{L}^2(\mathcal{T})$, $h \in H^{-\frac{1}{3}}(0,T)$, and $\underline{f} \in L^1(0,T;\mathbb{L}^2(\mathcal{T}))$, the problem* (2.29) *admits a unique solution $\underline{v} \in \mathbb{Y}_T$. Moreover, there exists $C > 0$ depending only on T and $\ell_1,\dots,\ell_n$ such that*

$$\|\underline{v}\|_{\mathbb{Y}_T} \leq C\left(\|h\|_{H^{-\frac{1}{3}}(0,T)} + \|\underline{f}\|_{L^1(0,T;\mathbb{L}^2(\mathcal{T}))} + \|\underline{v}^0\|_{\mathbb{L}^2(\mathcal{T})}\right).$$

2.2. Nonlinear problem. With all the tools developed in the last sections we are ready to prove the global well-posedness result established on Theorem 1.3; the main ingredients of this proof are the regularity obtained in the linear cases, energy and multiplier estimates, and a fixed point argument. Let $T > 0$ and define $\mathbb{X}_T = \mathbb{L}^2(\mathcal{T}) \times H^{-\frac{1}{3}}(0,T)$.

Proof of Theorem 1.3. Let $(\underline{u}^0, 0) \in \mathbb{X}_T$ and $R,\ \theta > 0$ that will be chosen after. Consider the closed ball $B_{\mathbb{Y}_\theta}(0,R) := \{\underline{v} \in \mathbb{Y}_\theta,\ \|\underline{v}\|_{\mathbb{Y}_\theta} \le R\}$. Then $B_{\mathbb{Y}_\theta}(0,R)$ is a complete metric space. Consider the map $\Phi : \mathbb{Y}_\theta \to \mathbb{Y}_\theta$ defined by $\Phi(\underline{v}) = \underline{u}$, where $\underline{u}$ is the solution of

(2.31)

$$\begin{cases} (\partial_t u_n + \partial_x^3 u_n)(t,x) = -(\partial_x v_n + v_n \partial_x v_n)(t,x) & \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N,\\ u_n(t,0) = u_{n'}(t,0) & \forall n,n' = 1,\dots,N,\\ \sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha v_1(t,0) - \dfrac{N}{3}(v_1(t,0))^2, & t>0,\\ u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0, & t>0,\ n=1,\dots,N,\\ u_n(0,x) = u_n^0, & x \in (0,\ell_n). \end{cases}$$

Clearly, $\underline{u} \in \mathbb{Y}_\theta$ is solution of (KdV-N) if $\underline{u}$ is a fixed point of Φ. Now we write two lemmas to deal with the source term and boundary conditions. ∎

Lemma 2.6 (Lemma 3.1 of [3]). *There exists a constant $C > 0$ such that for any $T > 0$ and $u,\ v \in Y_T$*

$$\int_0^T \|u(t,\cdot)\partial_x v(t,\cdot)\|_{L^2(0,L)} dt \le C(T^{1/2} + T^{1/3})\|u\|_{Y_T}\|v\|_{Y_T},$$

where Y_T is $\mathbb{Y}_T$ for $N = 1$.

Lemma 2.7 (Lemma 3.2 of [12]). *There exist of constants $C,\ \beta > 0$ such that for any $T > 0$ and $g_1,\ g_2 \in H^{\frac{1}{3}}(0,T)$, it holds that, $g_1 g_2 \in H^{-\frac{1}{3}}(0,T)$ and*

$$\|g_1 g_2\|_{H^{-\frac{1}{3}}(0,T)} \le C T^\beta \|g_1\|_{H^{\frac{1}{3}}(0,T)} \|g_2\|_{H^{\frac{1}{3}}(0,T)}.$$

From Proposition 2.5 and Lemmas 2.6 and 2.7 we get for all $\underline{v} \in \mathbb{Y}_\theta$

$$\begin{aligned} \|\Phi(\underline{v})\|_{\mathbb{Y}_\theta} =\le C\Bigg(&\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \left\| -\alpha v_1(t,0) - \frac{N}{3}(v_1(t,0))^2 \right\|_{H^{-\frac{1}{3}}(0,\theta)} \\ &+ \int_0^\theta \|\partial_x \underline{v}(t,\cdot)\|_{\mathbb{L}^2(\mathcal{T})} dt + \int_0^\theta \|\underline{v}(t,\cdot)\partial_x \underline{v}(t,\cdot)\|_{\mathbb{L}^2(\mathcal{T})} dt \Bigg) \\ \le C\Big(&\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} + \theta^\beta(\|\underline{v}\|_{\mathbb{Y}_\theta} + \|\underline{v}\|^2_{\mathbb{Y}_\theta}) + (\theta^{1/2} + \theta^{1/3})\|\underline{v}\|^2_{\mathbb{Y}_\theta} + \theta^{1/2}\|\underline{v}\|_{\mathbb{Y}_\theta}\Big). \end{aligned}$$

We consider Φ restricted to the closed ball $B_{\mathbb{Y}_\theta}(0,R)$ and choose $\theta,\ R > 0$ such that

$$\begin{cases} R = 3C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})},\\ C(\theta^\beta + \theta^{1/2}) \le \frac{1}{3},\\ C(\theta^\beta + \theta^{1/2} + \theta^{1/3})R \le \frac{1}{6}. \end{cases} \tag{2.32}$$

Thus, for $\underline{u} \in B_{\mathbb{Y}_\theta}(0,R)$, Φ maps $B_{\mathbb{Y}_\theta}(0,R)$ into itself. Take now $\underline{v}$ and $\underline{\widetilde{v}} \in B_{\mathbb{Y}_\theta}(0,R)$, then $\underline{w} = \Phi(\underline{v}) - \Phi(\underline{\widetilde{v}})$ solves the equation

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n = -(\partial_x v_n - \partial_x \widetilde{v_n}) \\ -\frac{1}{2}\partial_x((v_n - \widetilde{v_n})(v_n + \widetilde{v_n})) & \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N, \\ w_n(t,0) = w_{n'}(t,0) & \forall n,n' = 1,\dots,N, \\ \displaystyle\sum_{n=1}^N \partial_x^2 w_n(t,0) = -\alpha(v_1(t,0) - \widetilde{v}_1(t,0)), \\ -\dfrac{N}{3}((v_1(t,0) - \widetilde{v}_1(t,0))(v_1(t,0) + \widetilde{v}_1(t,0))), & t>0, \\ w_n(t,\ell_n) = \partial_x w_n(t,\ell_n) = 0, & t>0,\ n=1,\dots,N, \\ w_n(0,x) = 0, & x \in (0,\ell_n). \end{cases}$$

