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ASYMPTOTIC CONTROLLABILITY AND ROBUST ASYMPTOTIC STABILIZABILITY*

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Abstract. This paper deals with asymptotically controllable systems for which there exists no smooth stabilizing state feedback. To investigate the robustness asymptotic stabilization property, a new class of hybrid feedbacks (with a continuous component and a discrete one) is introduced: the hybrid patchy feedbacks. The notion of solutions is a generalization of π-solutions and Euler solutions. It is proved that the origin of all globally asymptotically controllable systems can be globally asymptotically stabilized via a hybrid feedback with robustness with respect to measurement noise, actuator errors, and external disturbances.

Key words. control systems, feedback stabilization, controllability, measurement noise

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1. Introduction.

Let us consider the system

$$\dot{x} = f(x, u), \tag{1}$$

assuming that the control set $K \subset \mathbb{R}^m$ is a compact subset of $\mathbb{R}^m$ and that the map $f : \mathbb{R}^n \times K \to \mathbb{R}^n$ is locally Lipschitz in x, uniformly with respect to u, and continuous in u. We focus our study on systems that are asymptotically controllable, i.e., that satisfy, for every initial point x_0 in $\mathbb{R}^n$, there exists a measurable $u : [0, +\infty) \to K$ such that the (Carathéodory) solution of

$$\dot{x} = f(x, u(t)), \qquad x(0) = x_0,$$

is defined for all $t \geq 0$ and tends to the origin as t tends to infinity; and that satisfy a stability property (see Definition 2.5).

The general problem under consideration in this paper is the asymptotic stabilization via state feedback. Let us recall that *asymptotic stabilization* means that the following two properties hold:

- stability of the origin of the closed-loop system and
- convergence to the origin of all the solutions.

There exists a necessary condition [6, Theorem 1, (iii)] for the existence of a continuous control law which makes the origin globally asymptotically stable. But there are asymptotically controllable systems which do not satisfy this necessary condition and hence for which there does not exist a continuous stabilizing feedback [23, 6] (consider, e.g., the so-called Brockett's example).

Therefore we must consider discontinuous controllers to stabilize all asymptotically controllable systems. The first result concerning the use of such controllers is [24], but the author assumes that the system is analytic and completely controllable. The following property is proved in [8]:

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($\mathcal{P}$) Any asymptotically controllable systems can be asymptotically stabilized by a discontinuous controller.

The notion of solutions used by the authors is the notion of π-solutions (i.e., solutions with a feedback computed with an arbitrary small sampling schedule) [19]. In [1], the authors prove the property ($\mathcal{P}$) for all Carathéodory solutions by exhibiting a patchy feedback.

The controllers in [8, 1] are robust with respect to actuator and external disturbances (i.e., all systems perturbed by small actuator and external disturbances are asymptotically stable) but are not robust with respect to arbitrary small measurement noise. One way to robustly stabilize the system (1) is to enlarge the class of controllers as in [14], where the authors introduced the notion of a dynamic hybrid controller, which is computed with an external model. This controller compares, at suitable sampling times, the predicted state with the measured state. Due to the measurement noise these can differ substantially; therefore, as remarked in [21], it requires a resetting of the controller which may be difficult to construct. Moreover, with this controller, the origin is a robustly globally asymptotically stable equilibrium for π-solutions only. Here we prove also the existence of a hybrid controller (in the sense that it has a continuous component and a discrete one) which renders the origin a robustly globally asymptotically stable equilibrium for a *larger* class of solutions and, moreover, our feedback does not need a *resetting*.

In [21, 7], the authors proved the existence, for all asymptotically controllable systems, of a controller that is robust with respect to measurement noise and makes the origin of system (1) be a semiglobal *practical* stable equilibrium (i.e., driving all states in a given compact set of initial conditions into a specified neighborhood of the origin). (The case of the state-constraint stabilization is studied separately in [10].) It is proved in [22, section 5.4] that one can get a more general result: one can prove the existence of a sampling feedback making the origin be a robust global asymptotically stable equilibrium for all π-solutions with a sampling rate *sufficiently slow*. We exhibit in this paper a robust global asymptotically stabilizing controller for π-solutions with *any fast enough* sampling schedule, so for a larger class of solutions than those considered in [22].

The main result of this paper is Theorem 2.7: if (1) is asymptotically controllable, then there exists a hybrid feedback which makes the origin be a globally asymptotically stable equilibrium and with robustness with respect to measurement noise, actuator errors, and external disturbances. The class of solutions under consideration in this result includes π-solutions, Euler solutions (i.e., the limit of π-solutions as the sampling schedules tend to zero), and the *generalized* solutions (defined in [11, 12]).

To prove this result, we use some techniques of [1] to deduce from the asymptotic controllability a family of nested patchy vector fields, and we introduce hysteresis between an infinite number of controllers as it is done in [16] for two controllers. This allows us to define a *hybrid patchy feedback*. This gives rise to a hybrid system for which we rewrite the notion of solutions of [5] in the context of π-solutions (see Definition 2.1).

Note that this method was used in [18], where the authors used the special geometry of the chained system in dimension n. (In dimension 3 it is equivalent to the Brockett's example by a change of coordinates.) They exhibit a simple hybrid feedback (with only one discrete variable) making the origin of the chained system be a globally *exponentially* stable equilibrium with a robustness with respect to noise.

The paper is organized as follows. In section 2 we introduce the class of solutions of a system in closed loop with a hybrid feedback and we state our main result. In section 3 we define the class of hybrid patchy feedbacks and we give properties of π-solutions of systems in closed loop with such a feedback in section 4. Finally we prove our main result in section 5.

2. Definitions and statement of the main result. In this section we make more precise the notions of controller and solutions under consideration.

Let $\mathcal{A}$ be a nonempty totally ordered index set. The controllers under consideration in this paper admit the following description (see [25, 5]):

$$u = u(x, s_d), \qquad s_d = k_d(x, s_d^-), \tag{2}$$

where s_d evolves in the set $\{1,2\}^{\mathcal{A}}$, $u : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to K$ is continuous in x for each fixed s_d, $k_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to \{1,2\}^{\mathcal{A}}$ is a function, and s_d^- is defined, at this stage only formally, as

$$s_d^-(t) = \lim_{s<t,\ s\to t} s_d(s). \tag{3}$$

For this to make sense, we equip $\{1,2\}^{\mathcal{A}}$ with the discrete topology, i.e., every set is an open set. We say that the above controller is hybrid because it has a continuous component and a discrete one. Moreover, there is a delay since to evaluate $s_d^-(t)$ at time t, we need to know the past values of $s_d(t)$. Note that time cannot be reversed.

In this paper we are interested in a notion of robustness with respect to small noise. To this end, consider three functions satisfying our *standing regularity assumptions*, i.e.,

- ξ and ζ in $\mathcal{L}^\infty_{loc}(\mathbb{R}^n \times \mathbb{R}_{\geq 0}; \mathbb{R}^n)$ which are continuous in x in $\mathbb{R}^n$ for each t in $\mathbb{R}_{\geq 0}$,
- ψ in $\mathcal{L}^\infty_{loc}(\mathbb{R}^n \times \mathbb{R}_{\geq 0}; \mathbb{R}^m)$ which is continuous in x in $\mathbb{R}^n$ for each t in $\mathbb{R}_{\geq 0}$.

We introduce these functions as a measurement noise ξ, an actuator noise ψ, and an external noise ζ of (1) and study the following perturbed system:

$$\begin{cases} \dot{x}(t) = f(x(t), u(x(t) + \xi(x,t), s_d(t)) + \psi(x,t)) + \zeta(x,t), \\ s_d(t) = k_d(x(t) + \xi(x,t), s_d^-(t)). \end{cases} \tag{4}$$

As noted in [14, Remark 1.4], with the presence of ζ and the continuity of f in u, we can omit any explicit reference to actuator errors. So in the following we suppose that, for all x in $\mathbb{R}^n$ and for all $t \geq 0$, we have

$$\psi(x,t) = 0.$$

We have to clarify what we mean by a solution of the corresponding differential equation. The notion of solution is given in detail in [5] but, here, we want to study the implementation of the controller (2). Therefore we consider π-solutions that have a meaningful physical interpretation: it is an accurate model of the process in computer control. These π-solutions are studied in [8, 21, 15, 22, 13] in the case of an ordinary differential equation. Let π be a sampling schedule of $\mathbb{R}$, i.e., a sequence $(t_n)_{n\in\mathbb{Z}}$ such that, for all n in $\mathbb{Z}$, we have $t_n < t_{n+1}$ and $\lim_{n\to+\infty} t_n = \lim_{n\to-\infty} -t_n = +\infty$. Note that the upper and lower diameters of the sampling schedule are defined by (see [21])

$$\overline{d}(\pi) = \sup_{i\in\mathbb{Z}}(t_{i+1} - t_i), \qquad \underline{d}(\pi) = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i).$$

We rewrite the notion of solution given in [5] in the context of π-solutions.

DEFINITION 2.1. *Let π be a sampling schedule of $\mathbb{R}$, t_0 in π, $T > t_0$, and $(x_0, s_0) \in \mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$. We say that $(X, S_d) : [t_0, T) \to \mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$ is a* π-solution *of* (4) *on $[t_0, T)$ with initial condition (x_0, s_0) if*

1. *The map X is absolutely continuous on $[t_0, T)$.*
2. *We have, for all t in $[t_0, \min(t_1, T))$,*

$$S_d(t) = S_d(t_0), \tag{5}$$

for all i in $\mathbb{N}_{>0}$ and for all t in $[\min(t_i, T), \min(t_{i+1}, T))$,

$$S_d(t) = k_d(X(t_i) + \xi(X(t_i), t_i), S_d(t_{i-1})). \tag{6}$$

3. *We have, for all i in $\mathbb{N}$ and for almost all t in $[\min(t_i, T), \min(t_{i+1}, T))$,*

$$\dot{X}(t) = f\left(X(t), u(X(t_i) + \xi(X(t_i), t_i), S_d(t_i))\right) + \zeta(X(t), t).$$

4. *We have*

$$X(t_0) = x_0, \qquad S_d(t_0) = k_d(x_0 + \xi(x_0, t_0), s_0). \tag{7}$$

As usual we define Euler solutions as the limits of π-solutions as the sampling schedules tend to zero. More precisely, we have the following definition.

DEFINITION 2.2. *Given t_0 in $\mathbb{R}$, $T > t_0$ and $x_0 \in \mathbb{R}^n$, we say that $X : [t_0, T) \to \mathbb{R}^n$ is an* Euler solution *starting from x_0 of* (4) *on $[t_0, T)$ if, for each compact subinterval J of $[t_0, T)$, there exists a sequence π^n of sampling schedules of $\mathbb{R}$ and a sequence (X^n, S_d^n) of π^n-solutions of* (4) *defined on J such that*

$$\lim_{n\to\infty} \left(\sup_J |X^n - X| + \overline{d}(\pi^n) \right) = 0$$

and such that we have

$$X(t_0) = x_0. \tag{8}$$

Actually we are interested in a notion of solutions which is robust with respect to disturbances. For this reason we introduce a notion of generalized solutions (see [11, 12, 17]).

