OUTPUT FEEDBACK CONTROL OF A CASCADE SYSTEM OF LINEAR KORTEWEG-DE VRIES EQUATIONS*

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Abstract. This paper is about the stabilization of a cascade system of n linear Korteweg-de Vries equations in a bounded interval. It considers an output feedback control placed at the left endpoint of the last equation, while the output involves only the solution to the first equation. The boundary control problems investigated include two cases: a classical control on the Dirichlet boundary condition and a less standard one on its second-order derivative. The feedback control law utilizes the estimated solutions of a high-gain observer system and the output feedback control leads to stabilization for any n for the first boundary conditions case and for $n = 2$ for the second one.

Key words. Korteweg-de Vries equation, cascade systems, output feedback control

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction. In this paper, we study the following cascade system of n linear Korteweg-de Vries (KdV for short) equations posed in a bounded interval of length L

$$v_t + v_x + v_{xxx} = (A_n - B)v, \text{ in } (0,\infty)\times(0,L), \tag{1.1}$$

where $v = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^\top$ is the state and

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}, \qquad B = \operatorname{diag}(1,1,\dots,1,-1).$$

Let us consider two different types of boundary conditions, where the input control u in both of them is placed on the left side and only acts on the n-th coordinate of the state.

Boundary conditions A (BC-A):

$$\begin{aligned} v_i(t,0) &= 0, i = 1,\dots,n-1, \text{ for all } t>0, \\ v_n(t,0) &= u(t), \text{ for all } t>0, \\ v(t,L) &= 0, v_x(t,L) = 0, \text{ for all } t>0. \end{aligned} \tag{1.2a}$$

*Submitted to the editors 27/05/2021.

Funding: This work has been partially supported by ECOS-CONICYT C16E06, Fondecyt grant 1180528, ANID-Basal Project FB0008 AC3E.

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Boundary conditions B (BC-B):

$$
\begin{aligned}
v_{i,xx}(t,0) &= 0, i = 1, \ldots, n-1, \text{ for all } t > 0,\\
v_{n,xx}(t,0) &= u(t), \text{ for all } t > 0,\\
v(t,L) = 0, v_x(t,L) &= 0, \text{ for all } t > 0.
\end{aligned}
\tag{1.2b}
$$

In order to complete our control system, we add an initial condition given by

$$v(0,x) = v^0(x), x \in (0,L) \tag{1.3}$$

and a distributed measurement given by the following output

$$
\begin{aligned}
y(t,x) =& Cv(t,x);\\
C =& \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.
\end{aligned}
\tag{1.4}
$$

The nonlinear version of a single KdV equation describes propagation of waters with small amplitude in closed channels. It was introduced in 1895 and since then its properties have gained much consideration, see for instance [6]. Surveys on recent progresses and open problems on control and stabilization of such models can be found in [26] and [3].

The aim of the present work is to stabilize the cascade system (1.1) considering any of the boundary control problems (BC-A) and (BC-B) and by utilizing the knowledge of the first state only, while the other states are estimated via an observer. Notice that this system is unstable due to the instability of the trajectory corresponding to its last equation, as it can be seen by following classical energy arguments. In the recent decades, stabilization of single KdV equations has gained significant interest, see for instance [4], where backstepping method is used for feedback controls placed on the left boundary, see also [29, 7, 27, 13]. Output feedback laws for single linearized and nonlinear KdV equations have been already established via boundary observers in [21, 23] (see also [28, 11, 14, 2]), by means of backstepping and Lyapunov techniques. In these two works, the measurement injected in the observer involves the right endpoint of the domain, more precisely, the second derivative of the boundary or the Dirichlet condition, depending on the boundary conditions. Output feedback control laws for systems written in the cascade form considered here have not yet appeared in the literature, while controllability of coupled KdV equations but with couplings, different from the ones studied here (particularly, internal couplings in one-order derivatives), describing strong interactions of weakly nonlinear long waves, have been investigated for instance in [5]. However, coupling in zero-order internal terms, with coupling coefficient A_n, might result from the linearization of coupled nonlinear KdV equations of some forms appearing in [20], describing oceanic and atmospehric phenomena, such as the atmospheric blockings, the interactions between the atmosphere and ocean, the oceanic circulations, and hurricanes (see system (27)-(28) and model 5, therein, according to the well-known Painlevé classification), see also the Hirota-Satsuma model [12] and [10] for multicomponent KdV equations (related to the weak nonlinear dispersion). For these systems, it is often difficult to control and observe directly all the equations. Also, general settings of coupled infinite-dimensional systems with couplings in zero-order terms, as the ones considered here, have been studied with respect to their controllability and observability properties, when considering reduced numbers of controls and observations, see [1], [19]. In those works, the authors have shown that the problem of control of underactuated systems with reduced number of observations is quite challenging. Furthermore, placing the control on the second derivative of the left boundary, as in the considered second boundary control problem is even more original and its investigation exhibits some technical difficulties, for which solutions are proposed in the present work. To the best of our knowledge, boundary control problems of this second type have not appeared in the literature.

Here, we aim at observing the full state of a system of KdV equations written in a cascade form and finally controlling it, by considering a single observation. Observer design for nonlinear systems of partial differential equations written in such a form, based on the well-known high-gain methodology, have been considered, for instance in [17, 16, 18], in the framework of first-order hyperbolic systems, extending results for finite-dimensional systems [15]. A similar form considered here, in its linearized version, allows an observer design, which relies on a choice of a sufficiently large parameter in its equations, while appropriate choice of the latter leads simultaneously to the closed-loop output feedback stabilization. In summary, the contribution of the present work first lies in stabilizing the trajectory of the last equation by means of an observer relying on the measurement of the first state only. The control placed on the left boundary, combined with the observer gain, brings this trajectory asymptotically to zero in an arbitrarily fast manner (first part of Theorem 3.2). Subsequently, it is proven that 1) the whole cascade system becomes asymptotically stable for any $L > 0$, when boundary condition (BC-A) is considered, and 2) this result holds for boundary condition (BC-B), only when $n = 2$ (number of equations), noting that for $n > 2$, stabilization is achieved for quite small L (last part of Theorem 3.2). The methodology relies on backstepping techniques and appropriate Lyapunov analysis. Exponential stabilization for (BC-B) is proven here to be linked to the solvability of an ordinary differential equations problem, similar to the differential equation satisfied by the eigenvectors of the associated differential operator to these KdV equations, and being subject to some constraints.

In Section 2 we prove a preliminary result on the stability of a single damped KdV equation and then we prove the full state stabilization of the cascade system for both boundary condition problems. In Section 3, we first present the observer design for the coupled system and finally the main output feedback stabilization result. In Section 4, we provide conclusions and some perspectives.

2. Full State Feedback Stabilization.

In this section, we study the full state feedback stabilization of system (1.1) for boundary control problems (BC-A) and (BC-B).

2.1. Stability of a single KdV equation.

Prior to the stabilization of the cascade system, we present a preliminary result about the stability of a single damped linear KdV equation, which will be invoked in the sequel. Consider a single KdV equation in the domain $(0, L)$

$$w_t + w_x + w_{xxx} + \lambda w = 0, \text{in } (0, \infty) \times (0, L), \tag{2.1}$$

satisfying one of the following distinct cases of boundary conditions

$$w(t, 0) = w(t, L) = w_x(t, L) = 0, t > 0, \tag{2.2a}$$

$$w_{xx}(t, 0) = w(t, L) = w_x(t, L) = 0, t > 0, \tag{2.2b}$$

and initial condition of the form

$$w(0, x) = w^0(x), x \in (0, L). \tag{2.3}$$

The stability result for solutions w to the above problem is presented in the following propositions. Although asymptotic stability assuming boundary conditions (2.2a) is ensured for every $\lambda > 0$, for (2.2b) asymptotic stability is guaranteed only when $\lambda \geq \lambda_0$ for some $\lambda_0 > 0$. These results are stated precisely in the next two propositions and will be used throughout this work.

PROPOSITION 2.1. *Consider system (2.1) with boundary conditions (2.2a) and initial condition $w^0 \in L^2(0,L)$. Then for all $\lambda > 0$, we have*

$$\|w(t,\cdot)\|_{L^2(0,L)} \leq e^{-\lambda t}\|w^0(\cdot)\|_{L^2(0,L)}, t \geq 0, \tag{2.4}$$

for every $L > 0$.

Proposition 2.1 concerning boundary conditions (2.2a) is a standard result and can be derived from energy estimates. Well-posedness of this equation is presented in Appendix B.1. Let us note here, that asymptotic stability for this case can be proven even when the damping is not constant in the domain but localized to a part of it, see for instance [24], and even when the damping is saturated, see [22].

To proceed to the stability result for boundary conditions (2.2b), we utilize the following lemma.