Now from Proposition 2.5 we obtain

$$\begin{aligned} \|\Phi(\underline{v}) - \Phi(\underline{\widetilde{v}})\|_{\mathbb{Y}_\theta} \le C \Big(& \theta^{1/2}\|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta} + \frac{1}{2}(\theta^{1/2} + \theta^{1/3})\|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta}\|\underline{v} + \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta} \\ & + \theta^\beta \|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta} + \theta^\beta \|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta}\|\underline{v} + \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta}\Big) \\ \le C \Big(& (\theta^{1/2} + \theta^\beta)\|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta} + \frac{1}{2}(\theta^{1/2} + \theta^{1/3} + 2\theta^\beta)\|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta} 2R \Big); \end{aligned}$$

then with (2.32)

$$\|\Phi(\underline{v}) - \Phi(\underline{\widetilde{v}})\|_{\mathbb{Y}_\theta} \le \left(\frac{1}{3} + \frac{1}{3}\right)\|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta} = \frac{2}{3}\|\underline{v} - \underline{\widetilde{v}}\|_{\mathbb{Y}_\theta}.$$

That means that the map Φ is a contraction on $B_{\mathbb{Y}_\theta}$ and by the Banach fixed point theorem has a unique fixed point $\underline{u} \in \mathbb{Y}_\theta$. It gives the local-in-time well-posedness for bounded initial data. Now taking $T > 0$, we can check using integration by parts and the boundary conditions that every solution of (KdV-N) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\left(\alpha - \frac{N}{2}\right)|u_1(t,0)|^2 - \frac{1}{2}\sum_{n=1}^N |\partial_x u_n(t,0)|^2 \le 0 \tag{2.33}$$

since $N \le 2\alpha$. This dissipation law tells us that the energy is a nonincreasing function of the time variable, that means

$$E(t) \le E(\theta) \le E(0) = \frac{1}{2}\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \quad \forall t > \theta > 0. \tag{2.34}$$

From here, taking the maximum for $t \in [0,T]$ we can see that

$$\|\underline{u}\|_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \le \|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}. \tag{2.35}$$

Finally, following [16, 10] we multiply (KdV-N) by $q_n u_n$, integrate over $(0,T)\times(0,\ell_n)$, and sum over $n = 1,\dots,N$ to obtain the following equality:

$$\begin{aligned} \sum_{n=1}^N \int_0^{\ell_n} q_n(t,x)|u_n(t,x)|^2 dx \,\Big|_0^T - \sum_{n=1}^N \int_0^T \int_0^{\ell_n} (\partial_t q_n + \partial_x q_n + \partial_x^3 q_n)|u_n|^2 dxdt \\ + 3\sum_{n=1}^N \int_0^T \int_0^{\ell_n} |\partial_x u_n|^2 \partial_x q_n dxdt - \frac{2}{3}\sum_{n=1}^N \int_0^T \int_0^{\ell_n} |u_n|^3 \partial_x q_n dxdt \end{aligned} \tag{2.36}$$

$$= \sum_{n=1}^{N} \int_0^T \Big[(q_n + \partial_x^2 q_n)|u_n|^2 + 2q_n u_n \partial_x^2 u_n$$
$$-2\partial_x q_n u_n \partial_x u_n - q_n |\partial_x u_n|^2 + \frac{2}{3} q_n |u_n|^3 \Big] (t,0) dt.$$

- Taking $q_n = 1$ in (2.36) we can derive

$$\sum_{n=1}^{N} \int_0^T |\partial_x u_n(t,0)|^2 dt \leq \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}. \tag{2.37}$$

- If we take $q_n = \frac{x(2\ell_n - x)}{\ell_n^2}$ in (2.36), defining $L = \max_{n=1,\cdots,N} \ell_n$ and $\ell = \min_{n=1,\cdots,N} \ell_n$, we can obtain

$$\frac{2N}{L^2} \|u_1(\cdot,0)\|^2_{L^2(0,T)} \leq \frac{2T}{\ell^2} \|\underline{u}\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))} - 2\int_0^T u_1(t,0) \sum_{n=1}^{N} \partial_x u_n(t,0) \frac{2}{\ell_n} dt$$
$$+ \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \frac{4}{3\ell} \sum_{n=1}^{N} \int_0^T \int_0^{\ell_n} u_n^3(t,x) dx dt.$$

Using (2.35)–(2.37) and Young's inequality we derive

$$\|u_1(t,0)\|^2_{L^2(0,T)} \leq C(T+1)\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + C\sum_{n=1}^{N} \int_0^T \int_0^{\ell_n} u_n^3(t,x) dx dt. \tag{2.38}$$

As $H^1(0,\ell_n)$ embeds compactly into $C([0,\ell_n])$ we get

$$\sum_{n=1}^{N} \int_0^T \int_0^{\ell_n} |u_n|^3 dx dt \leq CT^{1/2} \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \|\underline{u}\|_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))}$$

and then with (2.38)

$$\|u_1(t,0)\|^2_{L^2(0,T)} \leq C(T+1)\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + CT^{1/2}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \|\underline{u}\|_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))}. \tag{2.39}$$

- Finally, considering $q_j = x$ and using (2.35)–(2.37)–(2.39)

$$\|\partial_x \underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} \leq C(T+1)\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + CT^{1/2}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \|\underline{u}\|_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))}.$$

Using Young's inequality, we can find $C > 0$ which does not depend on $T > 0$ such that

$$\|\partial_x \underline{u}\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} \leq C(T+1)\left(\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{u}^0\|^4_{\mathbb{L}^2(\mathcal{T})} \right), \tag{2.40}$$

which concludes the proof of Theorem 1.3. □

To obtain a well-posedness result for the systems (KdV-S) and (LKdV-S) we can use the same idea presented in Theorem 1.3 and Lemma A.3 to take into account the saturation. It is very important that in Lemma A.3, time appears on the right-hand side; this estimate gives us the possibility of using small time in the fixed point approach. Then to derive the global-in-time well-posedness similar estimates to (2.35)–(2.40) can be obtained.

THEOREM 2.8. *Let* $(\ell_n)_{n=1,\dots,N} \in (0,\infty)^N$, $\alpha \geq \dfrac{N}{2}$, *and* $T > 0$. *Then, for all* $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$, *there exists a unique solution* $\underline{u} \in \mathbb{B}_T$ *of* (KdV-S) *or* (LKdV-S). *Moreover, there exist* $0 < T^* \leq T$ *and* $C > 0$ *such that* $\underline{u} \in \mathbb{Y}_{T^*}$ *and* $\|\underline{u}\|_{\mathbb{Y}_{T^*}} \leq C\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})}$.