DEFINITION 2.3. *Let t_0 in $\mathbb{R}$, $T > t_0$ and x_0 in $\mathbb{R}^n$. We say that $X : [t_0, T) \to \mathbb{R}^n$ is a* generalized solution *starting from x_0 of* (4) *if we have* (8) *and if, for each J compact subinterval of $[t_0, T)$, there exist two sequences $(e^n)_{n\in\mathbb{N}}$ and $(d^n)_{n\in\mathbb{N}}$ of measurable functions $[t_0, +\infty) \to \mathbb{R}^n$ and a sequence $(X^n, S_d^n)_{n\in\mathbb{N}}$ of π-solutions of*

$$\begin{cases} \dot{x}(t) = f(x(t), u(x(t) + \xi(x(t), t), s_d(t)) + \zeta(x, t) + d^n(t), \\ s_d(t) = k_d(x(t) + \xi(x, t) + e^n(t), s_d^-(t)) \end{cases} \tag{9}$$

such that we have

$$\lim_{n\to+\infty} \left(\sup_J |X^n - X| + \sup_J |e^n| + \operatorname*{esssup}_J |d^n| \right) = 0. \tag{10}$$

By invoking Zorn's lemma exactly as in the proof of [20, Proposition 1], one can prove that every π-solution can be extended to a maximal solution. More precisely, we define the maximal extension taking account of all sufficiently fast sampling schedules π of $[0, +\infty)$.

Definition 2.4. *Let t_0 in $\mathbb{R}$, $T > t_0$, (x_0, s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$ and $d_0 > 0$. We say that $(X, S_d) : [t_0, T) \to \mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$ is a* d_0-maximal solution *starting from (x_0, s_0) of* (4) *on $[t_0, T)$, if the following properties hold:*

- *For all $T' < T$, there exists a sampling schedule π of $[0, +\infty)$ such that*

$$\overline{d}(\pi) \leq d_0 \tag{11}$$

and such that (X, S_d) is a π-solution starting from x_0 of (4) *on $[t_0, T')$.*

- *For all $T' > T$ and for all sampling schedules π of $[0, +\infty)$ such that* (11)*, there does not exist any π-solution (X', S'_d) starting from (x_0, s_0) and defined on $[t_0, T')$ such that the restriction of (X', S'_d) to $[t_0, T)$ is (X, S_d).*

We say that $X : [t_0, T) \to \mathbb{R}^n$ is a maximal Euler solution *starting from x_0 of* (4) *on $[t_0, T)$ if the following properties hold:*

- *For all $T' < T$, X is an Euler solution starting from x_0 of* (4) *on $[t_0, T')$.*
- *For all $T' > T$, there does not exist any Euler solution X' starting from x_0 of* (4) *on $[t_0, T')$ such that the restriction of X' to $[t_0, T)$ is X.*

We say that $X : [t_0, T) \to \mathbb{R}^n$ is a d_0-maximal generalized solution *starting from x_0 of* (4) *on $[t_0, T)$ if the following properties hold:*

- *For all $T' < T$, X is a generalized solution obtained as limit of π-solutions whose sampling schedule satisfies* (11) *starting from x_0 of* (4) *on $[t_0, T')$.*
- *For all $T' > T$, there does not exist any generalized solution X' obtained as limit of π-solutions whose sampling schedule satisfies* (11) *starting from x_0 of* (4) *on $[t_0, T')$ and such that the restriction of X' to $[t_0, T)$ is X.*

Let us recall that a function of class $\mathcal{K}_\infty$ is a function $\delta : [0, +\infty) \to [0, +\infty)$ which is continuous, strictly increasing, satisfying $\delta(0) = 0$ and $\lim_{\varepsilon \to +\infty} \delta(\varepsilon) = +\infty$. In the following we denote the closed ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ by $B(x, r)$. In our context our definition of robust global asymptotic stability is as follows (see [3]).

Definition 2.5. *The origin is said to be a* robustly globally asymptotically stable equilibrium *of the system* (4) *if the following properties hold:*

1. Existence of solutions: *For all $C > 0$, there exists $\chi_0 = \chi_0(C) > 0$ such that for all ξ, ζ satisfying our regularity assumptions and such that*

$$\sup_{x \in \mathbb{R}^n,\, t \geq 0} |\xi(x,t)| \leq \chi_0, \qquad \operatorname*{esssup}_{x \in \mathbb{R}^n,\, t \geq 0} |\zeta(x,t)| \leq \chi_0, \tag{12}$$

for all (x_0, s_0) in $B(0, C) \times \{1,2\}^{\mathcal{A}}$, and for all sampling schedules π of $\mathbb{R}$, there exists a π-solution of (4) *(resp., an Euler solution, resp., a generalized solution) starting from (x_0, s_0) (resp., starting from x_0) at $t_0 = 0$.*

2. Completeness: *Moreover, there exists $d_0 = d_0(C)$ such that all the d_0-maximal solutions (resp., maximal Euler solutions, resp., d_0-maximal generalized solutions) of* (4) *are defined on $[0, +\infty)$.*

3. Global stability: *There exists δ of class $\mathcal{K}_\infty$ such that, for all $\varepsilon > 0$, there exist $\chi_0 = \chi_0(\varepsilon) > 0$ and $d_0 = d_0(\varepsilon) > 0$ such that, for all ξ, ζ satisfying our regularity assumptions and* (12)*, for all (x_0, s_0) in $B(0, \delta(\varepsilon)) \times \{1,2\}^{\mathcal{A}}$, and for every d_0-maximal solution (X, S_d) of* (4) *(resp., maximal Euler solution X, resp., d_0-maximal generalized solution) starting from (x_0, s_0) (resp., starting from x_0) at $t_0 = 0$, one has*

$$X(t) \in B(0, \varepsilon) \quad \forall t \geq 0. \tag{13}$$

4. Global attractivity: *For all* $\varepsilon > 0$ *and for all* $C > 0$*, there exist* $T > 0$*,* $\chi_0 > 0$*, and* $d_0 > 0$ *such that, for all* ξ*,* ζ *satisfying our regularity assumptions and* (12)*, for each* (x_0, s_0) *in* $B(0, C) \times \{1, 2\}^{\mathcal{A}}$*, and for* d_0*-every maximal solution* (X, S_d) *of* (4) *(resp., maximal Euler solution* X*, resp.,* d_0*-maximal generalized solution) starting from* (x_0, s_0) *(resp., starting from* x_0*) at* $t_0 = 0$*, one has*

$$X(t) \in B(0, \varepsilon) \quad \forall t \geq T. \tag{14}$$

We recall the definition of global asymptotic controllability of the system (1).

Definition 2.6. *The system* (1) *is said to be* globally asymptotically controllable to the origin *if the following properties hold:*

1. *For each* x_0 *in* $\mathbb{R}^n$*, there exists an admissible control* u_0 *(i.e., a measurable function* $[0, +\infty) \to K$*) such that the maximal Carathéodory solution* X *starting from* x_0 *of*

$$\dot{x} = f(x, u_0) \tag{15}$$

is defined for all $t \geq 0$ *and satisfies* $X(t) \to 0$ *as* $t \to +\infty$*.*

2. *For each* $\varepsilon > 0$ *there exists* $C > 0$ *such that for each* x_0 *in* $B(0, C)$*, there is an admissible control* u_0 *as in* 1 *such that*

$$X(t) \in B(0, \varepsilon) \quad \forall t \geq 0.$$

Our main result is as follows.

Theorem 2.7. *Let* (1) *be a globally asymptotically controllable system to the origin. Then there exists a feedback control,* $u : \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}} \to K$*,* $k_d : \mathbb{R}^n \times \{1, 2\}^{\mathbb{N}} \to \{1, 2\}^{\mathbb{N}}$ *such that the origin is a robustly globally asymptotically stable equilibrium for the system* (4)*.*

Remark 2.8.

1. Note that in Theorem 2.7 we have the robust global asymptotic stability for π-solutions for *any fast enough* sampling rate since the only constraint on the sampling schedule is (11).

In [21, 7], only for the π-solutions with a sampling rate *sufficiently slow* are considered since, in these papers, it is assumed moreover that the lower diameters of the sampling schedules have a strictly positive lower bound.

See in particular the assumption in [21, Theorem 1],

$$|\xi(t)| \leq \underline{d}(\pi) \quad \forall t \geq 0. \tag{16}$$

Thus the class of solutions under consideration in Theorem 2.7 is *larger* than those considered in [21, 7].

Let us compare (12) and the inequality (16). Given a sampling schedule whose lower diameter is close to zero, this restriction forces the measurement noise to be close to zero. In our context the measurement noise and the lower diameter are completely independent.

Note that the controller given by [21] is not robust with respect to noise which does not satisfy (16) (consider the example of Artstein's circles). See also the discussion given in [22, section 4].

2. Note that Theorem 2.7 is false if in (12) the supremum *sup* is relaxed by *esssup*. See [17, Theorem 4.2], where it is proved, in an analogous situation, that there exists a noise ξ such that $\operatorname{esssup} |\xi| = 0$, $\sup |\xi| \neq 0$ and such that the origin of the perturbed closed-loop system is not an attractive equilibrium.

To prove Theorem 2.7 we need to introduce a class of hybrid patchy feedbacks (see section 3) whose continuous component is derived from a family of nested patchy vector fields (a slight generalization of patchy vector fields defined in [1]) and whose discrete component allows us to unite vector fields, as done in [17] for two vector fields, with robustness with respect to noise. Then we give basic properties of π-solutions of system (1) with a hybrid patchy feedback in section 4 and we prove Theorem 2.7 in section 5.

3. Definition of the hybrid patchy feedbacks.

Let Ω be a nonempty open connected subset of $\mathbb{R}^n$. The closure, the interior, and the boundary of Ω are written as $\operatorname{clos}(\Omega)$, $\operatorname{int}(\Omega)$, and $\partial\Omega$, respectively. We define the set $\mathcal{F} = \{1, \ldots, 7\}$. Let $\mathcal{A}$ be a nonempty totally ordered index set. Given a set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, we can define the solutions X of the differential inclusion

$$\dot{x} \in F(x)$$

as all absolutely continuous functions satisfying $\dot{X}(t) \in F(X(t))$ almost everywhere. We follow the ideas of [1, Definition 2.1], but we extend the definition to allow nested sets (as in [16]).

DEFINITION 3.1. *We say that* $(\Omega, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{A}})$ *is a* family of nested patchy vector fields *if*

1. *for all* $(\alpha, l) \in \mathcal{A} \times \mathcal{F}$*;* $\Omega_{\alpha,l}$ *is an open bounded subset of* $\mathbb{R}^n$*,*
2. *for all* $\alpha \in \mathcal{A}$ *and for all* $m > l \in \mathcal{F}$

$$\Omega_{\alpha,l} \subsetneq \operatorname{clos}(\Omega_{\alpha,l}) \subsetneq \Omega_{\alpha,m}; \tag{17}$$

3. *for all* α *in* $\mathcal{A}$*,* g_α *is a smooth vector field defined in a neighborhood of* $\operatorname{clos}(\Omega_{\alpha,7})$ *taking values in* $\mathbb{R}^n$*;*
4. *for all compact subsets* C *of* $\mathbb{R}^n$*, there exist* $r = r(C) > 0$ *and* $T = T(C) > 0$ *such that for all* $(\alpha, l) \in \mathcal{A} \times \mathcal{F}$ *satisfying* $\Omega_{\alpha,l} \subset C$*, all solutions* X *of*

$$\dot{x} \in g_\alpha(x) + B(0, r) \tag{18}$$

starting in $\partial\Omega_{\alpha,l} \setminus \bigcup_{\beta>\alpha} \Omega_{\beta,1}$ *are such that*

$$X(t) \in \operatorname{clos}(\Omega_{\alpha,l}) \quad \forall t \in [0, T].$$

5. *The sets* $(\Omega_{\alpha,1})_{\alpha\in\mathcal{A}}$ *form a locally finite covering of* Ω*.*

Remark 3.2. Some observations are in order.

