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Approximate controllability of a reaction-diffusion system

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ABSTRACT

An open-loop control for a system coupling a reaction-diffusion system and an ordinary differential equation is proposed in this study. We use a flatness-like property, indeed, the solution can be expressed in terms of an infinite series depending on a flat output, its derivatives and its integrals. This series is shown to be convergent if the flat output is Gevrey of order $1 < a \leq 2$. Approximate controllability of the system is then proved.

1. Introduction

We study the approximate controllability of the following reaction-diffusion system

$$\begin{cases} \mathbf{y}_t = \mathbf{y}_{xx} + \mu \mathbf{y}_x + A\mathbf{y} + Bz, \\ z_t = \delta z_{xx} + C\mathbf{y} + dz, \\ \mathbf{y}_x(0,t) = \mathbf{0}, \quad G\mathbf{y}_x(L,t) + \mathbf{y}(L,t) = \mathbf{u}(t), \\ \mathbf{y}(x,0) = \mathbf{0}, \quad z(x,0) = 0, \end{cases} \tag{1}$$

where, $\mathbf{y} \in \mathbb{R}^p$ is the vector of diffusing species and $z \in \mathbb{R}$ the stored one. The parameters $A, G \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times 1}$, $C \in \mathbb{R}^{1\times p}$, $\mu, \delta, d \in \mathbb{R}$ are given. The time dependent vector function $\mathbf{u}$ is the control in $L^2(0,T)^p$.

This type of system appears in varied domains such as chemistry, electrophysiology, genetics, combustion...The degenerate case, $\delta = 0$ models, for example, the linearized FitzHugh Nagumo equations in electrophysiology [3,13] where y is the electrical potential and z the chemical concentration. Furthermore, the governing equations for chemical reactions in a tubular reactor model can exactly be written as (1) where y stands for the temperature ($p = 1$) and z for the solid fuel concentration. In solid combustion, the reactants do not diffuse and thus $\delta = 0$, [11]. Note also that gas electrodes may be modelled by (1) where $\mathbf{y}$ is the concentration of diffusive species with $p \leq 4$ and z is the concentration of the stored one, [5].

The approximate controllability of the non degenerate case, i.e. $\delta \neq 0$, has been studied in detail in [4,8] for the linear case and in [12] for a special nonlinear case, see also [2]. They have introduced some Gevrey functions in order to exhibit a "flat" output for their equations. The null controllability of the non degenerate case has also been studied with Carleman estimates in [6,7]. A previous study of controllability of the FitzHugh–Nagumo system has been done in [10] where an optimal control result is obtained.

In this paper we study the approximate controllability for the system (1), with $\delta = 0$. The following assumptions are needed for our control objectives

$$C \neq \mathbf{0}, \quad \sum_{k=1}^{p} B_k C_k \neq 0, \quad \text{and} \quad A(BC) = (BC)A. \tag{2}$$

Let us note that these assumptions are satisfied as soon as $p = 1$ and $BC \neq 0$. For the scalar case, the hypothesis $C \neq 0$ is also necessary for a control property since if $C = 0$ then the z-subsystem is completely decoupled from the control $\mathbf{u}$. If $A(BC) \neq (BC)A$, the results remain valid but the computations are much more difficult (in particular (6)).

Our main result (see Theorem 7) states that the system (1) with $\delta = 0$ is approximately controllable in $L^2(0,1)^{p+1}$. More precisely, for all $T > 0$, for all $\epsilon > 0$, for all final conditions $(\mathbf{y}_T, z_T)$ in $L^2(0,1)^{p+1}$ of (1) with $\delta = 0$, we may specify a control law $\mathbf{u}$ in $C^\infty([0,T], L^2(0,1)^p)$ such that the solution of (1) with $\delta = 0$ satisfies $\|\mathbf{y}(.,T) - \mathbf{y}_T\|_{L^2(0,1)} \leq \epsilon$ and $\|z(.,T) - z_T\|_{L^2(0,1)} \leq \epsilon$.

To do this, we introduce a "flat-like" property, where the trajectory of the system depends not only on a flat output and

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its derivatives but also on its integrals. This is due to the integro-differential character of the system under consideration.

The outline of the paper is as follows. In Section 2, we prove that the system (1) is flat-like. Then we prove in Section 3 the convergence of the series. The next section is devoted to the approximate controllability of our system and the last one to numerical tests. In Appendices A and B we give some technical proofs.

2. Formal solution

We consider the degenerate model introduced in Section 1, where $(\mathbf{y}, z) \in \mathbb{R}^p \times \mathbb{R}$, $A, G \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times 1}$, $C \in \mathbb{R}^{1\times p}$, $\mu, d \in \mathbb{R}$, $L = 1$ and $C \neq 0$ and satisfy (2).

$$\begin{cases} \mathbf{y}_t = \mathbf{y}_{xx} + \mu\mathbf{y}_x + A\mathbf{y} + Bz, & z_t = C\mathbf{y} + dz, \\ \mathbf{y}_x(0,t) = \mathbf{0}, & G\mathbf{y}_x(1,t) + \mathbf{y}(1,t) = \mathbf{u}(t), \\ \mathbf{y}(x,0) = \mathbf{0}, & z(x,0) = 0. \end{cases} \tag{3}$$

