Systems & Control Letters 164 (2022) 105238

Contents lists available at ScienceDirect

# Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

# Output feedback stabilization of reaction–diffusion PDEs with a non-collocated boundary condition ☆

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ARTICLE INFO

*Article history:*
Received 19 October 2021
Received in revised form 24 February 2022
Accepted 9 April 2022
Available online xxxx

*Keywords:*
Reaction–diffusion PDEs
Non-collocated boundary conditions
Output feedback
Finite-dimensional control

ABSTRACT

This paper addresses the control design problem of output feedback stabilization of a reaction–diffusion PDE with a non-collocated boundary condition. More precisely, we consider a reaction–diffusion equation with a boundary condition describing a proportional relationship between the left and right Dirichlet traces. Such a boundary condition naturally emerges, e.g., in the context of reaction–diffusion partial differential equations presenting a transport term and with a periodic Dirichlet boundary condition. The control input takes the form of the left Neumann trace. Finally, the measurement is selected as a pointwise Dirichlet measurement located either in the domain or at the boundary. The adopted control strategy takes the form of a finite-dimensional controller coupling a state feedback and a finite-dimensional observer. The stability of the closed-loop system is obtained provided the order of the observer is selected to be large enough. Finally, we extend this result to the establishment of an input-to-state stability estimate with respect to an additive perturbation in the application of the boundary control.



## 1. Introduction

Control design for 1-D reaction–diffusion partial differential equations (PDEs) with collocated boundary conditions has been widely studied in the literature in a great variety of configurations [1,2]. By collocated, we mean here that the main reaction–diffusion PDE is accompanied with a set of two boundary conditions, each one describing the behavior of the system at one of the two boundaries. In contrast, a non-collocated boundary condition refers to a single condition mixing the behavior of the system at both boundaries simultaneously. In the case of reaction–diffusion PDEs, such non-collocated boundary conditions can be used, e.g., to describe the dynamics of the heat distribution on a ring [3], closed circuit cooling or heat transfer in heterogeneous materials [4]. Even if non-collocated boundary conditions have been widely investigated in the context of the boundary control of hyperbolic systems, see [5–7] and references therein, the case of parabolic PDEs remains essentially unexplored. One of the main reasons is that the control design of such parabolic PDEs is particularly challenging due to inherent technical difficulties in the application of backstepping control design procedures [2] (which are particularly successful in the collocated setting) in the presence of non-collocated boundary conditions. To the best of our knowledge, this type of control design problem for a reaction–diffusion PDE (with a collocated boundary condition different from the one studied in this paper) was solely addressed in [8,9] in the case of a state-feedback using spectral reduction methods [1,10,11]. The problem of output feedback stabilization of the linearized Kuramoto–Sivashinsky PDE by means of bounded input and output operators was reported in [12]. It is worth noting that a non-collocated boundary condition can sometimes emerge due to the application of a static output feedback control strategy when the input and the output are non-collocated; see for example [13] for such a situation in the case of a wave equation. In this case, the non-collocated boundary condition helps to achieve the stabilization of the plant. In sharp contrast, the non-collocated boundary condition considered in this paper is one of the two sources of instability for the plant. Hence, its harmful effect needs to be mitigated by an adequate control strategy. In this general context, we solve for the first time the problem of boundary output feedback stabilization of a reaction–diffusion PDE presenting a non-collocated boundary condition by means of a pointwise measurement.

From a general perspective, it is well-known that the collocated setting confers strong structural properties to 1-D reaction–diffusion PDEs. Indeed, the Sturm–Liouville theory [14] shows

☆ The work of H. Lhachemi has been partially supported by ANR PIA funding: ANR-20-IDEES-0002. The work of C. Prieur has been partially supported by MIAI@Grenoble Alpes (ANR-19-P3IA-0003).

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https://doi.org/10.1016/j.sysconle.2022.105238
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that the unbounded operators associated with such PDEs are self-adjoint, present real eigenvalues, and the corresponding unit eigenvectors form a Hilbert basis of the state-space. In contrast, such structural properties may be lost when considering non-collocated boundary conditions. In particular, the underlying unbounded operators are (in general) not self-adjoint anymore. This may give rise to interesting phenomena such as time-domain oscillations induced by complex conjugate eigenvalues.

We address in this paper the control design problem of output feedback stabilization of reaction–diffusion PDEs with a boundary condition describing a proportional relationship between the left and right Dirichlet traces. As we shall see, such a boundary condition can emerge, e.g., after a change of variable in the context of reaction–diffusion PDEs with a transport term and a periodic Dirichlet boundary condition. The control input takes the form of a Neumann boundary trace. The measurement is selected as an arbitrarily located pointwise Dirichlet measurement. The adopted control strategy takes the form of a finite-dimensional observer [15–19] coupled with a finite-dimensional state-feedback. The design of the finite-dimensional observer is performed by leveraging ideas in terms of controller architecture initially reported in [19] coupled with the LMI-based approach initiated in [20], and more precisely on the enhanced procedures described in [21,22] that allow to handle both Dirichlet and Neumann boundary measurements while performing, for very general 1-D reaction–diffusion PDEs, the control design directly with the actual boundary control input instead of its time-derivative; see [23, Sec. 3.3] for generalities on boundary control systems. These procedures have been developed for parabolic PDEs with collocated boundary conditions (in addition of the above references for reaction–diffusion PDEs, see also [24] for the case of the Kuramoto–Sivashinsky equation) for which the underlying unbounded operator is self-adjoint and the associated unit eigenvectors form a Hilbert basis. Due to the non-collocated nature of the boundary condition considered in this paper for the reaction–diffusion PDE, we adapt these procedures to the case of an underlying unbounded operator that is not self-adjoint, with all the eigenvalues but one that are complex conjugate, while the unit eigenvectors do not form a Hilbert basis but a Riesz basis. We show that the proposed control strategy always achieves the exponential stabilization of the plant provided the order of the observer is selected large enough. Beyond the sole output feedback stabilization of the plant, we show that the procedure developed in this paper also allows the establishment of an input-to-state stability (ISS) estimate with respect to an additive boundary perturbation in the application of the boundary control. Note that ISS estimates with respect to unmatched disturbances can also be obtained in our framework for additive perturbations applied in the domain of the PDE (see, e.g., [25] for the study of such a case in the context of collocated reaction–diffusion PDE with bounded input and output operators while using an infinite-dimensional observer) or in the measurement. Note however that such input perturbations apply to the closed-loop system dynamics as inputs of bounded operators. In this context, it is well-known that the establishment of ISS estimates with respect to boundary perturbations is much more challenging compared to perturbations applied through bounded operators [26]. This is why we focus the presentation of the results on the case of a boundary disturbance.

The paper is organized as follows. The control design problem addressed in this paper is introduced in Section 2. The structural properties of the underlying unbounded operator and the subsequent spectral reduction of the PDE are reported in Section 3. The proposed control strategy is described in Section 4. The stability analysis of the resulting closed-loop system is then carried out in Section 5. A numerical illustration is reported in Section 6. Finally, concluding remarks regarding possible extensions of the control strategy are formulated in Section 7.

**Notations.** Real spaces $\mathbb{R}^n$ are equipped with the usual Euclidean norm denoted by $\|\cdot\|$. The associated induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors $X$ and $Y$, $\mathrm{col}(X,Y)$ represents the vector $[X^\top, Y^\top]^\top$. The space of square integrable functions on $(0,1)$ is denoted by $L^2(0,1)$ and is endowed with the inner product $\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x$. The associated norm is denoted by $\|\cdot\|_{L^2}$. For an integer $m \geq 1$, $H^m(0,1)$ stands for the $m$-order Sobolev space and is endowed with its usual norm $\|\cdot\|_{H^m}$. For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) indicates that $P$ is positive semi-definite (resp. positive definite).

## 2. Problem description and abstract representation

### 2.1. Problem description

We consider in this paper the boundary control of the reaction–diffusion system described by

$$z_t(t,x) = p z_{xx}(t,x) + r z(t,x) \tag{1a}$$

$$z(t,1) = s z(t,0) \tag{1b}$$

$$z_x(t,0) = u_d(t) \triangleq u(t) + d(t) \tag{1c}$$

$$y_D(t) = z(t,\xi) \tag{1d}$$

$$z(0,x) = z_0(x) \tag{1e}$$

for $t > 0$ and $x \in (0,1)$. Here $p > 0$ is the diffusion coefficient and $r \in \mathbb{R}$ is the reaction coefficient. Boundary condition (1b) is non-collocated with coefficient $s > 1$. The control input $u(t) \in \mathbb{R}$ applies to the left Neumann trace (1c) with unknown boundary disturbance $d(t) \in \mathbb{R}$. Throughout the paper, we assume that $d \in \mathcal{C}^2(\mathbb{R}_+)$. The system output is selected as the pointwise Dirichlet measurement $y_D(t)$ defined by (1d) for some given $\xi \in [0,1]$. Finally, the initial condition (1e) is characterized by $z_0$. It is worth noting that for $r \geq 0$ and $s > 1$, the open-loop PDE (1) is unstable; see Lemma 1 for further details.

The control objective is to design a finite-dimensional control strategy that achieves the output feedback exponential stabilization of (1).

