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Singular perturbation approximation by means of a H^2 Lyapunov function for linear hyperbolic systems

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ABSTRACT

A linear hyperbolic system of two conservation laws with two time scales is considered in this paper. The fast time scale is modeled by a small perturbation parameter. By formally setting the perturbation parameter to zero, the full system is decomposed into two subsystems, the reduced subsystem (representing the slow dynamics) and the boundary-layer subsystem (standing for the fast dynamics). The solution of the full system can be approximated by the solution of the reduced subsystem. This result is obtained by using a H^2 Lyapunov function. The estimate of the errors is the order of the perturbation parameter for all initial conditions belonging to H^2 and satisfying suitable compatibility conditions. Moreover, for a particular subset of initial conditions, more precise estimates are obtained. The main result is illustrated by means of numerical simulations.

1. Introduction

Singular perturbations were introduced in control engineering in the late 1960s and have become a tool for analysis and design of control systems [1–4]. Singularly perturbed systems often occur naturally due to the presence of small parasitic parameters, for example, inductance in DC motors model, high gain amplifier in voltage regulators [5]. The singular perturbation method is a way of neglecting the fast transitions and considering them in a separate fast time scale. The significant advantage of this technique is to reduce the system order. Singularly perturbed partial differential equations have been considered in research works from late 1980s. Such systems are interesting since they describe many phenomena in various fields in physics and engineering, see [6] as a survey.

Tikhonov theorem is a fundamental tool for the analysis of singularly perturbed systems. It describes the limiting behavior of solutions of the perturbed system. Tikhonov theorem has been studied for finite dimensional systems modeled by ODEs in many research works [7,8]. The approximation of the full system by the reduced subsystem on a finite time interval requires only the exponential stability of the boundary-layer subsystem. Furthermore, the approximation on an infinite time interval is achieved based on the exponential stability of both subsystems.

In the previous work [9] it has been considered a Tikhonov theorem for linear singularly perturbed hyperbolic systems. The approximation has been achieved by using a L^2 Lyapunov function. Different from the previous work, we consider a H^2 Lyapunov function for a more sophisticated system which consists of the error between the slow dynamics of the full system and the reduced subsystem, the fast dynamics of the full system and the dynamics of the reduced subsystem in this work. The contribution of the present work is a more precise approximation result obtained by Lyapunov methods. More specifically, two cases are considered. In the first case, for any initial conditions belonging to H^2 satisfying suitable compatibility conditions, the difference between the slow dynamics of the full system and the reduced subsystem in L^2-norm is estimated of the order of the small perturbation parameter ϵ. Furthermore, the estimate of the fast dynamics in H^2-norm is also of the order of ϵ. In the second case, where the equilibrium point is chosen as the initial condition of the fast dynamics, the two estimates are obtained of the order of ϵ^2.

The paper is organized as follows. In Section 2, the singularly perturbed linear hyperbolic system under consideration is introduced, and the reduced subsystem is computed. Section 3 contains the main result on singular perturbation approximation of solutions for the full system by that for the subsystem. A numerical simulation is provided in Section 4 to illustrate the main result. Finally, concluding remarks end the paper.

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Notation. For a partitioned symmetric matrix M in $\mathbb{R}^{n\times n}$, the symbol $*$ stands for symmetric block. $M > 0$, $M < 0$ mean that M is positive definite and negative definite respectively. M^{-1} and $M^\top$ represent the inverse and the transpose matrix of M. $|\ |$ denotes the usual Euclidean norm and $\|\ \|$ is the associated matrix norm. $\|\ \|_{L^2}$ denotes the associated norm in $L^2(0,1)$ space, defined by $\|f\|_{L^2} = \left(\int_0^1 |f|^2 dx\right)^{\frac{1}{2}}$ for all functions $f \in L^2(0,1)$. Similarly, the associated norm in $H^2(0,1)$ space is denoted by $\|\ \|_{H^2}$, defined for all functions $f \in H^2(0,1)$, by $\|f\|_{H^2} = \left(\int_0^1 |f|^2 + |f_x|^2 + |f_{xx}|^2 dx\right)^{\frac{1}{2}}$. Following [10], we introduce the notation, $\rho_1(M) = \inf\{\|\Delta M \Delta^{-1}\|, \Delta \in D_{n,+}\}$, where $D_{n,+}$ denotes the set of diagonal positive matrices in $\mathbb{R}^{n\times n}$.

2. Linear singularly perturbed hyperbolic system of two conservation laws

Let us consider a 2×2 linear singularly perturbed hyperbolic system as follows

$$\begin{aligned} &y_t(x,t) + y_x(x,t) = 0,\\ &\epsilon z_t(x,t) + z_x(x,t) = 0, \end{aligned} \tag{1}$$

where $x \in [0,1]$, $t \in [0,+\infty)$, $y : [0,1]\times[0,+\infty) \to \mathbb{R}$, $z : [0,1]\times[0,+\infty) \to \mathbb{R}$ and $0 < \epsilon < 1$ is a small perturbation parameter.

We consider the following boundary conditions for system (1)

$$\begin{pmatrix} y(0,t)\\ z(0,t)\end{pmatrix} = G\begin{pmatrix} y(1,t)\\ z(1,t)\end{pmatrix}, \quad t\in[0,+\infty), \tag{2}$$

where $G = \begin{pmatrix} G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}$ is a 2×2 constant matrix.

Given two functions $y^0 : [0,1] \to \mathbb{R}$ and $z^0 : [0,1] \to \mathbb{R}$, the initial conditions are

$$\begin{pmatrix} y(x,0)\\ z(x,0)\end{pmatrix} = \begin{pmatrix} y^0(x)\\ z^0(x)\end{pmatrix}, \quad x\in[0,1]. \tag{3}$$

Remark 1. According to Proposition 2.1 in [10], for every $(y^0\ z^0)^\top \in H^2(0,1)$ satisfying the following compatibility conditions

$$\begin{pmatrix} y^0(0)\\ z^0(0)\end{pmatrix} = G\begin{pmatrix} y^0(1)\\ z^0(1)\end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} y^0_x(0)\\ \frac{1}{\epsilon} z^0_x(0)\end{pmatrix} = G\begin{pmatrix} y^0_x(1)\\ \frac{1}{\epsilon} z^0_x(1)\end{pmatrix}, \tag{5}$$

the system (1) and (2) has a unique maximal classical solution $(y\ z)^\top \in C^0([0,+\infty), H^2(0,1))$.