Let us perform the following change of variables, $\bar{z} = \mathrm{e}^{-dt+\mu x/2} z$, $\bar{\mathbf{y}} = \mathrm{e}^{-dt+\mu x/2}\mathbf{y}$ and $\bar{\mathbf{u}} = \mathrm{e}^{-dt+\mu/2}\mathbf{u}$. System (3) becomes, with $\bar{A} = (d + \frac{\mu^2}{4})I_p - A$, $\bar{B} = -B$,

$$\begin{cases} \bar{\mathbf{y}}_t = \bar{\mathbf{y}}_{xx} - \bar{A}\bar{\mathbf{y}} - \bar{B}\bar{z}, & \bar{z}_t = C\bar{\mathbf{y}}, \\ \bar{\mathbf{y}}_x(0,t) = \mu\bar{\mathbf{y}}(0,t)/2, \\ G\bar{\mathbf{y}}_x(1,t) + (I_p - G\mu/2)\bar{\mathbf{y}}(1,t) = \bar{\mathbf{u}}(t), \\ \bar{\mathbf{y}}(x,0) = \mathbf{0}, & \bar{z}(x,0) = 0. \end{cases} \tag{4}$$

We seek a formal solution, $\bar{\mathbf{y}}(x,t) = \sum_{k=0}^{+\infty} \bar{\mathbf{y}}_k(t)\frac{x^k}{k!}$. The convergence of this series will be described below. We assume for the time being that this series converges and that the $\bar{\mathbf{y}}_k$ functions are sufficiently smooth. We get with (4),

$$\forall k \in \mathbb{N}, \quad \bar{\mathbf{y}}_{k+2}(t) = \dot{\bar{\mathbf{y}}}_k(t) + \bar{A}\bar{\mathbf{y}}_k(t) + \bar{B}C\int_0^t \bar{\mathbf{y}}_k(\tau)\mathrm{d}\tau.$$

Let us define the function $\mathbf{S} : [0,\infty) \to \mathbb{R}^p$ by, for all $t \geq 0$,

$$\mathbf{S}(t) = \bar{\mathbf{y}}(0,t). \tag{5}$$

Assume, for the time being, that $\mathbf{S}$ is a smooth function, and define the following notation, for all $t \geq 0$,

$$\mathbf{S}^{(0)}(t) = \mathbf{S}(t), \qquad \mathbf{S}^{(n)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{S}^{(n-1)})(t),$$

$$\mathbf{S}^{(-n)}(t) = \int_0^t \mathbf{S}^{(-n+1)}(\tau)\mathrm{d}\tau, \quad \text{for } n \geq 1.$$

Let us note that for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, we have $\bar{\mathbf{y}}_{x^k t^n}(0,0) = \mathbf{0}$. Consequently the function $\mathbf{S}$ satisfies for all $n \in \mathbb{N}$, $\mathbf{S}^{(n)}(0) = \mathbf{0}$, and thus $(\mathbf{S}^{(n)})^{(m)} = \mathbf{S}^{(n+m)}$ for all $(n,m) \in \mathbb{Z}$. With hypothesis (2) we obtain by recurrence the formal solution of (4),

$$\forall t \geq 0, \forall n \in \mathbb{N}, \quad \begin{cases} \bar{\mathbf{y}}_{2n}(t) = \displaystyle\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}^{(n-k-j)}(t), \\ \bar{\mathbf{y}}_{2n+1}(t) = \mu\bar{\mathbf{y}}_{2n}(t)/2, \end{cases}$$

and thus, for all $(x,t) \in [0,1] \times [0,+\infty)$

$$\begin{cases} \bar{\mathbf{y}}(x,t) = \displaystyle\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}^{(n-k-j)}(t)\right] \\ \qquad \times \left(\dfrac{x^{2n}}{(2n)!} + \dfrac{\mu}{2}\dfrac{x^{2n+1}}{(2n+1)!}\right), \\ \bar{z}(x,t) = C\displaystyle\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}^{(n-k-j-1)}(t)\right] \\ \qquad \times \left(\dfrac{x^{2n}}{(2n)!} + \dfrac{\mu}{2}\dfrac{x^{2n+1}}{(2n+1)!}\right). \end{cases} \tag{6}$$

The formal control of (4) is then defined for all $t \geq 0$ by,

$$\begin{aligned} \bar{\mathbf{u}}(t) &= (I_p - G\mu/2)\bar{\mathbf{y}}(1,t) + G\bar{\mathbf{y}}_x(1,t), \\ &= (I_p - G\mu/2)\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}^{(n-k-j)}(t)\right] \\ &\quad \times \left(\frac{4n+2+\mu}{2(2n+1)!}\right) \\ &\quad + G\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}^{(n-k-j)}(t)\right]\left(\frac{4n+\mu}{2(2n)!}\right). \end{aligned} \tag{7}$$

Remark 1. Using the terminology of [8], we may understand this parametrization of the smooth solution of (4) as a flat result. Indeed, the solutions are written in terms of an infinite number of time-derivatives of $\mathbf{S}$ defined by (5). Note however that, in contrast to [8] where only time-derivatives are used, we need to integrate an infinite number of times to compute exactly the solutions of (4).

The convergence of the series in (6) and (7) is precisely stated in Section 3.

3. Convergence of the series (6) and (7)

In this section we make more precise the computations which have been informally done in the previous section. To do that, denoting

$$\alpha = \|\bar{A}\|_\infty, \qquad \beta = \|\bar{B}C\|_\infty, \tag{8}$$

$\|y\|_\infty = \sup_{i\in\{1..p\}} |y_i|$, for all $\mathbf{y} = (y_1,\ldots,y_p) \in \mathbb{R}^p$, and given $T > 0$, we need to recall some definitions on Gevrey functions (see e.g. [9, page 132] or [1, Definition 1.1]).