**Remark 1.** For $p > r$ and $d = 0$, if we assume that $z(t,0)$ and $z_x(t,1)$ are available for feedback control, the exponential stabilization of (1a)–(1c) can be achieved by setting $u(t) = s z_x(t,1) + k z(t,0)$ for $k > 0$ sufficiently large positive. Indeed, defining $V(t) = \frac{1}{2}\int_0^1 |z(t,x)|^2\,\mathrm{d}x$, we infer that $\dot{V}(t) = r\int_0^1 |z(t,x)|^2\,\mathrm{d}x + p\int_0^1 z(t,x) z_{xx}(t,x)\,\mathrm{d}x$. An integration by parts gives $\int_0^1 z(t,x)z_{xx}(t,x)\,\mathrm{d}x = z(t,0)\{s z_x(t,1) - u(t)\} - \int_0^1 |z_x(t,x)|^2\,\mathrm{d}x = -k|z(t,0)|^2 - \int_0^1 |z_x(t,x)|^2\,\mathrm{d}x$. Let $\epsilon > 0$ be such that $p > r(1+\epsilon^{-1})$. Then using Cauchy–Schwarz and Young inequalities, we infer that $\int_0^1 |z(t,x)|^2\,\mathrm{d}x \leq (1+\epsilon)|z(t,0)|^2 + (1+\epsilon^{-1})\int_0^1 |z_x(t,x)|^2\,\mathrm{d}x$. Hence, we have

$$\dot{V}(t) \leq -\{kp - r(1+\epsilon)\}\,|z(t,0)|^2 - \left\{p - r(1+\epsilon^{-1})\right\}\int_0^1 |z_x(t,x)|^2\,\mathrm{d}x$$

So if we select $k > \frac{r}{p}(1+\epsilon)$, we obtain the existence of $\alpha = \min\left(kp - r(1+\epsilon), p - r(1+\epsilon^{-1})\right) > 0$ and $\kappa > 0$ so that $\dot{V}(t) \leq -\alpha\left\{|z(t,0)|^2 + \int_0^1 |z_x(t,x)|^2\,\mathrm{d}x\right\} \leq -2\kappa V(t)$. This ensures the exponential stabilization of the plant. Note however the following structural limitations. First, this approach requires the structural constraint $p > r$. Second, while the above approach requires the knowledge of both $z_x(t,1)$ and $z(t,0)$, such a strategy cannot be applied in the case of the sole point measurement (1d). Finally, this approach is not easily extendable to settings presenting, for instance, input nonlinearities or long input/output/state delays. The method developed in this paper allows to remove all these limitations.

**Remark 2.** The occurrence of the $s$-parameter in the non-collocated boundary condition (1b) naturally arises in the context of reaction–diffusion equations with a transport term and a periodic Dirichlet boundary conditions. More specifically, let us consider the system described by

$$y_t(t,x) = \alpha y_{xx}(t,x) + \beta y_x(t,x) + \gamma y(t,x) \tag{2a}$$

$$y(t,1) = y(t,0) \tag{2b}$$

$$y_x(t,0) = v(t) \tag{2c}$$

$$y_D(t) = y(t,1) \tag{2d}$$

where $\alpha > 0$ and $\beta, \gamma \in \mathbb{R}$. In order to obtain an equivalent formulation of (2) without transport term, we introduce the classical change of variable formula $z(t,x) = e^{\frac{\beta}{2\alpha}x} y(t,x)$. Then we have that

$$z_t(t,x) = \alpha z_{xx}(t,x) + \left\{ \gamma - \frac{\beta^2}{4\alpha} \right\} z(t,x) \tag{3a}$$

$$z(t,1) = e^{\frac{\beta}{2\alpha}} z(t,0) \tag{3b}$$

$$-\frac{\beta}{2\alpha} z(t,0) + z_x(t,0) = v(t) \tag{3c}$$

$$y_D(t) = z(t,0) \tag{3d}$$

Hence, if we set $v(t) = -\frac{\beta}{2\alpha} y_D(t) + u(t)$ we infer that $z_x(t,0) = u(t)$, giving (1) with $p = \alpha$, $r = \gamma - \frac{\beta^2}{4\alpha}$, $s = e^{\frac{\beta}{2\alpha}}$, $\xi = 0$, and $d = 0$.

### 2.2. Preliminary change of variable and abstract form

Let us consider the change of variable

$$w(t,x) = z(t,x) - \left( x + \frac{1}{s-1} \right) u_d(t). \tag{4}$$

This allows us to derive the following equivalent homogeneous representation of (1) described by

$$\dot{u}(t) = v(t) \tag{5a}$$

$$w_t(t,x) = p w_{xx}(t,x) + r w(t,x) + a(x)u(t) + b(x)v(t) + a(x)d(t) + b(x)\dot{d}(t) \tag{5b}$$

$$w(t,1) = s w(t,0), \quad w_x(t,0) = 0 \tag{5c}$$

$$\tilde{y}_D(t) = w(t,\xi) \tag{5d}$$

$$w(0,x) = w_0(x) \tag{5e}$$

where $a(x) = r\left(x + \frac{1}{s-1}\right)$ and $b(x) = -\left(x + \frac{1}{s-1}\right)$ while $\tilde{y}_D(t) = y_D(t) - \left(\xi + \frac{1}{s-1}\right) u_d(t)$ and the initial condition $w_0(x) = z_0(x) - \left(x + \frac{1}{s-1}\right) u_d(0)$.

Let us now define the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$ defined by $\mathcal{A}f = f''$ on the domain $D(\mathcal{A}) = \{f \in H^2(0,1) : f(1) = sf(0), f'(0) = 0\}$. Hence (5a)–(5c) can be written under the following abstract form:

$$\dot{u}(t) = v(t) \tag{6a}$$

$$w_t(t,\cdot) = (p\mathcal{A} + rI_{L^2}) w(t,\cdot) + au(t) + bv(t) + ad(t) + b\dot{d}(t) \tag{6b}$$

## 3. Structural properties and spectral reduction

### 3.1. Riesz spectral properties of $\mathcal{A}$

We start by the following lemma describing the point spectrum of $\mathcal{A}$.

**Lemma 1.** *Let $s > 1$ and define $\tau = s + \sqrt{s^2-1} > 1$. The eigenvalues $\mu_n \in \mathbb{C}$ of $\mathcal{A}$ and the corresponding eigenvectors $\phi_n \in L^2(0,1)$, with $n \in \mathbb{Z}$, are described by*

$$\mu_n = (\log\tau)^2 - 4n^2\pi^2 + 4in\pi \log\tau,$$

$$\phi_n(x) = \cosh\left((\log\tau + 2in\pi)x\right).$$

**Proof.** We are looking for $\mu \in \mathbb{C}$ and a non zero $f \in H^2(0,1)$ so that $f'' - \mu f = 0$, $f(1) = sf(0)$, and $f'(0) = 0$. Since $s \neq 1$, the resolution of the above ODE in the case $\mu = 0$ gives $f = 0$. Hence the case $\mu = 0$ is discarded. Let $\sqrt{\mu} \neq 0$ denote one of the two square roots of $\mu$. Function $f$ must be of the form $f(x) = \alpha e^{\sqrt{\mu}x} + \beta e^{-\sqrt{\mu}x}$. The condition $f'(0) = 0$ along with $\sqrt{\mu} \neq 0$ implies $\alpha = \beta$ hence $f(x) = 2\alpha\cosh(\sqrt{\mu}x)$. Now the condition $f(1) = sf(0)$ along with $\alpha \neq 0$ (because we are looking for a non zero function $f$) gives $\cosh(\sqrt{\mu}) = s$. Writing cosh in terms of exponentials, this latter identity is equivalent to $(e^{\sqrt{\mu}})^2 - 2se^{\sqrt{\mu}} + 1 = 0$, implying that $e^{\sqrt{\mu}} = \tau \triangleq s + \sqrt{s^2-1}$ or $e^{\sqrt{\mu}} = \tau_- \triangleq s - \sqrt{s^2-1}$. So we have either $\sqrt{\mu} = \log\tau + 2in\pi$ or $\sqrt{\mu} = \log\tau_- + 2in\pi$ for some $n \in \mathbb{Z}$. Thus we have $\mu = (\log\tau + 2in\pi)^2$ or $\mu = (\log\tau_- + 2in\pi)^2$ for some $n \in \mathbb{Z}$. Noting that $\tau_- = 1/\tau$ hence $\log\tau_- = -\log\tau$, this implies that $\mu = (\log\tau + 2in\pi)^2 = (\log\tau)^2 - 4n^2\pi^2 + 4in\pi\log\tau$ for some $n \in \mathbb{Z}$. This concludes the proof.

In order to further study the properties of $\{\phi_n\}_{n\in\mathbb{Z}}$, we shall need to establish the existence, and determine the expression, of a family $\{\psi_n\}_{n\in\mathbb{Z}}$ that is biorthogonal to $\{\phi_n\}_{n\in\mathbb{Z}}$, i.e., $\langle \phi_n, \psi_m \rangle = \delta_{n,m}$ for all $n, m \in \mathbb{Z}$ where $\delta_{n,m} \in \{0,1\}$ with $\delta_{n,m} = 1$ if and only if $n = m$. To do so, we first compute $\mathcal{A}^*$, the adjoint operator of $\mathcal{A}$.

**Lemma 2.** *The adjoint operator $\mathcal{A}^*$ is described by $\mathcal{A}^*f = f''$ on the domain $D(\mathcal{A}^*) = \{f \in H^2(0,1) : f(1) = 0, f'(0) = sf'(1)\}$.*

**Proof.** We first note that $\mathcal{A}$ is invertible with inverse given for all $g \in L^2(0,1)$ by $(\mathcal{A}^{-1}g)(x) = \int_0^x \int_0^{\xi_1} g(\xi_2)\,d\xi_2\,d\xi_1 + \frac{1}{s-1}\int_0^1 \int_0^{\xi_1} g(\xi_2)\,d\xi_2\,d\xi_1$, showing that $0 \in \rho(\mathcal{A})$. Hence we know that $(\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^*$ (see [23, Lem. A.3.65]). Direct computations give, for all $w \in L^2(0,1)$, $((\mathcal{A}^{-1})^*w)(x) = \int_x^1 \int_{\xi_1}^1 w(\xi_2)\,d\xi_2\,d\xi_1 + \frac{1-x}{s-1}\int_0^1 w(\xi)\,d\xi$. The inversion of the latter operator gives the claimed result.

We now obtain the required biorthogonal sequence $\{\psi_n\}_{n\in\mathbb{Z}}$ by studying the eigenstructures of $\mathcal{A}^*$.