Due to Section 2.1 in [11], for all $(y^0\ z^0)^\top \in L^2(0,1)$, there exists a unique maximal weak solution $(y\ z)^\top \in C^0([0,+\infty), L^2(0,1))$ to (1) and (2). $\diamond$

The exponential stability of the linear system (1)–(2) in L^2-norm and H^2-norm is defined as follows

Definition 1. The linear system (1)–(2) is *exponentially stable to the origin in L^2-norm* if there exist $\gamma_1 > 0$ and $C_1 > 0$, such that for every $(y^0\ z^0)^\top \in L^2(0,1)$, the solution to the system (1)–(2) satisfies

$$\left\|\begin{pmatrix} y(.,t)\\ z(.,t)\end{pmatrix}\right\|_{L^2} \leqslant C_1 e^{-\gamma_1 t}\left\|\begin{pmatrix} y^0\\ z^0\end{pmatrix}\right\|_{L^2}, \quad t\in[0,+\infty).$$

The linear system (1)–(2) is *exponentially stable to the origin in H^2-norm* if there exist $\gamma_2 > 0$ and $C_2 > 0$, such that for every $(y^0\ z^0)^\top \in H^2(0,1)$ satisfying the compatibility conditions (4)–(5), the solution to the system (1)–(2) satisfies

$$\left\|\begin{pmatrix} y(.,t)\\ z(.,t)\end{pmatrix}\right\|_{H^2} \leqslant C_2 e^{-\gamma_2 t}\left\|\begin{pmatrix} y^0\\ z^0\end{pmatrix}\right\|_{H^2}, \quad t\in[0,+\infty).$$

Generalizing the approach in [5] to infinite dimensional systems, let us compute the reduced subsystem of the infinite dimensional systems. By setting $\epsilon = 0$ in system (1), we get

$$y_t(x,t) + y_x(x,t) = 0, \tag{6a}$$

$$z_x(x,t) = 0. \tag{6b}$$

Substituting (6b) into the boundary conditions (2) and assuming $G_{22} \neq 1$ yields

$$\begin{aligned} &y(0,t) = \left(G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}\right) y(1,t),\\ &z(.,t) = \frac{G_{21}}{1-G_{22}} y(1,t). \end{aligned} \tag{7}$$

The reduced subsystem is then calculated as

$$\bar{y}_t(x,t) + \bar{y}_x(x,t) = 0, \quad x\in[0,1],\ t\in[0,+\infty), \tag{8}$$

with the boundary condition

$$\bar{y}(0,t) = G_r \bar{y}(1,t), \quad t\in[0,+\infty), \tag{9}$$

where $G_r = G_{11} + \frac{G_{12}G_{21}}{1-G_{22}}$. The bar is used to indicate that the variable belongs to the system with $\epsilon = 0$.

The initial condition of the reduced subsystem (8)–(9) is chosen as the same as for the full system (1)–(2), i.e.

$$\bar{y}^0(x) = y^0(x), \tag{10}$$

and the compatibility conditions for existence of H^2 solutions are given by

$$\begin{aligned} &\bar{y}^0(0) = G_r \bar{y}^0(1),\\ &\bar{y}^0_x(0) = G_r \bar{y}^0_x(1). \end{aligned} \tag{11}$$

Let us recall the stability result for linear hyperbolic system (1)–(2) given in [10,12].

Theorem 1 ([10,12]). *If $\rho_1(G) < 1$ (resp. $\rho_1(G_r) < 1$), the linear system (1)–(2) (resp. the reduced subsystem (8)–(9)) is exponentially stable to the origin in L^2-norm and H^2-norm.*

3. Tikhonov approach for linear singularly perturbed hyperbolic systems

This section presents an approximation theorem for system (1)–(2). More precisely, the difference of the solution between the full system (1)–(2) and the reduced subsystem (8)–(9) is firstly estimated of the order of ϵ. Then for particular initial conditions, it is estimated of the order of ϵ^2. Therefore, the solution of the full system can be approximated by the solution of the reduced subsystem. A H^2 Lyapunov function is used to prove this result.

Let us first perform a change of variable

$$\eta = y - \bar{y},$$

where y is the solution of the full system and $\bar{y}$ is the solution of the reduced subsystem.

By considering the fast dynamics z in (1)–(2) and the dynamics $\bar{y}$ in (8)–(9), let us study the following system

$$\begin{aligned} &\eta_t + \eta_x = 0,\\ &\epsilon z_t + z_x = 0,\\ &\bar{y}_t + \bar{y}_x = 0, \end{aligned} \tag{12}$$

with the following boundary conditions, for all $t \geqslant 0$,

$$\eta(0,t) = G_{11}\eta(1,t) + G_{12}z(1,t) - \frac{G_{12}G_{21}}{1-G_{22}}\bar{y}(1,t), \tag{13a}$$

$$z(0,t) = G_{21}\eta(1,t) + G_{22}z(1,t) + G_{21}\bar{y}(1,t), \tag{13b}$$

$$\bar{y}(0,t) = G_r\bar{y}(1,t). \tag{13c}$$

The compatibility conditions (4), (5) and (11) are rewritten as follows

$$\begin{aligned} \eta^0(0) &= G_{11}\eta^0(1) + G_{12}z^0(1) - \frac{G_{12}G_{21}}{1-G_{22}}\bar{y}^0(1),\\ z^0(0) &= G_{21}\eta^0(1) + G_{22}z^0(1) + G_{21}\bar{y}^0(1),\\ \bar{y}^0(0) &= G_r\bar{y}^0(1). \end{aligned} \tag{14}$$

$$\begin{aligned} \eta_x^0(0) &= G_{11}\eta_x^0(1) + \frac{G_{12}}{\epsilon}z_x^0(1) - \frac{G_{12}G_{21}}{1-G_{22}}\bar{y}_x^0(1),\\ z_x^0(0) &= \epsilon G_{21}\eta_x^0(1) + G_{22}z_x^0(1) + \epsilon G_{21}\bar{y}_x^0(1),\\ \bar{y}_x^0(0) &= G_r\bar{y}_x^0(1). \end{aligned} \tag{15}$$

Let us introduce a candidate Lyapunov function for system (12)–(13)

$$\begin{aligned} V(\eta,z,\bar{y}) = {} & a\int_0^1 e^{-\mu x}\left(\frac{\eta^2}{\epsilon} + \epsilon\eta_x^2 + \epsilon^3\eta_{xx}^2\right)dx\\ & + b\int_0^1 e^{-\mu x}\left(\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 + z_x^2 + z_{xx}^2\right)dx\\ & + \epsilon c\int_0^1 e^{-\mu x}\left(\bar{y}^2 + \bar{y}_x^2 + \bar{y}_{xx}^2\right)dx, \end{aligned} \tag{16}$$

where a, b, c and μ are positive values. If initial conditions satisfy the compatibility conditions (14) and (15), then $V(\eta,z,\bar{y})$ is well defined along the solutions of (12) and (13).