Definition 2. A smooth vector function $\mathbf{y} : t \in [0,T] \mapsto \mathbf{y}(t) \in \mathbb{R}^p$ is Gevrey of order a if

$$\exists M, R > 0, \forall n \in \mathbb{N}, \quad \sup_{t\in[0,T]}\left\|\frac{\partial^n \mathbf{y}}{\partial t^n}(t)\right\|_\infty \leq M\frac{(n!)^a}{R^n}.$$

A smooth vector function $\mathbf{y} : (x,t) \in [0,1]\times[0,T] \mapsto \mathbf{y}(x,t) \in \mathbb{R}^p$ is Gevrey of order a in t and b in x if

$$\exists M, R, S > 0, \forall k, n \in \mathbb{N}, \quad \sup_{(x,t)\in[0,1]\times[0,T]}\left\|\frac{\partial^{k+n}\mathbf{y}}{\partial x^k \partial t^n}(x,t)\right\|_\infty \leq M\frac{(n!)^a (k!)^b}{R^n S^k}.$$

Remark that the scaling, integration, addition, multiplication and composition of Gevrey functions of order $a > 1$ is of order a.

Let us define the bump function used in the sequel,

$$\phi_\sigma(t) = \begin{cases} 0 & \text{if } t \notin (0,1), \\ \exp\left(\dfrac{-1}{((1-t)t)^\sigma}\right) & \text{if } t \in (0,1). \end{cases} \tag{9}$$

This function is Gevrey of order $1 + 1/\sigma$ whatever $\sigma > 0$, [14]. Similarly,

$$\Phi_\sigma(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t \geq 1, \\ \dfrac{\int_0^t \phi_\sigma(\tau)\mathrm{d}\tau}{\int_0^1 \phi_\sigma(\tau)\mathrm{d}\tau} & \text{if } t \in (0,1), \end{cases} \tag{10}$$

has order $1 + 1/\sigma$. The bump function is strictly increasing from 0 to 1 for $t \in [0,1]$ and all its derivatives vanish at $t = 0$ and $t = 1$. Let us prove the following proposition stating the convergence of the series (6) and (7) and some properties of the solutions and the control of (4).

Proposition 3. *When* $\mathbf{S}$ *is Gevrey of order* $1 \leq a < 2$, *(or* $a = 2$ *with conditions on matrices* $\bar{A}$ *and* $\bar{B}C$*), the formal solutions defined by (6) are Gevrey of order a in t and 1 in x and the formal control (7) is Gevrey of order a.*

Proof. We assume that $\mathbf{S}$ is Gevrey of order $a \geq 1$. Thus, with Definition 2, there exist $M_1, R > 0$ such that for all $n \in \mathbb{N}$, $\sup_{t\in[0,T]} \left\|\mathbf{S}^{(n)}(t)\right\|_\infty \leq M_1 \frac{(n!)^a}{R^n}$. We easily get, for all $n \in \mathbb{N}$, $\sup_{t\in[0,T]} \left\|\mathbf{S}^{(-n)}(t)\right\|_\infty \leq M_1 \frac{T^n}{n!}$. For T and R given, there exists $M_2 \geq 1$ such that, for all $n \in \mathbb{N}$, $\frac{T^n}{n!} \leq M_2 R^n$. With $M = M_1 M_2$, we get, for all $n \in \mathbb{N}$,

$$\sup_{t\in[0,T]} \left\|\mathbf{S}^{(-n)}(t)\right\|_\infty \leq M \frac{1}{R^{-n}} \quad \text{and} \quad \sup_{t\in[0,T]} \left\|\mathbf{S}^{(n)}(t)\right\|_\infty \leq M \frac{(n!)^a}{R^n}.$$

We formally differentiate $\bar{\mathbf{y}}$ given by (6),

$$\frac{\partial^{k+n}\bar{\mathbf{y}}}{\partial x^k \partial t^n} = \sum_{2i\geq k}^{\infty} \left[\sum_{m=0}^{i}\sum_{j=0}^{m} C_i^m C_m^j \bar{A}^{m-j}(\bar{B}C)^j \mathbf{S}^{(i+n-m-j)}(t)\right] \times \left(\frac{x^{2i-k}}{(2i-k)!} + \frac{\mu}{2}\frac{x^{2i+1-k}}{(2i+1-k)!}\right),$$

and we recall the estimation, $\sup_{t\in[0,T]} \left\|\mathbf{S}^{(i+n-m-j)}(t)\right\|_\infty \leq M((i+n)!)^a \frac{1}{R^{i+n-m-j}}$. With (8), we get

$$\sup_{t\in[0,T]} \left\|\sum_{j=0}^{m} C_m^j \bar{A}^{m-j}(\bar{B}C)^j \mathbf{S}^{(i+n-m-j)}(t)\right\|_\infty \leq M \frac{((i+n)!)^a}{R^{i+n-m}}(\alpha + \beta R)^m.$$

Thus we have

$$\sup_{t\in[0,T]} \left\|\sum_{m=0}^{i}\sum_{j=0}^{m} C_i^m C_m^j \bar{A}^{m-j}(\bar{B}C)^j \mathbf{S}^{(i+n-m-j)}(t)\right\|_\infty \leq M \frac{((i+n)!)^a}{R^{i+n}}(1 + \alpha R + \beta R^2)^i.$$

So we get, with the usual inequality $(l+m)! \leq 2^{l+m} l! m!$ and the Stirling's formula, $m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}$ as $m \to \infty$,

$$\sup_{(t,x)\in[0,T]\times[0,1]} \left\|\left[\frac{1}{(n!)^a k!}\sum_{m=0}^{i}\sum_{j=0}^{m} C_i^m C_m^j \bar{A}^{m-j}(\bar{B}C)^j \mathbf{S}^{(i+n-m-j)}(t)\right] \times \left(\frac{x^{2i-k}}{(2i-k)!} + \frac{\mu}{2}\frac{x^{2i+1-k}}{(2i+1-k)!}\right)\right\|_\infty \leq M\left(1+\frac{\mu}{2}\right)\frac{2^{na}}{R^n}\left(\frac{2^{ia}}{R^i}(i!)^{a-2}\sqrt{i\pi}(1+\alpha R+\beta R^2)^i\right).$$