LEMMA 2.2. *There exists $\lambda_0 > 0$, such that the following assertions hold true.*

Assertion 1: For every $\lambda \geq \lambda_0$, there exist $\pi(\cdot)$ in $C^3(0,\infty)$ and $b > 0$, such that the following holds for all $x \geq 0$

$$\begin{cases} \pi'''(x) + \pi'(x) - 2\lambda\pi(x) = -2b\pi(x), \\ \pi''(0)\pi(0) + (\pi'(0))^2 + \pi^2(0) \leq 0, \\ \pi(x) > 0, \\ \pi'(x) \geq 0. \end{cases} \tag{2.5}$$

Assertion 2: For every $\lambda \in (0,\lambda_0)$, there exist $\bar{L}, b > 0$ and $\pi(\cdot)$ in $C^3(0,\infty)$ satisfying (2.5) for all $x \in [0,\bar{L}]$.

Proof. See Appendix A ∎

The following proposition concerns the second case of boundary conditions.

PROPOSITION 2.3. *Consider system (2.1) with boundary conditions (2.2b) and initial condition $w^0 \in L^2(0,L)$. Then, there exists $\lambda_0 > 0$, such that:*

1) For all $\lambda \geq \lambda_0$, there exist $a, b > 0$, such that the solution to (2.1)-(2.3)-(2.2b) satisfies the following:

$$\|w(t,\cdot)\|_{L^2(0,L)} \leq ae^{-bt}\|w^0(\cdot)\|_{L^2(0,L)}, t \geq 0, \tag{2.6}$$

for every $L > 0$.

2) For all $\lambda \in (0,\lambda_0)$, there exist $\bar{L}$, $a, b > 0$ such that (2.6) is satisfied for all $L \in (0,\bar{L}]$.

Proof. In this context, we are interested by unique solutions w beloging to $C\left(0,\infty; L^2(0,L)\right)$. Well-posedness of the initial boundary value problem (2.1)-(2.3)-(2.2b) can be easily proven by invoking classical arguments, although these boundary conditions are less common in the literature. More details about the well-posedness of such systems are presented in Appendix B.1.

To prove the stability result, let us consider the following weighted L^2-norm

$$E(t) := \int_0^L \pi(x)w^2(x)\mathrm{d}x$$

along the L^2 solutions to (2.1)-(2.3)-(2.2b), for some appropriate choice of positive $\pi(\cdot) \in C^3[0,L]$. Calculating its time-derivative and applying integrations by parts, we obtain

$$\begin{aligned}\dot{E}(t) = &\int_0^L \left(\pi'''(x) + \pi'(x) - 2\lambda\right) w^2(x)\mathrm{d}x - 3\int_0^L \pi'(x) w_x^2(x)\mathrm{d}x \\ &+ \left[(-\pi''(x) - \pi(x))\, w^2(x) - 2\pi(x) w_{xx}(x) w(x) + \pi(x) w_x^2(x)\right. \\ &\left.+2\pi'(x) w_x(x) w(x)\right]_0^L .\end{aligned}$$

Substituting boundary conditions (2.2b) we get

$$\begin{aligned}\dot{E}(t) = &\int_0^L \left(\pi'''(x) + \pi'(x) - 2\lambda\right) w^2(x)\mathrm{d}x - 3\int_0^L \pi'(x) w_x^2(x)\mathrm{d}x \\ &- \begin{pmatrix} w_x(0) & w(0)\end{pmatrix} \begin{pmatrix} \pi(0) & -\pi'(0) \\ -\pi'(0) & -\pi(0) - \pi''(0)\end{pmatrix} \begin{pmatrix} w_x(0) \\ w(0)\end{pmatrix}.\end{aligned}$$

To ensure the exponential decay of $E(t)$, we invoke assertions of Lemma 2.2 for $\pi(\cdot)$, for which we assume that it satisfies (2.5). By Assertions 1 and 2, the second integral and the third boundary term of the above equation become nonpositive and we obtain the existence of a constant $b > 0$, such that

$$\dot{E}(t) \leq -2bE(t)$$

and, therefore, (2.6) holds with $a = \sqrt{\frac{\pi(L)}{\pi(0)}}$. This completes the proof of Proposition 2.3. ∎

2.2. Full state stabilization. Following the previous results, we are in a position to study the closed-loop stabilization. Here, the considered state feedback controls, which are placed in the last equation, will be proven to be of the following form for each of the problems (BC-A) and (BC-B)

$$\text{(BC-A): } u(t) = \int_0^L p(0,y) v_n(t,y)\mathrm{d}y, \tag{2.7a}$$

$$\text{(BC-B): } u(t) = -\frac{\omega+1}{3} L v_n(t,0) + \int_0^L p_{xx}(0,y) v_n(t,y)\mathrm{d}y, \tag{2.7b}$$

with $\omega > 0$ to be chosen appropriately and kernel function $p : \Pi \to \mathbb{R}$ depending on ω, where $\Pi := \{(x,y); x \in [0,L], y \in [x,L]\}$.

We now present the exponential decay result of the solution v to the cascade system (1.1) via the control (2.7), which utilizes the full state. The proof uses backstepping tools appearing in [4], [3] for single KdV equations.

THEOREM 2.4. *Consider system (1.1) with boundary conditions (BC-A) or (BC-B), feedback control laws of the form (2.7a) or (2.7b), respectively, and initial condition $v^0 \in L^2(0,L)^n$.*

a) If (BC-A) holds and $n \geq 2$, then for every $L > 0$, there exist constants $c, d > 0$, such that the solution v to (1.1) satisfies the following

$$\|v\|_{L^2(0,L)^n} \leq c e^{-dt} \|v^0\|_{L^2(0,L)^n}, \forall t \geq 0. \tag{2.8}$$

b) If (BC-B) holds and $n = 2$, then for every $L > 0$, there exist constants $c, d > 0$, such that solution v to (1.1) satisfies (2.8).

c) If (BC-B) holds and $n > 2$, then there exists $\bar{L} > 0$, such that (2.8) is guaranteed for all $L \in (0, \bar{L}]$.

Proof. The well-posedness for controlled system (1.1) with boundary conditions (BC-A) or (BC-B) is shown in Appendix B.1.

We first prove a preliminary result concerning the exponential stability of v_n. Let us apply a Volterra transformation $\mathcal{T} : L^2(0,L) \to L^2(0,L)$ of the form

$$z(x) = \mathcal{T}[v_n](x) := v_n(x) - \int_x^L p(x,y)v_n(y)\mathrm{d}y \tag{2.9}$$

to the solution to the last equation of the cascade system, with p defined on Π. Under appropriate choice of $p(\cdot,\cdot)$, we prove that this transformation maps solution v_n to the trajectory z satisfying the following target equation in $[0,\infty)\times[0,L]$

$$\begin{aligned} & z_t + z_x + z_{xxx} + \omega z = 0, \\ & (\text{BC-A}, z) : z(t,0) = z(t,L) = z_x(t,L) = 0, \\ & (\text{BC-B}, z) : z_{xx}(t,0) = z(t,L) = z_x(t,L) = 0, \end{aligned} \tag{2.10}$$

with control given by (2.7). Indeed, performing standard differentiations and integrations by parts (for more intuition about such operations, the reader can refer to [4]), we derive the following equations

$$\begin{aligned} & z_t(t,x) + z_x(t,x) + z_{xxx}(t,x) + \omega z(t,x) = \\ & -\int_x^L \left(p_{xxx}(x,y) + p_{yyy}(x,y) + p_y(x,y) + (\omega+1)p(x,y)\right) v_n(t,y)\mathrm{d}y \\ & + p(x,L)v_{n,xx}(t,L) + p(x,L)v_n(t,L) + p_{yy}(x,L)v_n(t,L) - p_y(x,L)v_{n,x}(t,L) \\ & + \left(\omega + 1 + \frac{\mathrm{d}^2}{\mathrm{d}x^2}p(x,x) + \frac{\mathrm{d}}{\mathrm{d}x}p_x(x,x) + p_{xx}(x,x) - p_{yy}(x,x)\right) v_n(t,x) \\ & + \left(p_x(x,x) + p_y(x,x) + 2\frac{\mathrm{d}}{\mathrm{d}x}p(x,x)\right) v_{n,x}(t,x). \end{aligned}$$

By choosing $p(\cdot,\cdot)$ satisfying the following equations

$$\left\{ \begin{array}{ll} p_{xxx} + p_{yyy} + p_x + p_y + (\omega+1)p = 0, & (x,y)\in\Pi \\ p(x,x) = p(x,L) = 0, & x\in[0,L] \\ p_x(x,x) = \frac{\omega+1}{3}(L-x), & x\in[0,L] \end{array} \right. \tag{2.11}$$

we achieve to obtain target system (2.10) for both boundary problems (BC-A, z) and (BC-B, z). Solutions to (2.11) are proven in [4] to be unique in the space $C^3(\Pi)$, by following successive approximation methods. The feedback control u is easily checked to satisfy (2.7), if we use (2.9) and also calculate the value of the second derivative, viz.

$$\begin{aligned} z_{xx}(x) = & v_{n,xx}(x) + \frac{\mathrm{d}}{\mathrm{d}x}p(x,x)v_n(x) + p(x,x)v_{n,x}(x) + p_x(x,x)v_n(x) \\ & - \int_0^L p_{xx}(x,y)v_n(y)\mathrm{d}y \end{aligned}$$

for $x = 0$.