3. Stabilization. In this section, we are going to prove our stabilization results inspired by [14]. The proofs are based on observability inequalities for (KdV-S) and (LKdV-S), respectively. These inequalities imply the exponential stability. First, note that, given $T > 0$, we can check that every solution of (KdV-S) and (LKdV-S) has a nonincreasing energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\left(\alpha - \frac{N}{2}\right)|u_1(t,0)|^2 - \frac{1}{2}\sum_{n=1}^{N}|\partial_x u_n(t,0)|^2 - \sum_{n=1}^{N}\int_0^{\ell_n} u_n \mathfrak{sat}(a_n u_n)dx \leq 0. \tag{3.1}$$

3.1. Stability of (KdV-S). We start by studying (KdV-S). First, note that multiplying (KdV-S) by u_n and integrating on $(0,s)\times(0,\ell_n)$ we get

$$\sum_{n=1}^{N}\int_0^{\ell_n}|u_n(s,x)|^2dx + \sum_{n=1}^{N}\int_0^s\int_0^{\ell_n}\mathfrak{sat}(a_nu_n)u_ndxdt + (2\alpha - N)\int_0^s |u_1(t,0)^2dt \\ + \sum_{n=1}^{N}\int_0^s|\partial_x u_n(t,0)|^2dt = \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})}.$$

Integrating again this expression with respect to time on $(0,T)$ we obtain

$$\begin{aligned} T\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq & \int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}dt + (2\alpha - N)T\int_0^T|u_1(t,0)|^2dt \\ & + T\sum_{n=1}^{N}\int_0^T|\partial_x u_n(t,0)|^2dt + \ T\sum_{n=1}^{N}\int_0^T\int_0^{\ell_n}\mathfrak{sat}(a_nu_n)u_ndxdt. \end{aligned} \tag{3.2}$$

Our goal here is to prove the following observability inequality:

$$\begin{aligned} \|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C\Bigg(& (2\alpha - N)\int_0^T|u_1(t,0)|^2dt + \sum_{n=1}^{N}\int_0^T|\partial_x u_n(t,0)|^2dt \\ & + \sum_{n=1}^{N}\int_0^T\int_0^{\ell_n}\mathfrak{sat}(a_nu_n)u_ndxdt\Bigg). \end{aligned} \tag{Obs}$$

Note that (Obs) is quite similar to (3.2). From (3.2) we can observe that to get (Obs) it is enough to prove the following inequality:

$$\begin{aligned} \int_0^T \|\underline{u}(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}dt \leq C\Bigg(& (2\alpha - N)\int_0^T|u_1(t,0)|^2dt + \sum_{n=1}^{N}\int_0^T|\partial_x u_n(t,0)|^2dt \\ & + \sum_{n=1}^{N}\int_0^T\int_0^{\ell_n}\mathfrak{sat}(a_nu_n)u_ndxdt\Bigg). \end{aligned}$$

Suppose that it is false and take $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \leq R$, then we can find $(\underline{u}^{0,j})_{j\in\mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ and

$$\lim_{j\to\infty}\frac{\|\underline{u}^j\|^2_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}}{(2\alpha - N)\|u_1^j(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x\underline{u}^j(\cdot,0)\|^2_{L^2(0,T)} + \sum_{n=1}^{N}\int_0^T\int_0^{\ell_n}\mathfrak{sat}(a_nu_n^j)u_n^jdxdt} = \infty,$$

where $\underline{u}^j$ is the corresponding solution of (KdV-S) with initial data $\underline{u}^{0,j}$. Note now that using (2.33), we deduce

$$\|\underline{u}^j(t,\cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq \|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq R. \tag{3.3}$$

Take $\lambda^j = \|\underline{u}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))}$, then $\lambda^j \leq T^{1/2}\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} \leq T^{1/2}R$. Thus $(\lambda^j)_{j\in\mathbb{N}} \subset \mathbb{R}$ is bounded. Taking $v_n^j = \frac{u_n^j}{\lambda^j}$, then $\underline{v}^j$ fulfills

$$\begin{cases} \left(\partial_t v_n^j + \partial_x v_n^j + \partial_x^3 v_n^j + \lambda^j v_n^j \partial_x v_n^j + \dfrac{\mathfrak{sat}(a_n\lambda^j v_n^j)}{\lambda^j}\right)(t,x) = 0 & \forall x \in (0,\ell_n),\ t>0, \\ & n = 1,\dots,N, \\ v_n^j(t,0) = v_{n'}^j(t,0) & \forall n,n' = 1,\dots,N, \\ \displaystyle\sum_{n=1}^N \partial_x^2 v_n^j(t,0) = -\alpha v_1^j(t,0) - \lambda^j \frac{N}{3}(v_1^j(t,0))^2, & t>0, \\ v_n^j(t,\ell_n) = \partial_x v_n^j(t,\ell_n) = 0, & t>0,\ n=1,\dots,N, \\ \|\underline{v}^j\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1, \end{cases} \tag{3.4}$$

and satisfies

$$(2\alpha - N)\|v_1^j(t,0)\|^2_{L^2(0,T)} + \|\partial_x \underline{v}^j(t,0)\|^2_{L^2(0,T)} + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j}\mathfrak{sat}(a_n\lambda^j v_n^j) v_n^j\, dxdt \to 0. \tag{3.5}$$

First, note that multiplying (3.4) by v_n^j and integrating on $(0,s)\times(0,\ell_n)$ we get

$$\begin{aligned} \sum_{n=1}^N \int_0^{\ell_n} |v_n^j(s,x)|^2 dx + \sum_{n=1}^N \int_0^s \int_0^{\ell_n} \frac{1}{\lambda^j}\mathfrak{sat}(a_n\lambda^j v_n^j) v_n^j\, dxdt + (2\alpha - N)\int_0^s |v_1^j(t,0)|^2 dt \\ + \sum_{n=1}^N \int_0^s |\partial_x v_n^j(t,0)|^2 dt = \|\underline{v}^j(0,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}, \end{aligned} \tag{3.6}$$

which gives us, using that $\mathfrak{sat}$ is odd,

$$\|\underline{v}^j\|^2_{C([0,T];\mathbb{L}^2(\mathcal{T}))} \leq \|\underline{v}^j(0,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}, \quad \|\partial_x \underline{v}^j(t,0)\|^2_{L^2(0,T)} \leq \|\underline{v}^j(0,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})}. \tag{3.7}$$

Now integrating (3.6) again with respect to time on $(0,T)$ we obtain

$$\begin{aligned} T\|\underline{v}^j(0,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} \leq \int_0^T \|\underline{v}^j(t,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} dt + (2\alpha - N)T\int_0^T |v_1^j(t,0)|^2 dt \\ + T\sum_{n=1}^N \int_0^T |\partial_x v_n^j(t,0)|^2 dt + 2T\sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j}\mathfrak{sat}(a_n\lambda^j v_n^j) v_n^j\, dxdt. \end{aligned} \tag{3.8}$$