To simplify the analysis of time derivative of $V(\eta,z,\bar{y})$, we rewrite it in the following way

$$V(\eta,z,\bar{y}) = V_0(\eta,z,\bar{y}) + V_1(\eta,z,\bar{y}) + V_2(\eta,z,\bar{y}), \tag{17}$$

with $V_0(\eta,z,\bar{y})$, $V_1(\eta,z,\bar{y})$ and $V_2(\eta,z,\bar{y})$ defined respectively by the 0th, 1st and 2nd space derivative of the solutions, that are

$$\begin{aligned} V_0(\eta,z,\bar{y}) = {} & \frac{a}{\epsilon}\int_0^1 e^{-\mu x}\eta^2dx\\ & + b\int_0^1 e^{-\mu x}\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2dx\\ & + \epsilon c\int_0^1 e^{-\mu x}\bar{y}^2dx, \end{aligned} \tag{18}$$

$$\begin{aligned} V_1(\eta,z,\bar{y}) = {} & \epsilon a\int_0^1 e^{-\mu x}\eta_x^2dx + b\int_0^1 e^{-\mu x}z_x^2dx\\ & + \epsilon c\int_0^1 e^{-\mu x}\bar{y}_x^2dx, \end{aligned} \tag{19}$$

$$\begin{aligned} V_2(\eta,z,\bar{y}) = {} & \epsilon^3 a\int_0^1 e^{-\mu x}\eta_{xx}^2dx + b\int_0^1 e^{-\mu x}z_{xx}^2dx\\ & + \epsilon c\int_0^1 e^{-\mu x}\bar{y}_{xx}^2dx. \end{aligned} \tag{20}$$

First, we estimate the time derivative of $V_0(\eta,z,\bar{y})$ along the classical C^1-solution of system (12) with the boundary condition (13). Lemma 1 gives an estimate of $\dot{V}_0(\eta,z,\bar{y})$.

Lemma 1. *If the boundary conditions matrix satisfies $\rho_1(G) < 1$, then there exist positive real values a, b and μ such that for all positive c and ϵ, along the solutions to (12)–(13), it holds,*

$$\begin{aligned} \dot{V}_0(\eta,z,\bar{y}) \leqslant {} & -\frac{a\mu}{\epsilon}\int_0^1 e^{-\mu x}\eta^2dx\\ & - \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x}\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2dx\\ & - \epsilon c\mu\int_0^1 e^{-\mu x}\bar{y}^2dx + \frac{2bG_{21}}{1-G_{22}}\\ & \times\int_0^1 e^{-\mu x}\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\bar{y}_x(1)dx. \end{aligned} \tag{21}$$

In order to handle the term $\bar{y}_x(1)$ in (21), let us consider the following estimate

$$\begin{aligned} |\bar{y}_x(1)| &= \left|\int_0^1\left(x\bar{y}_{xx} + \bar{y}_x\right)dx\right| \leqslant \int_0^1\left(|\bar{y}| + |\bar{y}_x| + |\bar{y}_{xx}|\right)dx\\ &\leqslant \sqrt{3}\left(\int_0^1\left(\bar{y}^2 + \bar{y}_x^2 + \bar{y}_{xx}^2\right)dx\right)^{\frac{1}{2}}. \end{aligned} \tag{22}$$

It appears clearly from (22) that it is necessary to check the dynamics of $\bar{y}_x$ and $\bar{y}_{xx}$. This is the reason why we consider a H^2 Lyapunov function. Differentiating system (12) with respect to x yields

$$\begin{aligned} &\eta_{xt} + \eta_{xx} = 0,\\ &\epsilon z_{xt} + z_{xx} = 0,\\ &\bar{y}_{xt} + \bar{y}_{xx} = 0. \end{aligned} \tag{23}$$

Differentiating (13) with respect to t and using (12), we obtain the following boundary conditions, for all $t \geqslant 0$,

$$\begin{aligned} &\eta_x(0,t) = G_{11}\eta_x(1,t) + \frac{G_{12}}{\epsilon}z_x(1,t) - \frac{G_{12}G_{21}}{1-G_{22}}\bar{y}_x(1,t),\\ &z_x(0,t) = \epsilon G_{21}\eta_x(1,t) + G_{22}z_x(1,t) + \epsilon G_{21}\bar{y}_x(1,t),\\ &\bar{y}_x(0,t) = G_r\bar{y}_x(1,t). \end{aligned} \tag{24}$$

Next, we compute the time derivative of $V_1(\eta,z,\bar{y})$ along the classical C^1-solution of system (23) with the boundary condition (24). The estimate on $\dot{V}_1(\eta,z,\bar{y})$ is given in Lemma 2.

Lemma 2. *Assume $\rho_1(G) < 1$ and let a, b and μ as in Lemma 1. Then there exists a positive real value c' such that for all $c > c'$ and $\epsilon > 0$, along the solutions to (12)–(13), it holds,*

$$\begin{aligned} \dot{V}_1(\eta,z,\bar{y}) \leqslant {} & -\epsilon a\mu\int_0^1 e^{-\mu x}\eta_x^2dx - \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x}z_x^2dx\\ & - \epsilon c\mu\int_0^1 e^{-\mu x}\bar{y}_x^2dx. \end{aligned} \tag{25}$$

To consider the dynamics of $\bar{y}_{xx}$, let us differentiate system (23) with respect to x. It follows

$$\begin{aligned} &\eta_{xxt} + \eta_{xxx} = 0,\\ &\epsilon z_{xxt} + z_{xxx} = 0,\\ &\bar{y}_{xxt} + \bar{y}_{xxx} = 0. \end{aligned} \tag{26}$$

Moreover, differentiating (24) with respect to t and using (23), the boundary conditions are given as follows, for all $t \geqslant 0$,

$$\begin{aligned} &\eta_{xx}(0,t) = G_{11}\eta_{xx}(1,t) + \frac{G_{12}}{\epsilon^2}z_{xx}(1,t) - \frac{G_{12}G_{21}}{1-G_{22}}\bar{y}_{xx}(1,t),\\ &z_{xx}(0,t) = \epsilon^2G_{21}\eta_{xx}(1,t) + G_{22}z_{xx}(1,t) + \epsilon^2G_{21}\bar{y}_{xx}(1,t),\\ &\bar{y}_{xx}(0,t) = G_r\bar{y}_{xx}(1,t). \end{aligned} \tag{27}$$

Lastly, we compute the time derivative of $V_2(\eta, z, \bar{y})$ along the C^1-solution of system (26) with the boundary condition (27). The estimate on $\dot{V}_2(\eta, z, \bar{y})$ is given in Lemma 3.

Lemma 3. *Assume $\rho_1(G) < 1$ and let a, b, c' and μ as in Lemmas 1 and 2. Then there exists a positive real value ϵ^*, such that for all $0 < \epsilon < \epsilon^*$ and $c > c'$, along the solutions to (12)–(13), it holds*

$$\dot{V}_2(\eta, z, \bar{y}) \leqslant -\epsilon^3 a\mu \int_0^1 e^{-\mu x}\eta_{xx}^2 dx - \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x} z_{xx}^2 dx - \epsilon c\mu \int_0^1 e^{-\mu x}\bar{y}_{xx}^2 dx. \tag{28}$$

Remark 2. Due to [12], it may be deduced that for every $(\eta_{xx}^0, \bar{y}_{xx}^0, z_{xx}^0)^\top \in L^2(0,1)$, the Cauchy problem (26)–(27) has a weak L^2-solution issuing from $(\eta_{xx}^0, \bar{y}_{xx}^0, z_{xx}^0)^\top$. Similarly, the time derivative of V_2 has to be understood in a weak sense (see i.e. [13, Chapter 4]). ◇

The proofs of Lemmas 1–3 are given in the Appendix. We are now ready to state and prove our main result.