**Lemma 3.** *Let $s > 1$ and define $\tau = s + \sqrt{s^2-1} > 1$. The eigenvalues $\mu_n^{\mathrm{ad}} \in \mathbb{C}$ of $\mathcal{A}^*$ and the corresponding eigenvectors $\psi_n \in L^2(0,1)$, with $n \in \mathbb{Z}$, are described by*

$$\mu_n^{\mathrm{ad}} = \overline{\mu_n} = (\log\tau)^2 - 4n^2\pi^2 - 4in\pi\log\tau,$$

$$\psi_n(x) = -\frac{2}{\sqrt{s^2-1}} \sinh\left((\log\tau - 2in\pi)(x-1)\right).$$

*Moreover, $\{\psi_n\}_{n\in\mathbb{Z}}$ is biorthogonal to $\{\phi_n\}_{n\in\mathbb{Z}}$.*

**Proof.** We are looking for $\mu \in \mathbb{C}$ and a non zero $f \in H^2(0,1)$ so that $f'' - \mu f = 0$, $f(1) = 0$, and $f'(0) = sf'(1)$. Since $s \neq 1$, the resolution of the above ODE in the case $\mu = 0$ gives $f = 0$. Hence the case $\mu = 0$ is discarded. Let $\sqrt{\mu} \neq 0$ denote one of the two square roots of $\mu$. Function $f$ must be of the form $f(x) = \alpha e^{\sqrt{\mu}x} + \beta e^{-\sqrt{\mu}x}$. The conditions $f(1) = 0$ and $f'(0) = sf'(1)$ along with $\sqrt{\mu} \neq 0$ give

$$\begin{bmatrix} e^{\sqrt{\mu}} & e^{-\sqrt{\mu}} \\ 1 - se^{\sqrt{\mu}} & -1 + se^{-\sqrt{\mu}} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0.$$

Hence, the function $f$ is non zero if and only if the determinant of the above $2 \times 2$ matrix is zero, i.e., $\cosh(\sqrt{\mu}) = s$. We now obtained the claimed closed-form for $\mu_n^{\mathrm{ad}}$ by following the same steps that the ones of Lemma 1. Moreover, since $f(1) = 0$ implies that $\beta = -\alpha e^{2\sqrt{\mu}}$, we infer that $f(x) = \alpha\left(e^{\sqrt{\mu}x} - e^{2\sqrt{\mu}}e^{-\sqrt{\mu}x}\right) = 2\alpha e^{\sqrt{\mu}}\sinh(\sqrt{\mu}(x-1))$. We obtain the claimed closed form for $\psi_n$ by setting $\alpha = -\frac{e^{-\sqrt{\mu}}}{\sqrt{s^2-1}}$.

To complete the proof, it remains to show that $\{\psi_n\}_{n\in\mathbb{Z}}$ is biorthogonal to $\{\phi_n\}_{n\in\mathbb{Z}}$. Note first that $\mu_n\langle\phi_n, \psi_m\rangle = \langle\mathcal{A}\phi_n, \psi_m\rangle = \langle\phi_n, \mathcal{A}^*\psi_m\rangle = \left\langle\phi_n, \mu_m^{\mathrm{ad}}\psi_m\right\rangle = \mu_m\langle\phi_n, \psi_m\rangle$. For $n \neq m$ we have $\mu_n \neq \mu_m$ hence $\langle\phi_n, \psi_m\rangle = 0$. Finally, explicit computations show that $\langle\phi_n, \psi_n\rangle = 1$ for all $n \in \mathbb{Z}$.

We are now in position to show that $\{\phi_n\}_{n\in\mathbb{Z}}$ forms a Riesz basis of $L^2(0,1)$. This means that the vector space spanned by $\{\phi_n\}_{n\in\mathbb{Z}}$ is dense in $L^2(0,1)$ and that there exist $m_R, M_R > 0$ so that, for any $N \geq 0$ and any $\alpha_i \in \mathbb{C}$, $m_R\sum_{|n|\leq N}|\alpha_n|^2 \leq \|\sum_{|n|\leq N}\alpha_n\phi_n\|^2 \leq M_R\sum_{|n|\leq N}|\alpha_n|^2$. In this case, we have for any $f \in L^2(0,1)$ the series expansion $f = \sum_{n\in\mathbb{Z}}\langle f, \psi_n\rangle\phi_n$ and

$$m_R\sum_{n\in\mathbb{Z}}|\langle f, \psi_n\rangle|^2 \leq \|f\|_{L^2}^2 \leq M_R\sum_{n\in\mathbb{Z}}|\langle f, \psi_n\rangle|^2. \tag{7}$$

**Lemma 4.** *The family $\{\phi_n\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(0,1)$.*

**Proof.** Using the characterization of Riesz bases reported in [27, Chap. 1, Thm. 9], we need to show that 1) the two vector spaces spanned by $\{\phi_n\}_{n\in\mathbb{Z}}$ and its biorthogonal family $\{\psi_n\}_{n\in\mathbb{Z}}$ are both dense in $L^2(0,1)$; and 2) both families $\{\phi_n\}_{n\in\mathbb{Z}}$ and $\{\psi_n\}_{n\in\mathbb{Z}}$ form each a Bessel sequence. We recall that a family $(\varphi_n)_{n\in\mathbb{N}}$ of $L^2(0,1)$ is said to be a Bessel sequence if for any $f \in L^2(0,1)$ we have $(\langle f, \varphi_n\rangle)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$.

We start by studying the properties of $\{\phi_n\}_{n\in\mathbb{Z}}$. Let $f \in L^2(0,1)$ be such that $\langle f, \phi_n\rangle = 0$ for all $n \in \mathbb{Z}$. We recall that $\phi_n$ is given in closed form by Lemma 1. The case $n = 0$ gives $\langle f\cosh((\log\tau)\cdot), 1\rangle = 0$. In the case $n \geq 1$ we have

$$\begin{aligned} 0 = \langle f, \phi_n\rangle &= \langle f\cosh((\log\tau)\cdot), \cos(2n\pi\cdot)\rangle \\ &\quad - i\,\langle f\sinh((\log\tau)\cdot), \sin(2n\pi\cdot)\rangle \\ 0 = \langle f, \phi_{-n}\rangle &= \langle f\cosh((\log\tau)\cdot), \cos(2n\pi\cdot)\rangle \\ &\quad + i\,\langle f\sinh((\log\tau)\cdot), \sin(2n\pi\cdot)\rangle \end{aligned}$$

hence we deduce that $\langle f\cosh((\log\tau)\cdot), \cos(2n\pi\cdot)\rangle = 0$ and $\langle f\sinh((\log\tau)\cdot), \sin(2n\pi\cdot)\rangle = 0$. Using these results, the Fourier series of $f\cosh((\log\tau)\cdot) \in L^2(0,1)$ reads

$$f\cosh((\log\tau)\cdot) = \sum_{n\geq 1}\alpha_n\sin(2n\pi\cdot)$$

where $\alpha_n \in \mathbb{C}$. This implies that

$$\begin{aligned} &\langle f\cosh((\log\tau)\cdot), f\sinh((\log\tau)\cdot)\rangle \\ &\qquad = \sum_{n\geq 1}\alpha_n\overline{\langle f\sinh((\log\tau)\cdot), \sin(2n\pi\cdot)\rangle} = 0. \end{aligned}$$

We deduce that

$$\begin{aligned} 0 &= \int_0^1 |f(x)|^2\sinh((\log\tau)x)\cosh((\log\tau)x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 |f(x)|^2\sinh(2(\log\tau)x)\,\mathrm{d}x. \end{aligned}$$

Owing to $\tau > 1$, we have $\sinh(2(\log\tau)x) > 0$ for all $x > 0$ hence $f = 0$ in $L^2(0,1)$. This shows that the vector space spanned by $\{\phi_n\}_{n\in\mathbb{Z}}$ is dense in $L^2(0,1)$.

Let us now show that $\{\phi_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence. To do so, let $f \in L^2(0,1)$ be arbitrarily fixed. Noting that $\phi_n(x) = \cosh((\log\tau)x)\cos(2n\pi x) + i\sinh((\log\tau)x)\sin(2n\pi x)$, we deduce using the triangular and Young's inequalities that $\sum_{n\in\mathbb{Z}}|\langle f, \phi_n\rangle|^2 \leq 4\sum_{n\geq 0}|\langle f\cosh((\log\tau)\cdot), \cos(2n\pi\cdot)\rangle|^2 + 4\sum_{n\geq 1}|\langle f\sinh((\log\tau)\cdot), \sin(2n\pi\cdot)\rangle| < \infty$ where the right hand side of the inequality is finite because $f\cosh((\log\tau)\cdot) \in L^2(0,1)$ and $f\sinh((\log\tau)\cdot) \in L^2(0,1)$ and owing to the fact that the Fourier coefficients of elements of $L^2(0,1)$ are square summable. This shows that $\{\phi_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence.

Using similar arguments, one can show that $\{\psi_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence and that the vector space spanned by this family is dense in $L^2(0,1)$. This completes the proof.

Finally, using the terminology of [23, Def. 2.3.4], $\mathcal{A}$ is a Riesz spectral operator. Moreover, since $\sup_{n\in\mathbb{Z}}\operatorname{Re}\mu_n = (\log\tau)^2 < \infty$, we obtain from [23, Thm. 2.3.5] the following result.

**Lemma 5.** *$\mathcal{A}$ is a Riesz spectral operator that generates a $C_0$-semigroup on $L^2(0,1)$. Its domain is characterized by $D(\mathcal{A}) = \{f \in L^2(0,1) : \sum_{n\in\mathbb{Z}}|\mu_n|^2|\langle f, \psi_n\rangle|^2 < \infty\}$ with $\mathcal{A}f = \sum_{n\in\mathbb{Z}}\mu_n\langle f, \psi_n\rangle\phi_n$ for all $f \in D(\mathcal{A})$.*

Let $f \in D(\mathcal{A})$ be arbitrarily given. From the Riesz basis property we have $f = \sum_{n\in\mathbb{Z}}\langle f, \psi_n\rangle\phi_n$ with convergence of the series in $L^2(0,1)$ norm. From the previous lemma, we also infer that $f'' = \sum_{n\in\mathbb{Z}}\langle f, \psi_n\rangle\phi_n''$ with also convergence of the series in $L^2(0,1)$ norm. Finally, from the definition of the domain of the operator $\mathcal{A}$, we have $f \in H^2(0,1)$ and $f'(0) = 0$. Invoking Poincaré's inequality, we infer that $f' = \sum_{n\in\mathbb{Z}}\langle f, \psi_n\rangle\phi_n'$ in $L^2(0,1)$ norm. Hence we have that $f = \sum_{n\in\mathbb{Z}}\langle f, \psi_n\rangle\phi_n$ with convergence of the series in $H^2(0,1)$ norm. The continuous embedding $H^1(0,1) \subset L^\infty(0,1)$ implies that the latter series converges in $L^\infty(0,1)$ norm.