Let $\tilde{M} = M(1+\frac{\mu}{2})\sum_{2i\geq k}\left(\frac{2^{ia}}{R^i}(i!)^{a-2}\sqrt{i\pi}(1+\alpha R+\beta R^2)^i\right)$, and let $\tilde{R} = \frac{R}{2^a}$. Note that $\tilde{M}$ is finite when $a < 2$. If $a = 2$, then $\tilde{M}$ is finite when $4(1+\alpha R+\beta R^2) < R$. We get the result

$$\sup_{(t,x)\in[0,T]\times[0,1]} \left\|\frac{\partial^{k+n}\bar{\mathbf{y}}}{\partial x^k \partial t^n}(t,x)\right\|_\infty \leq \tilde{M}\frac{(n!)^a k!}{\tilde{R}^n 1^k}.$$

Therefore the function $\bar{\mathbf{y}}$ is Gevrey of order 1 in x and a in t, and the control $\bar{\mathbf{u}}$ is Gevrey of order a. We can easily prove the same result for $\bar{z}$. This concludes the proof of Proposition 3. ■

4. Approximate controllability

In this section we state our main result: The system (3) is approximately controllable. To do that, we first approximate the final conditions with well-chosen polynomials and then we state some estimations on the function $\mathbf{S}$ defined by (5) which are useful to apply Proposition 3.

We assume that the initial and final conditions of (4) satisfy $(\bar{\mathbf{y}}_0, \bar{z}_0) = (\mathbf{0}, 0), (\bar{\mathbf{y}}_T, \bar{z}_T) \in L^2(0,1)^{p+1}$. We approach the functions $\bar{\mathbf{y}}_T$ and $\bar{z}_T$ with special polynomials. The proof of the following lemma is postponed until Appendix A.2.

Lemma 4. *For all* $\epsilon > 0$, *there exist an integer* $K > 0$, *and finite sequences* $(\boldsymbol{\gamma}_k^T)_{0\leq k\leq K} \in (\mathbb{R}^p)^{K+1}$, *and* $(\delta_k^T)_{0\leq k\leq K} \in \mathbb{R}^{K+1}$ *such that, defining*

$$\begin{cases} \boldsymbol{\gamma}_T(x) = \sum_{k=0}^{K} \boldsymbol{\gamma}_k^T \left(\frac{x^{2k}}{(2k)!} + \frac{\mu}{2}\frac{x^{2k+1}}{(2k+1)!}\right), \\ \delta_T(x) = \sum_{k=0}^{K} \delta_k^T \left(\frac{x^{2k}}{(2k)!} + \frac{\mu}{2}\frac{x^{2k+1}}{(2k+1)!}\right), \end{cases} \tag{11}$$

we have

$$\|\bar{\mathbf{y}}_T - \boldsymbol{\gamma}_T\|_{L^2(0,1)^p} \leq \epsilon, \quad \textit{and} \quad \|\bar{z}_T - \delta_T\|_{L^2(0,1)} \leq \epsilon. \tag{12}$$

We define for all $k > K$, $\boldsymbol{\gamma}_k^T = \mathbf{0}$, $\delta_k^T = 0$ and for all $k \in \mathbb{N}$, $\boldsymbol{\gamma}_k^0 = \mathbf{0}$, $\delta_k^0 = 0$. Let $N > K$. Due to (5) and (6) we want to find a function $\mathbf{S}_N : [0,T] \to \mathbb{R}^p$ Gevrey of order a such that

$$\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}_N^{(n-k-j)}(0) = \boldsymbol{\gamma}_n^0 = \mathbf{0}, \quad \forall n = 0,\ldots,N, \tag{13}$$

$$C\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}_N^{(n-k-j-1)}(0) = \delta_n^0 = 0, \quad \forall n = 0,\ldots,N-1, \tag{14}$$

$$\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}_N^{(n-k-j)}(T) = \boldsymbol{\gamma}_n^T, \quad \forall n = 0,\ldots,N, \tag{15}$$

$$C\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{S}_N^{(n-k-j-1)}(T) = \delta_n^T, \quad \forall n = 0,\ldots,N-1. \tag{16}$$

To do that let us state the following result which is proved in Appendix A.1.

Lemma 5. *There exists a sequence* $(\mathbf{s}_n)_{n\in\mathbb{Z}} \in (\mathbb{R}^p)^{\mathbb{Z}}$ *solution of the system*

$$\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{s}_{n-k-j} = \boldsymbol{\gamma}_n^T, \quad \forall n \in \mathbb{N}, \tag{17}$$

$$C\sum_{k=0}^{n}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{s}_{n-k-j-1} = \delta_n^T, \quad \forall n \in \mathbb{N}. \tag{18}$$

Let us define $\mathbf{S}_N$ as follows, where the sequence $\mathbf{s}_i$ satisfies (17)–(18) and Φ_σ is defined in (10), for all $t \in [0,T]$,

$$\mathbf{S}_N(t) = \frac{d^N}{dt^N}\left(\sum_{i=0}^{2N} \mathbf{s}_{i-N}\frac{(t-T)^i}{i!}\Phi_\sigma\left(\frac{t}{T}\right)\right). \tag{19}$$

Such a function $\mathbf{S}_N$ satisfies (13)–(16) and is Gevrey of order $1 \leq a < 2$ when $\sigma > 1$. The following proposition is the key result to state our main result.