- Roughly speaking, property 4 states that a part of $\operatorname{clos}(\Omega_{\alpha,l})$ is positively invariant in $[0, T]$ relative to the system (18). Note that we can characterize this property in terms of proximal normal by [9, Theorem 4.3.8] and we can redefine the notion of the patchy vector fields by using this concept of nonsmooth analysis as done in [4].
- On the one hand, given any compact set C, the positive real number $r(C)$ allows us to get robustness with respect to external disturbances. On the other hand, the gap between the different patches given by (17) allow us to get robustness with respect to measurement noise. See Definition 3.7 for a precise statement of admissible radius of measurement noise and external disturbances.
- Let us explain shortly why, to state our main result, we need to consider a family of seven nested patchy vector fields. Patches 2 and 6 define the dynamics of the discrete component of our hybrid controller (see Definition 3.4). Due to the

measurement noise, the switches (this notion will be precisely introduced in Definition 4.1) of the discrete variable can be located only in a neighborhood of Ω_2 and Ω_6 which are described by patches 1-3 and 5-7 (see Lemma 4.2). Patch 4 is only needed to describe π-solutions after the first switch (see Lemma 4.11). These seven patches are enough to state our main result and we show, for Artstein's circles, that we need to use so many patches (see Example 4.10).

EXAMPLE 3.3. *Let us give an example of such a family of nested patchy vector fields.*

We can construct a family of nested patchy vector fields for Artstein's circles. This system is one of the simplest which is not stabilizable by a continuous feedback and which admits a (nonrobust) discontinuous stabilizing feedback. This system is studied in several papers (see, e.g., [2, 21, 22, 16]*) and is defined by*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_1^2 + x_2^2 \\ -2x_1 x_2 \end{pmatrix}. \tag{19}$$

The integral curves of (19) *are*

- *the origin,*
- *the circles centered on the x_2-axis and tangent to the x_1-axis,*
- *the x_1-axis.*

Let us define the three smooth vector fields g_a, g_b, and $g_c : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} g_a(x_1, x_2) &= (-x_1^2 + x_2^2, -2x_1x_2)', \\ g_b(x_1, x_2) &= -g_a(x_1, x_2), \\ g_c(x_1, x_2) &= (0, 0). \end{aligned}$$

Let θ be in $\mathbb{R}$ the polar angle of a point $(x_1, x_2) \neq (0, 0)$. For all l in $\mathcal{F}$, let us define the open bounded sets $\Omega_{a,l}$, $\Omega_{b,l}$, and $\Omega_{c,l} \subset \mathbb{R}^2$ by

$$\begin{aligned} \Omega_{a,l} = &\left\{ x \in \mathbb{R}^2, -\frac{3\pi}{4} - \frac{l\pi}{30} < \theta < \frac{3\pi}{4} l + \frac{l\pi}{30} \right\} \cap \left\{ |x| > 1 - \frac{l}{14} \right\} \\ &\cap \left\{ \left(x_1 < 0 \text{ and } x_1^2 + \left(x_2 - 10 - \frac{l}{14} \right)^2 < \left(10 + \frac{l}{14} \right)^2 \right) \right. \\ &\qquad \left. \text{or } \left(x_1 \geq 0 \text{ and } x_1^2 + x_2^2 < \left(20 + \frac{l}{7} \right)^2 \right) \right\}, \\ \Omega_{b,l} = &\ \mathrm{sym}_{x_2}(\Omega_{a,l}), \\ \Omega_{c,l} = &\left\{ x \in \mathbb{R}^2, |x| < 1 + \frac{l}{7} \right\}, \end{aligned}$$

where sym_{x_2} is the symmetry with respect to the x_2-axis. Let $\mathcal{A} = \{a, b, c\}$ be lexicographically ordered ($a < b < c$) and $\Omega = \mathrm{int}(B(0, 10))$. It is easy to prove that

$$\left(\Omega, ((\Omega_{\alpha,l})_{l \in \mathcal{F}}, g_\alpha)_{\alpha \in \mathcal{A}}\right) \tag{20}$$

is a family of nested patchy vector fields. This is depicted in Figure 3.1*. To make the figure clearer, only two open sets and some values of the vector field g_a are shown.*

With such a family of nested patchy vector fields, we can define a class of hybrid controllers as those considered in section 2. To do this, we denote for all $s_d \in \{1, 2\}^{\mathcal{A}}$ the αth element of s_d by $s_{d,\alpha}$.

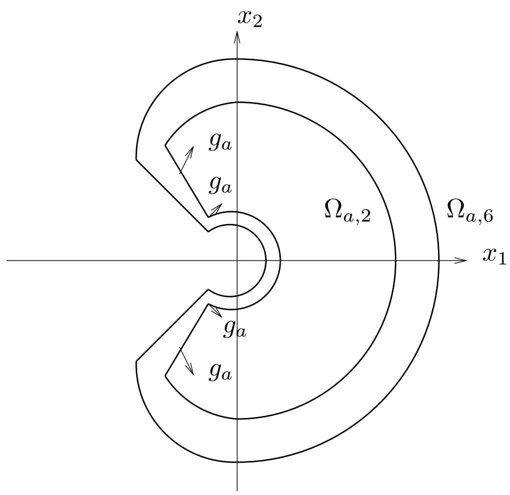

FIG. 3.1. *Artstein's circles as a family of nested patchy vector fields.*

DEFINITION 3.4. *Let $(\Omega, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{A}})$ be a family of nested patchy vector fields. Assume that for each α in $\mathcal{A}$, we can find a point k_α in K such that for each x in $\Omega_{\alpha,7}$, we have*

$$g_\alpha(x) = f(x, k_\alpha). \tag{21}$$

Let k_0 be an arbitrary point in K. Let (u, k_d) be the map defined by

$$\begin{array}{rcll} u : \{1,2\}^{\mathcal{A}} & \to & K, & \\ s_d & \mapsto & k_0 & \text{if } \{\beta \in \mathcal{A},\ s_{d,\beta} = 1\} \text{ is empty or infinite}, \\ & & k_\alpha & \text{if } \alpha = \max\{\beta \in \mathcal{A}, s_{d,\beta} = 1\}, \end{array} \tag{22}$$

and

$$\begin{array}{rcl} k_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} & \to & \{1,2\}^{\mathcal{A}}, \\ (x, s_d) & \mapsto & t_d, \end{array} \tag{23}$$

where t_d is the sequence defined, for all α in $\mathcal{A}$, by

$$\begin{array}{rl} t_{d,\alpha} = 1 & \text{if } x \in \mathrm{clos}(\Omega_{\alpha,2}), \\ t_{d,\alpha} = s_{d,\alpha} & \text{if } x \in \Omega_{\alpha,6} \setminus \mathrm{clos}(\Omega_{\alpha,2}), \\ t_{d,\alpha} = 2 & \text{if } x \notin \Omega_{\alpha,6}. \end{array} \tag{24}$$

We say that (u, k_d) is a hybrid patchy feedback *on Ω.*

Remark 3.5. This hybrid controller takes advantage of the existence of regions where different controllers k_α exist and, roughly speaking, allows the hybrid variable to choose between the different controllers. This is the main idea of the hysteresis as done in [17] to unite two controllers. Moreover, for any s_d in $\{1,2\}^{\mathcal{A}}$, the function $k_d(., s_d)$ is continuous except on the boundary of the sets defining the hysteresis. This remark is very helpful in particular to establish Lemma 4.2.

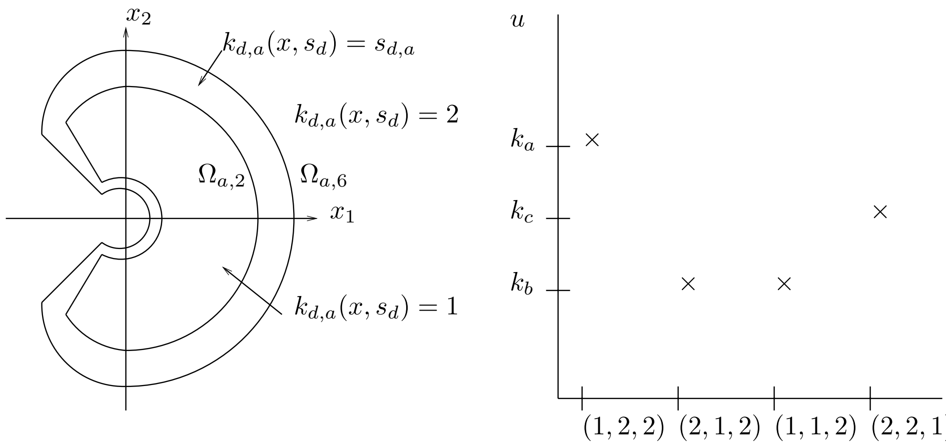

FIG. 3.2. *A hybrid patchy feedback. On the left is the $k_{d,a}$-component and on the right is the u-component.*

EXAMPLE 3.6. *Let us use the family of nested patchy vector fields* (20) *to define a hybrid patchy feedback for Artstein's circles.*

Let us define the controlled Artstein's circles by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} u(-x_1^2 + x_2^2) \\ -2ux_1x_2 \end{pmatrix} = f((x_1, x_2), u) \tag{25}$$

with u in $\mathbb{R}$. We remark that by denoting $k_a = 1$, $k_b = -1$, and $k_c = 0$, we have (21) *and thus we can define a hybrid patchy feedback, depicted in Figure* 3.2. *The $k_{d,a}$ component is on the left and the u-component (for some values) is on the right.*

Given a family of nested patchy vector fields $(\Omega, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{A}})$ it is easy to check from Definition 3.1 that for all x in $\mathbb{R}^n$, the set $C_x \subset \mathbb{R}^n$ defined by

$$C_x = \operatorname{clos}\left(\bigcup_{\alpha\in\mathcal{A},\ x\in\Omega_{\alpha,7}} \Omega_{\alpha,7} \right) \tag{26}$$

is a compact set. To investigate the robustness with respect to noise with a family of nested patchy vector fields we generalize [1, Definition 2.3] to a family of nested patchy vector fields and we introduce the next definition.