Now, as we saw in Propositions 2.1 and 2.3 of the previous subsection, solution z to target system (2.10) is asymptotically stable for every length $L > 0$, if $\omega > 0$ and under boundary conditions (BC-A, z) and if $\omega \geq 1$ under boundary conditions

(BC-B, z). This implies the asymptotic stability of v_n, solution to (1.1), with control given by (2.7) for each of the boundary problems (BC-A) and (BC-B). The latter follows from the fact that, as proven in [4], transformation (2.9), mapping solution v_n to z, is bounded and invertible with bounded inverse. So, for every $\bar{d} > 0$, there exist $\omega_0, \bar{c} > 0$, such that for all $\omega \geq \omega_0$, we have

$$\|v_n\|_{L^2(0,L)} \leq \bar{c}e^{-\bar{d}t}\|v_n^0\|_{L^2(0,L)}, \forall t \geq 0. \tag{2.12}$$

To prove the asymptotic stability of the full state, consider vector $v_{[n-1]} := \begin{pmatrix} v_1 & \cdots & v_{n-1} \end{pmatrix}^\top$. Then, $v_{[n-1]}$ satisfies the following equations

$$\begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell v_n, \\ (\text{BC-A}, v_{[n-1]}) : v_{[n-1]}(t,0) = v_{[n-1]}(t,L) = v_{[n-1],x}(t,L) = 0, \\ (\text{BC-B}, v_{[n-1]}) : v_{[n-1],xx}(t,0) = v_{[n-1]}(t,L) = v_{[n-1],x}(t,L) = 0, \end{cases} \tag{2.13}$$

where

$$\ell := \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\top .$$

To prove stability of this system, consider a Lyapunov functional of the following form

$$W(t) = \int_0^L \pi(x)|v_{[n-1]}(x)|^2 \mathrm{d}x$$

along the $L^2(0,L)^{n-1}$ solutions $v_{[n-1]}$ to the last equations, where $\pi(\cdot)$ is a positive increasing C^3 function to be chosen. After substistuting the above equations satisfied by $v_{[n-1]}$ and applying integrations by parts, we obtain for the time-derivative of W

$$\begin{aligned} \dot{W}(t) = &\int_0^L (\pi'''(x) + \pi'(x))|v_{[n-1]}(x)|^2 \mathrm{d}x - 3\int_0^L \pi'(x)|v_{[n-1],x}(x)|^2 \mathrm{d}x \\ &- \int_0^L \pi(x)v_{[n-1]}^\top(x)\left(2I_{n-1} - A_{n-1}^\top - A_{n-1}\right)v_{[n-1]}(x)\mathrm{d}x \\ &+ 2\int_0^L \pi(x)v_{n-1}(x)v_n(x)\mathrm{d}x + W_0, \end{aligned}$$

with

$$\begin{aligned} W_0 := \Big[&-(\pi''(x) + \pi(x))|v_{[n-1]}(x)|^2 + \pi(x)\left(|v_{[n-1],x}(x)|^2 - 2v_{[n-1],xx}^\top(x)v_{[n-1]}(x)\right) \\ &+2\pi'(x)v_{[n-1],x}^\top(x)v_{[n-1]}(x)\Big]_0^L. \end{aligned} \tag{2.14}$$

Matrix $2I_{n-1} - A_{n-1}^\top - A_{n-1}$ is positive definite and its eigenvalues are given by

$$\rho := 2 - 2\cos\frac{\pi j}{n}, j = 1, \dots, n-1.$$

Consequently, its minimal eigenvalue is given by

$$\rho_n := \lambda_{\min}(2I_{n-1} - A_{n-1}^\top - A_{n-1}) = 2 - 2\cos\frac{\pi}{n}, \mathbb{N} \ni n \geq 2. \tag{2.15}$$

Since $\pi'(x) \geq 0$, by use of Young's inequality we obtain

$$\begin{aligned} \dot{W}(t) \leq &\int_0^L (\pi'''(x) + \pi'(x) - \rho_n\pi(x))|v_{[n-1]}(x)|^2 \mathrm{d}x \\ &+ 2\delta\int_0^L \pi(x)|v_{[n-1]}(x)|^2 \mathrm{d}x + \frac{1}{2\delta}\int_0^L \pi(x)v_n^2 \mathrm{d}x + W_0, \end{aligned}$$

and $\delta > 0$ is chosen sufficiently small as in (A.4) in the proof of Lemma 2.2 of the previous subsection.

Now, we choose $\pi(\cdot)$ for each of the two boundary problems as follows.

For (BC-A, $v_{[n-1]}$) we choose

$$\pi(\cdot) = 1.$$

From this, taking also into account the exponential stability of v_n (2.12), we get for the case (BC-A, $v_{[n-1]}$) the following estimate

$$\dot{W}(t) \leq -2dW(t) + \frac{1}{2\delta}\pi(L)\bar{c}^2 e^{-2\bar{d}t}\|v_n^0\|^2_{L^2(0,L)} \tag{2.16}$$

with $d = \rho_n/2 - \delta$.

For (BC-B, $v_{[n-1]}$) we choose a positive and increasing $\pi(\cdot)$ satisfying (2.5) (see Assertion 1 in Lemma 2.2) with $\lambda = \frac{\rho_n}{2} - \delta$ and $b > 0$. It turns out by Assertion 1 that there is $\pi(\cdot)$ and $b > 0$ satisfying this equation for any $L > 0$, when $\lambda = 1 - \delta$, corresponding to $\rho_2 = 2$ (for $n = 2$). Then, the exponential decay of the Lyapunov functional is ensured similarly as in Proposition 2.3. More precisely, there exists $d > 0$, such that for all $L > 0$, (2.16) is satisfied for (BC-B, $v_{[n-1]}$) as well. Also, as shown in Proposition 2.3, for $n > 2$, which renders $\rho_n < 2$, (2.16) is satisfied for some $\pi(\cdot)$, $b > 0$, only when $0 < L \leq \bar{L}$, with $\bar{L}$ depending on n.

Combining the above results, from (2.16), which holds for both (BC-A, $v_{[n-1]}$) and (BC-B, $v_{[n-1]}$), we derive by Gronwall's inequality

$$W(t) \leq e^{-2dt}W(0) + \frac{\pi(L)\bar{c}^2}{4\delta(d-\bar{d})}\left(e^{-2\bar{d}t} - e^{-2dt}\right)\|v_n^0\|^2_{L^2(0,L)}, \tag{2.17}$$

recalling also, that $\bar{d}$ depending on the parameter ω of the control laws, can be chosen, such that $\bar{d} > d$. Combining (2.17) and (2.12), we get

$$\begin{aligned}\|v\|_{L^2(0,L)^n} \leq &\|v_{[n-1]}\|_{L^2(0,L)^{n-1}} + \|v_n\|_{L^2(0,L)} \leq \sqrt{\frac{\pi(L)}{\pi(0)}}e^{-dt}\|v_{[n-1]}(0,\cdot)\|_{L^2(0,L)^{n-1}} \\ &+ \frac{\bar{c}\sqrt{\pi(L)}}{2\sqrt{\pi(0)\delta(\bar{d}-d)}}\sqrt{e^{-2dt} - e^{-2\bar{d}t}}\|v_n^0\|_{L^2(0,L)} + \bar{c}e^{-\bar{d}t}\|v_n^0\|_{L^2(0,L)}.\end{aligned}$$

The last inequality leads to (2.8) for a suitable choice of c.

This concludes the proof and shows also, that although the exponential convergence to zero of v_n can become arbitrarily fast by the choice of parameter ω inside the controls, solution v to the whole cascade system has a fixed convergence rate. ∎

Remark 2.5. Note that in the above proof, parameter ρ_n in (2.15), depending on n, does not permit the stabilization of the closed-loop system for any number of equations n, when the length of the domain L is arbitrary. As it was shown in Proposition 2.3, the damped KdV equation in the case of boundary conditions of the type (BC-B), requires a damping with coefficient λ larger than a critical damping coefficient λ_0. The parameter ρ_n, which appears in the stabilization of the closed-loop system corresponding to the damping coefficient, is decreasing with n. For $n > 2$ the stabilization cannot be ensured for any $L > 0$, since, because of ρ_n, the damping coefficient becomes lower than the critical one, while for $n = 2$, the damping coefficient of the coupled equation is exactly equal to the critical one.