### 3.2. Spectral reduction

We introduce the coefficients of projection defined by $z_n(t) = \langle z(t,\cdot), \psi_n\rangle$, $w_n(t) = \langle w(t,\cdot), \psi_n\rangle$, $a_n = \langle a, \psi_n\rangle$, and $b_n = \langle b, \psi_n\rangle$. Then we have from (4) that

$$w_n(t) = z_n(t) + b_n u_d(t). \tag{8}$$

Moreover, the projection of (6) onto the Riesz basis $\{\phi_n\}_{n\in\mathbb{Z}}$ gives

$$\dot{u}(t) = v(t) \tag{9a}$$

$$\dot{w}_n(t) = \lambda_n w_n(t) + a_n u(t) + b_n v(t) + a_n d(t) + b_n\dot{d}(t),\ n \in \mathbb{Z} \tag{9b}$$

with $\lambda_n = p\mu_n + r$. Finally, using (8) into (9) to return to the original coordinate $z$, the projection of (1) reads

$$\dot{z}_n(t) = \lambda_n z_n(t) + \beta_n u_d(t), \quad n \in \mathbb{Z} \tag{10}$$

with $\beta_n = a_n + \lambda_n b_n$. From Lemma 3 we have $\mu_n b_n = \langle b, \mathcal{A}^*\psi_n\rangle$. Using an integration by parts, direct computations show that $\mu_n b_n = -2$ for all $n \in \mathbb{Z}$, hence $b_n = -2/\mu_n$ and $a_n = -rb_n = 2r/\mu_n$. This implies that $\beta_n = -2p \in \mathbb{R}$ for all $n \in \mathbb{Z}$, hence is a constant independent of $n$.

**Remark 3.** We observe that $|\mu_n||b_n|^2 = \frac{4}{|\mu_n|} \sim \frac{1}{n^2\pi^2}$ hence $\sum_{n\in\mathbb{Z}}|\mu_n||b_n|^2 < \infty$. Consequently, if for $t \geq 0$ we have $w(t,\cdot) \in D(\mathcal{A})$, we infer from (8) that $\sum_{n\in\mathbb{Z}}|\mu_n||z_n(t)|^2 < \infty$.

We observe that the ODEs (10) are complex-valued. However, the trajectories of the original problem described by (2) are real-valued. So, to perform the control design and obtain a real-valued control strategy, i.e. $u(t) \in \mathbb{R}$, we need to derive a real-valued version of (10). Since $\overline{\lambda_n} = \lambda_{-n}$, $\overline{\phi_n} = \phi_{-n}$, $\overline{\psi_n} = \psi_{-n}$, $z(t,x) \in \mathbb{R}$, and $d(t) \in \mathbb{R}$, we have $z_0(t) \in \mathbb{R}$ and, for all $n \geq 1$, $\overline{z_n(t)} = z_{-n}(t)$ with

$$\overbrace{\begin{bmatrix}\operatorname{Re} z_n \\ \operatorname{Im} z_n\end{bmatrix}}^{\cdot}(t) = \underbrace{\begin{bmatrix}\operatorname{Re}\lambda_n & -\operatorname{Im}\lambda_n \\ \operatorname{Im}\lambda_n & \operatorname{Re}\lambda_n\end{bmatrix}}_{=\mathfrak{A}_n} \underbrace{\begin{bmatrix}\operatorname{Re} z_n \\ \operatorname{Im} z_n\end{bmatrix}(t)}_{=\mathfrak{z}_n(t)} + \underbrace{\begin{bmatrix}-2p \\ 0\end{bmatrix}}_{=\mathfrak{H}_n} u_d(t) \tag{11}$$

where all the quantities appearing in the latter identity are real-valued. Moreover, introducing $\mathfrak{w}_n = \begin{bmatrix}\operatorname{Re} w_n & \operatorname{Im} w_n\end{bmatrix}^\top$ and owing to (8), we have for all $n \geq 1$

$$\mathfrak{w}_n(t) = \mathfrak{z}_n(t) + \mathfrak{G}_n u_d(t). \tag{12}$$

with $\mathfrak{G}_n = \begin{bmatrix}\operatorname{Re} b_n & \operatorname{Im} b_n\end{bmatrix}^\top$.

When considering classical solutions, the discussion after Lemma 5 shows that the system output $\tilde{y}_D(t)$ given by (5d) can be expressed as the following series expansions:

$$\begin{aligned}\tilde{y}_D(t) &= w(t,\xi) = \sum_{n\in\mathbb{Z}} w_n(t)\phi_n(\xi) \\ &= w_0(t)\phi_0(\xi) + 2\sum_{n=1}^{N} \operatorname{Re}\{w_n(t)\phi_n(\xi)\} + \sum_{|n|\geq N+1} w_n(t)\phi_n(\xi) \\ &= w_0(t)\phi_0(\xi) + \sum_{n=1}^{N} \mathfrak{C}_n \mathfrak{w}_n(t) + \zeta(t)\end{aligned} \tag{13}$$

for any given $N \geq 1$, where $\mathfrak{C}_n = 2\begin{bmatrix}\operatorname{Re}\phi_n(\xi) & -\operatorname{Im}\phi_n(\xi)\end{bmatrix}$ and $\zeta(t) = \sum_{|n|\geq N+1} w_n(t)\phi_n(\xi)$.

## 4. Control design

### 4.1. Control strategy

Let $\delta > 0$ and $N_0 \geq 0$ be such that $\operatorname{Re}\lambda_n < -\delta$ for all $|n| \geq N_0 + 1$. Let $N \geq N_0 + 1$ be arbitrarily fixed and that will be specified later. We consider the control strategy described by

$$\hat{w}_0(t) = \hat{z}_0(t) + b_0 u(t) \tag{14a}$$

$$\hat{\mathfrak{w}}_n(t) = \hat{\mathfrak{z}}_n(t) + \mathfrak{G}_n u(t), \quad 1 \leq n \leq N \tag{14b}$$

$$\hat{y}_D(t) = \hat{w}_0(t)\phi_0(\xi) + \sum_{k=1}^{N} \mathfrak{C}_k \hat{\mathfrak{w}}_k(t) \tag{14c}$$

$$\begin{aligned}\dot{\hat{z}}_0(t) &= \lambda_0 \hat{z}_0(t) + \beta_0 u(t) \\ &\quad - l_0\left\{\hat{y}_D(t) - (y_D(t) + b(\xi)u(t))\right\}\end{aligned} \tag{14d}$$

$$\begin{aligned}\dot{\hat{\mathfrak{z}}}_n(t) &= \mathfrak{A}_n \hat{\mathfrak{z}}_n(t) + \mathfrak{H}_n u(t) \\ &\quad - \mathfrak{L}_n\left\{\hat{y}_D(t) - (y_D(t) + b(\xi)u(t))\right\}, \quad 1 \leq n \leq N_0\end{aligned} \tag{14e}$$

$$\dot{\hat{\mathfrak{z}}}_n(t) = \mathfrak{A}_n \hat{\mathfrak{z}}_n(t) + \mathfrak{H}_n u(t), \quad N_0 + 1 \leq n \leq N \tag{14f}$$

$$u(t) = k_0 \hat{z}_0(t) + \sum_{k=1}^{N_0} \mathfrak{K}_k \hat{\mathfrak{z}}_k(t) \tag{14g}$$

where $l_0 \in \mathbb{R}$ and $\mathfrak{L}_n \in \mathbb{R}^2$ are the observer gains while $k_0 \in \mathbb{R}$ and $\mathfrak{K}_k \in \mathbb{R}^{1\times 2}$ stand for the feedback gains. Note that the controller dynamics (14) does not involve the unknown boundary perturbation $d(t)$.

**Remark 4.** In the disturbance-free case $d = 0$, the output $\tilde{y}_D(t)$ of (5)) is directly accessible because $\tilde{y}_D(t) = y_D(t) + b(\xi)u(t)$ where $y_D(t)$ is the actual measurement and $u(t)$ the applied command input. Hence the correction term due to the error of estimation appearing in (14d)–(14e) reduces to $\hat{y}_D(t) - (y_D(t) + b(\xi)u(t)) = \hat{y}_D(t) - \tilde{y}_D(t)$. In the disturbed case (i.e., $d \neq 0$), $\tilde{y}_D(t) = y_D(t) + b(\xi)u_d(t)$ with $u_d(t) = u(t) + d(t)$ where $d(t)$ is assumed to be unknown. Hence $\tilde{y}_D(t)$ cannot be used to implement the control strategy. This is why $\tilde{y}_D(t)$ is approximated by $y_D(t) + b(\xi)u(t)$ in (14d)–(14e).

**Remark 5.** The well-posedness in terms of classical solutions (defined for all $t \geq 0$) of the closed-loop system, formed by the plant in homogeneous coordinates (5) and the controller (14), for initial conditions $w_0 \in D(\mathcal{A})$, $\hat{z}_0(0) \in \mathbb{R}$, and $\hat{\mathfrak{z}}_n(0) \in \mathbb{R}^2$, and a boundary disturbance $d \in \mathcal{C}^2(\mathbb{R}_+)$, is a direct consequence of [28, Thm. 6.3.1 and Thm. 6.3.3]. Invoking the change of variable formula (4), this implies the well-posedness in terms of classical solutions of the closed-loop system composed of the plant in original coordinates (1) and the controller (14) for any initial conditions $z_0 \in H^2(0,1)$, $\hat{z}_0(0) \in \mathbb{R}$, and $\hat{\mathfrak{z}}_n(0) \in \mathbb{R}^2$, and any boundary disturbance $d \in \mathcal{C}^2(\mathbb{R}_+)$, all such that $z_0(1) = s z_0(0)$ and $z_0'(0) = k_0\hat{z}_0(0) + \sum_{k=1}^{N_0} \mathfrak{K}_k \hat{\mathfrak{z}}_k(0) + d(0)$.