Proposition 6. *There exist $M_K = M(K) > 0$ with K defined in Lemma 4 and $r > 0$ such that,*

$$\forall 0 \le n \le N, \quad \begin{cases} \mathbf{S}_N^{(n)}(0) = \mathbf{S}_N^{(-n)}(0) = \mathbf{0}, \\ \left\| \mathbf{S}_N^{(n)}(T) \right\|_\infty \le M_K r^n n!, \\ \left\| (\bar{B}C)^n \mathbf{S}_N^{(-n)}(T) \right\|_\infty \le M_K r^{n-1}(n-1)!, \end{cases}$$

and

$$\forall n \ge N+1, \quad \begin{cases} \mathbf{S}_N^{(n)}(0) = \mathbf{S}_N^{(-n)}(0) = \mathbf{S}_N^{(n)}(T) = \mathbf{0}, \\ \left\| \mathbf{S}_N^{(-n)}(T) \right\|_\infty \le M_K r^N N!. \end{cases}$$

The proof of this key result is in Appendix B.1. We are now in a position to state our main result proved in Appendix B.2.

Theorem 7. *The system (3) is approximately controllable in $L^2(0,1)^{p+1}$. More precisely, for all $\epsilon > 0$, for all $T > 0$, and for all $(\mathbf{y}_T, z_T) \in L^2(0,1)^p \times L^2(0,1)$, there exists $N_0 \in \mathbb{N}$ such that, for all $N \ge N_0$, considering $\mathbf{S}_N$ given by (19), the control $\mathbf{u}_N : [0,T] \to \mathbb{R}^p$ defined by, for all $t \in [0,T]$,*

$$\begin{aligned} \mathbf{u}_N(t) = \mathrm{e}^{\mathrm{d}t-\mu/2} \Bigg[& G \sum_{n=1}^{N} \left[\sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j} (-BC)^j \mathbf{S}_N^{(n-k-j)}(t) \right] \\ & \times \left(\frac{4n+\mu}{2(2n)!} \right) + (I_p - G\mu/2) \\ & \times \sum_{n=0}^{N} \left[\sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j} (-BC)^j \mathbf{S}_N^{(n-k-j)}(t) \right] \\ & \times \left(\frac{4n+2+\mu}{2(2n+1)!} \right) \Bigg] \end{aligned}$$

where $\bar{A} = (d + \mu^2/4) I_p - A$, approximately steers system (3) from the initial state $(\mathbf{0}, 0)$ to the final state $(\mathbf{y}_T, z_T)$ in time T, i.e. we have

$$\|\mathbf{y}_T - \mathbf{y}(.,T)\|_{L^2(0,1)^p} \le C(T)\epsilon, \qquad \|z_T - z(.,T)\|_{L^2(0,1)} \le C(T)\epsilon.$$

where $C(T)$ is a constant depending only on T.

Remark 8. Let us note that, given $\epsilon > 0$, the proof of Theorem 7 exhibits the computation of the integers N and K. Indeed, given $\epsilon > 0$, we choose K such that (12) holds, and using (B.4), (B.22), (B.23) and (B.26), the integer N is such that the conclusion of Theorem 7 holds. The assumption $A(BC) \neq (BC)A$ is required in our proof. How to remove this assumption seems to be an open question.

5. Rest-to-rest motion and numerical simulations

In this section we focus on the following particular system (obtained by letting $A = -1$, $B = 1$, $C = 1$, $d = -1$, $G = 1$, $\mu = 0$, $\delta = 0$ and $L = 1$ in (1)):

$$\begin{cases} y_t = y_{xx} - y + z, \\ z_t = y - z, \\ y_x(0,t) = 0, \quad y_x(1,t) + y(1,t) = u(t), \\ y(x,0) = 0, \quad z(x,0) = 0. \end{cases} \tag{20}$$

5.1. Rest-to rest motion

We show in this section how to steer system (20) from zero to a rest profile. A rest profile is such that, for a given $c \in \mathbb{R}$, $y(x,t) = z(x,t) = c$ for all $(x,t) \in [0,1] \times (0,+\infty)$. The control problem of rest profiles is very important for applications, see e.g. [5] for the use of the rest profile of (20) for a gas electrode in a catalytic converter. The aim of this section is to state the following

Proposition 9. *For all $T > 0$, for all $\sigma \le 2$, for all $c \in \mathbb{R}$ and for all $\epsilon > 0$, there exists N_0, such that for all $N \ge N_0$, the control*

$$\begin{aligned} u(t) = \mathrm{e}^{-t} \Bigg[& \sum_{n=0}^{N} \left(\sum_{k=0}^{n} C_n^k (-1)^k S_N^{(n-2k)}(t) \right) \frac{1}{(2n)!} \\ & + \sum_{n=1}^{N} \left(\sum_{k=0}^{n} C_n^k (-1)^k S_N^{(n-2k)}(t) \right) \frac{1}{(2n-1)!} \Bigg], \end{aligned} \tag{21}$$

where

$$S_N(t) = c \frac{\mathrm{d}^N}{\mathrm{d}t^N} \left(\mathrm{e}^t \Phi_\sigma \left(\frac{t}{T} \right) \right), \quad \sigma \ge 1, \tag{22}$$

approximately steers $(0,0)$ to (c,c) along the solutions of (20) in time T, i.e. the solution of (20) with the control law (21) satisfies

$$\|y(.,T) - c\|_{L^2(0,1)} \le C(T)\epsilon, \qquad \|z(.,T) - c\|_{L^2(0,1)} \le C(T)\epsilon,$$

where $C(T)$ is a constant depending only on T.