3. Observer Design and Output Feedback Stabilization. In this section, we first present the proposed observer, along with its convergence proof for each of the boundary control problems (BC-A) and (BC-B). Then, we study the output feeedback stabilization of system (1.1) with controls placed on the left boundaries as described in each of problems (BC-A) and (BC-B). We note here that, even though the considered system is linear, the use of the high-gain observer design is instrumental in the output feedback control in the two following manners and is based on the methodology [17], introduced for quasilinear hyperbolic systems. 1) For (BC-B), the choice of the high-gain parameter is needed to establish convergence of the observer, contrarily to a simpler Luenberger observer design, which would be sufficient for (BC-A); 2) The high-gain parameter is used in the stabilization of the closed-loop system for both boundary control problems (BC-A) and (BC-B).

In the following subsection we present the observer for the cascade system, whose exponential stability relies on the result presented in Proposition 2.3 of Section 2.

3.1. Observer. Define, first, diagonal matrix Θ_n by

$$\Theta_n := \operatorname{diag}\left(\theta, \theta^2, \ldots, \theta^n\right),$$

where $\theta > 0$ represents a gain, which will be selected later. Consider a vector gain $K_n = \begin{pmatrix} k_1 & \cdots k_n \end{pmatrix}^\top$ and let $P \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix satisfying a quadratic Lyapunov equation of the form

$$P\left(A_n + K_n C\right) + \left(A_n + K_n C\right)^\top P = -I_n. \tag{3.1}$$

The previous equation is always feasible, due to the observability of the pair (A_n, C).

Then, our observer is defined to satisfy the following equations in $(0,\infty)\times(0,L)$

$$\hat{v}_t(t,x) + \hat{v}_x(t,x) + \hat{v}_{xxx}(t,x) = (A_n - B)\hat{v}(t,x) - \Theta_n K_n\left(y(t,x) - C\hat{v}(t,x)\right) \tag{3.2}$$

with boundary conditions for each of (BC-A) and (BC-B) as follows

$$\text{(BC-A):} \quad \begin{array}{l} \hat{v}_i(t,0) = 0, i = 1,\ldots,n-1, \text{for all } t > 0,\\ \hat{v}_n(t,0) = u(t), \text{for all } t > 0,\\ \hat{v}(t,L) = \hat{v}_x(t,L) = 0, \text{for all } t > 0, \end{array} \tag{3.3a}$$

$$\text{(BC-B):} \quad \begin{array}{l} \hat{v}_{i,xx}(t,0) = 0, i = 1,\ldots,n-1, \text{for all } t > 0,\\ \hat{v}_{n,xx}(t,0) = u(t), \text{for all } t > 0,\\ \hat{v}(t,L) = \hat{v}_x(t,L) = 0, \text{for all } t > 0. \end{array} \tag{3.3b}$$

and initial condition

$$\hat{v}(0,x) = \hat{v}^0(x), x \in (0,L).$$

The main observer result is stated in the following theorem.

THEOREM 3.1. *Consider system (1.1) with output (1.4) and boundary conditions satisfying (1.2) ((BC-A) or (BC-B)) and $v^0 \in L^2(0,L)^n, u \in L^2_{loc}(0,\infty)$. Consider, also, P and K_n satisfying a Lyapunov equation as in (3.1). Then, (3.2), with boundary conditions (3.3) and initial condition $\hat{v}^0 \in L^2(0,L)^n$ is an observer for solution of (1.1), in the sense that for θ large it estimates the state v arbitrarily fast. More precisely, for every $\kappa > 0$, there exists θ_0, such that for every $\theta > \theta_0$, the following holds for all $v^0, \hat{v}^0 \in L^2(0,L)^n$, $t \geq 0$*

$$\|\hat{v}(t,\cdot) - v(t,\cdot)\|_{L^2(0,L)^n} \leq \nu\theta^{n-1}e^{-\kappa t}\|\hat{v}^0(\cdot) - v^0(\cdot)\|_{L^2(0,L)^n}, \tag{3.4}$$

with $\nu > 0$, depending on n and L.

Proof. First, we prove in Appendix B.2 that observer system (3.2)-(3.3) is well-posed. Then, to prove its asymptotic convergence to the state v, let us define a scaled observer error ε by

$$\varepsilon = \Theta_n^{-1}(\hat{v} - v). \tag{3.5}$$

Then, ε satisfies the following equations

$$\varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = \theta(A_n + K_n C)\varepsilon - B\varepsilon \tag{3.6}$$

and boundary conditions for each of the cases (BC-A) and (BC-B) as follows

$$\varepsilon(t,0) = \varepsilon(t,L) = \varepsilon_x(t,L) = 0, \tag{3.7a}$$
$$\varepsilon_{xx}(t,0) = \varepsilon(t,L) = \varepsilon_x(t,L) = 0. \tag{3.7b}$$

We expect that solutions to the previous coupled equations can approach zero exponentially fast, since $A_n + K_n C$ being Hurwitz will exhibit a damping effect (as in the single KdV equation), with its magnitude being controlled by θ. Indeed, to prove exponential stability, we choose the following Lyapunov functional defined on the $L^2(0,L)^n$ solutions to the observer error equations

$$V(t) := \int_0^L \mu(x)\varepsilon^\top(x)P\varepsilon(x)\mathrm{d}x, \tag{3.8}$$

with positive $\mu(\cdot) \in C^3[0,L]$ to be chosen suitably for each of the boundary conditions cases. Taking its time-derivative and substituting (3.6) and Lyapunov equation (3.1), yields

$$\begin{aligned}\dot{V}(t) = \int_0^L \mu(x) \big[&-\partial_x^3\left(\varepsilon^\top(x)P\varepsilon(x)\right) - \partial_x(\varepsilon^\top(x)P\varepsilon(x)) + 3\partial_x(\varepsilon_x^\top(x)P\varepsilon_x(x))\\ &-\theta\varepsilon^\top(x)\varepsilon(x) - 2\varepsilon^\top(x)PB\varepsilon(x)\big]\,\mathrm{d}x.\end{aligned}$$

Performing successive integrations by parts, we obtain

$$\begin{aligned}\dot{V}(t) \leq &\int_0^L \left(\mu'''(x) + \mu'(x) + \left(-\theta\frac{1}{|P|} + 2\frac{|P|}{\lambda_{\min}(P)}\right)\mu(x)\right)\varepsilon^\top(x)P\varepsilon(x)\mathrm{d}x\\ &- 3\int_0^L \mu'(x)\varepsilon_x^\top(x)P\varepsilon_x(x)\mathrm{d}x + V_0,\end{aligned}$$

where

$$\begin{aligned}V_0 := \big[&\left(-\mu''(x) - \mu(x)\right)\varepsilon^\top(x)P\varepsilon(x) - \mu(x)\left(\varepsilon_{xx}^\top(x)P\varepsilon(x) + \varepsilon^\top(x)P\varepsilon_{xx}(x)\right)\\ &+\mu(x)\varepsilon_x^\top(x)P\varepsilon_x(x) + \mu'(x)(\varepsilon_x^\top(x)P\varepsilon(x) + \varepsilon^\top(x)P\varepsilon_x(x))\big]_0^L\end{aligned} \tag{3.9}$$

and $\lambda_{\min}(P)$ is the minimal eigenvalue of P.

Let us now choose μ for boundary conditions case (3.7a) as follows

$$\mu(\cdot) := 1, \tag{3.10}$$

for which we obtain

$$V_0 = -\varepsilon_x^\top(0)P\varepsilon_x(0) \leq 0.$$

Note that given (3.10) for the boundary conditions case (3.7a), for every $\theta > \theta_{0,A}$, with

$$\theta_{0,A} := 2\frac{|P|^2}{\lambda_{\min}(P)},$$

we get

$$\dot{V}(t) \leq -2\kappa_A V(t), t \geq 0, \tag{3.11}$$

for some $\kappa_A > 0$.

Considering boundary conditions of case (3.7b), (3.9) is written as

$$V_0 := -\begin{pmatrix}\varepsilon_x^\top(0) & \varepsilon^\top(0)\end{pmatrix}\begin{pmatrix}P\mu(0) & P\mu'(0)\\ P\mu'(0) & -P\left(\mu''(0)+\mu(0)\right)\end{pmatrix}\begin{pmatrix}\varepsilon_x(0)\\ \varepsilon(0)\end{pmatrix}$$

For these boundary conditions (3.7b), we see that for all

$$\theta \geq \theta_{0,B} := 2\frac{|P|^2}{\lambda_{\min}(P)} + 2|P|,$$

Assertion 1 ((2.5) in Lemma 2.2) is satisfied with $\mu(\cdot)$ in the place of $\pi(\cdot)$, $\lambda = \theta\frac{1}{2|P|} - \frac{|P|}{\lambda_{min}(P)}$, $\lambda_0 = 1$ and $b = \kappa_B$ for some $\kappa_B > 0$ depending on θ. For all $\theta \geq \theta_{0,B}$, we choose, therefore, $\pi(\cdot) = \mu(\cdot)$ satisfying (2.5) and we derive again (3.11) with κ_A substituted by κ_B.