This last inequality implies that $(\underline{v}^j(0,\cdot))_{j\in\mathbb{N}}$ is bounded in $\mathbb{L}^2(\mathcal{T})$. Again using that $\mathfrak{sat}$ is odd and similar estimates in (2.37)–(2.39)–(2.40) we conclude

$$\|\underline{v}^j\|^2_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))} \leq C\left(\|\underline{v}^j(0,\cdot)\|^2_{\mathbb{L}^2(\mathcal{T})} + \|\underline{v}^j(0,\cdot)\|^4_{\mathbb{L}^2(\mathcal{T})}\right). \tag{3.9}$$

Thus $(\underline{v}^j)_{j\in\mathbb{N}} \subset L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))$ is bounded and it holds that

$$\|v_n^j \partial_x v_n^j\|_{L^2(0,T;L^1(0,\ell_n))} \leq \|\underline{v}^j\|_{C([0,T],\mathbb{L}^2(\mathcal{T}))}\|\underline{v}^j\|_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))},$$

which implies that $(v_n^j \partial_x v_n^j)_{j\in\mathbb{N}}$ is a subset of $L^2(0,T;L^1(0,\ell_n))$. Using Lemma A.1 we have

$$\left\| \frac{\mathfrak{sat}(a_n \lambda^j v_n^j)}{\lambda^j} \right\|_{L^2(0,T;L^2(0,\ell_n))} \leq 3\|a_n\|_{L^\infty(0,\ell_n)} \ell_n^{1/2} \|\underline{v}^j\|_{L^2(0,T;\mathbb{H}_e^1(\mathcal{T}))},$$

and then $(\frac{\mathfrak{sat}(a_n\lambda^j v_n^j)}{\lambda^j})_{j\in\mathbb{N}}$ is a subset of $L^2(0,T;L^2(0,\ell_n))$. From this, we can see that $\partial_t v_n^j = -(\partial_x^3 v_n^j + \partial_x v_n^j + \lambda^j v_n^j \partial_x v_n^j + \frac{\mathfrak{sat}(a_n\lambda^j v_n^j)}{\lambda^j})$ is bounded in $L^2(0,T;H^{-2}(0,\ell_n))$. Hence, by the Aubin–Lions lemma ([24, Chapter III, Proposition 1.3]) we can deduce that $(\underline{v}^j)_{j\in\mathbb{N}}$ is relatively compact in $L^2(0,T;\mathbb{L}^2(\mathcal{T}))$ and we can assume that $\underline{v}^j$ converges strongly at $\underline{v}$ in $L^2(0,T;\mathbb{L}^2(\mathcal{T}))$ with $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$. Now we are going to study the case $\mathfrak{sat} = \mathfrak{sat}_2$ and $\mathfrak{sat} = \mathfrak{sat}_{\mathrm{loc}}$ separately.

3.1.1. Case $\mathfrak{sat} = \mathfrak{sat}_2$. First, we consider the case $\mathfrak{sat} = \mathfrak{sat}_2$. We know that by (3.3), $\|\underline{u}^j(t,\cdot)\|_{\mathbb{L}^2(\mathcal{T})} \leq R$ and then by Lemma A.2 we have that

$$0 \leq \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dx dt \leq \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_2(a_n \lambda^j v_n^j) v_n^j,$$

which gives us using (3.5), as $j \to \infty$,

$$(2\alpha - N)\|v_1^j(t,0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{v}^j(t,0)\|_{L^2(0,T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R)|v_n^j|^2 dx dt \to 0. \tag{3.10}$$

Furthermore, passing to the limit in (3.10) we get

$$\begin{aligned} &(2\alpha - N)\|v_1(t,0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{v}(t,0)\|_{L^2(0,T)}^2 + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R)|v_n|^2 dx dt \\ &\quad \leq \liminf \Bigg((2\alpha - N)\|v_1^j(t,0)\|_{L^2(0,T)}^2 + \|\partial_x \underline{v}^j(t,0)\|_{L^2(0,T)}^2 \\ &\qquad\qquad + \sum_{n=1}^N \int_0^T \int_0^{\ell_n} a_n k_n(R)|v_n^j|^2 dx dt \Bigg) = 0. \end{aligned}$$

Thus, $v_n(t,x) = 0$ in $(0,T)\times\omega_n$ and $(2\alpha - N)v_1(t,0) = \partial_x v_n(t,0) = 0$ in $(0,T)$ for all $n = 1,\dots,N$. Additionally, as $(\lambda^j)_{j\in\mathbb{N}}$ is bounded and nonnegative, we can extract a convergent subsequence such that $\lambda^j \to \lambda \geq 0$, consequently $\underline{v}$ satisfies $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$ and solves the following system:

$$\begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n + \lambda v_n \partial_x v_n = 0 & \forall x \in (0,\ell_n),\ t > 0,\ n = 1,\dots,N, \\ v_n(t,\ell_n) = \partial_x v_n(t,\ell_n) = \partial_x v_n(t,0) = 0, & t \in (0,T)\ \forall n = 1,\dots,N, \\ (2\alpha - N)v_n(t,0) = 0, & t\in(0,T), \\ v_n(t,x) = 0, & (t,x) \in (0,T)\times\omega_n. \end{cases} \tag{3.11}$$

1. If $\lambda = 0$ the system satisfied by $\underline{v}$ is linear, then we can use Holmgrem's theorem as in [18] to conclude that $\underline{v} = 0$, which contradicts the fact that $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$.

2. If $\lambda > 0$, we have to prove that $v_n \in L^2(0,T;H^3(0,\ell_n))$ in order to apply [23, Theorem 4.2]. Consider $w_n = \partial_t v_n$ then

$$\begin{cases} \partial_t w_n + \partial_x w_n + \partial_x^3 w_n + \lambda w_n \partial_x v_n + \lambda v_n \partial_x w_n = 0 \\ \hfill \forall x \in (0,\ell_n),\ t>0,\ n=1,\dots,N, \\ w_n(t,\ell_n) = \partial_x w_n(t,\ell_n) = \partial_x w_n(t,0) = 0 \\ \hfill \forall n = 1,\dots,N, \\ (2\alpha - N) w_n(t,0) = 0, \\ \hfill t \in (0,T)\ \forall j = 1,\dots,N, \\ w_n(t,x) = 0, \\ \hfill (t,x) \in (0,T)\times\omega_n, \\ w_n(0,x) = -v_n'(0,x) - v_n'''(0,x) - \lambda v_n(0,x) v_n'(0,x) \in H^{-3}(0,\ell_n), \\ \hfill x \in (0,\ell_n),\ j=1,\dots,N. \end{cases}$$

With [9, Lemma A.2] we can get that $w_n(0,x) \in L^2(0,\ell_n)$ and $w_n \in C([0,T], L^2(0,\ell_n)) \cap L^2(0,T;H^1(0,\ell_n))$. Thus, $\partial_x^3 v_n = -(\partial_t v_n - \partial_x v_n - \lambda v_n \partial_x v_n) \in L^2(0,T;L^2(0,\ell_n))$ which implies $v_n \in L^2(0,T;H^3(0,\ell_n))$. Applying [23, Theorem 4.2] we obtain that $v_n = 0$ for all $j = 1,\dots,N$ that contradicts the fact that $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$.