Note that this result can be seen as a particular case of Theorem 7. However for the system (20) and the particular control problem of the rest-to-rest motion, the proof is easier and can be stated independently.

Proof. We follow the change of variables of Section 2, i.e. we consider

$$\begin{cases} \bar{y}_t = \bar{y}_{xx} + \bar{z}, \quad \bar{z}_t = \bar{y}, \\ \bar{y}_x(0,t) = 0, \quad \bar{y}_x(1,t) + \bar{y}(1,t) = \bar{u}(t), \\ \bar{y}(x,0) = 0, \quad \bar{z}(x,0) = 0. \end{cases} \tag{23}$$

Thus with the change of variables, we could approximately go from zero to $\bar{y}(x,T) = \bar{z}(x,T) = \mathrm{e}^T c$, for a given c and for all $x \in [0,1]$, along the solutions of (23). For all $\sigma \ge 1$, let us consider the function S_N given by (22). The function S_N is Gevrey of order $1 + 1/\sigma$ and we compute $S_N^{(i)}(0) = 0$, for all $i \in \mathbb{Z}$, and $S_N^{(i)}(T) = c\mathrm{e}^T$, for all $i \ge -N$. Furthermore we have, for all $i \in \mathbb{N}$, $i \le N$, $S_N^{(-i)}(t) = \frac{\mathrm{d}^{N-i}}{\mathrm{d}t^{N-i}}(c\mathrm{e}^t \Phi_\sigma(\frac{t}{T}))$. In particular we have $0 \le S_N^{(-N)}(t) = c\mathrm{e}^t \Phi_\sigma(\frac{t}{T}) \le c\mathrm{e}^t$. Moreover we may deduce by recurrence that, for all $t \in [0,T]$ and for all $i \in \mathbb{N}$, $0 \le S_N^{(-N-i)}(t) \le c\mathrm{e}^t$. We deduce the following estimates,

$$\begin{aligned} \sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j S_N^{(n-k-j)}(0) &= \sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j S_N^{(n-k-j-1)}(0) \\ &= 0 \quad \text{for } n \in \mathbb{N}, \end{aligned}$$

$$\sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j S_N^{(n-k-j)}(T) = 0 \quad \text{for } 0 < n \le N,$$

$$\begin{aligned} \sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j S_N^{(n-k-j-1)}(T) &\le \sum_{k=0}^{n} \sum_{j=0}^{k} C_n^k C_k^j c\mathrm{e}^T \\ &= c\mathrm{e}^T 3^n \quad \text{for } n > N. \end{aligned}$$

Thus, for all $\epsilon > 0$, there exists N_0 such that, for all $N \ge N_0$,

$$\begin{aligned} & \left\| c\mathrm{e}^T - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left[\sum_{j=0}^{k} C_n^k C_k^j S_N^{(n-k-j)}(T) \right] \frac{x^{2n}}{(2n)!} \right\|_{L^2(0,1)} \\ & = \left\| \sum_{n=N+1}^{\infty} \left[c\mathrm{e}^T 3^n \right] \frac{x^{2n}}{(2n)!} \right\|_{L^2} \\ & \le c\mathrm{e}^T \sum_{n=N+1}^{\infty} \frac{3^n}{(2n)!} \le \epsilon, \end{aligned}$$

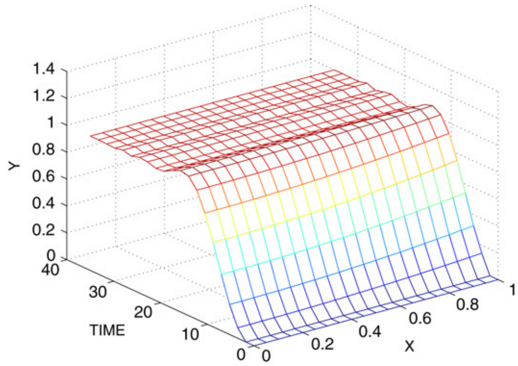

Fig. 1. Time-evolution of the y-component of the state.

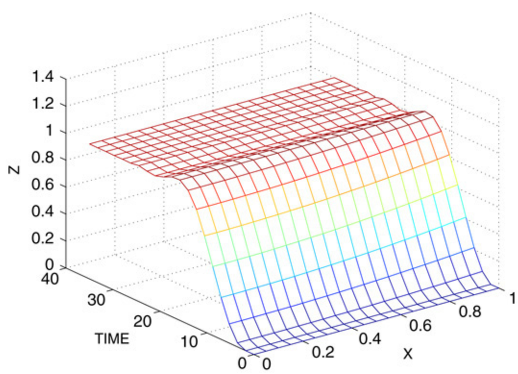

Fig. 2. Time-evolution of the z-component of the state.

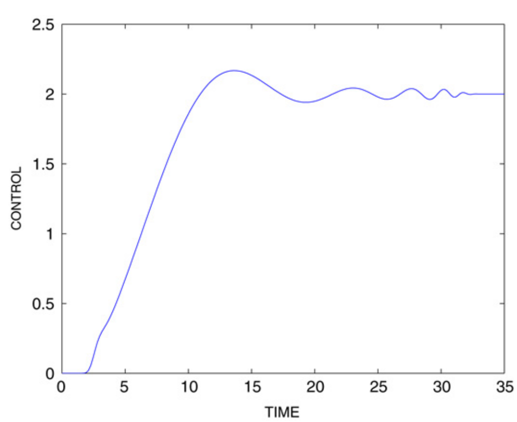

Fig. 3. Time-evolution of the control law u.

and in the same way $\|ce^T - \sum_{n=0}^{\infty}\sum_{k=0}^{n}[\sum_{j=0}^{k} C_n^k C_k^j S_N^{(n-k-j-1)}(T)] \frac{x^{2n}}{(2n)!}\|_{L^2(0,1)} \le \epsilon$.