Theorem 2. *Consider the linear singularly perturbed hyperbolic system of two conservation laws (1)–(2). If the condition $\rho_1(G) < 1$ is satisfied, there exist positive values ϵ^*, a, b, c, θ and μ such that for all $0 < \epsilon < \epsilon^*$, for all $(y^0\ z^0)^T \in H^2(0,1)$ satisfying the compatibility conditions (4), (5) and (11) with $\bar{y}^0 = y^0$ and for all $t \geqslant 0$, the following holds*

$$\|y(.,t) - \bar{y}(.,t)\|_{L^2}^2 \leqslant \epsilon \frac{be^{\mu}}{a} e^{-\epsilon\theta t}\left\| z^0 - \frac{G_{21}}{1-G_{22}}\bar{y}^0(1)\right\|_{H^2}^2 + \epsilon^2 \frac{ce^{\mu}}{a} e^{-\epsilon\theta t}\|\bar{y}^0\|_{H^2}^2, \tag{29}$$

$$\int_0^{+\infty} \left\| z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2 dt \leqslant \epsilon \frac{2e^{\mu}}{\mu}\left\| z^0 - \frac{G_{21}}{1-G_{22}}\bar{y}^0(1)\right\|_{H^2}^2 + \epsilon^2 \frac{2ce^{\mu}}{b\mu}\|\bar{y}^0\|_{H^2}^2. \tag{30}$$

Proof. Let us compute the time derivative of $V(\eta, z, \bar{y})$, $\dot{V}(\eta, z, \bar{y}) = \dot{V}_0(\eta, z, \bar{y}) + \dot{V}_1(\eta, z, \bar{y}) + \dot{V}_2(\eta, z, \bar{y})$. By Lemmas 1–3, there exist positive constants a, b, μ given by Lemma 1, c' given by Lemma 2 and ϵ^* given by Lemma 3 such that the following holds for $c > c'$, and for $0 < \epsilon < \epsilon^*$

$$\begin{aligned}\dot{V}(\eta, z, \bar{y}) &\leqslant -a\mu \int_0^1 e^{-\mu x}\left(\frac{\eta^2}{\epsilon} + \epsilon\eta_x^2 + \epsilon^3\eta_{xx}^2\right)dx \\ &- \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x}\left(\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 + z_x^2 + z_{xx}^2\right)dx \\ &- \epsilon c\mu \int_0^1 e^{-\mu x}(\bar{y}^2 + \bar{y}_x^2 + \bar{y}_{xx}^2)dx + \frac{2bG_{21}}{1-G_{22}} \\ &\times \int_0^1 e^{-\mu x}\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\bar{y}_x(1)dx.\end{aligned}$$

Applying Young's inequality, for all positive values κ_1, it follows

$$\begin{aligned}\dot{V}(\eta, z, \bar{y}) &\leqslant -a\mu \int_0^1 e^{-\mu x}\left(\frac{\eta^2}{\epsilon} + \epsilon\eta_x^2 + \epsilon^3\eta_{xx}^2\right)dx \\ &- \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x}\left(\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 + z_x^2 + z_{xx}^2\right)dx \\ &- \epsilon c\mu \int_0^1 e^{-\mu x}(\bar{y}^2 + \bar{y}_x^2 + \bar{y}_{xx}^2)dx \\ &+ \kappa_1 b\left|\frac{G_{21}}{1-G_{22}}\right||\bar{y}_x(1)|^2 + \frac{b\left|\frac{G_{21}}{1-G_{22}}\right|}{\kappa_1} \\ &\times \int_0^1 e^{-\mu x}\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 dx.\end{aligned}$$

Using the estimate of $|\bar{y}_x(1)|$ in (22), it follows

$$\begin{aligned}\dot{V}(\eta, z, \bar{y}) &\leqslant -a\mu \int_0^1 e^{-\mu x}\left(\frac{\eta^2}{\epsilon} + \epsilon\eta_x^2 + \epsilon^3\eta_{xx}^2\right)dx \\ &- \left(\frac{b\mu}{\epsilon} - \frac{b\left|\frac{G_{21}}{1-G_{22}}\right|}{\kappa_1}\right)\int_0^1 e^{-\mu x} \\ &\times \left(\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 + z_x^2 + z_{xx}^2\right)dx \\ &- \left(\epsilon c\mu - 3e^{\mu}\kappa_1 b\left|\frac{G_{21}}{1-G_{22}}\right|\right) \\ &\times \int_0^1 e^{-\mu x}(\bar{y}^2 + \bar{y}_x^2 + \bar{y}_{xx}^2)dx.\end{aligned}$$

Moreover, choosing $\kappa_1 = 2\epsilon\left|\frac{G_{21}}{1-G_{22}}\right|/\mu$, it yields

$$\begin{aligned}&\dot{V}(\eta, z, \bar{y}) \\ &\leqslant -a\mu \int_0^1 e^{-\mu x}\left(\frac{\eta^2}{\epsilon} + \epsilon\eta_x^2 + \epsilon^3\eta_{xx}^2\right)dx \\ &- \frac{b\mu}{2\epsilon}\int_0^1 e^{-\mu x}\left(\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 + z_x^2 + z_{xx}^2\right)dx \\ &- \epsilon c\left(\mu - \frac{6e^{\mu}b\left|\frac{G_{21}}{1-G_{22}}\right|^2}{c\mu}\right)\int_0^1 e^{-\mu x}(\bar{y}^2 + \bar{y}_x^2 + \bar{y}_{xx}^2)dx.\end{aligned} \tag{31}$$

Let $c^* = \max\left(c', 6e^{\mu}b\left|\frac{G_{21}}{1-G_{22}}\right|^2/\mu^2\right)$ such that for all $c > c^*$, we may define $\theta = \mu - \frac{6e^{\mu}b\left|\frac{G_{21}}{1-G_{22}}\right|^2}{2c^*\mu}$, it is deduced from (31)

$$\dot{V}(\eta, z, \bar{y}) \leqslant -\epsilon\theta V(\eta, z, \bar{y}).$$

We get the following inequality for all $t \geqslant 0$.

$$V(\eta, z, \bar{y}) \leqslant e^{-\epsilon\theta t}V(\eta^0, z^0, \bar{y}^0). \tag{32}$$

Using the fact that

$$\begin{aligned}&\frac{ae^{-\mu}}{\epsilon}\|\eta\|_{L^2}^2 + be^{-\mu}\left\|z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right\|_{H^2}^2 + \epsilon ce^{-\mu}\|\bar{y}\|_{H^2}^2 \\ &\leqslant V(\eta, z, \bar{y}) \leqslant \frac{a}{\epsilon}\|\eta\|_{H^2}^2 \\ &+ b\left\|z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right\|_{H^2}^2 + \epsilon c\|\bar{y}\|_{H^2}^2,\end{aligned} \tag{33}$$

it follows

$$\begin{aligned}\|\eta\|_{L^2}^2 &\leqslant \frac{\epsilon e^{\mu}}{a}V(\eta,z,\bar{y}) \leqslant \frac{\epsilon e^{\mu}}{a}e^{-\epsilon\theta t}V(\eta^0,z^0,\bar{y}^0)\\ &\leqslant \frac{\epsilon e^{\mu}}{a}e^{-\epsilon\theta t}\left(\frac{a}{\epsilon}\|\eta^0\|_{H^2}^2 + b\left\|z^0 - \frac{G_{21}}{1-G_{22}}\bar{y}^0(1)\right\|_{H^2}^2\right.\\ &\quad \left. + \epsilon c\|\bar{y}^0\|_{H^2}^2\right).\end{aligned}$$

Since the initial condition $y^0 = \bar{y}^0$ i.e. $\eta^0 = 0$, therefore (29) holds.