DEFINITION 3.7. *Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a continuous map such that for all $x \neq 0$, $\chi(x) > 0$.*

- *We say that χ is an admissible radius for the measurement noise if for all x in $\mathbb{R}^n$ and for all α in $\mathcal{A}$ such that x in $\Omega_{\alpha,7}$, we have*

$$\chi(x) < \frac{1}{2} \min_{l\in\{1,\dots,6\}} d(\mathbb{R}^n \setminus \Omega_{\alpha,l+1}, \Omega_{\alpha,l}). \tag{27}$$

- *We say that χ is an admissible radius for the external disturbances if for all x in $\mathbb{R}^n$, we have $\chi(x) \leq r(C_x)$, where C_x is defined by* (26) *and the corresponding $r > 0$ is guaranteed by* 4 *in Definition* 3.1.

There exists an admissible radius for the measurement noises and for the external disturbances. (Note that with (17), the right-hand side of inequality (27) is strictly positive.)

In Definition 3.4, u does not depend on x. Therefore only the function k_d depend on the measurement noise. Thus the notions of the admissible radius for the measurement noise and for the external disturbances are completely independent.

We need to consider sufficiently fast π-solutions. To define *sufficiently fast* π-solutions, let us introduce the following definition.

DEFINITION 3.8. *Let $p : \mathbb{R}^n \to \mathbb{R}_{>0}$ be a function continuous on $\mathbb{R}^n \setminus \{0\}$. We say that the sampling schedule π of a π-solution (X, S_d) defined on $[t_0, T)$ is* subordinate *to p if for all $i \in \mathbb{N}$ and for all $t \in [\min(t_i, T), \min(t_{i+1}, T))$, we have*

$$t_{i+1} - t_i \leq p(X(t_i) + \xi(X(t_i), t_i)). \tag{28}$$

Now we study the properties of π-solutions.

4. Properties of π-solutions. In this section we study the properties of π-solutions of a system in closed loop with a hybrid patchy feedback. Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ and let

$$(\Omega, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{A}})$$

be a family of nested patchy vector fields such that (21) holds. Let (u, k_d) be the hybrid patchy feedback on Ω defined by (22)–(24). Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be an admissible radius for the measurement noise and the external disturbances. Consider ξ and ζ satisfying our standing regularity assumptions and such that

$$\forall x \in \mathbb{R}^n, \quad \sup_{t\geq 0} |\xi(x,t)| \leq \chi(x), \quad \operatorname*{esssup}_{t\geq 0} |\zeta(x,t)| \leq \chi(x). \tag{29}$$

The perturbed system under consideration is

$$\begin{cases} \dot{x} = f(x, u(s_d)) + \zeta, \\ s_d = k_d(x + \xi, s_d^-). \end{cases} \tag{30}$$

Let $p : \mathbb{R}^n \to \mathbb{R}$ be a function continuous on $\mathbb{R}^n \setminus \{0\}$ and such that for all (ξ, ζ) with our regularity assumptions and (29), the following inequalities hold:[1]

A1. For all x in $\mathbb{R}^n$, $p(x) > 0$.

A2. For all x in $\mathbb{R}^n$,

$$p(x + \xi(x, 0)) < \frac{1}{4} \min_{l\in\{1,\ldots,6\}} \min_{\alpha\in\mathcal{A},\ x+\xi(x,0)\in\Omega_{\alpha,7}} \frac{d(\mathbb{R}^n \setminus \Omega_{\alpha,l+1}, \Omega_{\alpha,l})}{\sup_{y\in\Omega_{\alpha,7},\ u\in K} |f(y,u)|}.$$

A3. For all x in $\mathbb{R}^n$, we have $p(x + \xi(x, 0)) < T(C_x)$, where T is defined by 4 in Definition 3.1 and C_x is defined by (26).

The existence of such a function p results from the fact that for all l in $\{1, \ldots, 7\}$, $(\Omega_{\alpha,l})_{\alpha\in\mathcal{A}}$ is locally finite and results from (17).

DEFINITION 4.1. *A map $S_{d,\alpha} : [t_0, T) \to \{1, 2\}$ is said to have a* switch *at time t if $S_{d,\alpha}$ is not continuous at t.*

Given a sampling schedule π of $\mathbb{R}$ and a π-solution (X, S_d) of (30) and t in $[t_0, T)$, we denote the αth element of $S_d(t)$ by $S_{d,\alpha}(t)$. We start by locating the points where there exists α in $\mathcal{A}$ such that $S_{d,\alpha}$ may have a switch. Note that switches can occur only at sampling times, i.e., if there is a switch at time t, then there exists $i \in \mathbb{N}_{>0}$ such that $t = t_i$, $S_{d,\alpha}(t_i) \neq S_{d,\alpha}(t_{i-1})$.

[1] If $\sup |f(y, u)| = 0$, then assumption A2 forces no condition on p.

Lemma 4.2. *Let* (X, S_d) *be a* π*-solution of* (30) *whose sampling schedule is subordinate to* p *and such that* $S_{d,\alpha}$ *has a switch at time* $t_i \in (t_0, T)$.

- *If the switch is such that* $S_{d,\alpha}(t_{i-1}) = 1$ *and* $S_{d,\alpha}(t_i) = 2$*, then, for all* t *in* $[t_i, \min(t_{i+1}, T))$*,* $X(t)$ *is in* $\mathrm{clos}(\Omega_{\alpha,7}) \setminus \Omega_{\alpha,5}$.
- *If the switch is such that* $S_{d,\alpha}(t_{i-1}) = 2$ *and* $S_{d,\alpha}(t_i) = 1$*, then, for all* t *in* $[t_i, \min(t_{i+1}, T))$*,* $X(t)$ *is in* $\mathrm{clos}(\Omega_{\alpha,3}) \setminus \Omega_{\alpha,1}$.

Proof. Let α in $\mathcal{A}$ and $i \in \mathbb{N}_{>0}$ such that $S_{d,\alpha}(t_{i-1}) = 1$ and $S_{d,\alpha}(t_i) = 2$. Then due to (6) and (23)–(24), $X(t_{i-1}) + \xi(X(t_{i-1}), t_{i-1})$ is in $\Omega_{\alpha,6}$. Thus with (27), assumption A2, and (29), it follows directly that, for all t in $[t_{i-1}, \min(t_{i+1}, T))$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha,7})$. Similarly we prove that, for all t in $[t_i, \min(t_{i+2}, T))$, $X(t) \notin \Omega_{\alpha,5}$. Thus we obtain that, for all t in $[t_i, t_{i+1})$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha,7}) \setminus \Omega_{\alpha,5}$.

The case $S_d(t_{i-1}) = 2$ and $S_d(t_i) = 1$ is established in the same way. □

Let us claim a result of existence.

Lemma 4.3. *For all* (x_0, s_0) *in* $\mathbb{R}^n \times \{1, 2\}^{\mathcal{A}}$ *and for all sampling schedules* π *of* $\mathbb{R}$*, there exists a* π*-solution of* (30) *starting from* (x_0, s_0).

Proof. Let (x_0, s_0) be in $\mathbb{R}^n \times \{1, 2\}^{\mathcal{A}}$. Let $s_1 = k_d(x_0 + \xi(x_0, t_0), s_0)$ and α be in $\mathcal{A}$ such that $k_\alpha = u(s_1)$. From our regularity assumptions on f and ζ, the Carathéodory conditions are satisfied for the system

$$\dot{X} = f(X, k_\alpha) + \zeta, \qquad X(t_0) = x_0. \tag{31}$$

Let $t_0 \leq T \leq t_1$ and X defined on $[t_0, T)$ be a Carathéodory solution of (31). Let S_d be defined, for all t in $[t_0, T)$, by $S_d(t) = s_1$ for all t in $[t_0, T)$. Thus (X, S_d) is a π-solution of (30) starting from (x_0, s_0). □

We note that, as usual, maximal solutions of (30) must blow up if their domains of definition are bounded.

Lemma 4.4. *Let* $d_0 > 0$*,* ξ*, and* ζ *satisfy our regularity assumptions and* (29)*. Let* $T > t_0$ *and* (X, S_d) *be a* d_0*-maximal solution of* (30) *defined on* $[t_0, T)$*. Suppose that* $T < +\infty$*; then*

$$\limsup_{t \to T} \left(|X(t)| + \frac{1}{d(X(t), \partial\Omega)} \right) = +\infty.$$

Proof. Consider $d_0 > 0$, ξ, ζ satisfying our regularity assumptions and (29), $T > t_0$ and (X, S_d) a d_0-maximal solution defined on $[t_0, T)$. Suppose that the conclusion of Lemma 4.4 does not hold; i.e., there exists a compact subset C of Ω and times t_n in $[t_0, T)$ tending monotonically to T such that $(X(t_n), S_d(t_n))$ is in $C \times \{1, 2\}^{\mathcal{A}}$ for all n. We first establish the following.

Claim 4.5. For some n sufficiently large, for all $t \in [t_n, T)$, $X(t)$ is in the bounded open set $C + \mathrm{int}(B(0, 1))$.

Proof of Claim 4.5. If the conclusion of Claim 4.5 is not true, the continuity of X implies the existence of $s_n \in (t_n, T)$ such that

$$|X(t_n) - X(s_n)| = 1 \quad \text{and} \quad |X(t_n) - X(t)| < 1 \quad \forall t \in [t_n, s_n).$$

It follows that $X(t)$ is in the compact set $C + B(0, 1)$ for all t in $[t_n, s_n]$. Let

$$\rho = \max_{x \in C + B(0,1)} |\chi(x)|, \qquad \sigma = \sup_{\zeta \in B(0,\rho),\ x \in C + B(0,1),\ u \in K} |f(x, u) + \zeta|.$$

Then we have, for all t, s in $[t_n, s_n]$, $|X(t) - X(s)| \leq \sigma |t - s|$. Therefore, for n sufficiently large,

$$1 = |X(t_n) - X(s_n)| \leq \sigma |s_n - t_n| \leq \sigma |T - t_n|.$$

This cannot hold for n large enough and proves Claim 4.5. □

Claim 4.5 implies that there exists σ in $\mathbb{R}_{\geq 0}$ such that, for all (s,t) in $[t_n,T)$, we have

$$|X(s)-X(t)| \leq \sigma|s-t|.$$

By invoking the Cauchy criterion, it follows that $X(t)$ has a limit x_0 when t tends to T. Note moreover that by Definition 2.1, there exists $i \in \mathbb{N}$ such that T is in $(t_i,t_{i+1}]$ and thus, for all α in $\mathcal{A}$, $\lim_{t\to T,\, t<T} S_{d,\alpha}(t)$ exists. We denote $s_0 = S_d^-(T)$. Due to Lemma 4.3, there exists a π-solution $(\widetilde{X},\widetilde{S_d})$ starting from (x_0,s_0) and defined on $[t_0,\widetilde{T})$ with $\widetilde{T} > t_0$. Note that (X',S'_d) defined by

$$\begin{aligned} \forall t \in [t_0,T), \quad X'(t) = X(t), \quad S'_d(t) = S_d(t),\\ \forall t \in (T,T+\widetilde{T}), \quad X'(t) = \widetilde{X}(t-T), \quad S'_d(t) = \widetilde{S_d}(t-T), \end{aligned}$$

is a $\widetilde{\pi}$-solution of (30) defined on $[t_0,T+\widetilde{T})$ for the sampling schedule $\widetilde{\pi} = \pi \cup \{T\}$ whose restriction to $[t_0,T)$ is (X,S_d). Moreover $\widetilde{\pi}$ satisfies (11). So we have obtained a contradiction with the fact that (X,S_d) is a d_0-maximal solution. □