We conclude the proof in an analogous way as the proof of Theorem 7 and we get the approximate controllability as stated by Proposition 9. ■

5.2. Numerical simulations

In this section we illustrate the result of Section 5.1 by performing some numerical simulations. Let us choose $\sigma = 1.8$, and let us consider S_N given by (22). The function S is Gevrey of order $\sigma < 2$, and due to Proposition 3, the formal solutions defined by (6) are Gevrey of order a in t and 1 in x and the formal control (7) is Gevrey of order a. We illustrate Proposition 9 by choosing the final time $T = 35$ and $c = 1$ as the rest-profile we want to reach. We consider the control law (21) computed with about $N = 10$ terms. Figs. 1 and 2 display the time-evolution of the solution of (20) when the control law depicted in Fig. 3 is applied. We check that we reach the rest profile $(y, z) = (c, c)$ within time T. The simulation code can be downloaded from http://www.laas.fr/~cprieur/Papers/code-diffusion.zip. The value of T may be theoretically freely chosen. However to avoid numerical problems when dealing with too large control, T should be sufficiently large in the simulations. Also, because of Theorem 7, N should be sufficiently large. But to avoid Runge's phenomenon when using polynomials of high degree, N should be not too large for numerical simulations.

Acknowledgement

We thank Michel Sorine for having drawn our attention to this controllability problem and Pierre Rouchon for useful discussions.

Appendix A. Proof of two technical results

A.1. Proof of Lemma 4

We only have to prove that the set $\{P(x) = \sum_k a_k(\frac{x^{2k}}{(2k)!} + \frac{\mu}{2}\frac{x^{2k+1}}{(2k+1)!}), a_k \in \mathbb{R}\}$ is dense in $C^1(0,1)$. Let $f \in C^1(0,1)$. Let us introduce the function $g : [0,1] \to \mathbb{R}$ by letting, for all $x \in [0,1]$, $g(x) = \int_0^x e^{(\xi - x)\mu/2} f'(\xi)d\xi + e^{-\mu x/2} f(0)$. We compute, for all $x \in [0,1]$, $g'(x) = f'(x) - \frac{\mu}{2} g(x)$, and thus $f(x) = g(x) + \frac{\mu}{2}\int_0^x g(\xi)d\xi$. Let $\epsilon > 0$. Thanks to the Stone-Weierstrass theorem [15, Theorem 7.32], there exists $K \in \mathbb{N}$ and a polynomial function $P(x) = \sum_{k=1}^{K} a_k \frac{x^{2k}}{(2k)!}$ such that $\|g(x) - P(x)\|_{L^2(0,1)} \le \epsilon$ which implies $\left\| f(x) - \sum a_k(\frac{x^{2k}}{(2k)!} + \frac{\mu}{2}\frac{x^{2k+1}}{(2k+1)!})\right\|_{L^2(0,1)} \le \epsilon(1 + \mu/2)$. ■

A.2. Proof of Lemma 5

First note that $\mathbf{s}_0 = \boldsymbol{\gamma}_0^T$, $C\mathbf{s}_{-1} = \delta_0^T$. We choose $\mathbf{s}_{-1,i} = 0$ if $i \ne i_C$, and $\mathbf{s}_{-1,i_C} = \frac{\delta_0^T}{C_{i_C}}$, where $i_C \in \{1, \ldots, p\}$ is such that $C_{i_C} \ne 0$. Such integer exists due to (2). We prove the existence of $\{\mathbf{s}_n, \ldots, \mathbf{s}_{-n}\}$ by recurrence on n. Let $n \in \mathbb{N}$, suppose that $\{\mathbf{s}_n, \mathbf{s}_{n-1}, \ldots, \mathbf{s}_{-n}\}$ are known. Then $\mathbf{s}_{-n-1}$ and $\mathbf{s}_{n+1}$ are defined by

$$C(\bar{B}C)^n \mathbf{s}_{-n-1} = \delta_n^T - C\sum_{k=0}^{n-1}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{s}_{n-k-j-1} - C\sum_{j=0}^{n-1} C_n^j \bar{A}^{n-j}(\bar{B}C)^j \mathbf{s}_{-j-1},$$

$$\mathbf{s}_{n+1} = \boldsymbol{\gamma}_{n+1}^T - \sum_{k=1}^{n+1}\sum_{j=0}^{k} C_{n+1}^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{s}_{n+1-k-j}.$$

We easily prove that for all $j \in \{1, \ldots, p\}$ and for all $n \in \mathbb{N}[C(\bar{B}C)^n]_j = (\sum_{i=1}^{p} \bar{B}_i C_i)^n C_j$. We choose, thanks to hypothesis (2) the relations given in Box I.

Thus, by recurrence, there exists a sequence $(\mathbf{s}_n) \in \mathbb{R}^p$ solution of (17) and (18). ■

Appendix B. Proofs of Proposition 6 and of Theorem 7

In this appendix, we prove Proposition 6 which is the key step of our approach and we prove our main result Theorem 7. To do that, we split Proposition 6 into two lemmas (see Lemmas 10 and 12), which are proved separately.

$$\begin{cases} \mathbf{s}_{-n-1,i} = 0 \quad \text{if } i \neq i_C, \\ \mathbf{s}_{-n-1,i_C} = \dfrac{\left(\delta_n^T - C\sum_{k=0}^{n-1}\sum_{j=0}^{k} C_n^k C_k^j \bar{A}^{k-j}(\bar{B}C)^j \mathbf{s}_{n-k-j-1} - C\sum_{j=0}^{n-1} C_n^j \bar{A}^{n-j}(\bar{B}C)^j \mathbf{s}_{-j-1}\right)}{C_{i_C}\left(\sum_{i=1}^{p}\bar{B}_i C_i\right)^n} \end{cases}$$

Box I.