3.1.2. Case $\mathfrak{sat} = \mathfrak{sat}_{\texttt{loc}}$. Let us consider the case where $\mathfrak{sat} = \mathfrak{sat}_{\texttt{loc}}$, by the injection of $H^1(0,\ell_n)$ into $C([0,\ell_n])$, we can derive using similar estimate as in (3.9),

$$\int_0^T |u_n^j(t,x)|^2 dt \leq \ell_n \|\underline{u}^j\|^2_{L^2(0,T;\mathbb{H}^1_e(\mathcal{T}))} \leq \ell_n \beta \tag{3.12}$$

for $\beta = (R^2 + R^4)$. Now, inspired by [14], take $\Omega_{n,i} \subset [0,T]$ defined as follows:

$$\Omega_{n,i} = \left\{ t \in [0,T],\ \sup_{x\in[0,\ell_n]} |u_n(t,x)| > i \right\}. \tag{3.13}$$

Then denote $\Omega_{n,i}^c$ as the complement of $\Omega_{n,i}$ and observe that

$$\int_0^T \sup_{x\in[0,\ell_n]} |u_n^j(t,x)|^2 dt \geq \int_{\Omega_{n,i}} \sup_{x\in[0,\ell_n]} |u_n^j(t,x)|^2 dt \geq i^2 \nu(\Omega_{n,i})$$

for $\nu(\Omega_{n,i})$ the Lebesgue measure of $\Omega_{n,i}$. Thus, using (3.12) we obtain $\nu(\Omega_{n,i}) \leq \dfrac{\ell_n \beta}{i^2}$. Hence,

$$\max\left(T - \frac{\ell_n\beta}{i^2}, 0\right) \leq \nu(\Omega_{n,i}^c) \leq T. \tag{3.14}$$

Now using Lemma A.2

$$\begin{aligned} \sum_{n=1}^N \int_0^T \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\texttt{loc}}(a_n \lambda^j v_n^j) v_n^j dxdt &= \sum_{n=1}^N \int_{\Omega_{n,i}} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\texttt{loc}}(a_n \lambda^j v_n^j) v_n^j dxdt \\ &\quad + \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\texttt{loc}}(a_n \lambda^j v_n^j) v_n^j \\ &\geq \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} \frac{1}{\lambda^j} \mathfrak{sat}_{\texttt{loc}}(a_n \lambda^j v_n^j) v_n^j \\ &\geq \sum_{n=1}^N \int_{\Omega_{n,i}^c} \int_0^{\ell_n} a_n k_n(R) |v_n^j|^2 dxdt, \end{aligned}$$

which gives us, using (3.5),

$$(2\alpha - N)\|v_1^j(t,0)\|^2_{L^2(0,T)} + \|\partial_x \underline{v}^j(t,0)\|^2_{L^2(0,T)} + \sum_{n=1}^{N} \int_{\Omega^c_{n,i}} \int_0^{\ell_n} a_n k_n(R)|v_n^j|^2 dxdt \to 0. \tag{3.15}$$

Thus, the limit function v satisfies $(2\alpha - N)v_1(t,0) = \partial_x v_n(t,0) = 0$ in $(0,T)$ for all $n = 1,\dots,N$ and $v_n(t,x) = 0$ in $\cup_{i\in\mathbb{N}}\Omega^c_{n,i} \times \omega_n$. Using (3.14), we know that $\nu(\cup_{i\in\mathbb{N}}\Omega^c_{n,i}) = T$, thus we get that, for almost every $t \in [0,T]$, $v_n(t,x) = 0$ for $x \in \omega_n$. Last $\underline{v}$ is a solution to (3.11) and we conclude as we do in the case $\mathfrak{sat} = \mathfrak{sat}_2$.

Finally, we obtain that (Obs) is valid for a solution (KdV-S) with $\|u_n^n\|_{\mathbb{L}^2(\mathcal{T})} \le R$.

***Proof of Theorem* 1.4**. The proof closely follows [16] (see also [14]). Note that for $\underline{u}^0 \in \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^0\|_{\mathbb{L}^2(\mathcal{T})} \le R$ using that the energy is nonincreasing using (3.1) and (Obs) we argue the existence of $C = C(R) > 0$ such that.

$$E(T) \le \gamma E(0) \quad \text{with } \gamma = \frac{C}{1+C} < 1. \tag{3.16}$$

Now as the system is invariant by translation in time, we can repeat this argument on $[(m-1)T, mT]$ for $m = 1, 2, \dots$ to obtain

$$E(mT) \le \gamma E((m-1)T) \le \cdots \le \gamma^m E(0).$$

Hence we have $E(mT) \le e^{-\mu m T}E(0)$, where $\mu = \frac{1}{T}\ln(\frac{1}{\gamma}) > 0$. Let $t > 0$ then there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \le mT$, and then again using the nonincreasing property of the energy we get

$$E(t) \le E((m-1)T) \le e^{-\mu(m-1)T}E(0) \le \frac{1}{\gamma}e^{-\mu t}E(0).$$

This concludes the proof of Theorem 1.4. ☐

3.2. Stability (LKdV-S). Now we study the stabilization of (LKdV-S). For doing that, we follow the approach of section 3.1, and we prove the following observability inequality

$$\begin{aligned}\|\underline{u}^0\|^2_{\mathbb{L}^2(\mathcal{T})} \le C \Bigg(&(2\alpha - N)\int_0^T |u_1(t,0)|^2 dt + \sum_{j=1}^{N}\int_0^T |\partial_x u_n(t,0)|^2 dt \\ &+ \sum_{j\in I_c^*}\int_0^T\int_0^{\ell_n} \mathfrak{sat}(a_n u_n)u_n dxdt\Bigg)\end{aligned} \tag{Obs2}$$

for any solution $\underline{u}$ of (LKdV-S). Suppose that it is false, then there exists a sequence $(\underline{u}^{0,j})_{j\in\mathbb{N}} \subset \mathbb{L}^2(\mathcal{T})$ such that $\|\underline{u}^{0,j}\|_{\mathbb{L}^2(\mathcal{T})} = 1$ and the corresponding solution of (LKdV-S) satisfies

$$(2\alpha - N)\|u_1^j(\cdot,0)\|^2_{L^2(0,T)} + \|\partial_x \underline{u}^j(\cdot,0)\|^2_{L^2(0,T)} + \sum_{n\in I_c^*}\int_0^T\int_0^{\ell_n} \mathfrak{sat}(a_n u_n^j)u_n^j dxdt \to 0,$$

as $j \to \infty$. Using the same arguments as in Theorem 1.4 we can find a nontrivial solution $\underline{v} \in \mathbb{B}_T$ of (LKdV-S) such that

$$\begin{cases} (2\alpha - N)\,\|v_1(\cdot,0)\|_{L^2(0,T)} = 0, \\ \|\partial_x \underline{v}(\cdot,0)\|_{L^2(0,T)} = 0, \\ v_n = 0 \quad \text{in } (0,T)\times\omega_n, \ n \in I_c^*, \\ \|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1. \end{cases}$$