Now we can study the behavior of π-solutions between two switches. For all α in $\mathcal{A}$, let

$$\tau_\alpha = \sup\big\{T, X \text{ is a Carathéodory solution of } \dot{x} = f(x,k_\alpha) + B(0,\chi(x)) \text{ with } X(t) \in \Omega_{\alpha,7}\ \forall t \in [0,T)\big\}. \tag{32}$$

Note that there may exist α in $\mathcal{A}$ such that (s.t.) $\tau_\alpha = +\infty$. Let M be the subset of $\Omega \times \{1,2\}^{\mathcal{A}}$ defined by

$$M = \left\{(x,s_d), \text{s.t.} \left\{\begin{matrix} \{\beta \in \mathcal{A},\ s_{d,\beta} = 1\} \text{ is empty or infinite} \\ \text{or} \\ x \in \Omega_{\alpha,5},\ \text{where } \alpha = \max\{\beta,\ s_{d,\beta} = 1\} \end{matrix}\right\}\right\}. \tag{33}$$

Note that we have the property

$$\forall x_0 \in \Omega, \quad \exists s_0 \in \{1,2\}^{\mathcal{A}}, \quad (x_0,s_0) \in M. \tag{34}$$

EXAMPLE 4.6. *Let us particularize the set M for Artstein's circles. We have*

$$\begin{aligned} M = \Omega_{a,5} \times \{(1,2,2)\} \cup \Omega_{b,5} \times \{(s_d,1,2),\ s_d \in \{1,2\}\}\\ \cup\ \Omega_{c,5} \times \{(s_d,s'_d,1),\ s_d,s'_d \in \{1,2\}\} \cup \Omega \times \{(2,2,2)\}. \end{aligned}$$

In the following we denote $\overline{m} = \{0,\dots,m\}$ if $m \in \mathbb{N}$ and $\overline{m} = \mathbb{N}$ if $m = +\infty$.

LEMMA 4.7. *Let $0 < T \leq \infty$ and (X,S_d) be a π-solution of* (30) *whose sampling schedule is subordinate to p, defined on $[0,T)$ and starting in M. Then, there exist $m \in \mathbb{N} \cup \{+\infty\}$, an increasing sequence of time instants $(T_j)_{j\in\overline{m}}$ in $[0,T)$, a sequence $(\alpha_j)_{j\in\overline{m}}$ in $\mathcal{A}$, and a sequence $(k_j)_{j\in\overline{m}}$ in K such that if we let $T_0 = 0$ and $T_{m+1} = T$ (if $m < +\infty$), we have for all $j \in \overline{m}$ the following:*

1. *For all t in (T_j,T_{j+1}), $u(S_d(t)) = k_{\alpha_j}$.*
2. *The map X is a Carathéodory solution of $\dot{x} = f(x,k_{\alpha_j}) + \zeta$ on (T_j,T_{j+1}).*
3. *For all t in $[T_0,T_1)$, $X(t)$ is in $\Omega_{\alpha_0,5}$.*
4. *For all t in $[T_j,T_{j+1})$, $X(t)$ is in $\operatorname{clos}(\Omega_{\alpha_j,3})$, if $j \geq 1$.*
5. *The sequence $(\alpha_j)_{j\in\overline{m}}$ is strictly increasing.*
6. *The inequality $T_{j+1} - T_j < \tau_{\alpha_j}$ holds.*

Proof. Note first that the switches may occur only at a sampling time. Thus we can define $m \in \mathbb{N} \cup \{+\infty\}$ and a sequence of sampling times $(T_j)_{j \in \overline{m}}$ in $[0, T)$ at which switches occur. Between two switches, S_d is constant and thus there exist a sequence α_j in $\mathcal{A}$ and a sequence of admissible controls such that the statements 1 and 2 hold. We denote again by $(T_j)_{j \in \overline{m}}$ the subsequence of $(T_j)_{j \in \overline{m}}$ such that we have, for all $j \in \overline{m}$,

$$\alpha_j \neq \alpha_{j+1}. \tag{35}$$

Let us prove statement 3 and $\alpha_0 < \alpha_1$.

Due to Definition 3.1, there exists a finite number of α in $\mathcal{A}$ such that $X(T_0)$ is in $\Omega_{\alpha,1}$; then due to (23), (27), and (29), there exists α in $\mathcal{A}$ such that $S_{d,\alpha}(T_0) = 1$, and thus by (22), we have $\alpha_0 = \max\{\alpha, S_{d,\alpha}(T_0) = 1\}$.

This implies with (33) that $X(T_0)$ is in $\Omega_{\alpha_0,5}$. Similarly, we can prove that, for all β in $\mathcal{A}$ such that $\alpha < \beta$, we have $X(T_0)$ is not in $\Omega_{\beta,1}$. Thus (29), the fact that χ is an admissible radius for the external noise, (28), and assumption A3 on the function p yield, for all t in $[T_0, T_1)$, $X(t)$ is in $\Omega_{\alpha_0,5}$. Therefore with Lemma 4.2, we deduce that S_{d,α_0} cannot switch at time T_1 and, for all t in $[T_1, T_2)$, $S_{d,\alpha_0}(t) = 1$.

Moreover, due to (22), for all t in (T_1, T_2), we have $S_{d,\alpha_1}(t) = 1$. So, due to (22) and (35), $\alpha_0 < \alpha_1$.

Let us prove the following Claim 4.8, which implies statements 4 and 5 of Lemma 4.7.

Claim 4.8. For all $j > 0$, $j \in \overline{m}$, and for all t in $[T_j, T_{j+1})$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha_j,3})$ and $\alpha_j < \alpha_{j+1}$.

Proof of Claim 4.8. Let us prove Claim 4.8 by induction.

The inequality $\alpha_0 < \alpha_1$ implies with (22) that $S_{d,\alpha_1}(T_0) = 2$. Thus with Lemma 4.2, (28), and assumption A3 we have, for all t in $[T_1, T_2)$, $X(t)$ is in $\mathrm{clos}(\Omega_{\alpha_1,3}) \setminus \Omega_{\alpha_1,1}$. Thus with Lemma 4.2, S_{d,α_1} cannot switch at time T_2 and we have, for all t in $[T_2, T_3)$, $S_{d,\alpha_1}(t) = 1$. Moreover due to (22), for all t in (T_2, T_3), we have $S_{d,\alpha_2}(t) = 1$. So, due to (22) and (35), $\alpha_1 < \alpha_2$.

One can inductively deduce statements 4 and 5 for $j \geq 2$ similarly. ∎

To complete the proof of Lemma 4.7, note that statement 6 is a consequence of (32) and statements 2, 3, and 4 ∎

Remark 4.9. Some observations are in order.

- Lemma 4.7 states that for all π-solutions starting in M, the sequence α is strictly increasing and there exists a bound on the time between two switches. This result is analogous to [1, Proposition 3.1]. However, for all π-solutions that do not start in M, the sequence can be nonincreasing. See Example 4.10. Thus we need to add an initial switch to make all solutions enter in M. This is the result stated in Lemma 4.11.
- M is forward invariant for the system (30) for all ξ and ζ satisfying our regularity assumptions and (29).

EXAMPLE 4.10. *Figure* 4.1 *shows two different π-solutions of the hybrid patchy vector field of Artstein's circles by taking account of Lemmas* 4.2 *and* 4.7. *On the left, the x-component of π-solutions is depicted, and, on the right, we have the evolution of the controllers.*

- *The π-solution (X, S_d), a solid line, starts in M (with $x_0 \in \Omega_{a,5}$ and $s_0 = (1, 2, 2)$) at $T_0 = 0$ and has one switch at time T_1 (with $X(T_1) + \xi(X(T_1), T_1) \in \Omega_{c,2}$ and $S_d(T_1) = (1, 2, 1)$). With the notations of Lemma* 4.7, *we have $(\alpha_1, \alpha_2) = (a, c)$ (see Figure* 3.2), *which is strictly increasing.*

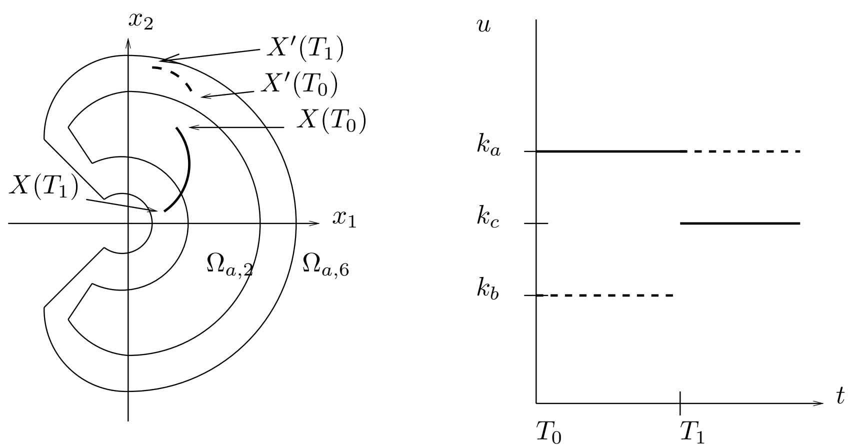

FIG. 4.1. *Two π-solutions with a hybrid patchy feedback. On the left is the x component and on the right is the evolution of the control.*

- *The other π-solution (X', S'_d), a dashed line, starts in the complement of M ($x_0 \in \Omega_{b,7} \setminus \Omega_{b,6}$ and $s_0 = (1,1,2)$) at $T_0 = 0$. There exists a measurement noise ξ satisfying* (29)*, vanishing for $t \neq T_0$, and such that $x_0 + \xi(x_0, T_0) \in \Omega_{b,6}$. Thus $S'_d(T_0) = (1,1,2)$. Therefore, with the notations of Lemma* 4.7*, we have $\alpha_1 = b$. There exists a time $T_1 > T_0$ such that $X'(T_1) \in \Omega_{b,7} \setminus \Omega_{b,6}$. Therefore $S'_d(T_1) = (1,2,2)$ and $\alpha_2 = a$. Thus $\alpha_1 > \alpha_2$ and the sequence is not strictly increasing.*

This example proves that the conclusions of Lemma 4.7 *do not hold for π-solutions which do not start in M (see statement* 5 *in Lemma* 4.7*).*

Due to property (34), we can add a switch to make all π-solutions enter in M. More precisely, let $(\Omega, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{A}})$ be a family of nested patchy vector fields. Assume that we have (21). Then we can define a map $u : \{1,2\}^{\mathcal{A}} \to K$ by (22) and $\widetilde{k}_d : \mathbb{R}^n \times \{1,2\}^{\mathcal{A}} \to \{1,2\}^{\mathcal{A}}$ by

$$
(36)\qquad \begin{aligned}
\widetilde{k}_d(x, s_d) \;&=\; k_d(x, s_d) \text{ if } \left\{ \begin{array}{c} \{\beta \in \mathcal{A},\ s_{d,\beta} = 1\} \text{ is empty or infinite} \\ \text{or} \\ x \in \Omega_{\alpha,4},\ \text{where } \alpha = \max\{\beta, k_{d,\beta}(x, s_d) = 1\} \end{array} \right. \\
&\text{else} \\
&=\; s_0,\ \text{where } s_0 \text{ is such that } x \in \Omega_{\alpha,2},\ \text{and } \alpha = \max\{\beta, s_{0,\beta} = 1\}.
\end{aligned}
$$

Consider now the system

$$
(37)\qquad \left\{ \begin{array}{l} \dot{x}(t) = f(x(t), u(s_d(t))) + \zeta(x, t), \\ s_d(t) = \widetilde{k}_d(x(t) + \xi(x, t), s_d^-(t)). \end{array} \right.
$$

We rewrite Lemma 4.7 for all initial conditions.