### 4.2. Truncated finite-dimensional model

Defining first $\hat{Z}^{N_0} = \begin{bmatrix}\hat{z}_0 & \hat{\mathfrak{z}}_1^\top & \ldots & \hat{\mathfrak{z}}_{N_0}^\top\end{bmatrix}^\top$, we obtain from (14g) that

$$u = K\hat{Z}^{N_0} \tag{15}$$

where $K = \begin{bmatrix}k_0 & \mathfrak{K}_1 & \ldots & \mathfrak{K}_{N_0}\end{bmatrix}$. Let the observation error be defined by $e_0 = z_0 - \hat{z}_0$ and $\mathfrak{e}_n = \mathfrak{z}_n - \hat{\mathfrak{z}}_n$ for all $1 \leq n \leq N$. Using (8), (12), and (13), the error of observation $\hat{y}_D(t) - (y_D(t) + b(\xi)u(t))$ of the controller (14) can be rewritten as

$$\begin{aligned}\hat{y}_D - (y_D + b(\xi)u) &= \hat{y}_D - \tilde{y}_D + b(\xi)d \\ &= -\phi_0(\xi)e_0 - \sum_{k=1}^{N} \mathfrak{C}_k e_k - \zeta - \nu d\end{aligned} \tag{16}$$

where $\nu = \phi_0(\xi)b_0 + \sum_{k=1}^{N} \mathfrak{C}_k \mathfrak{G}_k - b(\xi)$. Defining the scaled quantities $\tilde{\mathfrak{z}}_n = \hat{\mathfrak{z}}_n / n$ and $\tilde{\mathfrak{e}}_n = \sqrt{|\mu_n|}\mathfrak{e}_n$ and the vectors $E^{N_0} = \begin{bmatrix}e_0 & \mathfrak{e}_1^\top & \ldots & \mathfrak{e}_{N_0}^\top\end{bmatrix}^\top$, $\tilde{Z}^{N-N_0} = \begin{bmatrix}\tilde{\mathfrak{z}}_{N_0+1}^\top & \ldots & \tilde{\mathfrak{z}}_N^\top\end{bmatrix}^\top$, and $\tilde{E}^{N_0} = \begin{bmatrix}\tilde{\mathfrak{e}}_{N_0+1}^\top & \ldots & \tilde{\mathfrak{e}}_N^\top\end{bmatrix}^\top$, we infer from (14) that

$$\dot{\hat{Z}}^{N_0} = (A_0 + \mathfrak{B}_0 K)\hat{Z}^{N_0} + LC_0 E^{N_0} + L\tilde{C}_1 \tilde{E}^{N-N_0} + L\zeta + \nu L d \tag{17a}$$

$$\dot{E}^{N_0} = (A_0 - LC_0)E^{N_0} - L\tilde{C}_1 \tilde{E}^{N-N_0} - L\zeta + (\mathfrak{B}_0 - \nu L)d \tag{17b}$$

$$\dot{\tilde{Z}}^{N-N_0} = A_1 \tilde{Z}^{N-N_0} + \tilde{\mathfrak{B}}_1 K \hat{Z}^{N_0} \tag{17c}$$

$$\dot{\tilde{E}}^{N-N_0} = A_1 \tilde{E}^{N-N_0} + \tilde{\mathfrak{B}}_2 d \tag{17d}$$

where the different matrices are defined by

$$\begin{aligned}A_0 &= \operatorname{diag}(\lambda_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_{N_0}), \\ A_1 &= \operatorname{diag}(\mathfrak{A}_{N_0+1}, \ldots, \mathfrak{A}_N), \\ \mathfrak{B}_0 &= \begin{bmatrix}\beta_0 & \mathfrak{H}_1^\top & \ldots & \mathfrak{H}_{N_0}^\top\end{bmatrix}^\top, \\ \tilde{\mathfrak{B}}_1 &= \begin{bmatrix}\frac{1}{N_0+1}\mathfrak{H}_{N_0+1}^\top & \ldots & \frac{1}{N}\mathfrak{H}_N^\top\end{bmatrix}^\top, \\ \tilde{\mathfrak{B}}_2 &= \begin{bmatrix}\sqrt{|\mu_{N_0+1}|}\mathfrak{H}_{N_0+1}^\top & \ldots & \sqrt{|\mu_N|}\mathfrak{H}_N^\top\end{bmatrix}^\top, \\ C_0 &= \begin{bmatrix}\phi_0(\xi) & \mathfrak{C}_1 & \ldots & \mathfrak{C}_{N_0}\end{bmatrix}, \\ \tilde{C}_1 &= \begin{bmatrix}\frac{1}{\sqrt{|\mu_{N_0+1}|}}\mathfrak{C}_{N_0+1} & \ldots & \frac{1}{\sqrt{|\mu_N|}}\mathfrak{C}_N\end{bmatrix}, \\ L &= \begin{bmatrix}l_0 & \mathfrak{L}_1^\top & \ldots & \mathfrak{L}_{N_0}^\top\end{bmatrix}^\top.\end{aligned}$$

In particular, noting that $|\phi_n(\xi)| \leq \cosh((\log\tau)\xi)$, we have $\|\tilde{\mathfrak{B}}_1\| = O(1)$ and $\|\tilde{C}_1\| = O(1)$ as $N \to +\infty$.

Introducing the vector

$$X = \operatorname{col}\left(\hat{Z}^{N_0}, E^{N_0}, \tilde{Z}^{N-N_0}, \tilde{E}^{N-N_0}\right), \tag{18}$$

the reduced model (17) can be rewritten as

$$\dot{X} = FX + \mathcal{L}\zeta + \mathcal{L}_d d \tag{19}$$

where

$$F = \begin{bmatrix} A_0 + \mathfrak{B}_0 K & LC_0 & 0 & L\tilde{C}_1 \\ 0 & A_0 - LC_0 & 0 & -L\tilde{C}_1 \\ \tilde{\mathfrak{B}}_1 K & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix},$$

$$\mathcal{L} = \mathrm{col}(L, -L, 0, 0),$$

$$\mathcal{L}_d = \mathrm{col}(\nu L, \mathfrak{B}_0 - \nu L, 0, \tilde{\mathfrak{B}}_2).$$

Moreover we have

$$u = \tilde{K} X \tag{20}$$

with the matrix $\tilde{K} = \begin{bmatrix} K & 0 & 0 & 0 \end{bmatrix}$.

**Remark 6.** Both pairs $(A_0, \mathfrak{B}_0)$ and $(A_0, C_0)$ satisfy the Kalman condition. This can be easily observed from the Hautus test using the fact that the eigenvalues $\lambda_n$, $|n| \leq N_0$, of $A_0$ are simple. The case of the pair $(A_0, \mathfrak{B}_0)$ now follows from the fact that $\beta_0 \neq 0$ and that the pairs $(\mathfrak{A}_n, \mathfrak{H}_n)$ satisfy the Kalman condition because $p \neq 0$ and $\operatorname{Im} \lambda_n \neq 0$ for $n \geq 1$. Regarding the pair $(A_0, C_0)$, the conclusion follows from the facts that $\operatorname{Im} \lambda_n \neq 0$ for $n \geq 1$ and $\phi_n(\xi) \neq 0$ for all $n \in \mathbb{Z}$. The latter is because $|\phi_n(\xi)|^2 = |\cosh((\log\tau)\xi)|^2 \cos^2(2n\pi\xi) + |\sinh((\log\tau)\xi)|^2 \sin^2(2n\pi\xi) \geq |\sinh((\log\tau)\xi)|^2 > 0$ for $\xi \neq 0$ while $\phi_n(0) = 1$ for all $n \in \mathbb{Z}$.

## 5. Exponential stability assessment

### 5.1. Stability of the disturbance-free system

We assume throughout this subsection that the boundary disturbance is zero, i.e. $d = 0$. Introducing $\tilde{X} = \mathrm{col}(X, \zeta)$ we obtain from (15) and (17a) that

$$v = \dot{u} = K\dot{\hat{Z}}^{N_0} = E\tilde{X} \tag{21}$$

with the matrix $E = K \begin{bmatrix} A_0 + \mathfrak{B}_0 K & LC_0 & 0 & L\tilde{C}_1 & L \end{bmatrix}$.

The main result of this subsection is stated by the following theorem.