Noting that, for all $c > c^*$, the third term in the right hand part of (31) is always negative, as well as the first term, we can rewrite (31) as follows

$$\dot{V}(\eta,z,\bar{y}) \leqslant -\frac{b\mu}{2\epsilon}\int_0^1 e^{-\mu x}\left(\left(z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 + z_x^2 + z_{xx}^2\right)dx. \tag{34}$$

Performing the time integration of both sides from 0 to $+\infty$ yields

$$\begin{aligned}&\frac{b\mu e^{-\mu}}{2\epsilon}\int_0^{+\infty}\left\|z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right\|_{H^2}^2 dt\\ &\leqslant V(\eta^0,z^0,\bar{y}^0) - \lim_{t\to+\infty}V(\eta,z,\bar{y}),\end{aligned}$$

and since $\lim_{t\to+\infty}V(\eta,z,\bar{y}) = 0$, it follows

$$\int_0^{+\infty}\left\|z - \frac{G_{21}}{1-G_{22}}\bar{y}(1)\right\|_{H^2}^2 dt \leqslant \epsilon\frac{2e^{\mu}}{b\mu}V(\eta^0,z^0,\bar{y}^0). \tag{35}$$

Due to (33) and $\eta^0 = 0$, the inequality (30) holds. This concludes the proof of Theorem 2. □

Selecting particular initial conditions, we can establish more precise estimates.

Corollary 1. *If $\rho_1(G) < 1$, there exist positive values ϵ^*, a, b, c, θ and μ, such that for all $\epsilon \in (0,\epsilon^*)$ and for all $y^0 \in H^2(0,1)$ satisfying the compatibility conditions (4), (5) and (11) with $\bar{y}^0 = y^0$, $z^0 = \frac{G_{21}}{1-G_{22}}y^0(1)$, it holds for all $t \geqslant 0$,*

$$\|y(.,t) - \bar{y}(.,t)\|_{L^2}^2 \leqslant \epsilon^2\frac{ce^{\mu}}{a}e^{-\epsilon\theta t}\|\bar{y}^0\|_{H^2}^2, \tag{36}$$

$$\int_0^{+\infty}\left\|z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2 dt \leqslant \epsilon^2\frac{2ce^{\mu}}{b\mu}\|\bar{y}^0\|_{H^2}^2, \tag{37}$$

$$\left\|z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2 \leqslant \epsilon\frac{ce^{\mu}}{b}e^{-\epsilon\theta t}\|\bar{y}^0\|_{H^2}^2. \tag{38}$$

Proof. The proof of this corollary is based on Theorem 2. We get that (36) and (37) hold by considering the initial condition $z^0 = \frac{G_{21}}{1-G_{22}}y^0(1)$ in (29) and (30). It is deduced from (32) and (33) that, for all $t \geqslant 0$,

$$\begin{aligned}&\left\|z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2\\ &\leqslant \frac{e^{\mu}e^{-\epsilon\theta t}}{b}\left(\frac{a}{\epsilon}\|\eta^0\|_{H^2}^2 + b\left\|z^0 - \frac{G_{21}}{1-G_{22}}\bar{y}^0(1)\right\|_{H^2}^2\right.\\ &\quad\left. + \epsilon c\|\bar{y}^0\|_{H^2}^2\right).\end{aligned} \tag{39}$$

Table 1
Estimates of the errors with the initial conditions $(y^0\ z^0)^\top \in H^2$.

ϵ	10^{-3}	10^{-2}	10^{-1}
$\|y - \bar{y}(.,t=15)\|_{L^2}^2$	1.8×10^{-6}	9.9×10^{-5}	3.5×10^{-3}
$\int_0^{50}\left\|z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2 dt$	1.7×10^{-6}	1.3×10^{-4}	2.3×10^{-2}

With the initial conditions $\eta^0 = 0$ and $z^0 = \frac{G_{21}}{1-G_{22}}y^0(1)$, we get that (38) holds. This concludes the proof of Corollary 1. □

Remark 3. For the simplicity, we consider a 2×2 system throughout the whole paper. However, the main result in this paper can be extended, in a straightforward way, to systems of $(n+m)$ equations with $\epsilon > 0$, where $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$. ◇

4. Numerical example

In this section, we give a numerical example to illustrate the main result. Let us consider the boundary conditions matrix $G = \begin{pmatrix}0.6 & 1.5\\ 0.2 & -0.5\end{pmatrix}$ for system (1)–(2). With $\Delta = \text{diag}(\sqrt{0.1}, 1)$, it holds $\|\Delta G\Delta^{-1}\| < 1$. Thus $\rho_1(G) < 1$ and Theorem 2 applies.

Let us check the numerical solutions. We use a two-step variant of the Lax–Wendroff method, which is presented in [14,15], to discretize the equation. More precisely, we divide the space domain [0, 1] into 100 intervals of identical length, and 50 as final time. We choose a time-step $dt = 0.9\epsilon dx$ that satisfies the CFL condition, $\lambda dt/dx < 1$ where λ is the maximum value of the transport velocities (in the present work $\lambda = 1/\epsilon$), for the stability. Let us select the following initial conditions $y(x,0) = \bar{y}(x,0) = 1 - \cos(4\pi x)$, $z(x,0) = \cos(6\pi x) - 1$, for all $x \in [0,1]$.

Remark 4. The singular perturbation approximation decreases the simulation cost. Precisely, instead of simulating the full system by using a small time-step which depends on ϵ and satisfies the CFL condition $dt < \epsilon dx$, we simulate the reduced system where a longer time-step can be chosen satisfying the CFL condition $dt < dx$. ◇

The values of perturbation parameter are chosen as $\epsilon = \{10^{-3}, 10^{-2}, 10^{-1}\}$. Table 1 shows the evolution of $\|y - \bar{y}(.,t=15)\|_{L^2}^2$ and $\int_0^{50}\left\|z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2 dt$ with different values of ϵ, for the initial conditions $(y^0\ z^0)^\top \in H^2$ satisfying the compatibility conditions. It indicates that the two estimates are small and decrease as ϵ decreases, as expected from Theorem 2. However, in this simulation it is seen that the decay coefficient of the two estimates is roughly ϵ^2, it is different from the result in Theorem 2 which is ϵ. Fig. 1 shows the time evolutions of $\log\|y(.,t) - \bar{y}(.,t)\|_{L^2}^2$ for different ϵ. It is observed that $\|y(.,t) - \bar{y}(.,t)\|_{L^2}^2$ decreases as time increases.

Let us examine Corollary 1 by choosing the particular initial condition z^0, which is given as $y(x,0) = \bar{y}(x,0) = 1 - \cos(4\pi x)$, $z(.,0) = \frac{G_{21}}{1-G_{22}}y(1,0)$.