LEMMA 4.11. *Let $0 < T \leq \infty$ and let (X, S_d) be a π-solution of* (37) *whose sampling schedule is subordinate to p, defined on $[0, T)$ and starting in $\mathbb{R}^n \times \{1,2\}^{\mathcal{A}}$. Then, there exist $m \in \mathbb{N} \cup \{+\infty\}$, an increasing sequence of time-instants $(T_j)_{j\in\overline{m}}$ in $[0, T)$, a sequence $(\alpha_j)_{j\in\overline{m}}$ in $\mathcal{A}$, and a sequence $(k_j)_{j\in\overline{m}}$ in K such that if we let $T_0 = 0$ and $T_{m+1} = T$ (if $m < +\infty$), we have, for all j in $\overline{m}$, the following:*

1. *For all t in (T_j, T_{j+1}), $u(S_d(t)) = k_{\alpha_j}$.*
2. *The map X is a Carathéodory solution of $\dot{x} = f(x, k_{\alpha_j}) + \zeta$ on (T_j, T_{j+1}).*
3. *For all t in $[T_0, T_1)$, $X(t)$ is in $\Omega_{\alpha_0,4}$.*
4. *For all t in $[T_j, T_{j+1})$, $X(t)$ is in $\operatorname{clos}(\Omega_{\alpha_j,3})$ if $j \geq 1$.*
5. *The sequence $\alpha_1, \ldots, a_{m+1}$ is strictly increasing.*
6. *The inequality $T_{j+1} - T_j < \tau_{\alpha_j}$ holds.*

Proof. The proof of statements 1 and 2 of Lemma 4.11 is analogous of the proof of statements 1 and 2 of Lemma 4.7.

Due to (36), $X(T_0) + \xi(X(T_0), T_0)$ is in $\Omega_{\alpha_0,4}$. Then due to (27) and (29), we have $X(T_0)$ is in $\Omega_{\alpha_0,5}$. Similarly, we can prove that, for all β in $\mathcal{A}$ such that $\alpha < \beta$, we have $X(T_0)$ is not in $\Omega_{\beta,1}$. Therefore with (29), the fact that χ is an admissible radius for the external noise, (28), and assumption A3 on the function p, we have statement 3.

This implies that $X(T_1)$ is in $\Omega_{\alpha_0,4}$ and therefore $(X(T_1), S_d(T_1))$ is in M and we deduce statements 4 to 6 of Lemma 4.11 from statements 4 to 6 of Lemma 4.7. ∎

Remark 4.12. Note that if there exists a switch (i.e., if $m > 0$ in Lemma 4.11), then, after the first switch, we have $\widetilde{k}_d(X(t) + \xi(X(t), t)) = k_d(X(t) + \xi(X(t), t))$. And thus after the first switch (if it exists), every π-solution of (37) is a π-solution of (30) and in particular we have the conclusion of Lemma 4.4.

5. Use of the asymptotic controllability. Now we use properties of a differential system in closed loop with a hybrid patchy feedback. The purpose of this section is to prove Theorem 2.7. Let us prove a generalization of [1, Proposition 4.1] which yields a feedback that is robust with respect to measurement noise.

PROPOSITION 5.1. *Let* (1) *be globally asymptotically controllable to the origin. Then for every $0 < r < s$, there exist $T, R, \chi, d > 0$, an open subset of $\mathbb{R}^n$, $D^{r,s}$, and a feedback control, $u = u^{r,s} : \{1,2\}^{\mathbb{N}} \to K$, $k_d = k_d^{r,s} : \mathbb{R}^n \times \{1,2\}^{\mathbb{N}} \to \{1,2\}^{\mathbb{N}}$ satisfying*

$$B(0,s) \setminus \operatorname{int}(B(0,r)) \subset D^{r,s} \subset B(0,R) \tag{38}$$

such that for any measurable maps $\zeta, \xi : [0,+\infty) \to \mathbb{R}^n$ satisfying

$$\sup_{t\geq 0} |\xi(t)| \leq \chi, \qquad \operatorname*{esssup}_{t\geq 0} |\zeta(t)| \leq \chi, \tag{39}$$

and for any initial state x_0 in $D^{r,s} \setminus \operatorname{int}(B(0,r))$, and for any s_0 in $\{1,2\}^{\mathbb{N}}$, the perturbed system

$$\begin{cases} \dot{x} = f(x, u(s_d)) + \zeta, \\ s_d = k_d(x + \xi, s_d^-) \end{cases} \tag{40}$$

admits a π-solution (X, S_d) starting from (x_0, s_0). Moreover, for all (x_0, s_0) in $\mathbb{R}^n \times \{1,2\}^{\mathbb{N}}$ and for any d-maximal solution (X, S_d) starting from (x_0, s_0) and defined on $[0, T)$, there exists $t_{X,S_d} \leq T$, such that

$$|X(t_{X,S_d})| < r. \tag{41}$$

Proof. We follow the proof of [1, Proposition 4.1] and we prove Proposition 5.1 in four steps.

Step 1. Fix $0 < r < s$. For each x_0 in $B(0,s)$, there exist a piecewise constant admissible control $u_0 = u_{x_0}$ and some constant $T_0 = T_{x_0}$ such that there exists a solution $X_0 = x(.; x_0, u_0)$ of $\dot{x} = f(x, u_0)$ for which the inequality

$$|X_0(T_0)| < \frac{r}{2} \tag{42}$$

holds. Moreover, by continuity, we can assume that we have

$$m_0 := \inf_{t\in[0,T_0]} |X_0(t)| > \frac{r}{4} \tag{43}$$

and, by possibly redefining u_0, we may assume that X_0 takes different values at any two different points t, t' in $[0, T_0]$. Let $\tau_{0,0} = 0 < \cdots < \tau_{0,N_0} = T_0$ be the points of discontinuity for u_0 on $[0, T_0]$ and $k_{0,j}$ in K be the corresponding values of u_0, i.e., we suppose that, for all j in $\{0, \ldots, N_0 - 1\}$ and for all t in $(\tau_{0,j}, \tau_{0,j+1})$, we have $u_0(t) = k_{0,j}$. Define

$$M_0 = M_{x_0} = \sup_{t\in[0,T_0]} |X_0(t)|. \tag{44}$$

There exist some strictly positive constants $c_0 = c_{x_0}$, $\overline{\rho}_0 = \overline{\rho}_{x_0}$ and $\overline{\chi}_0 = \overline{\chi}_{x_0}$ such that, for any fixed τ in $[0, T_0]$, any strictly positive radius $\rho \leq \overline{\rho}_0$ and $\chi \leq \overline{\chi}_0$, any initial point $\overline{x}$ in $B(X_0(\tau), \rho)$, and any Carathéodory solution $X_{\rho,\chi}(.)$ of

$$\dot{x} \in f(x, u_0(t)) + B(0, \chi) \tag{45$_\chi$}$$

starting from $\overline{x}$ at time $t = \tau$, we have

$$\sup_{[\tau,T_0+\rho]} |X_{\rho,\chi}(t) - X_0(t)| < c_0(\rho + \chi). \tag{46}$$

Let two strictly positive reals $\rho_0 \leq \overline{\rho}_0$ and $\chi_0 \leq \frac{\overline{\chi}_0}{2}$ be such that, letting

$$\rho_{x_0,1} = \rho_{0,1} = \rho_0, \tag{47}$$

and for all j in $\{2, \ldots, N_0 + 1\}$,

$$\rho_{x_0,j} = \rho_{0,j} = \Sigma_{k=0}^{j-2} 7^k c_0^{k+1} 2\chi_0 + 7^{j-1} c_0^{j-1} \rho_0,$$

we have

$$7\rho_{x_0,N_0+1} < \frac{r}{2} \tag{48}$$

and

$$\max_j \rho_{x_0,j} < \frac{1}{7} \min\left(m_0 - \frac{r}{8}, \overline{\rho}_0\right), \tag{49}$$

where m_0 is defined by (43). Thus it follows directly by induction that if for any fixed $j = 1, \ldots, N_0$ and for any $\overline{x}$ such that

$$\overline{x} \in B(X_0(\tau_{0,j}), 7\rho_{0,j}),$$

we consider any Carathéodory solution $X_{\rho_{0,j},2\chi_0}(.)$ of $(45)_{2\chi_0}$ starting from $\overline{x}$ at time $\tau_{0,j}$, then one has

$$\sup_{t\in[\tau_{0,j},T_0+\rho_{0,j}]} |X_{\rho_{0,j},\chi_0}(t) - X_0(t)| < \rho_{0,j+1}. \tag{50}$$

Step 2. For j in $\{0, \ldots, N_0 - 1\}$ and for any $\overline{x}$ in $\mathbb{R}^n$, denote $\mathcal{A}_j(\overline{x}, t)$ the attainable set in time t for the Carathéodory solutions of $\dot{x} \in f(x, k_{0,j}) + B(0, 2\chi_0)$ starting from $\overline{x}$. Let us define for all l in $\{1, \ldots, 7\}$ and for all j in $\{1, \ldots, N_0 - 1\}$ the open sets

$$\Gamma_{x_0,j,l} = \bigcup_{\substack{\overline{x}\in \mathrm{int}(B(X_0(\tau_{0,j-1}), l\rho_{0,j})) \\ 0\leq t\leq \tau_{0,j}-\tau_{0,j-1}}} \mathcal{A}_j(\overline{x}, t)$$

and

$$\Gamma_{x_0,N_0,l} = \bigcup_{\substack{\overline{x}\in \operatorname{int}(B(X_0(\tau_{0,N_0-1}),l\rho_{0,j}))\\ 0\leq t\leq T_0+\rho-\tau_{0,N_0-1}}} \mathcal{A}_{N_0}(\overline{x},t).$$

Note that we have, for all j in $\{1,\dots,N_0\}$ and for all $l < l'$ in $\{1,\dots,7\}$,

$$\Gamma_{x_0,j,l} \subsetneq \operatorname{clos}(\Gamma_{x_0,j,l}) \subsetneq \Gamma_{x_0,j,l'}, \tag{51}$$

and, due to (44) and (50),

$$\Gamma_{x_0,j,l} \subset B(0, 7\rho_{x_0,j+1} + M_{x_0}). \tag{52}$$

Moreover (43), (49), and (50) yield

$$B\left(0, \frac{r}{8}\right) \subset \mathbb{R}^n \setminus \Gamma_{x_0,j,l}. \tag{53}$$