Combining the previous estimates, we directly obtain (3.4) with

$$\nu := \sqrt{\frac{\mu(L)}{\mu(0)}}\sqrt{\frac{|P|}{\lambda_{\min}(P)}}$$

and this concludes the proof of Theorem 3.1. ∎

3.2. Output feedback stabilization. Next, it is proven that plugging the observer's state considered in Theorem 3.1 in the feedback laws (2.7) of the previous section, the closed-loop system is stabilized. This is done in two steps. First, it is proven that the considered output feedback law stabilizes arbitrarily fast the solution of the last KdV equation and second, the stabilization of the whole cascade system of KdV equations follows. However, for system with boundary conditions (BC-B), stabilization for any L is only achieved when $n = 2$, corresponding to a cascade system of two equations, while for $n > 2$, stabilization is achieved for small L, following the result of Proposition 2.3 of the previous section. Even if this requirement is restrictive, we find several physical applications, where only two coupled equations appear in the model, see [20]. These statements are presented in the following theorem.

THEOREM 3.2. *Consider the closed-loop system (1.1)-(3.2), output (1.4), and boundary conditions being of the form (BC-A) or (BC-B). Then, for any $\bar{d} > 0$, there exist an output feedback law $u(t)$ of the form (2.7), where v is substituted by the observer state $\hat{v}$, and constants $\theta_0, \omega_0, \bar{c} > 0$, such that for any design parameters $\theta > \theta_0, \omega > \omega_0$ (with θ involved in the observer and ω involved in the control laws), the closed-loop system solution with $v^0, \hat{v}^0 \in L^2(0,L)^n$ satisfies the following stability inequality (on the estimation error and last observer state)*

$$\begin{aligned}\|\hat{v} - v\|_{L^2(0,L)^n} &+ \|\hat{v}_n\|_{L^2(0,L)}\\ &\leq \bar{c}e^{-\bar{d}t}\left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}\right), \forall t \geq 0.\end{aligned} \tag{3.12}$$

Moreover, whenever the previous assertion holds, we get the following (full state convergence):

a) When boundary conditions (BC-A) hold with $n \geq 2$, then for every $L > 0$, there exist constants $c, d > 0$, such that solutions $v, \hat{v}$ satisfy the following

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \leq ce^{-dt}\left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}\right), \forall t \geq 0, \tag{3.13}$$

with d depending on n.

b) When boundary conditions (BC-B) hold with $n = 2$, then for every $L > 0$, there exist constants $c, d > 0$, such that solutions $v, \hat{v}$ satisfy (3.13)*.*

c) When (BC-B) holds, with $n > 2$ there exists $\bar{L} > 0$ small, such that asymptotic stability (3.13) *is guaranteed for all $L \in (0, \bar{L}]$.*

Proof. To address the closed-loop control problem, let us rewrite observer error and observer coupled equations, viz. (see (3.6), (3.2))

$$\begin{cases} \varepsilon_t + \varepsilon_x + \varepsilon_{xxx} = & \theta(A_n + K_n C)\varepsilon - B\varepsilon, \\ \hat{v}_t + \hat{v}_x + \hat{v}_{xxx} = & (A_n - B)\hat{v} + \theta\Theta_n K_n \varepsilon_1, \end{cases} \tag{3.14}$$

with boundary conditions (3.7), (3.3).

Let us perform a Volterra transformation to the solution of the n-th equation of the observer, which by (3.2) is written as

$$\hat{v}_{n,t} + \hat{v}_{n,x} + \hat{v}_{n,xxx} = \hat{v}_n + k_n\theta^{n+1}\varepsilon_1. \tag{3.15}$$

The Volterra transformation

$$q(x) = \mathcal{T}[\hat{v}_n](x) := (k_n\theta^{n+1})^{-1}\hat{v}_n(x) - (k_n\theta^{n+1})^{-1}\int_x^L p(x,y)\hat{v}_n(y)\mathrm{d}y, \tag{3.16}$$

under appropriate choice of $p(\cdot,\cdot)$ maps (3.15) into target system

$$q_t + q_x + q_{xxx} = -\omega q + \varepsilon_1 - \int_x^L p(x,y)\varepsilon_1(t,y)\mathrm{d}y, \tag{3.17}$$

with ω a constant involved in the controller, and boundary conditions for each of the two considered cases as follows

$$q(t,0) = q(t,L) = q_x(t,L) = 0, \tag{3.18a}$$

$$q_{xx}(t,0) = q(t,L) = q_x(t,L) = 0. \tag{3.18b}$$

Then, the kernel functions $p(\cdot,\cdot)$ satisfy (2.11) for both problems (3.18a), (3.18b). It is easy to check this if we apply successive differentiations of (3.16) as in Theorem 2.4, we obtain the above target system, by choosing $p(\cdot,\cdot)$ satisfying (2.11). Subsequently, the output feedback control $u(\cdot)$ for (BC-A) is given by

$$u(t) = \int_0^L p(0,y)\hat{v}_n(t,y)\mathrm{d}y, \tag{3.19a}$$

and for (BC-B),

$$u(t) = -\frac{\omega+1}{3}L\hat{v}_n(t,0) + \int_0^L p_{xx}(0,y)\hat{v}_n(t,y)\mathrm{d}y. \tag{3.19b}$$

As noticed in proof of Theorem 2.4, it has been proven that the kernel equations (2.11) are solvable in Π and the corresponding Volterra transformation is bounded and injective with bounded inverse.

Consider now the Lyapunov function

$$
\begin{aligned}
&U_1(t) = U_{1,1}(t) + U_{1,2}(t);\\
&U_{1,1}(t) := \int_0^L \mu(x)\varepsilon^\top(x)P\varepsilon(x)\mathrm{d}x, U_{1,2}(t) := \int_0^L \sigma(x)q^2(x)\mathrm{d}x,
\end{aligned}
$$

along the solutions to (3.6)-(3.7) and (3.17)-(3.18), where $U_{1,1}$ is the same as (3.8) of Theorem 3.1 and $\sigma(\cdot)$ is a positive C^3 increasing function in $[0, L]$ to be chosen later.

Taking the time-derivative of $U_{1,2}$ and substituting (3.17), we infer

$$
\begin{aligned}
\dot{U}_{1,2}(t) = &\int_0^L \left(\sigma'''(x) + \sigma'(x) - 2\omega\right) q^2(x)\mathrm{d}x - 3\int_0^L \sigma'(x)q_x^2(x)\mathrm{d}x\\
&+ 2\int_0^L \sigma(x)q(x)\varepsilon_1(x)\mathrm{d}x - 2\int_0^L \sigma(x)q(x)\int_x^L p(x,y)\varepsilon_1(y)\mathrm{d}y\mathrm{d}x\\
&+ \left[\left(-\sigma''(x) - \sigma(x)\right) q^2(x) - 2\sigma(x)q_{xx}(x)q(x) + \sigma(x)q_x^2(x)\right.\\
&\left.+2\sigma'(x)q_x(x)q(x)\right]_0^L .
\end{aligned}
$$

By using

$$
\begin{aligned}
&2\int_0^L \sigma(x)q(x)\int_x^L p(x,y)\varepsilon_1(y)\mathrm{d}y\mathrm{d}x \leq U_{1,2}(t) + \sigma(L)\int_0^L \left(\int_x^L p(x,y)\varepsilon_1(y)\mathrm{d}y\right)^2 \mathrm{d}x\\
&\leq U_{1,2}(t) + L^2\sigma(L)\max_{x,y\in[0,L]} p^2(x,y)\int_0^L \varepsilon_1^2(y)\mathrm{d}y\\
&\leq U_{1,2}(t) + L^2\frac{\sigma(L)}{\mu(0)\lambda_{\min}(P)}\max_{x,y\in[0,L]} p^2(x,y)U_{1,1}(t),
\end{aligned}
$$

we get

$$
\begin{aligned}
\dot{U}_{1,2}(t) \leq &\int_0^L \left(\sigma'''(x) + \sigma'(x) - 2(\omega-1)\right) q^2(x)\mathrm{d}x - 3\int_0^L \sigma'(x)q_x^2(x)\mathrm{d}x + hU_{1,1}(t)\\
&+ \left[\left(-\sigma''(x) - \sigma(x)\right) q^2(x) - 2\sigma(x)q_{xx}(x)q(x) + \sigma(x)q_x^2(x)\right.\\
&\left.+2\sigma'(x)q_x(x)q(x)\right]_0^L , \qquad (3.20)
\end{aligned}
$$

where $h := \left(L^2 \max_{x,y\in[0,L]} p^2(x,y) + 1\right)\frac{\sigma(L)}{\mu(0)\lambda_{\min}(P)}$.