We distinguish three cases:

- For $n \in I_c^*$, $v_n = 0$ in $(0,T)\times\omega_n$. Then, $\partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0$ and thanks to Holmgrem's theorem, $v_n = 0$ for all $n \in I_c^*$. Note that this implies that $v_n(t,0) = 0$ for all $n \in I_c^*$ and by the continuity condition $v_n(t,0) = 0$ for all $n = 1,\dots,N$.
- For $n \in \{1,\dots,N\}\backslash I_c$, v_n is the solution to

$$\begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0, & x\in(0,\ell_n),\ t\in(0,T),\ n=1,\dots,N, \\ v_n(t,0) = 0, & t\in(0,T)\ \forall j=1,\dots,N, \\ \sum_{n=1}^N \partial_x^2 v_n(t,0) = 0, & t\in(0,T), \\ v_n(t,\ell_n) = \partial_x v_n(t,\ell_n) = 0, & t\in(0,T),\ n=1,\dots,N, \\ v_n(0,x) = v_n^0, & x\in(0,\ell_n). \end{cases}$$

  Then thanks to [1, Lemma 3.2], $v_n = 0$.
- For $n \in I_c\backslash I_c^*$, v_n then satisfies

$$\begin{cases} \partial_t v_n + \partial_x v_n + \partial_x^3 v_n = 0, & t\in(0,T)\ \forall x\in(0,\ell_n), \\ v_n(t,0) = \partial_x v_n(t,0) = \partial_x^2 v_n(t,0) = 0, & t\in(0,T), \\ v_n(t,\ell_n) = \partial_x v_n(t,\ell_n) = 0, & t\in(0,T), \\ v_n(0,x) = v_n^0, & x\in(0,\ell_n). \end{cases}$$

  Due to the three null conditions at the central node, we obtain that $v_n = 0$.

Thus $\underline{v} = 0$ and we get a contradiction, with $\|\underline{v}\|_{L^2(0,T;\mathbb{L}^2(\mathcal{T}))} = 1$ which ends the proof of (Obs2). As we have the observability inequality (Obs2), to derive the exponential decay of the energy of (LKdV-S) given in Theorem 1.5, it is enough to follow the proof of Theorem 1.4.

4. Conclusions and remarks. In this paper, the global well-posedness was studied and the exponential stability of a KdV equation on a star-shaped network with internal saturated feedback terms has been established. The well-posedness was addressed using the Laplace transform of the linearization and obtaining Kato smoothing properties which gave the local-in-time well-posedness, then using multiplier estimates the global-in-time result was deal with.

4.1. Generalization of the well-posedness result. In the work [7] a complete result for general linear boundary conditions for the KdV equation on a bounded domain was derived. In this work, homogeneous Dirichlet and Neumann right conditions ($u_n(t,\ell_n) = \partial_x u_n(t,\ell_n) = 0$) were considered. These conditions come from the problems studied in [1, 16], but in a more general framework the following problem could be studied:

(4.1)
$$\begin{cases} (\partial_t u_n + \partial_x u_n + u_n\partial_x u_n + \partial_x^3 u_n)(t,x) = f_n(t,x), & \forall x\in(0,\ell_n),\ t>0,\ n=1,\dots,N, \\ u_n(t,0) = u_{n'}(t,0) & \forall n,n'=1,\dots,N, \\ \displaystyle\sum_{n=1}^N \partial_x^2 u_n(t,0) = -\alpha u_1(t,0) - \frac{N}{3}u_1^2(t,0) + h(t), & t>0, \\ u_n(t,\ell_n) = g_n(t),\quad \partial_x u_n(t,\ell_n) = p_n(t), & t>0,\ n=1,\dots,N, \\ u_n(0,x) = u_n^0, & x\in(0,\ell_n). \end{cases}$$

We expected that adapting the ideas introduced in this paper and in [3], it could be possible to obtain the following result.

Conjecture 4.1. *Let* $(\ell_n)_{n=1,\dots,N} \in (0,\infty)^N$, $0 \leq s \leq 3$, *and* $T > 0$. *There exists* $0 < T^* \leq T$ *such that for all*

$$\underline{u}^0 \in \prod_{n=1}^N H^s(0,\ell_n), \quad (h,\underline{g},\underline{p}) \in H^{\frac{s-1}{3}}(0,T) \times \prod_{n=1}^N H^{\frac{s+1}{3}}(0,T) \times \prod_{n=1}^N H^{\frac{s}{3}}(0,T),$$
$$\underline{f} \in \prod_{n=1}^N W^{\frac{s}{3},1}(0,T;L^2(0,\ell_n)),$$

satisfying the compatibility condition,

$$\begin{cases} u_n^0(\ell_n) = g_n(0) & n=1,\dots,N \quad \text{if } \frac{1}{2} < s \leq 3, \\ \partial_x u_n^0(\ell_n) = p_n(0) & n=1,\dots,N \quad \text{if } \frac{3}{2} < s \leq 3, \\ \sum_{n=1}^N \partial_x^2 u_n^0(0) = h(0) \quad \text{if } \frac{5}{2} < s \leq 3, \end{cases}$$

there exists a unique solution $\underline{u} \in \prod_{n=1}^N C([0,T];H^s(0,\ell_n)) \cap L^2(0,T^*;H^{s+1}(0,\ell_n))$ *of* (4.1). *Moreover* $\partial_x^\kappa u_n \in L_x^\infty(0,\ell_n;H^{\frac{s+1-\kappa}{3}}(0,T^*))$ *for* $\kappa = 0,1,2$.

The complications would come from the study of the matrix, which is obtained by replacing the column $j+3(n-1)$ of A_N by $[0\ 1\ 0\cdots 0]^T$ for the g_n case and $[0\ 0\ 1\cdots 0]^T$ for the p_n case. It is not clear how to derive a result similar to (2.23).