Finally, for all N_0-tuple s with entries in $\{1,\dots,7\}$, let us define

$$\Delta_{x_0,s} = \bigcup_{j=1}^{N_0} \Gamma_{x_0,j,s(j)}, \qquad \Delta_{x_0} = \Delta_0 = \Delta_{x_0,\mathbf{1}_{x_0}},$$

where, for any point x_0 in $B(0,s)\setminus \operatorname{int}(B(0,r))$, we denote $\mathbf{1}_{x_0}$ (whose length depends on x_0) the following constant sequence:

$$\begin{array}{rcl}\mathbf{1}_{x_0} = \mathbf{1}_0 : \{1,\dots,N_0\} & \to & \{1,\dots,7\},\\ l & \mapsto & 1.\end{array}$$

Let $g_{0,j} = g_{x_0,j}$ be the vector field on $\mathbb{R}^n$ defined, for all j in $\{1,\dots,N_0-1\}$, by

$$g_{0,j}(x) = f(x, k_{0,j}). \tag{54}$$

Let $\mathcal{F}$ be the finite set $\mathcal{F} = \{1,\dots,7\}$. We can claim that

$$(\Delta_0, ((\Gamma_{0,j,l})_{l\in\mathcal{F}}, g_{0,j})_{j\in\{1,\dots,N_0\}})$$

is a family of nested patchy vector fields. Indeed note first that due to (51) we have (17); second, we have (18) because there exists $T > 0$ such that all solutions of $(45)_{2\chi_0}$ starting in $\partial\Gamma_{x_0,j,l}\setminus\bigcup_{j'>j}\Gamma_{x_0,j',1}$ stay in $\operatorname{clos}(\Gamma_{x_0,j,l})$ for all t in $[0,T)$; third, the other properties to fulfill Definition 3.1 are obvious.

Let $p_0 : \mathbb{R}^n \to \mathbb{R}$ satisfy assumptions A1 to A3 for this family of nested patchy vector fields and define

$$d_0 = \inf_{x\in\Delta_0} p_0(x).$$

Moreover, due to (54), we define a hybrid patchy feedback (u^0, k_d^0) as considered in Definition 3.4 and thus a feedback control $(u^0, \widetilde{k}_d^0)$ defined by (36). We take χ_0 smaller and suppose that

$$0 < \chi_0 < \frac{1}{2} \min_{j\in\{1,\dots,N_0\}} \min_{l\in\{1,\dots,6\}} d(\mathbb{R}^n \setminus \Gamma_{0,j,l+1}, \Gamma_{0,j,l})$$

and note that χ_0 is an admissible radius for the external disturbances. Consider two measurable maps $\zeta, \xi : [0, +\infty) \to \mathbb{R}^n$ satisfying

$$\sup_{t\geq 0} |\xi(t)| \leq \chi_0, \qquad \operatorname*{esssup}_{t\geq 0} |\zeta(t)| \leq \chi_0,$$

and let (x_0, s_0) be an initial condition in $\Delta_0 \setminus B(0,r) \times \{1,2\}^{N_0}$. Due to Lemma 4.3, there exists (X, S_d) a d_0-maximal solution of (40) in closed loop with $(u^0, \widetilde{k}_d^0)$ starting from (x_0, s_0) and defined on $[0,T)$. Moreover, due to Lemma 4.11, there exist $H \in \mathbb{N} \cup \{+\infty\}$, a sequence of points $t_0 = 0 < \cdots < t_H \leq 2T_0$, and a sequence of indices $j_0, \ldots, j_H$ in $\{1, \ldots, N_0\}$, such that, for all h in $\{1, \ldots, H-1\}$,

$$\forall t \in [t_h, t_{h+1}), \quad X(t) \in \Gamma_{x_0, j_h, 7}, \tag{55}$$

$$t_{h+1} - t_h \leq \tau_{0, j_h}. \tag{56}$$

Note that due to Lemma 4.11, the sequence $j_1, \ldots, j_H$ described below is strictly increasing. Due to (52) and (55), (X, S_d) cannot blow up in $\Gamma_{x_0, j_h, 7}$ for all h in $\{0, \ldots, H-1\}$, and due to Lemma 4.4, (42), (48), (52), and (55), there exists $T^0_{X,S_d} \leq 2T_0$ such that we have the inequalities

$$\forall t \in [0, T^0_{X,S_d}), \quad |X(t)| \leq 7 \max_{j\in\{1,\ldots,N_0\}} \rho_{0,j+1} + M_0, \tag{57}$$

$$|X(T^0_{X,S_d})| < r. \tag{58}$$

Step 3. Since x_0 is in Δ_{x_0}, the family of open tubes $\{\Delta_{x_0},\ r \leq |x_0| \leq s\}$ forms an open covering of the compact set $B(0,s) \setminus \operatorname{int}(B(0,r))$. Let

$$\{\Delta_i,\ i \in \{1, \ldots, N(r,s)\}\}, \quad \Delta_i = \bigcup_{j=1}^{N_i} \Gamma_{i,j,1}, \quad \Gamma_{i,j,1} = \Gamma_{x_i, j, \mathbf{1}_{x_i}(j)},$$

be a finite subcover. Denote

$$k_{i,j} = k_{x_i, j}$$

and the vector field

$$g_{i,j}(x) = f(x, k_{i,j}) \tag{59}$$

defined on $\mathbb{R}^n$. The index set

$$\mathcal{A} = \{(i,j),\ i \in \{1, \ldots, N(r,s)\},\ j \in \{1, \ldots, N_i\}\}$$

can be totally ordered by letting

$$(i,j) < (h,k) \quad \text{if either } i < h \text{ or else } i = h,\ j < k. \tag{60}$$

Let

$$D^{r,s} = \bigcup_{i=1}^{N(r,s)} \Delta_i.$$

We can now define a family of nested patchy vector fields on $\Omega = D^{r,s}$. Let, for all $(\alpha, l) \in \mathcal{A} \times \mathcal{F}$, $\Omega_{\alpha,l}$ be the open set $\Gamma_{i,j,l}$, where $(i,j) = \alpha$. Note that due to (51), for all $m > l$ in $\mathcal{F}$ and for all α in $\mathcal{A}$, (17) and (18) can be proved as in Step 2. Therefore we can claim that

$$(\Omega, ((\Omega_{\alpha,l})_{l\in\mathcal{F}}, g_\alpha)_{\alpha\in\mathcal{A}})$$

is a family of nested patchy vector fields. Let $p^{r,s} : \mathbb{R}^n \to \mathbb{R}$ satisfy assumptions A1 to A3 for this family of nested patchy vector fields and

$$d^{r,s} = \inf_{x\in\Omega} p^{r,s}(x). \tag{61}$$

Moreover, due to (59), we can define a hybrid patchy feedback $(u^{r,s}, k_d^{r,s})$ and thus a feedback control $(u^{r,s}, \widetilde{k}_d^{r,s})$ as in (36). We let

$$\chi^{r,s} = \min_{1\leq i\leq N(r,s)} \chi_{x_i}, \tag{62}$$

which is an admissible radius for the external disturbances. We can choose $\chi^{r,s}$ smaller and suppose that

$$0 < \chi^{r,s} < \frac{1}{2} \min_{(i,j)\in\mathcal{A}} \min_{l\in\{1,\ldots,6\}} d(\mathbb{R}^n \setminus \Gamma_{i,j,l+1}, \Gamma_{i,j,l}).$$

Then $\chi^{r,s}$ is an admissible radius for the measurement noise on $D^{r,s}$.

Step 4. For all x_0 in $B(0,s)\setminus B(0,r)$, let $T_{x_i} > 0$ be defined just at the beginning of Step 1, and let $\rho_{x_i} >$ be defined by $(45)_{\chi_0}$–(46). Moreover let $\chi^{r,s} > 0$ be defined by (62) and $d^{r,s}$ be defined by (61). Let

$$T = 2\sum_{i=1}^{N(r,s)} T_{x_i},$$

and consider two measurable maps ξ and $\zeta : [0,+\infty) \to \mathbb{R}^n$ such that

$$\sup_{t\geq 0} |\xi(t)| \leq \chi^{r,s}, \qquad \operatorname*{esssup}_{t\geq 0} |\zeta(t)| \leq \chi^{r,s}.$$

Let (x_0, s_0) be an initial condition in $D^{r,s} \setminus B(0,r) \times \{1,2\}^{\mathcal{A}}$. Due to Lemma 4.3, there exists (X, S_d) a $d^{r,s}$-maximal solution of (40) in closed loop with $(u^{r,s}, \widetilde{k}_d^{r,s})$ starting from (x_0, s_0). Moreover, due to properties established in Step 3 and Lemma 4.11, there exist $H \in \mathbb{N}\cup\{+\infty\}$, a sequence of points $t_0 = 0 < \cdots < t_H \leq T$, and a sequence of indices $\alpha_1, \ldots, \alpha_H$ in $\mathcal{A}$, such that, for all h in $\{0,\ldots,H-1\}$,

$$\forall t \in [t_h, t_{h+1}), \quad X(t) \in \Gamma_{\alpha_h,7}, \tag{63}$$

$$t_{h+1} - t_h < \tau_{\alpha_h}. \tag{64}$$

Note that due to Lemma 4.11, the sequence $\alpha_1, \ldots, \alpha_H$ described above is strictly increasing. Due to (52) and (63), (X, S_d) cannot blow up in $\Gamma_{\alpha_h,7}$ for all h in $\{0,\ldots,H-1\}$, and due to Lemma 4.4, (63), there exists $T_{X,S_d} \leq T$ such that we have the inequalities

$$\forall t \in [0,T], \quad |X(t)| < R, \tag{65}$$

$$|X(T_{X,S_d})| < r, \tag{66}$$

where R is defined by

$$R = \sup_{1 \le i \le N(r,s), 1 \le j \le N_i} \{7\rho_{x_i,j} + M_{x_i}\}. \tag{67}$$

This completes the proof of Proposition 5.1. □

PROPOSITION 5.2. *Let* (1) *be globally asymptotically controllable to the origin. Then for any fixed* $\varepsilon > 0$, *there exists* $\sigma > 0$ *such that for every* $0 < r < s \le \sigma$, *there exist* $T, R, \chi, d > 0$, *an open subset of* $\mathbb{R}^n$, $D^{r,s}$, *and a feedback control,* $u = u^{r,s} : 2^{\mathbb{N}} \to K$, $k_d = k_d^{r,s} : \mathbb{R}^n \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *as in Proposition* 5.1 *with*

$$R < \varepsilon. \tag{68}$$

Proof. The proof is similar to the proof of [1, Proposition 4.2] and consists of properly choosing the piecewise constant admissible control u_{x_0} for each point x_0 in $B(0, s) \setminus \mathrm{int}(B(0, r))$.

To do this, fix $\varepsilon > 0$. Since (1) is globally asymptotically controllable, there exists $\sigma = \sigma(\varepsilon) > 0$ such that, for any fixed $0 < r < s \le \sigma$, the conclusions of Proposition 5.1 hold together with

$$M_{x_0} < \frac{\varepsilon}{2},$$
$$\max_{j \in \{1,\ldots,N_0+1\}} \rho_{x_0,j} < \frac{\varepsilon}{14}$$

for all x_0 satisfying $r < |x_0| < s$. With (65) and (67), this implies (68). □

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7.