**Theorem 1.** *Let $p > 0$, $r \in \mathbb{R}$, and $s > 1$ be given. Let $\delta > 0$ and $N_0 \geq 0$ be such that $\operatorname{Re} \lambda_n < -\delta$ for all $|n| \geq N_0 + 1$. Let $K \in \mathbb{R}^{1 \times (2N_0+1)}$ and $L \in \mathbb{R}^{2N_0+1}$ be such that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq \max\left(N_0 + 1, \left\lfloor \frac{\log\tau}{2\pi} \right\rfloor\right)$, assume that there exist $P \succ 0$, $\alpha > 1/p$, and $\beta, \gamma > 0$ such that*

$$\Theta_1 \preceq 0, \quad \Theta_2 < 0 \tag{22}$$

*where*

$$\Theta_1 = \begin{bmatrix} F^\top P + PF + 2\delta P + \alpha\gamma \mathfrak{c}_\mu \mathcal{S}_a \tilde{K}^\top \tilde{K} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta \end{bmatrix} + \alpha\gamma \mathfrak{c}_\mu \mathcal{S}_b E^\top E \tag{23}$$

$$\Theta_2 = 2\gamma \left\{ -\left(p - \frac{1}{\alpha}\right) |\operatorname{Re} \mu_{N+1}| + r + \delta \right\} + \beta M_\phi$$

*with $\mathcal{S}_a = \sum_{|n| \geq N+1} |a_n|^2$, $\mathcal{S}_b = \sum_{|n| \geq N+1} |b_n|^2$, $M_\phi = \sum_{|n| \geq N+1} \frac{|\phi_n(\xi)|^2}{|\mu_n|}$, and $\mathfrak{c}_\mu = \frac{|\mu_{N+1}|}{|\operatorname{Re} \mu_{N+1}|} = \frac{4(N+1)^2\pi^2 + (\log\tau)^2}{4(N+1)^2\pi^2 - (\log\tau)^2}$. Then there exists a constant $M > 0$ such that for any initial conditions $z_0 \in H^2(0,1)$, $\hat{z}_0(0) \in \mathbb{R}$, and $\hat{\mathfrak{z}}_n(0) \in \mathbb{R}^2$ such that $z_0(1) = s z_0(0)$ and $z_0'(0) = K\hat{Z}^{N_0}(0)$, the trajectories of the disturbance-free (i.e., $d = 0$) closed-loop system composed of the plant (1) and the controller (14) satisfy*

$$\sum_{n \in \mathbb{Z}} |\mu_n| |z_n(t)|^2 + |\hat{z}_0(t)|^2 + \sum_{n=1}^{N} \|\hat{\mathfrak{z}}_n(t)\|^2$$

$$\leq M e^{-2\delta t} \left( \sum_{n \in \mathbb{Z}} |\mu_n| |z_n(0)|^2 + |\hat{z}_0(0)|^2 + \sum_{n=1}^{N} \|\hat{\mathfrak{z}}_n(0)\|^2 \right) \tag{24}$$

*for all $t \geq 0$. Moreover, the constraints (22) are always feasible for $N$ selected large enough.*

**Proof.** Since $\Theta_2 < 0$, we fix $\epsilon > 0$ so that

$$\Theta_{2,\epsilon} = 2\gamma \left\{ -\left(p - \frac{1}{\alpha}\right) |\operatorname{Re} \mu_{N+1}| + r + \delta \right\} + (1+\epsilon)\beta M_\phi \leq 0.$$

Following [22], it seems natural to consider the Lyapunov function candidate defined for $X \in \mathbb{R}^{2(2N+1)}$ and $w \in D(\mathcal{A})$ by

$$V_\infty(X, w) = X^\top P X + \gamma \sum_{|n| \geq N+1} |\mu_n| \, |\langle w, \psi_n \rangle|^2. \tag{25}$$

However, contrary to the case of reaction–diffusion PDEs with collocated boundary conditions such as the ones studied in [22], the underlying unbounded operator $\mathcal{A}$ is not self-adjoint. In this case, and contrary to the self-adjoint case, it is not straightforward to assess the continuous differentiability of the series appearing in (25) along the system trajectories. To avoid this technical difficulty, we introduce for any given $M \geq N+1$ the functional

$$V_M(X, w) = X^\top P X + \gamma \sum_{N+1 \leq |n| \leq M} |\mu_n| \, |\langle w, \psi_n \rangle|^2. \tag{26}$$

The computation of the time derivative of $V_M$ along the system trajectories (9) and (19) with $d = 0$ gives

$$\begin{aligned} \dot{V}_M &= X^\top \{F^\top P + PF\} X + 2X^\top P \mathcal{L} \zeta \\ &\quad + 2\gamma \sum_{N+1 \leq |n| \leq M} |\mu_n| \operatorname{Re}\left(\{\lambda_n w_n + a_n u + b_n v\} \overline{w_n}\right) \\ &\leq \tilde{X}^\top \begin{bmatrix} F^\top P + PF & P\mathcal{L} \\ \mathcal{L}^\top P & 0 \end{bmatrix} \tilde{X} \\ &\quad + 2\gamma \sum_{N+1 \leq |n| \leq M} |\mu_n| \operatorname{Re} \lambda_n |w_n|^2 \\ &\quad + 2\gamma \sum_{N+1 \leq |n| \leq M} |\mu_n| \{|a_n||u| + |b_n||v|\} |w_n|. \end{aligned}$$

Using Young's inequality, we infer that

$$2 \sum_{N+1 \leq |n| \leq M} |\mu_n| |a_n| |u| |w_n| \leq \frac{1}{\alpha \mathfrak{c}_\mu} \sum_{N+1 \leq |n| \leq M} |\mu_n|^2 |w_n|^2 + \alpha \mathfrak{c}_\mu \mathcal{S}_a |u|^2$$

$$2 \sum_{N+1 \leq |n| \leq M} |\mu_n| |b_n| |v| |w_n| \leq \frac{1}{\alpha \mathfrak{c}_\mu} \sum_{N+1 \leq |n| \leq M} |\mu_n|^2 |w_n|^2 + \alpha \mathfrak{c}_\mu \mathcal{S}_b |v|^2.$$

From Lemma 1 we have $|\mu_n| = (\log\tau)^2 + 4n^2\pi^2$ and $\operatorname{Re} \mu_n = (\log\tau)^2 - 4n^2\pi^2$. Since $N \geq \lfloor \frac{\log\tau}{2\pi} \rfloor > \frac{\log\tau}{2\pi} - 1$, we have $\operatorname{Re} \mu_n \leq \operatorname{Re} \mu_{N+1} = (\log\tau)^2 - 4(N+1)^2\pi^2 < 0$ for all $|n| \geq N+1$. Therefore we infer for $|n| \geq N+1$ that $\frac{|\mu_n|}{|\operatorname{Re} \mu_n|} \leq \frac{|\mu_{N+1}|}{|\operatorname{Re} \mu_{N+1}|} = \mathfrak{c}_\mu$ hence $|\mu_n| \leq \mathfrak{c}_\mu |\operatorname{Re} \mu_n|$. The combination of these estimates and the use of (20)–(21) imply

$$\begin{aligned} \dot{V}_M + 2\delta V_M &\leq \tilde{X}^\top \Theta_{1,\beta=0} \tilde{X} \\ &\quad + 2\gamma \sum_{N+1 \leq |n| \leq M} |\mu_n| \left\{ -\left(p - \frac{1}{\alpha}\right) |\operatorname{Re} \mu_n| + r + \delta \right\} |w_n|^2 \end{aligned}$$

where $\Theta_{1,\beta=0}$ is obtained from (23) by setting $\beta = 0$. Recalling that $\zeta = \sum_{|n|\geq N+1} w_n\phi_n(\xi)$, we infer that

$$\begin{aligned}\zeta^2 &\leq (1+\epsilon)\left(\sum_{N+1\leq|n|\leq M} w_n\phi_n(\xi)\right)^2 \\ &\quad + \left(1+\frac{1}{\epsilon}\right)\left(\sum_{|n|\geq M+1} w_n\phi_n(\xi)\right)^2 \\ &\leq (1+\epsilon) M_\phi \sum_{N+1\leq|n|\leq M} |\mu_n||w_n|^2 \\ &\quad + \left(1+\frac{1}{\epsilon}\right) R_{\phi,M} \sum_{|n|\geq M+1} |\mu_n||w_n|^2\end{aligned}$$

where $R_{\phi,M} = \sum_{|n|\geq M+1} \frac{|\phi_n(\xi)|^2}{|\mu_n|} \rightarrow 0$ when $M \rightarrow +\infty$ and where $\epsilon > 0$ has been fixed at the beginning of the proof. Combining the latter estimates, we obtain that

$$\begin{aligned}\dot{V}_M + 2\delta V_M &\leq \tilde{X}^\top \Theta_1 \tilde{X} + \sum_{N+1\leq|n|\leq M} |\mu_n|\Gamma_n|w_n|^2 \\ &\quad + \left(1+\frac{1}{\epsilon}\right)\beta R_{\phi,M}\sum_{n\in\mathbb{Z}}|\mu_n||w_n|^2\end{aligned}$$

where $\Gamma_n = 2\gamma\left\{-\left(p-\frac{1}{\alpha}\right)|\operatorname{Re}\mu_n| + r + \delta\right\} + (1+\epsilon)\beta M_\phi$. Since $\alpha > 1/p$, we note that $\Gamma_n \leq \Gamma_{N+1} = \Theta_{2,\epsilon} \leq 0$ for all $|n| \geq N+1$. Using in addition $\Theta_1 \preceq 0$, we deduce that

$$\dot{V}_M + 2\delta V_M \leq \left(1+\frac{1}{\epsilon}\right)\beta R_{\phi,M}\sum_{n\in\mathbb{Z}}|\mu_n||w_n|^2.$$

Integrating on the time interval $[0, t]$, we have

$$\begin{aligned}V_M(t) &\leq e^{-2\delta t}V_M(0) \\ &\quad + \left(1+\frac{1}{\epsilon}\right)\beta R_{\phi,M}\int_0^t e^{-2\delta(t-\tau)}\sum_{n\in\mathbb{Z}}|\mu_n||w_n(\tau)|^2\,\mathrm{d}\tau\end{aligned}$$

which implies, by letting $M \rightarrow +\infty$, that $V_\infty(t) \leq e^{-2\delta t}V_\infty(0)$. The claimed stability estimate (24) now easily follows from the definition of $V_\infty$, the use of (8) and Remark 3.

We now assess the feasibility of the constraints (22) for $N$ selected large enough. First, the application of the lemma reported in [21, Appendix] to the matrix $F + \delta I$, shows the existence of $P \succ 0$ so that $F^\top P + PF + 2\delta P = -I$ with $\|P\| = O(1)$ when $N \rightarrow +\infty$. We now fix arbitrarily $\alpha > 1/p$ and we set $\gamma = 1/N$ and $\beta = \sqrt{N}$. Hence we have $\Theta_2 \rightarrow -\infty$ when $N \rightarrow +\infty$. Moreover, noting that $\|\tilde{K}\| = \|K\|$ is a constant independent of $N$ and $\|E\| = O(1)$ as $N \rightarrow +\infty$, the use of Schur complement shows that $\Theta_1 \preceq 0$ for $N$ selected large enough. This completes the proof.