Table 2 gives the estimates of $\|y - \bar{y}(.,t = 15)\|_{L^2}^2$ and $\int_0^{50}\left\|z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|_{H^2}^2 dt$ with different ϵ. When ϵ decreases, both estimates tend to zero. Moreover, the decay coefficient is ϵ^2, as estimated in Corollary 1.

The time evolutions of $\log\|y(.,t)-\bar{y}(.,t)\|_{L^2}^2$ for different ϵ with initial condition $z^0 = \frac{G_{21}}{1-G_{22}}y^0(1)$ is given in Fig. 2. After $t = 5$, $\|y(.,t) - \bar{y}(.,t)\|_{L^2}^2$ decreases as time increases.

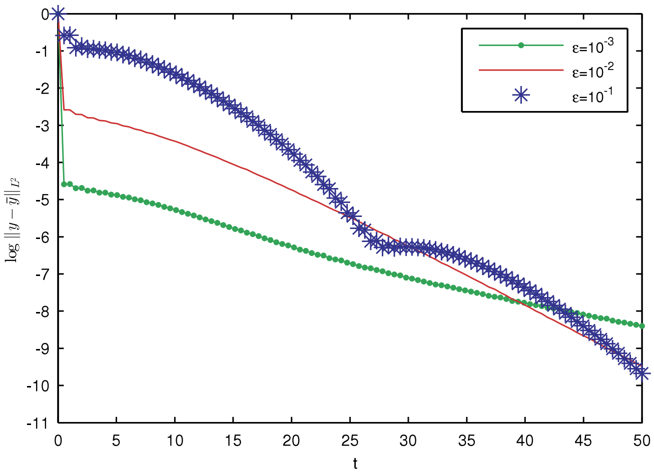

Fig. 1. Time evolutions of $\log \|y(.,t)-\bar{y}(.,t)\|^2_{L^2}$ for different values of ϵ with initial conditions belonging to H^2.

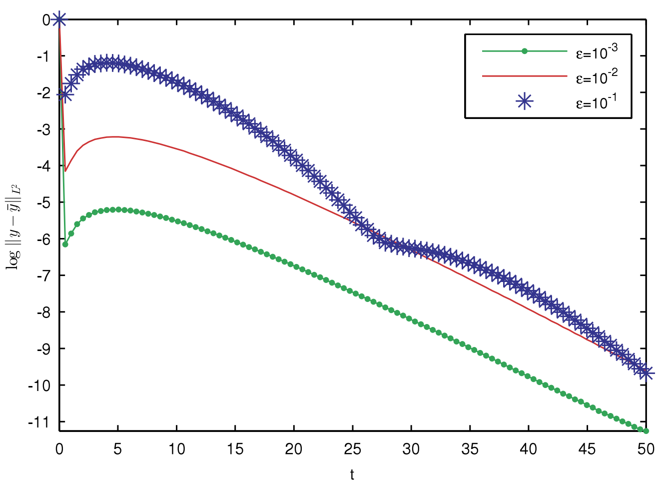

Fig. 2. Time evolutions of $\log \|y(.,t)-\bar{y}(.,t)\|^2_{L^2}$ for different values of ϵ with the particular initial condition $z^0 = \frac{G_{21}}{1-G_{22}} y^0(1)$.

Table 2
Estimates of the errors with the particular initial condition $z^0 = \frac{G_{21}}{1-G_{22}} y^0(1)$.

ϵ	10^{-3}	10^{-2}	10^{-1}
$\|y-\bar{y}(.,t=15)\|^2_{L^2}$	9.2×10^{-7}	8.4×10^{-5}	2.9×10^{-3}
$\int_0^{50} \left\| z(.,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right\|^2_{H^2} dt$	2.0×10^{-7}	3.2×10^{-5}	1.6×10^{-2}

5. Conclusion

This paper is concerned with a linear singularly perturbed hyperbolic system of two conservation laws. By setting the perturbation parameter ϵ to zero, the reduced subsystem is computed. The Tikhonov approximation for such infinite dimensional systems is achieved by a H^2 Lyapunov function. In Theorem 2, for all initial conditions belonging to H^2 satisfying the suitable compatibility conditions, the solution of the full system can be approximated by that of the reduced subsystem. Moreover, the error is estimated as the order of ϵ. Furthermore, by choosing the particular initial condition z^0, the estimate of the error between the full system and the reduced subsystem is the order of ϵ^2 as stated in Corollary 1. Applying this main result to some physical applications, like open channels as considered in [12] and gas flow through pipelines in [16] or [17], is a topic of future work.

Appendix

Proof of Lemma 1. Let us consider $V_0(\eta, z, \bar{y})$ defined in (18) and differentiate it with respect to t along the solutions to (12), it follows

$$\begin{aligned}
&\dot{V}_0(\eta,z,\bar{y})\\
&= \frac{a}{\epsilon}\int_0^1 e^{-\mu x}(-2\eta\eta_x)\,dx + \epsilon c\int_0^1 e^{-\mu x}(-2\bar{y}\bar{y}_x)\,dx\\
&\quad + \frac{b}{\epsilon}\int_0^1 e^{-\mu x}\left(-2\left(z-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)z_x\right)\,dx\\
&\quad + b\int_0^1 e^{-\mu x}\frac{2G_{21}}{1-G_{22}}\left(z-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\bar{y}_x(1)\,dx.
\end{aligned}$$

Performing an integration by parts yields $\dot{V}_0(\eta,z,\bar{y}) = V_{01}+V_{02}$, where

$$\begin{aligned}
V_{01} &= -\frac{a}{\epsilon}[e^{-\mu x}\eta^2]_0^1 - \frac{b}{\epsilon}\left[e^{-\mu x}\left(z-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2\right]_0^1\\
&\quad - \epsilon c[e^{-\mu x}\bar{y}^2]_0^1,\\
V_{02} &= -\frac{a\mu}{\epsilon}\int_0^1 e^{-\mu x}\eta^2 dx - \frac{b\mu}{\epsilon}\\
&\quad \times \int_0^1 e^{-\mu x}\left(z-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2 dx\\
&\quad - \epsilon c\mu\int_0^1 e^{-\mu x}\bar{y}^2 dx + \frac{2bG_{21}}{1-G_{22}}\\
&\quad \times \int_0^1 e^{-\mu x}\left(z-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\bar{y}_x(1)dx.
\end{aligned}$$

To simplify the computation, let us rewrite (13b) as follows

$$\begin{aligned}
&z(0,t) - \frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\\
&= G_{21}\eta(1,t) + G_{22}\left(z(1,t)-\frac{G_{21}}{1-G_{22}}\bar{y}(1,t)\right).
\end{aligned}\tag{40}$$

Using (40), (13a) and (13c), V_{01} is rewritten as

$$\begin{aligned}
V_{01} &= -\frac{a}{\epsilon}\Bigg[e^{-\mu}\eta^2(1)\\
&\quad - \left(G_{11}\eta(1) + G_{12}\left(z(1)-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\right)^2\Bigg]\\
&\quad - \frac{b}{\epsilon}\Bigg[e^{-\mu}\left(z(1)-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)^2\\
&\quad - \left(G_{21}\eta(1) + G_{22}\left(z(1)-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\right)^2\Bigg]\\
&\quad - \epsilon c[e^{-\mu}\bar{y}^2(1) - G_r^2\bar{y}^2(1)].
\end{aligned}$$