4.2. Exact controllability in the network. In the paper [1] the exact controllability of linearization around 0 of (KdV-N) was achieved by acting with $N+1$ boundary controls (N controls in the external nodes and one in the central node) if $\#\{\ell_n \in \mathcal{N}\} \leq 1$. Recently, in [10] the authors could reduce the numbers of controls (N controls acting on the external nodes), but the controllability holds for a large time and small lengths. This raises the question of what happens for the boundary control and how many components corresponding to the critical lengths one needs to control in the network case. In particular, we can mention the following open problems:

- Is the linearization around 0 of (KdV-N) exactly controllable with N controls acting in the external nodes for $T > 0$ and $\ell_n \notin \mathcal{N}$ for all $n \in \{1,\dots,N\}$?
- Is (KdV-N) exactly controllable from the boundary in the case where for some lengths we have $\ell_n \in \mathcal{N}$? A starting point could be, to consider the smallest critical lengths ($k=l=1$ or $k=l=2$).

4.3. Generalization of stabilization results. The stabilization results were obtained, proving appropriate observability inequalities working directly on the nonlinear systems. In the work [19] more general feedback laws were considered as cone bounded control laws. Note that Theorems 1.4 and 1.5 hold, replacing $\mathfrak{sat}$ by any odd nonlinearity that satisfies the properties given in Lemmas A.1, A.2, and A.3.

Appendix A. Useful lemmas. In this section, we present some technical lemmas about the regularity and sector condition of the saturation maps $\mathfrak{sat}$. Let $a : [0,L] \to \mathbb{R}$ such that

$$a^* \geq a \geq a_* > 0 \text{ in an open nonempty set } \omega \text{ of } (0,L). \tag{A.1}$$

Lemma A.1 (Lemma 3.2 of [14]). *For all* $(f, \widetilde{f}) \in L^2(0, L)$*, we have*

$$\|\mathfrak{sat}(f) - \mathfrak{sat}(\widetilde{f})\|_{L^2(0,L)} \leq 3\|f - \widetilde{f}\|_{L^2(0,L)}. \tag{A.2}$$

Lemma A.2 (Lemma 4.3 of [14]). *Let* r *be a positive value and* $a : [0, L] \to \mathbb{R}$ *be a function satisfying* (A.1) *and* $k(r)$ *defined by*

$$k(r) = \min\left\{\frac{M}{a_* r}, 1\right\}: \tag{A.3}$$

1. *Given* $\mathfrak{sat} = \mathfrak{sat}_2$ *and* $f \in L^2(0, L)$ *such that* $\|f\|_{L^2(0,L)} \leq r$*, we have*

$$(\mathfrak{sat}_2(a(x)f(x)) - k(r)a(x)f(x))f(x) \geq 0 \ \forall x \in [0, L]. \tag{A.4}$$

2. *Given* $\mathfrak{sat} = \mathfrak{sat}_{loc}$ *and* $f \in L^\infty(0, L)$ *such that* $\forall x \in [0, L]$*,* $|f(x)| \leq r$*, we have*

$$(\mathfrak{sat}_{loc}(a(x)f(x)) - k(r)a(x)f(x))f(x) \geq 0 \ \forall x \in [0, L]. \tag{A.5}$$

Lemma A.3 (Proposition 3.4 of [14]). *Let* $a : [0, L] \to \mathbb{R}$ *satisfy* (A.1)*. If* $y \in L^2(0, T; H^1(0, L))$*, then* $\mathfrak{sat}(ay) \in L^1(0, T; L^2(0, L))$ *is continuous and* $\forall y, z \in L^2(0, T; H^1(0, L))$ *we have*

$$\|\mathfrak{sat}(ay) - \mathfrak{sat}(az)\|_{L^1(0,T;L^2(0,L))} \leq 3L^{1/2}T^{1/2}a^*\|y - z\|_{L^2(0,T;H^1(0,L))}.$$

Appendix B. For all $s \neq 0$ with $\mathrm{Re}(s) \geq 0$, it holds that $\Delta^1(s) \neq 0$.

This property was stated in [7, Remark 2.5] without proof; here, for the sake of completeness, we give a proof based on [6]. Suppose that $\Delta^1(s) = 0$ for some s with $\mathrm{Re}(s) \geq 0$. Then, there exists $f \in H^3(0, \ell_1)$, a nontrivial solution of

$$\begin{cases} sf(x) + f'''(x) = 0, & x \in (0, \ell_1), \\ f''(0) = f'(\ell_1) = f(\ell_1) = 0. \end{cases} \tag{B.1}$$

Now, consider the conjugate of (B.1):

$$\begin{cases} \overline{sf(x)} + \overline{f'''(x)} = 0, & x \in (0, \ell_1), \\ \overline{f''(0)} = \overline{f'(\ell_1)} = \overline{f(\ell_1)} = 0. \end{cases} \tag{B.2}$$

Multiplying (B.1) by $\overline{f}$, integrating over $(0, \ell_1)$, and performing integration by parts, we get

$$s\int_0^{\ell_1} |f|^2 dx - \int_0^{\ell_1} f\overline{f'''} dx + |f'(0)|^2 = 0. \tag{B.3}$$

Similarly, multiplying (B.2) by f and integrating over $(0, \ell_1)$, we get

$$\overline{s}\int_0^{\ell_1} |f|^2 dx + \int_0^{\ell_1} \overline{f'''} f dx = 0. \tag{B.4}$$

Then adding (B.3) and (B.4) yields

$$2\mathrm{Re}(s)\int_0^{\ell_1} |f|^2 dx = -|f'(0)|^2. \tag{B.5}$$

As f is nontrivial and $\mathrm{Re}(s) \geq 0$, we get $f'(0) = 0$. Then, by (B.5) $\mathrm{Re}(s) = 0$. Thus, we can make the change of variable $s = i\rho^3$ for $\rho \in \mathbb{R}$. Multiplying (B.1) by $x\overline{f}$, integrating over $(0, \ell_1)$, and performing integration by parts, we get

$$i\rho^3 \int_0^{\ell_j} x|f|^2 dx + 3\int_0^{\ell_j} |f'|^2 dx - \int_0^{\ell_j} xf\overline{f'''}dx = 0. \tag{B.6}$$

Similarly, multiplying (B.2) by xf and integrating over $(0, \ell_j)$, we get

$$-i\rho^3 \int_0^{\ell_j} x|f|^2 dx + \int_0^{\ell_j} x\overline{f'''}f dx = 0. \tag{B.7}$$

Then, adding (B.6) and (B.7), we obtain $f' \equiv 0$. Using the boundary conditions of (B.1) we deduce $f \equiv 0$ which is a contradiction. Finally $f \equiv 0$ and $\Delta^1(s) \neq 0$ for all $s \neq 0$ with $\mathrm{Re}(s) \geq 0$.