**Remark 7.** Noting that $\min_{n\in\mathbb{Z}}|\mu_n| = (\log\tau)^2 > 0$, because $\tau > 1$, we infer from the stability estimate (24) and the Riesz basis inequalities (7) that $\|z(t,\cdot)\|_{L^2}$ exponentially decreases to zero.

**Remark 8.** It is worth mentioning that the result of Theorem 2 can be easily extended to the derivation of an ISS estimate with respect to a distributed perturbation $\phi(t,\cdot) \in L^2(0,1)$ and a perturbation $d_m(t)$ of the measurement. More precisely, (1a) is replaced by $z_t(t,x) = pz_{xx}(t,x) + rz(t,x) + \phi(t,x)$ and the measurement (1d) is replaced by $y_D(t) = z(t,\xi) + d_m(t)$. This result is easily obtained because the subsequent perturbations $\phi$ and $d_m$ act on the closed-loop system dynamics through bounded operators. The situation is much more involved in the case of the boundary perturbation $d(t)$ applying at the control input via (1c). This is because the establishment of ISS estimates with respect to boundary perturbations is much more challenging compared to perturbations applied through bounded operators [26]. Therefore, we focus our next subsection on the derivation of an ISS property with respect to the boundary perturbation $d$.

### 5.2. Input-to-state stability

We now consider the case of the closed-loop system in the presence of a boundary perturbation $d \neq 0$.

**Theorem 2.** *Let $p > 0$, $r \in \mathbb{R}$, and $s > 1$ be given. Let $\delta > 0$ and $N_0 \geq 0$ be such that $\operatorname{Re}\lambda_n < -\delta$ for all $|n| \geq N_0 + 1$. Let $K \in \mathbb{R}^{1\times(2N_0+1)}$ and $L \in \mathbb{R}^{2N_0+1}$ be such that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. For a given $N \geq \max\left(N_0+1, \lfloor\frac{\log\tau}{2\pi}\rfloor\right)$, assume that there exist $P \succ 0$, $\alpha > 1/(2p)$, and $\beta,\gamma > 0$ such that*

$$\Theta_1 \prec 0, \quad \Theta_2 < 0, \quad \Theta_3 > 0 \tag{27}$$

*where*

$$\begin{aligned}\Theta_1 &= \begin{bmatrix}\Theta_{1,1} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta\end{bmatrix} \\ \Theta_{1,1} &= F^\top P + PF + 2\delta P + (\alpha\gamma\mathcal{S}_1 + 2\beta M_\phi\mathcal{S}_2)\tilde{K}^\top\tilde{K} \\ \Theta_2 &= 2\gamma\left\{-\left(p-\frac{1}{2\alpha}\right)|\operatorname{Re}\mu_{N+1}| + r + \delta\right\} \\ &\quad + 2\beta M_\phi c_\mu^{3/4}|\operatorname{Re}\mu_{N+1}|^{3/4} \\ \Theta_3 &= 2\gamma\left(p-\frac{1}{2\alpha}\right) - \frac{2\beta M_\phi c_\mu^{3/4}}{|\operatorname{Re}\mu_{N+1}|^{1/4}}\end{aligned}$$

*with $M_\phi = \sum_{|n|\geq N+1}\frac{|\phi_n(\xi)|^2}{|\mu_n|^{3/4}}$, $\mathcal{S}_1 = \sum_{|n|\geq N+1}\frac{|\beta_n|^2}{|\operatorname{Re}\mu_n|}$, $\mathcal{S}_2 = \sum_{|n|\geq N+1}|\mu_n|^{3/4}|b_n|^2$, and $c_\mu = \frac{|\mu_{N+1}|}{|\operatorname{Re}\mu_{N+1}|} = \frac{4(N+1)^2\pi^2+(\log\tau)^2}{4(N+1)^2\pi^2-(\log\tau)^2}$. Then there exist constants $M_1, M_2 > 0$ such that for any initial conditions $z_0 \in H^2(0,1)$, $\hat{z}_0(0) \in \mathbb{R}$, and $\hat{\mathfrak{z}}_n(0) \in \mathbb{R}^2$, and any boundary disturbance $d \in \mathcal{C}^2(\mathbb{R}_+)$, all such that $z_0(1) = sz_0(0)$ and $z_0'(0) = K\hat{Z}^{N_0}(0) + d(0)$, the trajectories of the closed-loop system composed of the plant (1) and the controller (14) satisfy*

$$\begin{aligned}&\|z(t,\cdot)\|_{L^2}^2 + |\hat{z}_0(t)|^2 + \sum_{n=1}^N\|\hat{\mathfrak{z}}_n(t)\|^2 \\ &\leq M_1 e^{-2\delta t}\left(\|z_0\|_{L^2}^2 + |\hat{z}_0(0)|^2 + \sum_{n=1}^N\|\hat{\mathfrak{z}}_n(0)\|^2\right) \\ &\quad + M_2\sup_{\tau\in[0,t]} e^{-2\delta(t-\tau)}|d(\tau)|^2\end{aligned} \tag{28}$$

*for all $t \geq 0$. Moreover, the constraints (27) are always feasible for $N$ selected large enough.*

**Proof.** In view of (27), let $\delta' > \delta$ and $\epsilon > 0$ be fixed such that $\Theta_{1,\epsilon} \prec 0$, $\Theta_{2,\epsilon} < 0$, and $\Theta_{3,\epsilon} > 0$ where

$$\begin{aligned}\Theta_{1,\epsilon} &= \begin{bmatrix}\Theta_{1,1,\epsilon} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta\end{bmatrix} \\ \Theta_{1,1,\epsilon} &= F^\top P + PF + 2\delta' P \\ &\quad + (\alpha\gamma(1+\epsilon)\mathcal{S}_1 + (2+\epsilon)\beta M_\phi\mathcal{S}_2)\tilde{K}^\top\tilde{K} \\ \Theta_{2,\epsilon} &= 2\gamma\left\{-\left(p-\frac{1}{2\alpha}\right)|\operatorname{Re}\mu_{N+1}| + r + \delta'\right\} \\ &\quad + (2+\epsilon)\beta M_\phi c_\mu^{3/4}|\operatorname{Re}\mu_{N+1}|^{3/4} \\ \Theta_{3,\epsilon} &= 2\gamma\left(p-\frac{1}{2\alpha}\right) - \frac{(2+\epsilon)\beta M_\phi c_\mu^{3/4}}{|\operatorname{Re}\mu_{N+1}|^{1/4}}.\end{aligned}$$

We consider the Lyapunov functional defined by $V(X,z) = X^\top PX + \gamma \sum_{|n|\geq N+1} |\langle z, \psi_n\rangle|^2$. The computation of the time derivative of $V$ along the system trajectories (10) and (19) gives

$$\begin{aligned}\dot{V} &= X^\top\{F^\top P + PF\}X + 2X^\top P\mathcal{L}\zeta + 2X^\top P\mathcal{L}_d d \\ &\quad + 2\gamma \sum_{|n|\geq N+1} \operatorname{Re}\left(\{\lambda_n z_n + \beta_n u_d\}\,\overline{z_n}\right) \\ &\leq \tilde{X}^\top \begin{bmatrix} F^\top P + PF & P\mathcal{L} \\ \mathcal{L}^\top P & 0\end{bmatrix}\tilde{X} + 2\gamma \sum_{|n|\geq N+1} \operatorname{Re}\lambda_n |z_n|^2 \\ &\quad + 2X^\top P\mathcal{L}_d d + 2\gamma \sum_{|n|\geq N+1} |\beta_n||u_d||z_n|.\end{aligned}$$

The use of Young's inequality implies that

$$2\sum_{|n|\geq N+1} |\beta_n||u_d||z_n| \leq \alpha\mathcal{S}_1|u_d|^2 + \frac{1}{\alpha}\sum_{|n|\geq N+1} |\operatorname{Re}\mu_n||z_n|^2.$$

Moreover, from $u_d = u + d$ with (20), we deduce that $|u_d|^2 \leq (1+\epsilon)X^\top\tilde{K}^\top\tilde{K}X + (1+1/\epsilon)|d|^2$. Recalling the definition $\zeta = \sum_{|n|\geq N+1} w_n\phi_n(\xi)$ and in view of (8) and (20), we infer that

$$\begin{aligned}\zeta^2 &\leq M_\phi \sum_{|n|\geq N+1} |\mu_n|^{3/4}|w_n|^2 \\ &\leq (2+\epsilon)M_\phi \sum_{|n|\geq N+1} |\mu_n|^{3/4}|z_n|^2 + (2+\epsilon)M_\phi\mathcal{S}_2 X^\top\tilde{K}^\top\tilde{K}X \\ &\quad + (1+2/\epsilon)M_\phi\mathcal{S}_2|d|^2.\end{aligned}$$

Finally, since $N \geq \lfloor \frac{\log\tau}{2\pi}\rfloor > \frac{\log\tau}{2\pi} - 1$, we have $\operatorname{Re}\mu_n \leq \operatorname{Re}\mu_{N+1} = (\log\tau)^2 - 4(N+1)^2\pi^2 < 0$ for all $|n| \geq N+1$. Hence, for any $|n| \geq N+1$, we have $\operatorname{Re}\lambda_n = p\operatorname{Re}\mu_n + r = -p|\operatorname{Re}\mu_n| + r$ and we infer that $\frac{|\mu_n|}{|\operatorname{Re}\mu_n|} \leq \frac{|\mu_{N+1}|}{|\operatorname{Re}\mu_{N+1}|} = c_\mu$ thus $|\mu_n| \leq c_\mu|\operatorname{Re}\mu_n|$. Putting together all the above estimates, we deduce that

$$\dot{V} + 2\delta' V \leq \tilde{X}^\top\Theta_{1,\epsilon}\tilde{X} + \sum_{|n|\geq N+1} \Gamma_n|z_n|^2 + 2X^\top P\mathcal{L}_d d + c_d|d|^2$$

where $c_d = \alpha\gamma(1+1/\epsilon)\mathcal{S}_1 + (1+2/\epsilon)\beta M_\phi\mathcal{S}_2$ and

$$\begin{aligned}\Gamma_n &= 2\gamma\left\{-\left(p - \frac{1}{2\alpha}\right)|\operatorname{Re}\mu_n| + r + \delta'\right\} \\ &\quad + (2+\epsilon)\beta M_\phi c_\mu^{3/4}|\operatorname{Re}\mu_n|^{3/4}.\end{aligned}$$