Developing the above terms and reorganizing them, we get

$$\begin{aligned}
V_{01} &= -\begin{pmatrix} \eta(1)\\ \left(z(1)-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\\ \bar{y}(1)\end{pmatrix}^{\top}\left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_{01}} & \begin{array}{c}0\\0\end{array}\\ \hline 0 & 0 & M_{02}\end{array}\right)\\
&\quad \times \begin{pmatrix} \eta(1)\\ \left(z(1)-\frac{G_{21}}{1-G_{22}}\bar{y}(1)\right)\\ \bar{y}(1),\end{pmatrix},
\end{aligned}$$

with $M_{01} = \begin{pmatrix} \frac{ae^{-\mu} - (aG_{11}^2 + bG_{21}^2)}{\epsilon} & -\frac{aG_{11}G_{12} + bG_{21}G_{22}}{\epsilon} \\ * & \frac{be^{-\mu} - (bG_{22}^2 + aG_{12}^2)}{\epsilon} \end{pmatrix}$, $M_{02} = \epsilon c(e^{-\mu} - G_r^2)$.

To prove V_{01} is negative, let us check $\left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_{01}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & M_{02} \end{array}\right) > 0$.

Let us consider the matrix $\widehat{M} = e^{-\mu}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} - \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}^\top \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$. The straightforward calculations prove that (see e.g. [10, Section 4]).

Claim 1. *If $\rho_1(G) < 1$, then for a suitable choice of positive values a, b and μ, it holds $\widehat{M} > 0$.*

By using Claim 1, it holds $M_{01} = \epsilon^{-1}\widehat{M} > 0$. Due to Proposition 1 in [9], $\rho_1(G_r) < 1$ implies $e^{-\mu} - G_r^2 > 0$. Then, it holds $M_{02} > 0$. Since $\left(\begin{array}{cc|c} \multicolumn{2}{c|}{M_{01}} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & M_{02} \end{array}\right) > 0$, V_{01} is negative. We get that (21) holds. This concludes the proof of Lemma 1. $\diamond$

Proof of Lemma 2. Let us consider $V_1(\eta, z, \bar{y})$ defined in (19) and differentiate it with respect to t along the solutions to (23), it follows

$$\dot{V}_1(\eta, z, \bar{y}) = \epsilon a \int_0^1 e^{-\mu x}(-2\eta_x\eta_{xx})\, dx + \epsilon c \int_0^1 e^{-\mu x}(-2\bar{y}_x\bar{y}_{xx})\, dx + \frac{b}{\epsilon}\int_0^1 e^{-\mu x}(-2z_x z_{xx})\, dx. \quad (41)$$

Performing an integration by parts, we obtain $\dot{V}_1(\eta, z, \bar{y}) = V_{11} + V_{12}$, where

$$V_{11} = -\epsilon a[e^{-\mu x}\eta_x^2]_0^1 - \frac{b}{\epsilon}[e^{-\mu x}z_x^2]_0^1 - \epsilon c[e^{-\mu x}\bar{y}_x^2]_0^1,$$

$$V_{12} = -\epsilon a\mu \int_0^1 e^{-\mu x}\eta_x^2 dx - \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x}z_x^2 dx - \epsilon c\mu \int_0^1 e^{-\mu x}\bar{y}_x^2 dx.$$

Using the boundary conditions (24), V_{11} is rewritten as

$$V_{11} = -\epsilon a\left[e^{-\mu}\eta_x^2(1) - \left(G_{11}\eta_x(1) + \frac{G_{12}}{\epsilon}z_x(1) - \frac{G_{12}G_{21}}{1-G_{22}}\bar{y}_x(1)\right)^2\right] - \frac{b}{\epsilon}\left[e^{-\mu}z_x^2(1) - \left(\epsilon G_{21}\eta_x(1) + G_{22}z_x(1) + \epsilon G_{21}\bar{y}_x(1)\right)^2\right] - \epsilon c(e^{-\mu}\bar{y}_x^2(1) - G_r^2\bar{y}_x^2(1)).$$

Developing the above terms and reorganizing yield

$$V_{11} = -\begin{pmatrix} \eta_x(1) \\ z_x(1) \\ \bar{y}_x(1) \end{pmatrix}^\top \left(\begin{array}{c|c} M_{11} & M_{13} \\ \hline * & M_{12} \end{array}\right) \begin{pmatrix} \eta_x(1) \\ z_x(1) \\ \bar{y}_x(1) \end{pmatrix},$$

with $M_{11} = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}} \end{pmatrix}^\top \widehat{M} \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & \frac{1}{\sqrt{\epsilon}} \end{pmatrix}$, $M_{12} = \epsilon\left(c(e^{-\mu} - G_r^2) - bG_{21}^2 - a\left(\frac{G_{12}G_{21}}{1-G_{22}}\right)^2\right)$, $M_{13} = \begin{pmatrix} \epsilon\left(\frac{aG_{11}G_{12}G_{21}}{1-G_{22}} - bG_{21}^2\right) \\ \frac{aG_{12}^2G_{21}}{1-G_{22}} - bG_{21}G_{22} \end{pmatrix}$, where $\widehat{M}$ is defined in the proof of Lemma 1. To prove V_{11} is negative, let us check $\left(\begin{array}{c|c} M_{11} & M_{13} \\ \hline * & M_{12} \end{array}\right) > 0$. Due to Claim 1, we prove that $M_{11} > 0$. Let us compute the inverse of M_{11},

$$M_{11}^{-1} = \frac{1}{\beta}\begin{pmatrix} \frac{be^{-\mu} - (bG_{22}^2 + aG_{12}^2)}{\epsilon} & aG_{11}G_{12} + bG_{21}G_{22} \\ * & \epsilon\left(ae^{-\mu} - (aG_{11}^2 + bG_{21}^2)\right) \end{pmatrix},$$

where

$$\beta = [ae^{-\mu} - (aG_{11}^2 + bG_{21}^2)][be^{-\mu} - (bG_{22}^2 + aG_{12}^2)] - [aG_{11}G_{12} + bG_{21}G_{22}]^2.$$

Computing $M_{13}^\top M_{11}^{-1}M_{13}$ yields

$$\begin{aligned} M_{13}^\top M_{11}^{-1}M_{13} &= \frac{\epsilon}{\beta}\left\{\left[\frac{aG_{11}G_{12}G_{21}}{1-G_{22}} - bG_{21}^2\right]^2\left[be^{-\mu} - (bG_{22}^2 + aG_{12}^2)\right]\right. \\ &+ 2\left[\frac{aG_{11}G_{12}G_{21}}{1-G_{22}} - bG_{21}^2\right]\left[\frac{aG_{12}^2G_{21}}{1-G_{22}} - bG_{21}G_{22}\right] \\ &\times\left[aG_{11}G_{12} + bG_{21}G_{22}\right] \\ &\left.+ \left[\frac{aG_{12}^2G_{21}}{1-G_{22}} - bG_{21}G_{22}\right]^2\left[ae^{-\mu} - (aG_{11}^2 + bG_{21}^2)\right]\right\} = \epsilon L. \end{aligned}$$