We can prove that for each of the two cases of boundary conditions we get

$$
\dot{U}_1(t) \leq -2\bar{d}U_1(t). \qquad (3.21)
$$

Case (BC-A):

We choose $\mu(\cdot) = \sigma(\cdot) = 1$ and we obtain:

$$
\dot{U}_{1,2}(t) \leq -2(\omega-1)U_{1,2}(t) + hU_{1,1}(t).
$$

As seen in Theorem 3.1, for $\mu(\cdot) = 1$, we have:

$$
\dot{U}_{1,1}(t) \leq \left(-\theta\frac{1}{|P|} + 2\frac{|P|}{\lambda_{min}(P)}\right)U_{1,1}(t).
$$

Combining the last two equations, if we choose θ, ω as follows:

$$\theta > h|P| + 2\frac{|P|^2}{\lambda_{min}(P)}, \omega > 1, \tag{3.22}$$

we get a $\bar{d} > 0$, such that (3.21) holds.

Case (BC-B):

We see for this case of boundary conditions that for all

$$\theta \geq \theta_0 := 2\frac{|P|^2}{\lambda_{min}(P)} + h|P| + 2|P|, \tag{3.23}$$

Assertion 1 ((2.5) in Lemma 2.2) is satisfied with $\mu(\cdot)$ in the place of $\pi(\cdot)$, $\lambda = \theta\frac{1}{2|P|} - \frac{|P|}{\lambda_{min}(P)} - \frac{h}{2}$, $\lambda_0 = 1$. For all $\theta \geq \theta_0$, we choose, therefore, $\mu(\cdot)$ satisfying (2.5) and we get that the first term of the right hand side of

$$\begin{aligned}\dot{U}_1(t) \leq & \int_0^L \left(\mu'''(x) + \mu'(x) - 2\left(\theta\frac{1}{2|P|} - \frac{|P|}{\lambda_{min}(P)} - \frac{h}{2}\right)\mu(x)\right)\varepsilon^\top(x)P\varepsilon(x)\mathrm{d}x \\ & + \dot{U}_{1,2}(t), t \geq 0.\end{aligned}$$

becomes negative.

Similarly, for every

$$\omega \geq 2,$$

we can find $\sigma(\cdot) = \pi(\cdot)$ satisfying (2.5), with $\lambda = \omega - 1$ and $\lambda_0 = 1$ and by virtue of Proposition 2.3, right hand side of (3.20) becomes negative. Hence, returning to $\dot{U}_1$ and choosing $\theta \geq \theta_0$ and $\omega \geq 2$, we can always find $\mu(\cdot), \sigma(\cdot)$ as in Assertion 1 of Lemma 2.2, in a such way that we always get a $c_2 > 0$, satisfying again (3.21).

Consequently, for each of the two problems (BC-A) and (BC-B), for each $\bar{d} > 0$ we can find θ, ω, chosen as before in such a way that there exists constant $\gamma > 0$ depending polynomially on θ, such that

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|q\|_{L^2(0,L)} \leq \gamma e^{-\bar{d}t}\left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|q(0,\cdot)\|_{L^2(0,L)}\right), \forall t \geq 0.$$

Transformation $\mathcal{T}$ is bounded with bounded inverse (see the comments in Theorem 2.4) and, therefore, we obtain an inequality as (3.12).

Remark 3.3. The previous calculations indicate that gain θ appearing in observer system (3.2)-(3.3) is crucial in the stabilization of the closed-loop system. Indeed, in (3.23), we see that choice of θ compensates for some terms appearing therein. The dependence of the terms on the eigenvalues of matrix P indicates that a simpler Luenberger observer with pole placement would not suffice for the stabilization of the closed-loop system. These terms play the role of the nonlinearities, appearing in the Lyapunov derivative used for the observer error in high-gain observer designs for finite-dimensional systems. Although in finite dimension, a pole-placement observer is enough for linear systems, in the present framework of infinite dimension, a design similar to high-gain observers in finite-dimension is required.

We are now in a position to prove the closed-loop stability for the whole system following the methodology of Theorem 2.4. Let $\hat{v}_{[n-1]} := \left(\hat{v}_1 \cdots \hat{v}_{n-1}\right)^\top$. Then, $\hat{v}_{[n-1]}$ satisfies the following equations

$$\begin{aligned}&\hat{v}_{[n-1],t} + \hat{v}_{[n-1],x} + \hat{v}_{[n-1],xxx} = (A_{n-1} - I_{n-1})\hat{v}_{[n-1]} + \ell\hat{v}_n + \Theta_{n-1}K_{n-1}(\hat{v}_1 - v_1), \\ &(\text{BC-A}, \hat{v}_{[n-1]}) : \hat{v}_{[n-1]}(t,0) = \hat{v}_{[n-1]}(t,L) = \hat{v}_{[n-1],x}(t,L) = 0 \\ &(\text{BC-B}, \hat{v}_{[n-1]}) : \hat{v}_{[n-1],xx}(t,0) = \hat{v}_{[n-1]}(t,L) = \hat{v}_{[n-1],x}(t,L) = 0\end{aligned}$$

where $\ell := \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\top$ and Θ_{n-1}, K_{n-1} are involved in observer (3.2).

By choosing

$$U_2(t) = \int_0^L \pi(x)|\hat{v}_{[n-1]}(x)|^2 \mathrm{d}x$$

as a Lyapunov functional along the $L^2(0,L)^{n-1}$ solutions to the last equations, with $\pi(\cdot)$ a positive increasing C^3 function, we obtain

$$\begin{aligned}
\dot{U}_2(t) &= \int_0^L \left(\pi'''(x) + \pi'(x)\right) |\hat{v}_{[n-1]}(x)|^2 \mathrm{d}x - 3\int_0^L \pi'(x)|\hat{v}_{[n-1],x}(x)|^2 \mathrm{d}x \\
&\quad - 2\int_0^L \pi(x)\hat{v}_{[n-1]}^\top(x) \mathrm{Sym}\left(I_{n-1} - A_{n-1}\right) \hat{v}_{[n-1]}(x) \mathrm{d}x \\
&\quad + 2\int_0^L \pi(x)\hat{v}_{n-1}(x)\hat{v}_n(x)\mathrm{d}x + 2\int_0^L \pi(x)\hat{v}_{[n-1]}^\top \Theta_{n-1} K_{n-1}(\hat{v}_1 - v_1)\mathrm{d}x + U_{2,0},
\end{aligned}$$

where $U_{2,0}$ is as W_0 in (2.14) (see the proof of Theorem 2.4), while $v_{[n-1]}$ is substituded by $\hat{v}_{[n-1]}$. Applying Young's inequality, we get

$$\begin{aligned}
\dot{U}_2(t) &\leq \int_0^L \left(\pi'''(x) + \pi'(x) - (\rho_n - 2\delta)\pi(x)\right) |\hat{v}_{[n-1]}(x)|^2 \mathrm{d}x \\
&\quad + \frac{1}{\delta}\int_0^L \pi(x)\hat{v}_n^2(x)\mathrm{d}x + \frac{1}{\delta}\theta^{2n-2}|K_{n-1}|^2 \int_0^L \pi(x)|\hat{v}_1(x) - v_1(x)|^2 \mathrm{d}x + U_{2,0},
\end{aligned} \tag{3.24}$$

with $\delta > 0$ chosen sufficiently small, as in (A.4), determined in the proof of Lemma 2.2 of previous section, and ρ_n defined in (2.15).

Now, to ensure negativity of the Lyapunov derivative, we choose $\pi(\cdot)$ for each of the two boundary problems as follows.

Case (BC A, $\hat{v}_{[n-1]}$):

$$\pi(\cdot) = 1.$$

Then, in conjunction with the previously proven equation (3.12), we get from (3.24)

$$\dot{U}_2(t) \leq -2dU_2(t) + me^{-2\bar{d}t}\left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}\right)^2 \tag{3.25}$$

where $d := \rho_n - 2\delta > 0$ and

$$m := \frac{1}{\delta}\pi(L)\bar{c}^2 \max\left(1, \theta^{2n-2}|K_{n-1}|^2\right). \tag{3.26}$$

Case (BC B, $\hat{v}_{[n-1]}$):

For boundary conditions (BC B,$\hat{v}_{[n-1]}$), to obtain an asymptotic stability result, we first check that for $n = 2$, we have $\rho_n = 2$. For this ρ_2, proof of Lemma 2.2 suggests that there exists $\pi(\cdot)$ satisfying (2.5) for some $b > 0$, with the same $\pi(\cdot)$, $\lambda = \frac{\rho_2}{2} - \delta$. Then, a similar inequality as (3.25) is satisfied for all $L > 0$, $d = b$ and m as in (3.26). Additionally, following Assertion 2 in the proof of Lemma 2.2, we see that for any $n > 2$, implying $\rho_n < 2$, there exist again $\pi(\cdot)$, $d = b > 0$, such that (2.5) holds for $L \in (0, \bar{L}]$.

Now, we see that for both boundary problems (BC-A) and (BC-B), (3.25) gives

$$U_2(t) \leq e^{-2dt}W(0) + \frac{m}{2d - 2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt})\left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}\right)^2.$$

The latter implies

$$U_2(t) \le \left(e^{-2dt} + \frac{m}{2d-2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt})\right)\left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}\right)^2. \tag{3.27}$$

Recalling that $\bar{d}$ depends on the adjustable observer parameter θ, we suppose, without loss of generality, that can be chosen such that $\bar{d} > d$, so that the previous inequality has meaning.