Appendix C. For all $\rho>0$ and $j\in\{1,\dots,N\}$, it holds that $\det(D_j)\neq 0$. Let $j \in \{1, \dots, N\}$. Following [6] and Appendix B, suppose that $\det(D_j) = 0$ for some $\rho > 0$. Then, there exists $f \in H^3(0, \ell_j)$, a nontrivial solution of

$$\begin{cases} i\rho^3 f(x) + f'''(x) = 0, & x \in (0, \ell_j), \\ f(0) = f(\ell_j) = f'(\ell_j) = 0. \end{cases} \tag{C.1}$$

Now, consider the conjugate of (C.1),

$$\begin{cases} -i\rho^3\overline{f(x)} + \overline{f'''(x)} = 0, & x \in (0, \ell_j), \\ \overline{f(0)} = \overline{f(\ell_j)} = \overline{f'(\ell_j)} = 0. \end{cases} \tag{C.2}$$

Multiplying (C.1) by $\overline{f}$, integrating over $(0, \ell_j)$, and performing integration by parts, we get

$$i\rho^3 \int_0^{\ell_j} |f|^2 dx - \int_0^{\ell_j} f\overline{f'''}dx + |f'(0)|^2 = 0. \tag{C.3}$$

Similarly, multiplying (C.2) by f and integrating over $(0, \ell_j)$, we get

$$-i\rho^3 \int_0^{\ell_j} |f|^2 dx + \int_0^{\ell_j} \overline{f'''}f dx = 0. \tag{C.4}$$

Then, adding (C.3) and (C.4) yields $f'(0) = 0$. Multiplying (C.1) by $x\overline{f}$, integrating over $(0, \ell_j)$, and performing integration by parts, we get

$$i\rho^3 \int_0^{\ell_j} x|f|^2 dx + 3\int_0^{\ell_j} |f'|^2 dx - \int_0^{\ell_j} xf\overline{f'''}dx = 0. \tag{C.5}$$

Similarly, multiplying (C.2) by xf and integrating over $(0, \ell_j)$, we get

$$-i\rho^3 \int_0^{\ell_j} x|f|^2 dx + \int_0^{\ell_j} x\overline{f'''}f dx = 0. \tag{C.6}$$

Then, adding (C.5) and (C.6), we obtain $f' \equiv 0$. Using the boundary conditions of (C.1) we deduce $f \equiv 0$ which is a contradiction. Hence, $\det(D_j) \neq 0$ for all $\rho > 0$.

Appendix D. For all $\rho > 0$, it holds that $\sum_{j=1}^{N} \frac{\det(F_j)}{\det(D_j)} \neq 0$. Letting $j \in \{1, \ldots, N\}$, we are going to show that $\mathrm{Re}\left(\frac{\det(F_j)}{\det(D_j)}\right) < 0$. Using (2.20) and (2.21) we get

$$\frac{\det(F_j)}{\det(D_j)} = \frac{\sqrt{3}\rho^3\left(e^{-i\rho\ell_j} + e^{-\frac{1}{2}\rho(\sqrt{3}-i)\ell_j} + e^{-\frac{1}{2}\rho(-\sqrt{3}-i)\ell_j}\right)}{\sqrt{3}\rho\left(e^{-i\rho\ell_j} + \left(-\frac{1}{2} - \frac{\sqrt{\sqrt{3}}}{2}i\right)e^{\left(-\frac{\rho\sqrt{3}}{2}+i\frac{\rho}{2}\right)\ell_j} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)e^{\left(\frac{\rho\sqrt{3}}{2}+i\frac{\rho}{2}\right)\ell_j}\right)}.$$

$$= \frac{\rho^2\left(e^{-i\ell_j\rho} + 2e^{\frac{i\ell_j\rho}{2}}\cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right)\right)}{e^{-i\ell_j} - e^{\frac{i\ell_j\rho}{2}}\cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) + \sqrt{3}ie^{\frac{i\ell_j\rho}{2}}\sinh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right)}.$$

After some algebraic manipulations and writing the complex numbers in their binomial form ($\mathrm{Re} + i\mathrm{Im}$), we obtain

$$\frac{\det(F_j)}{\det(D_j)} = \frac{\rho^2\left(\cos\left(\frac{3\ell_j\rho}{2}\right) + 2\cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) - i\sin\left(\frac{3\ell_j\rho}{2}\right)\right)}{\cos\left(\frac{3\ell_j\rho}{2}\right) - \cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) + i\left(\sqrt{3}\sinh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) - \sin\left(\frac{3\ell_j\rho}{2}\right)\right)}.$$

Letting $\zeta = \cos\left(\frac{3\ell_j\rho}{2}\right) - \cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) + i\left(\sqrt{3}\sinh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) - \sin\left(\frac{3\ell_j\rho}{2}\right)\right)$, and multiplying the previous equation by $\frac{\bar{\zeta}}{\bar{\zeta}}$ we get

$$\mathrm{Re}\left(\frac{\det(F_j)}{\det(D_j)}\right) = \frac{\rho^2}{|\zeta|^2}\left(1 + \cos\left(\frac{3\ell_j\rho}{2}\right)\cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) - 2\cosh^2\left(\frac{\sqrt{3}\ell_j\rho}{2}\right)\right.$$
$$\left. -\sqrt{3}\sin\left(\frac{3\ell_j\rho}{2}\right)\sinh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right)\right).$$

By analyzing the function

$$F(\rho, \ell_j) = 1 + \cos\left(\frac{3\ell_j\rho}{2}\right)\cosh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right) - 2\cosh^2\left(\frac{\sqrt{3}\ell_j\rho}{2}\right)$$
$$- \sqrt{3}\sin\left(\frac{3\ell_j\rho}{2}\right)\sinh\left(\frac{\sqrt{3}\ell_j\rho}{2}\right),$$

it can be shown that for all $\rho > 0$, $\ell_j > 0$ it holds that $F(\rho, \ell_j) < 0$. Thus, $\mathrm{Re}(\sum_{j=1}^{N} \frac{\det(F_j)}{\det(D_j)}) < 0$, and thus $\sum_{j=1}^{N} \frac{\det(F_j)}{\det(D_j)} \neq 0$.

Remark 4. In the case $\ell_1 = \cdots = \ell_N$, the proof become easier. In fact,

$$\sum_{j=1}^{N} \frac{\det(F_j)}{\det(D_j)} = N\frac{\det(F_1)}{\det(D_1)} \neq 0$$

because, $\det(F_1) = \Delta^{1,+} \neq 0$ thanks to Appendix B. ○

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