Let c' be given as follows $c' = \frac{bG_{21}^2 + a\left(\frac{G_{12}G_{21}}{1-G_{22}}\right)^2 + L}{e^{-\mu} - G_r^2}$. Then, for all $c > c'$, it holds $M_{12} - M_{13}^\top M_{11}^{-1}M_{13} > 0$. Since $M_{11} > 0$ and $M_{12} - M_{13}^\top M_{11}^{-1}M_{13} > 0$, according to the Schur complement, it holds $\left(\begin{array}{c|c} M_{11} & M_{13} \\ \hline * & M_{12} \end{array}\right) > 0$. Thus (25) holds. This concludes the proof of Lemma 2. $\diamond$

Proof of Lemma 3. Let us consider $V_2(\eta, z, \bar{y})$ defined in (20) and differentiate it with respect to t along the solutions to (26), it follows

$$\dot{V}_2(\eta, z, \bar{y}) = \epsilon^3 a \int_0^1 e^{-\mu x}(-2\eta_{xx}\eta_{xxx})\, dx + \epsilon c \int_0^1 e^{-\mu x}(-2\bar{y}_{xx}\bar{y}_{xxx})\, dx + \frac{b}{\epsilon}\int_0^1 e^{-\mu x}(-2z_{xx}z_{xxx})\, dx. \quad (42)$$

Performing an integration by parts, we obtain $\dot{V}_2(\eta, z, \bar{y}) = V_{21} + V_{22}$, where

$$V_{21} = -\epsilon^3 a[e^{-\mu x}\eta_{xx}^2]_0^1 - \frac{b}{\epsilon}[e^{-\mu x}z_{xx}^2]_0^1 - \epsilon c[e^{-\mu x}\bar{y}_{xx}^2]_0^1,$$

$$V_{22} = -\epsilon^3 a\mu \int_0^1 e^{-\mu x}\eta_{xx}^2 dx - \frac{b\mu}{\epsilon}\int_0^1 e^{-\mu x}z_{xx}^2 dx - \epsilon c\mu \int_0^1 e^{-\mu x}\bar{y}_{xx}^2 dx.$$

Using the boundary conditions (27), V_{21} is rewritten as

$$V_{21} = -\epsilon^3 a\left[e^{-\mu}\eta_{xx}^2(1) - \left(G_{11}\eta_{xx}(1) + \frac{G_{12}}{\epsilon^2}z_x(1) - \left(\frac{G_{12}G_{21}}{1-G_{22}}\right)\bar{y}_{xx}(1)\right)^2\right] - \frac{b}{\epsilon}\left[e^{-\mu}z_{xx}^2(1) - \left(\epsilon^2 G_{21}\eta_{xx}(1) + G_{22}z_{xx}(1) + \epsilon^2 G_{21}\bar{y}_{xx}(1)\right)^2\right] - \epsilon c(e^{-\mu}\bar{y}_{xx}^2 - G_r^2\bar{y}_{xx}^2(1)).$$

Developing the above terms and reorganizing, we obtain

$$V_{21} = -\begin{pmatrix}\eta_{xx}(1)\\ z_{xx}(1)\\ \bar{y}_{xx}(1)\end{pmatrix}^\top \left(\begin{array}{c|c} M_{21} & M_{23} \\ \hline * & M_{22}\end{array}\right) \begin{pmatrix}\eta_{xx}(1)\\ z_{xx}(1)\\ \bar{y}_{xx}(1)\end{pmatrix},$$

with

$$M_{21} = \begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}^\top M_{11} \begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\sqrt{\epsilon^3} & 0\\ 0 & \frac{1}{\sqrt{\epsilon}}\end{pmatrix}^\top \widehat{M} \begin{pmatrix}\sqrt{\epsilon^3} & 0\\ 0 & \frac{1}{\sqrt{\epsilon}}\end{pmatrix},$$

$$M_{22} = \epsilon\left(c(e^{-\mu} - G_r^2) - \epsilon^2 bG_{21}^2 - a\epsilon^2\left(\frac{G_{12}G_{21}}{1-G_{22}}\right)^2\right),$$

$$M_{23} = \begin{pmatrix}\epsilon^3\left(\dfrac{aG_{11}G_{12}G_{21}}{1-G_{22}} - bG_{21}^2\right)\\ \epsilon\left(\dfrac{aG_{12}^2G_{21}}{1-G_{22}} - bG_{21}G_{22}\right)\end{pmatrix} = \begin{pmatrix}\epsilon^2 & 0\\ 0 & \epsilon\end{pmatrix} M_{13}$$

where $\widehat{M}$ is defined in the proof of Lemma 1 and M_{11}, M_{13} are defined in the proof of Lemma 2. To prove V_{21} is negative, let us check $\left(\begin{array}{c|c} M_{21} & M_{23} \\ \hline * & M_{22}\end{array}\right) > 0$. Due to Claim 1, $M_{21} > 0$. Computing the inverse of M_{21} yields

$$M_{21}^{-1} = \begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}^{-1} M_{11}^{-1} \begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}^{-1} = \frac{1}{\epsilon^2}\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix} M_{11}^{-1} \begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix},$$

and $M_{23}^\top M_{21}^{-1} M_{23}$ is computed as follows

$$\begin{aligned} M_{23}^\top M_{21}^{-1} M_{23} &= \frac{1}{\epsilon^2} M_{13}^\top \begin{pmatrix}\epsilon^2 & 0\\ 0 & \epsilon\end{pmatrix}^T \begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix} \\ &\quad \times M_{11}^{-1} \begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\epsilon^2 & 0\\ 0 & \epsilon\end{pmatrix} M_{13} = \epsilon^3 L, \end{aligned}$$

where L is given in the proof of Lemma 2.

Let ϵ^* be given as follows

$$\epsilon^* = \sqrt{\frac{c'(e^{-\mu} - G_r^2)}{bG_{21}^2 + a\left(\frac{G_{12}G_{21}}{1-G_{22}}\right)^2 + L}}, \quad \text{if } bG_{21}^2 + a\left(\frac{G_{12}G_{21}}{1-G_{22}}\right)^2 + L > 0,$$

and

$$\epsilon^* = +\infty, \quad \text{if } bG_{21}^2 + a\left(\frac{G_{12}G_{21}}{1-G_{22}}\right)^2 + L \leqslant 0.$$

Then, for all $0 < \epsilon < \epsilon^*$, it holds $M_{22} - M_{23}^\top M_{21}^{-1} M_{23} > 0$. Since $M_{21} > 0$ and $M_{22} - M_{23}^\top M_{21}^{-1} M_{23} > 0$, according to the Schur complement, it holds $\left(\begin{array}{c|c} M_{21} & M_{23} \\ \hline * & M_{22}\end{array}\right) > 0$. Therefore, we prove that (28) holds. This concludes the proof of Lemma 3. ◇

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