Now, using trivial inequalities and by virtue of (3.27) and (3.12), we easily get:

$$\begin{aligned}\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} &\le \|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} + \|\hat{v}_{[n-1]}\|_{L^2(0,L)^{n-1}} \\ &\le \left[\bar{c}e^{-\bar{d}t} + \frac{1}{\sqrt{\pi(0)}}\sqrt{e^{-2dt} + \frac{m}{2d-2\bar{d}}(e^{-2\bar{d}t} - e^{-2dt})}\right] \\ &\quad \times \left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}\right).\end{aligned}$$

The latter completes the proof of Theorem 3.2, suggesting also that the asymptotic rate of the whole closed-loop cascade system is no larger than d, which is decreasing with n, contrary to the asymptotic rate for the last state v_n, which is adjusted by the observer and control parameters. ∎

Remark 3.4. The considered stabilization problem of under-actuated and under-observed cascade systems of KdV equations was here limited to the linear case, and special forms of couplings. Even though stabilization results of the original nonlinear KdV equation can be found (see survey [3]), or observer results for some infinite-dimensional systems with nonlinearities (satisfying some "triangular structure"), as in [16, 18], extensions of our output feedback stabilization to more general couplings and/or nonlinearities are quite challenging, and are thus left for future studies: a strong difficulty comes from the coefficients of system's differential operator, where the presence of distinct elements raises problems related to the notion of algebraic solvability, which has been given attention in [1, 19] and other works of the same authors. In the Lyapunov-based approach we have considered, this problem translates into the lack of a commutative property between a Lyapunov matrix and coefficients of system's differential operator. Handling nonlinearities in one-order term (for instance terms $v_i v_{i,x}$ or even couplings of this type between the equations) and zero-order term at the same time is also part of the challenge, as this commutative property would not be fulfilled. We also refer to [9], where some open problems concerning such coupling are presented, while the reader can understand the difficulties in the controllability analysis of under-actuated systems with nontrivial coefficients of the differential operators and the presence of nonlinearities.

Notice yet that it could be possible to adopt an indirect approach, based on our previous approaches [18], to deal with the case where the one-order and dispersion terms would be of the form $A_1 v_x + A_2 v_{xxx}$, for some $A_1, A_2 \in \mathbb{R}^{n\times n}$. Consideration of linear lower triangular couplings of one-order and third-order terms would be feasible as well, but more general cases remain open.

Notice also that this under-observed problem being already challenging, the case when only a boundary measurement is available (instead of an internal one, at least localized to a part of the domain) is even more difficult: even though a solution does exist for a single equation and boundary measurement [23], it does not easily extend to the case of coupled KdV equations, via a backstepping and a single measurement instead of a distributed one. For the case of n coupled equations, a backstepping approach that would lead to an exponentially stable observer error system would fail, even for $n = 2$, if the observations were fewer than the states. In addition, the control problem of under-actuated systems by itself is a hard problem, and if 2×2 systems have local solutions via backstepping, see [8], they concern hyperbolic systems where a dissipative target system is feasible. For the coupled KdV equations, the exponentially stable target system for the observer error would be needed to be a damped system, which cannot be achieved by a single observation.

Some possible generalizations of the present framework, as the ones described before, will be a subject of our future works.

4. Conclusion. In this work, output feedback stabilization for a class of cascade system of linear KdV equations was achieved. Two boundary control problems, with controls placed on the left side of the last equation, were investigated. Distributed measurement of the first state was considered, which provided an estimation (using a high-gain observer) of the states fed in the control laws. The cascade system is stabilized for both boundary problems, but with a limitation on the number of equations and length of the domain for the second one.

Future developments might include the same stabilization framework, but with more general couplings (in zero-order and one-order derivative terms), including localized terms and nonlinearities.

Appendix A. Proof of Lemma 2.2. To prove Assertions 1 and 2 of Lemma 2.2, it is more convenient to write the characteristic equation of the differential equation in (2.5) as

$$r^3 + r - s^3 - s = 0 \tag{A.1}$$

(as in [25], a technique used to solve the characteristic equation of the KdV operator), where

$$s^3 + s = 2\lambda - 2b \tag{A.2}$$

and considering s being the real root of the latter equation. Then, solutions to (A.1) are given by

$$r_1 = s, r_2 = -\frac{s}{2} + i\frac{\sqrt{3s^2+4}}{2}, r_3 = -\frac{s}{2} - i\frac{\sqrt{3s^2+4}}{2}$$

and, therefore, unique solutions to the differential equation in (2.5) are given by

$$\pi(x) = \alpha e^{sx} + \beta e^{-\frac{s}{2}x} \cos\frac{\sqrt{3s^2+4}}{2}x + \gamma e^{-\frac{s}{2}x} \sin\frac{\sqrt{3s^2+4}}{2}x \tag{A.3}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$ chosen, such that restriction on initial conditions in (2.5) is satisfied. We can check numerically that there exists a number $\epsilon > 0$ near zero, such that for $s \geq 1 - \epsilon$, $\pi(\cdot)$ given by (A.3) with initial conditions $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$ (corresponding to $\alpha = 56/25, \beta = 44/25, \gamma = 8/25$) is positive and increasing and, therefore, it satisfies (2.5). Defining a small constant $\delta > 0$ by

$$\delta := \frac{\epsilon^3}{5} - \frac{3\epsilon^2}{5} + \frac{7\epsilon}{5}, \tag{A.4}$$

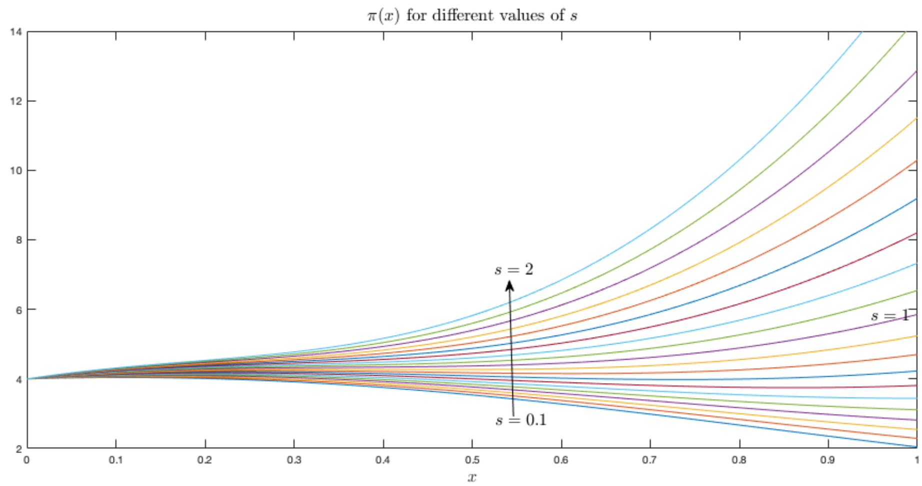

FIG. 1. *Solutions to (2.5) for different s*

we see from (A.2) that for $s \geq 1 - \epsilon$ we have $\lambda \geq \lambda_0 := 1 - \delta$ for choice $b = \frac{11}{16}\lambda$. Thus, for all $\lambda \geq \lambda_0$, there exists $b > 0$, such that conditions (2.5) are satisfied. Hence, Assertion 1 is proven to hold for $\lambda_0 = 1 - \delta$, where δ is defined above. Now, notice that for $s < 1 - \epsilon$, corresponding to $\lambda < \lambda_0$, and for any initial condition of $\pi(\cdot)$, satisfying second equation of (2.5), there is a $\bar{L} > 0$, such that for $x > \bar{L}$, $\pi(\cdot)$ becomes decreasing and, thus, fails to satisfy all conditions (2.5). This implies that for $0 < \lambda < \lambda_0$, Assertion 2 is satisfied for some small $\bar{L} > 0$. Letting $s \to 0^+$, and choosing initial conditions $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$ as before, π approaches the trajectory of $\pi(x) = -1 + 5\cos(x) + 2\sin(x)$, for which $\pi'(x) < 0$ for $L > \arctan(2/5)$. By this, for $\lambda \to 0^+$, $b \to 0^+$, Assertion 2 is satisfied with $\bar{L} = \arctan(2/5)$.

In Figure 1, we represent the evolution of $\pi(x)$ for choice of initial condition $\pi(0) = 4, \pi'(0) = 2, \pi''(0) = -5$ and various values of s, corresponding to various values of λ. For small values of s, corresponding to small values λ, $\pi(\cdot)$ is increasing until some point $x = \bar{L}$ quite small, but for $x > \bar{L}$, it is decreasing and, thus, fails to satisfy fourth equation of (2.5) after this point, in accordance with Assertion 2. We also see that for all $s \geq 1 - \epsilon$, for $\epsilon > 0$ small, given as before, $\pi(\cdot)$ is always increasing, verifying Assertion 1. The proof is complete.