Part 1: Definition of the feedback control. Let $(r_n)_{n\in\mathbb{Z}}$ and $(s_n)_{n\in\mathbb{Z}}$ be two decreasing sequences of strictly positive numbers such that

- for all n in $\mathbb{Z}$, we have $r_{n-1} < s_n$;
- s_n converges to zero as $n \to +\infty$;
- r_{-n} converges to infinity as $n \to +\infty$.

Let $T_n = T(r_n, s_n)$, $R_n = R(r_n, s_n)$, $\chi_n = \chi(r_n, s_n)$ be three sequences of strictly positive numbers and consider a sequence of hybrid patchy feedbacks

$$(D^{r_n,s_n}, u^{r_n,s_n}, k_d^{r_n,s_n}, ((\Gamma^n_{i,l})_{l\in\{1,\ldots,7\}}, k_i^n)_{i\in\{1,\ldots,N_n\}})$$

such that

$$R_n < \frac{1}{n} \quad \forall n \in \mathbb{N}_{>0} \tag{69}$$

as in Proposition 5.2. The index set

$$\mathcal{B} = \{(n, i),\ n \in \mathbb{Z},\ i \in \{1, \ldots, N_n\}\}$$

can be totally ordered with the same relation of order as (60), i.e., by letting

$$(n, i) < (m, j) \quad \text{if either } n < m \text{ or else } n = m,\ i < j.$$

Then we have the following family of nested patchy vector fields on $\mathbb{R}^n \setminus \{0\}$:

$$(\mathbb{R}^n \setminus \{0\}, ((\Gamma^m_{i,l})_{l\in\mathcal{F}}, k_i^m),\ (m, i) \in \mathcal{B}).$$

We can define a hybrid patchy feedback (u, k_d) on $\mathbb{R}^n \setminus \{0\}$ as in Definition 3.4. Let $\chi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_{>0}$ be a continuous map satisfying

$$\chi(x) \leq \min\left(\chi_n, \frac{|x|}{2}\right) \quad \text{if } x \in D^{r_n,s_n} \setminus \bigcup_{m>n} D^{r_m,s_m}. \tag{70}$$

We define $\chi(0) = 0$. The map χ is continuous at 0 and then χ is an admissible radius for the measurement noise and the external disturbances. Let $(u, \widetilde{k}_d)$ be the feedback control defined by (36) for the hybrid patchy feedback (u, k_d). Let us prove that $(u, \widetilde{k}_d)$ is a global robust stabilizing controller on $\mathbb{R}^n$, i.e., that the origin of system (37) is a robust globally asymptotically stable equilibrium as stated in Theorem 2.7.

Part 2: Theorem 2.7 for π-solutions. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a function continuous on $\mathbb{R}^n \setminus \{0\}$ satisfying the properties A1, A2, and A3.

Existence of π-solutions. Consider ξ, ζ satisfying our regularity assumptions. Let (x_0, s_0) be in $\mathbb{R}^n \times \{1, 2\}^{\mathcal{B}}$. Let $s_1 = \widetilde{k}_d(x_0 + \xi(x_0, 0), s_0)$ and α be in $\mathcal{B}$ such that $k_\alpha = u(s_1)$. From our regularity assumptions on f and ζ, the Carathéodory conditions are satisfied for system (31). Let $0 < T \leq t_1$ and $X : [0, T) \to \mathbb{R}^n$ be a Carathéodory solution of (31). Let S_d be defined by $S_d(t) = s_1$ for all t in $[0, T)$. The map (X, S_d) is a π-solution of (37) starting from (x_0, s_0).

Completeness and global stability for π-solutions. Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that $\varepsilon < R_{-n}$. Such an R_{-n} exists because we have $r_{-n} \leq R_{-n}$ and r_{-n} tends to infinity as $n \to +\infty$. Let $\chi_0 > 0$ be defined by

$$\chi_0 = \inf_{x \in B(0,R_{-n}) \setminus B(0,r_{-n})} \chi(x), \tag{71}$$

and let $d_0 > 0$ satisfy the inequalities

$$d_0 < \frac{d(\mathbb{R}^n \setminus B(0, s_{-n}), B(0, r_{-n}))}{\max_{x \in B(0,s_{-n}),\, u \in K} |f(x, u)|}, \tag{72}$$

and for all x in $B(0, R_{-n}) \setminus B(0, r_{-n})$, for all y in $B(0, \chi(x))$,

$$d_0 < p(x + y). \tag{73}$$

Note that due to (70), we have $d_0 > 0$ and $\chi_0 > 0$. Let ξ, ζ satisfy our regularity assumptions and (12). Let (X, S_d) be a d_0-maximal solution of (37) on $[0, T)$ starting from (x_0, s_0) with $|x_0| < s_{-n}$ and (11).

Note that due to (71)–(73), for all $i \in \mathbb{N}$ such that $X(t_i)$ is in $B(0, R_{-n}) \setminus B(0, r_{-n})$, we have (28) and, for all t such that $X(t)$ is in $B(0, R_{-n}) \setminus B(0, r_{-n})$, we have (29).

Therefore, due to Proposition 5.1 and to the definition of the feedback control, if there exists $i \in \mathbb{N}$ such that $X(t_i)$ is in $B(0, s_{-n}) \setminus B(0, r_{-n})$, then there exists $j > i$ such that $X(t_j)$ is in $B(0, r_{-n})$ and for all t in $[i, j]$, $X(t)$ is in $B(0, R_{-n})$. Moreover, due to (72), if there exists $i \in \mathbb{N}$ such that $X(t_i)$ is in $B(0, r_{-n})$, then, for all t in $[t_i, t_{i+1}]$, we have $X(t)$ is in $B(0, s_{-n})$.

Thus we have, for all t in $[0, T)$,

$$|X(t)| \leq R_{-n}. \tag{74}$$

Therefore the conclusion of Lemma 4.4 cannot hold ($\limsup_{t \to T} |X(t)| \neq +\infty$) and thus we have $T = +\infty$ and the maximality property.

Finally, note that $\delta(\varepsilon) = s_{-n}$ tends to $+\infty$ as ε tends to infinity because when ε tends to infinity, n tends to infinity, r_{-n} tends to infinity, and we have $r_{-n-1} < s_{-n}$. Thus we have the stability property.

Global attractivity for π-solutions. Let $\varepsilon > 0$ and $C > 0$. Let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \varepsilon$ and such that $\delta < r_{-n}$. Let $d_0 > 0$ and $\chi_0 > 0$ be defined, respectively, by

$$d_0 = \inf_{x \in B(0,R_{-n}) \setminus B(0,r_n)} p(x), \tag{75}$$

$$\chi_0 = \inf_{x \in B(0,R_{-n}) \setminus B(0,r_n)} \chi(x). \tag{76}$$

Let ξ, ζ satisfying our regularity assumptions and (12). Let (X, S_d) be a π-solution defined on $[0, +\infty)$, starting from (x_0, s_0) whose sampling schedule satisfies $\overline{d}(\pi) < d_0$ and whose initial condition satisfies $|x_0| < C$. Due to Proposition 5.1, there exists $\widetilde{T}$ in $[0, T_{-n} + T_{-n+1} + \cdots + T_n]$ such that $|X(\widetilde{T})| < r_n$. Let $T' = \inf\{t \in [0, \widetilde{T}],\ |X(t)| < s_n\}$. Then due to the stability property and as $R_n < \frac{1}{n}$, we have

$$\forall t \geq \widetilde{T}, \quad |X(t)| \leq \frac{1}{n}.$$

Therefore we have (14) with $T = T_{-n} + \cdots + T_n$.

Part 3: Theorem 2.7 for the generalized solutions.

Existence and completeness for the generalized solutions. This results from the fact that every π-solution of (37) is a generalized solution of (37).

Global stability and global attractivity for the generalized solutions. Let $\varepsilon > 0$. Let $\chi_0 > 0$, $d_0 > 0$ and δ of $\mathcal{K}_\infty$, be such that we have the stability property (13) for all π-solutions of (37) whose sampling schedule satisfies (11) and for all ξ, ζ satisfying our regularity assumptions and (12).

Let X be a generalized solution of (37) starting from $x_0 \in B(0, \frac{\delta(\varepsilon)}{2})$ with ξ, ζ satisfying our regularity assumptions and

$$\sup_{x \in \mathbb{R}^n,\, t \geq 0} |\xi(x,t)| \leq \frac{\chi_0}{2}, \qquad \operatorname*{esssup}_{x \in \mathbb{R}^n,\, t \geq 0} |\zeta(x,t)| \leq \frac{\chi_0}{2} \tag{77}$$

and obtained as limit of π-solutions (X^n, S_d^n) whose sampling schedule satisfies (11). Let us prove (13).

For n sufficiently large, we have

$$\sup_J |e_n(t)| + \operatorname*{esssup}_J |d_n(t)| < \frac{\chi_0}{2} \tag{78}$$

for all J compact subinterval of $[0, T)$. Then for n sufficiently large, (X^n, S_d^n) is a π-solution of (37) whose sampling schedule satisfies (11) with a disturbance satisfying (12). Then we have (13) for this sequence of π-solutions. Therefore we have (13) for the generalized solution X.

The global attractivity property can be proved similarly.

Part 4: Theorem 2.7 for Euler solutions.

Existence and completeness for Euler solutions. Let x_0, s_0 in $\mathbb{R}^n \times \{1,2\}^{\mathcal{B}}$ and π_n be a sequence of sampling schedules such that $\overline{d}(\pi_n) \to 0$ as n tends to infinity. Let (X^n, S_d^n) be a π_n-solution of (37), starting from (x_0, s_0) and defined on $[0, +\infty)$. Due to Part 2 of the proof of Theorem 2.7, this sequence exists for n sufficiently large and there exists R such that, for all t in $[0, +\infty)$ and for n sufficiently large, we have

$$|X_n(t)| < R.$$

Therefore with Ascoli's theorem, we can define X an Euler solution defined on $[0, +\infty)$ and starting from x_0.

Global stability and global attractivity for Euler solutions. Let $\varepsilon > 0$. Let $\chi_0 > 0$, $d_0 > 0$ and δ of $\mathcal{K}_\infty$ be such that we have the stability property (13) for all d_0-maximal solutions of (37) and for all ξ, ζ satisfying our regularity assumptions and (12).

Let X be an Euler solution of (37) starting from $x_0 \in B(0, \frac{\delta(\varepsilon)}{2})$ with ξ, ζ satisfying our regularity assumptions and (12) and obtained as limit of π-solutions (X^n, S_d^n) satisfying $\overline{d}(\pi_n) \to 0$ as n tends to infinity.

Let us prove (13).

For n sufficiently large, we have $\overline{d}(\pi_n) < d_0$. Then for n sufficiently large, (X^n, S_d^n) is a π-solution of (37) whose sampling schedule satisfies (11) with a disturbance satisfying (12). Then we have (13) for this sequence of π-solutions. Therefore we have (13) for the generalized solution X.

The global attractivity can be proved similarly.

This completes the proof of Theorem 2.7. □

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