We now note that $|\operatorname{Re}\mu_n| \geq |\operatorname{Re}\mu_{N+1}|$ for all $|n| \geq N+1$ hence $|\operatorname{Re}\mu_n|^{3/4} = \frac{|\operatorname{Re}\mu_n|}{|\operatorname{Re}\mu_n|^{1/4}} \leq \frac{|\operatorname{Re}\mu_n|}{|\operatorname{Re}\mu_{N+1}|^{1/4}}$. This implies that $\Gamma_n \leq -\Theta_{3,\epsilon}|\operatorname{Re}\mu_n| + 2\gamma\{r+\delta'\} \leq -\Theta_{3,\epsilon}|\operatorname{Re}\mu_{N+1}| + 2\gamma\{r+\delta'\} = \Theta_{2,\epsilon} \leq 0$ for all $|n| \geq N+1$, where we have used that $\Theta_{3,\epsilon} \geq 0$. This implies that $\dot{V} + 2\delta' V \leq \tilde{X}^\top\Theta_{1,\epsilon}\tilde{X} + 2X^\top P\mathcal{L}_d d + c_d|d|^2$. Since $\Theta_{1,\epsilon} \prec 0$, the use of Schur complement shows the existence of a sufficiently large constant $c_d' > 0$ so that $\dot{V} + 2\delta' V \leq c_d'|d|^2$. The claimed ISS estimate (28) now easily follows from the integration of the latter inequality, the definition of $V$, the use of (7), and the fact that $\delta' > \delta$.

It remains to assess the fact that the constraints (27) are feasible provided $N$ is selected large enough. To do so, we first apply the lemma reported in [21, Appendix] to the matrix $F + \delta I$, showing the existence of $P \succ 0$ so that $F^\top P + PF + 2\delta P = -I$ with $\|P\| = O(1)$ when $N \to +\infty$. We then arbitrarily fix $\alpha > 1/(2p)$ and we set $\gamma = 1$. We note that $M_\phi \neq 0$ for all $N \geq N_0 + 1$ (because $\phi_n(\xi) \neq 0$; see the end of Remark 6). Hence we can define $\beta = 1/\sqrt{M_\phi}$. We note that $\mathcal{S}_1, \mathcal{S}_2, M_\phi \to 0$, hence $\beta \to +\infty$ and $\beta M_\phi \to 0$, as $N \to +\infty$. Finally we have from Lemma 1 that $\operatorname{Re}\mu_n = (\log\tau)^2 - 4n^2\pi^2$. This shows that $\Theta_2 \to -\infty$ and $\Theta_3 \to 2\gamma\left(p - \frac{1}{2\alpha}\right) > 0$ when $N \to +\infty$. Moreover, the use of Schur complement gives $\Theta_1 \prec 0$ for $N$ selected large enough. This completes the proof.

**Remark 9.** It is worth noting that the procedure employed for the proof of Theorem 2 can also be used to establish an ISS estimate with respect to an additive boundary perturbation of the control input for the classical reaction–diffusion PDE (with collocated boundary conditions) studied in [22] in the specific case of a Dirichlet boundary measurement and a Neumann actuation. This is because the Neumann actuation setting gives $\beta_n = O(1)$. Note however that this approach fails for Dirichlet and Robin boundary actuations because, in that case, one has in general no better than $\beta_n = O(\sqrt{\lambda_n})$.

## 6. Numerical illustration

We illustrate the theoretical results of this paper by considering the reaction–diffusion PDE described by (1) with $p = 0.1$, $r = 1$, $s = 5$, and $\xi = 3/4$. The open-loop system is unstable. Moreover, note that $r > p$ hence the basic control strategy described in Remark 1 cannot be applied.

For a desired exponential decay rate $\delta = 1$, we set the feedback and observer gains as $K = 17.6276$ and $L = 1.9186$, respectively. The constraints (22) of Theorem 1 are found feasible for $N = 2$, ensuring the exponential stability of the disturbance-free (i.e., $d = 0$) closed-loop system in the sense of (24). In the presence of a boundary perturbation $d \neq 0$, the constraints (27) of Theorem 2 appear to be more stringent from a numerical perspective as they are found feasible for $N = 22$. This ensures the exponential input-to-state stability of the closed-loop system in the sense of (28).

For numerical simulation of the above results, we consider the initial condition $z_0(x) = 2 + 2(s-1)\sin\left(\frac{5\pi}{2}x\right)$ while the initial condition of the observer is set so that $z_0'(0) = K\hat{Z}^{N_0}(0) + d(0)$. The time domain evolution of the closed-loop system composed of the PDE (1) and the controller (14) without perturbation ($d = 0$) is depicted in Fig. 1. Considering the boundary disturbance $d(t) = 5\sin(t^2)$, the time domain evolution of the disturbed closed-loop system is depicted in Fig. 2. These results are compliant with the theoretical predictions of Theorem 1 and Theorem 2, respectively.

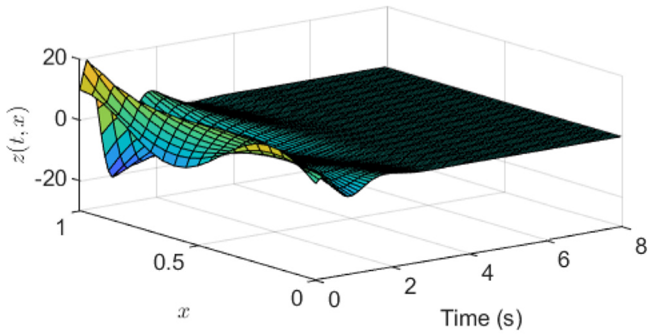
## 7. Conclusion

This paper addressed the topic of output feedback stabilization of reaction–diffusion PDEs with a non-collocated boundary condition. To the best of our knowledge, this is the first time that a solution is reported for such a control design problem. The control strategy couples a finite-dimensional observer and a state-feedback for which it was shown that the exponential stability of the resulting closed-loop system is always achieved when the order of the observer is selected to be large enough.

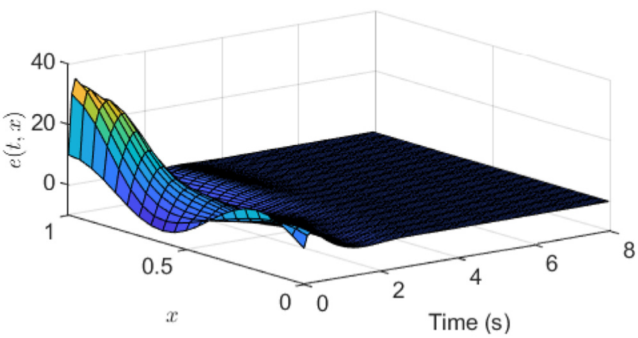
It is worth noting that, using control architectures similar to the one employed in this paper, the output feedback boundary control of general 1-D reaction–diffusion PDEs with collocated boundary conditions using Dirichlet/Neumann boundary measurements was achieved in the case of regulation control [29], input nonlinearities [22], and arbitrarily long input [30], output [31], and state [32] delays. Taking advantage of the procedure described in this paper, all these approaches can be adapted to the PDE described by (1) with non-collocated boundary condition.

Finally, the method employed in this paper to address the disturbance-free setting can also be used to address the dual problem to (1a)–(1c). More precisely, and in view of the adjoint operator $\mathcal{A}^*$ characterized by Lemma 2, we consider the PDE described by

$$\begin{aligned}&z_t(t,x) = pz_{xx}(t,x) + rz(t,x) \\ &z_x(t,0) = sz_x(t,1) \\ &z(t,1) = u(t)\end{aligned}$$

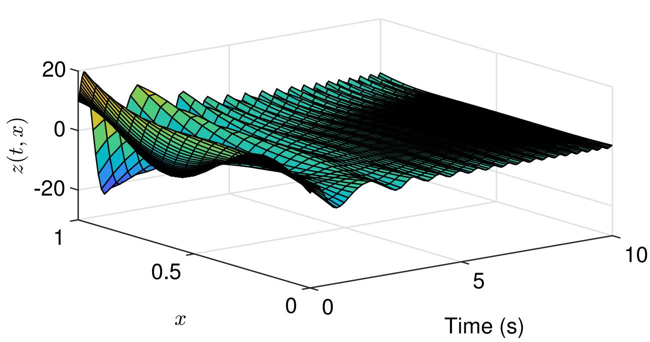



(a) State of the reaction-diffusion system $z(t, x)$

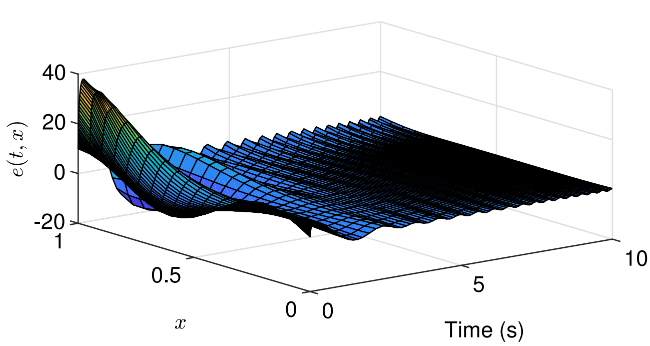



(b) Error of observation $e(t, x) = z(t, x) - \hat{z}(t, x)$

**Fig. 1.** Time evolution of the closed-loop system without perturbation.




(a) State of the reaction-diffusion system $z(t, x)$




(b) Error of observation $e(t, x) = z(t, x) - \hat{z}(t, x)$

**Fig. 2.** Time evolution of the closed-loop system in the presence of a boundary perturbation.

Selecting the system output as $y_D(t) = z(t, \xi)$ for some $\xi \in [0, 1)$, the exponential stabilization of the plant can be obtained using a similar procedure that the one described in this paper.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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