Appendix B. Well-posedness of system and observer.

We show here the well-posedness of both controlled system (1.1)-(1.3) and observer system (3.2)-(3.3).

B.1. Well-posedness of (1.1)-(1.3). First, for system (1.1), with boundary conditions (BC-A) or (BC-B), feedback control laws of the form (2.7a) or (2.7b), respectively, and initial condition $v^0 \in L^2(0, L)^n$, it is sufficient to prove the well-posedness of target system, which results after applying the isomorphic transformation $\mathcal{T}$, see (2.9), in conjunction with (2.10):

$$\begin{cases} v_{[n-1],t} + v_{[n-1],x} + v_{[n-1],xxx} = (A_{n-1} - I_{n-1})v_{[n-1]} + \ell\mathcal{T}^{-1}[z], \\ z_t + z_x + z_{xxx} + \omega z = 0 \\ \text{(BC-A)} : \eta(t, 0) = \eta(t, L) = \eta_x(t, L) = 0, \\ \text{(BC-B)} : \eta_{xx}(t, 0) = \eta(t, L) = \eta_x(t, L) = 0, \end{cases}$$

where $\eta := \begin{pmatrix} v_{[n-1]} & z \end{pmatrix}^\top$ is the target state and we adopt the same notation as in (2.13). We rewrite the above system as an abstract evolution system in $L^2(0,L)^n$ as

$$\dot{\zeta} = \mathcal{A}\zeta,$$

where $\mathcal{A} := \operatorname{diag}(\mathcal{A}_1, \dots, \mathcal{A}_n) : D(\mathcal{A}) \to L^2(0,L)^n$ is an linear unbounded operator defined as

$$\begin{aligned}
\mathcal{A}_i\zeta_i &= -\zeta_{i,x} - \zeta_{i,xxx} - \zeta_i + \zeta_{i+1}, i = 1, \dots, n-2 \\
\mathcal{A}_{n-1}\zeta_{n-1} &= -\zeta_{n-1,x} - \zeta_{n-1,xxx} - \zeta_{n-1} + \mathcal{T}^{-1}[\zeta_n] \\
\mathcal{A}_n\zeta_n &= -\zeta_{n,x} - \zeta_{n,xxx} - \omega\zeta_n,
\end{aligned}$$

with domain

$$\begin{aligned}
D(\mathcal{A}) = \{\zeta \in H^3(0,L)^n; \zeta(0) = \zeta(L) = \zeta'(L) = 0, \text{ for (BC-A)}, \\
\text{or } \zeta''(0) = \zeta(L) = \zeta'(L) = 0, \text{ for (BC-B)}\}.
\end{aligned}$$

Its adjoint operator satisfies

$$\begin{aligned}
\mathcal{A}_i^*\zeta_i =& \zeta_{i,x} + \zeta_{i,xxx} - \zeta_i + \zeta_{i+1}, i = 1, \dots, n-2 \\
\mathcal{A}_{n-1}^*\zeta_{n-1} =& \zeta_{n-1,x} + \zeta_{n-1,xxx} - \zeta_{n-1} + \mathcal{T}^{-1}[\zeta_n] \\
\mathcal{A}_n^*\zeta_n =& \zeta_{n,x} + \zeta_{n,xxx} - \omega\zeta_n,
\end{aligned}$$

with domain

$$\begin{aligned}
D(\mathcal{A}^*) = \{\zeta \in H^3(0,L)^n; \zeta(0) = \zeta'(0) = \zeta(L) = 0, \text{ for (BC-A)}, \\
\text{or } \zeta''(0) = -\zeta(0), \zeta(L) = \zeta'(0) = \zeta(L) = 0, \text{ for (BC-B)}\}.
\end{aligned}$$

Operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$ are closed with domains dense in $L^2(0,L)^n$. Furthermore, they are both dissipative. Indeed, from the stability proof of Theorem 2.4, we first see that $\mathcal{A}_n$ is dissipative. Then, the exponential stability of $\mathcal{T}[\zeta_n]$ in (2.12), in conjunction with (2.17) implies that

$$\left\langle \mathcal{A}_{[n-1]}\zeta_{[n-1]}, \pi\zeta_{[n-1]} \right\rangle_{L^2(0,L)^{n-1}} \leq 0, \tag{B.1}$$

where $\pi(\cdot)$ satisfies (2.5). Inequality (B.1) is satisfied for every $L > 0$, if $n \geq 2$ under boundary conditions (BC-A) and if $n = 2$ under boundary conditions (BC-B). For the latter case, the same inequality holds for $n > 2$, when $L \in (0, \bar{L}]$, for some $\bar{L} > 0$. This was shown in the stability proof of Theorem 2.4 and it implies that operator $\mathcal{A}$ is dissipative, namely

$$\langle \mathcal{A}\zeta, \zeta \rangle_{L^2(0,L)^n} \leq 0.$$

To show dissipativity of the adjoint operator $\mathcal{A}^*$, i.e., that $\langle \mathcal{A}^*\zeta, \zeta \rangle_{L^2(0,L)^n} \leq 0$, we can easily show first that $\mathcal{A}_n^*$ is dissipative, by applying integrations by parts. Then, we show that $\left\langle \mathcal{A}_{[n-1]}^*\zeta_{[n-1]}, \zeta_{[n-1]} \right\rangle_{L^2(0,L)^{n-1}} \leq 0$. This implies that $\mathcal{A}^*$ is dissipative.

Consequently, we can apply the Lumer-Phillips theorem and we conclude that $\mathcal{A}$ generates a C^0-semigroup of contractions and, thus, returning to the original coordinates via $\mathcal{T}^{-1}$, we have that for any initial condition $v^0 \in L^2(0,L)^n$, there exists a unique mild solution

$$v \in C^0\left(0, \infty; L^2(0,L)^n\right)$$

for system (1.1)-(1.3), noting also, that for (BC-A), the above holds for all $L > 0$ and $n \geq 2$, while for (BC-B), the above holds for all $L > 0$, when $n = 2$ and for all $L \in (0, \bar{L}]$, when $n > 2$, where $\bar{L}$ is given in Lemma 2.2.

The above well-posedness result for the n coupled equations holds also for the single damped KdV equation, see (2.1)-(2.3), as this system's operator is equal to $\mathcal{A}_n$, as defined above.

B.2. Well-posedness of (3.2)-(3.3). Observer system given by (3.2) with boundary conditions (3.3) is well posed. To see this, it suffices to show the well-posedness of the error system (3.6)-(3.7), invoking also the well-posedness of initial system (1.1)-(1.3) that we showed before. The differential operator for error system (3.6)-(3.7) is given by

$$\mathcal{A}\zeta = -\zeta_x - \zeta_{xxx} + \theta(A_n + K_n C)\zeta - B\zeta,$$

with domain

$$\begin{aligned} D(\mathcal{A}) = \big\{\zeta \in H^3(0,L)^n; \zeta(0) = \zeta(L) = \zeta'(L) = 0,\ &\text{for (BC-A)},\\ \text{or } \zeta''(0) = \zeta(L) = \zeta'(L) = 0,\ &\text{for (BC-B)}\big\}. \end{aligned}$$

and its adjoint operator is given by

$$\mathcal{A}^*\zeta = \zeta_x + \zeta_{xxx} + \theta(A_n + K_n C)\zeta - B\zeta,$$

with domain

$$\begin{aligned} D(\mathcal{A}^*) = \big\{\zeta \in H^3(0,L)^n; \zeta(0) = \zeta'(0) = \zeta(L) = 0,\ &\text{for (BC-A)},\\ \text{or } \zeta''(0) = -\zeta(0), \zeta(L) = \zeta'(0) = \zeta(L) = 0,\ &\text{for (BC-B)}\big\}. \end{aligned}$$

By the stability proof in Theorem 3.1, we see that

$$\langle \mathcal{A}\zeta, \mu P\zeta\rangle_{L^2(0,L)^n} \leq 0, \tag{B.2}$$

whenever $\theta > \theta_0$, where θ_0 is defined in the proof of Theorem 3.1 and where function $\mu(\cdot)$ and matrix P are defined in the proof of Theorem 3.1. Inequality (B.2) implies that $\mathcal{A}$ is dissipative. The adjoint $\mathcal{A}^*$ is also dissipative and it can be shown by proving that $\langle \mathcal{A}^*\zeta, P\zeta\rangle_{L^2(0,L)^n} \leq 0$, by applying successive integrations by parts. This, similarly as in Appendix B.1, proves the well-posedness of the error equations, which along with the well-posedness of the initial system results in the well-posedness of the observer system (3.2)-(3.3), namely for any initial condition $\hat{v}^0 \in L^2(0,L)^n$, there exists a unique mild solution

$$\hat{v} \in C^0\left(0, \infty; L^2(0,L)^n\right),$$

for all $\theta > \theta_0$.

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