B.1. Proof of Proposition 6

Lemma 10. *There exist $M_K = M(K) > 0$ and $r > 0$ such that, for all $0 \le n \le N$*

$$\mathbf{S}_N^{(n)}(0) = \mathbf{S}_N^{(-n)}(0) = \mathbf{0}, \tag{B.1}$$

$$\left\|\mathbf{S}_N^{(n)}(T)\right\|_\infty \le M_K r^n n!, \tag{B.2}$$

$$\left\|(\bar{B}C)^n \mathbf{S}_N^{(-n)}(T)\right\|_\infty \le M_K r^{n-1}(n-1)!. \tag{B.3}$$

Proof. Eq. (B.1) follows from (19). Let

$$M_K = \max\left(\max_{0\le n\le K}\|\mathbf{s}_n\|_\infty, \max_{0\le n\le K}\|(\bar{B}C)^n\mathbf{s}_{-n}\|_\infty\right). \tag{B.4}$$

Remark that $\mathbf{S}_N^{(k)}(T) = \mathbf{s}_k$, for all $k = -N-1,\dots,N$. Now we prove (B.2) and (B.3) by recurrence on n for $n > K$. First of all, using Lemma 5, we rewrite (15) and (16) as, for all $0 \le n \le N$,

$$\begin{aligned} \mathbf{s}_{2n} = \boldsymbol{\gamma}_{2n}^T - \Bigg[&\left(\sum_{j=0}^{n} C_{2n}^{j+n} C_{j+n}^{n-j}\bar{A}^{2j}(\bar{B}C)^{n-j}\right)\mathbf{s}_0 + (\bar{B}C)^{2n}\mathbf{s}_{-2n} \\ &+ \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right) \\ &\times(\mathbf{s}_{2n-2k-1} + (\bar{B}C)^{2n-2k-1}\mathbf{s}_{-2n+2k+1}) \\ &+ \sum_{k=1}^{n}\left(\sum_{j=0}^{k} C_{2n}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right) \\ &\times\ (\mathbf{s}_{2n-2k} + (\bar{B}C)^{2n-2k}\mathbf{s}_{-(2n-2k)})\Bigg], \end{aligned} \tag{B.5}$$

$$\begin{aligned} \mathbf{s}_{2n+1} = \boldsymbol{\gamma}_{2n+1}^T - \Bigg[&\left(\sum_{j=0}^{n} C_{2n+1}^{j+n+1} C_{j+n+1}^{n-j}\bar{A}^{2j+1}(\bar{B}C)^{n-j}\right)\mathbf{s}_0 \\ &+ (\bar{B}C)^{2n+1}\mathbf{s}_{-2n-1} + \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n+1}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right) \\ &\times(\mathbf{s}_{2n-2k} + (\bar{B}C)^{2n-2k}\mathbf{s}_{-(2n-2k)}) \\ &+ \sum_{k=1}^{n}\left(\sum_{j=0}^{k} C_{2n+1}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right) \\ &\times\ (\mathbf{s}_{2n+1-2k} + (\bar{B}C)^{2n-2k+1}\mathbf{s}_{-(2n+1-2k)})\Bigg], \end{aligned} \tag{B.6}$$

$$\begin{aligned} C(\bar{B}C)^{2n}\mathbf{s}_{-2n-1} = \delta_{2n}^T \\ - C\Bigg[&\mathbf{s}_{2n-1} + \left(\sum_{j=0}^{n} C_{2n}^{j+n}C_{j+n}^{n-j}\bar{A}^{2j}(\bar{B}C)^{n-j}\right)\mathbf{s}_{-1} \\ &+ \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right) \\ &\times(\mathbf{s}_{2n-(2k+1)-1} + (\bar{B}C)^{2n-2k-1}\mathbf{s}_{-(2n-(2k+1))-1}) \\ &+ \sum_{k=1}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right) \\ &\times\ (\mathbf{s}_{2n-2k-1} + (\bar{B}C)^{2n-2k}\mathbf{s}_{-(2n-2k)-1})\Bigg], \end{aligned} \tag{B.7}$$

$$\begin{aligned} C(\bar{B}C)^{2n+1}\mathbf{s}_{-2n-2} = \delta_{2n+1}^T \\ - C\Bigg[&\mathbf{s}_{2n} + \left(\sum_{j=0}^{n} C_{2n+1}^{j+n+1}C_{j+n+1}^{n-j}\bar{A}^{2j+1}(\bar{B}C)^{n-j}\right)\mathbf{s}_{-1} \\ &+ \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n+1}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right) \\ &\times(\mathbf{s}_{2n-2k-1} + (\bar{B}C)^{2n-2k}\mathbf{s}_{-(2n-2k)-1}) \\ &+ \sum_{k=1}^{n}\left(\sum_{j=0}^{k} C_{2n+1}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right) \\ &\times\ (\mathbf{s}_{2n-2k} + (\bar{B}C)^{2n-2k+1}\mathbf{s}_{-(2n+2-2k)})\Bigg]. \end{aligned} \tag{B.8}$$

We easily prove the following estimations:

Claim 11. *For all $k \in \mathbb{N}$, $M \ge 2k$, and $\lambda \ge 0$, we have*

$$(M-2k)!\sum_{j=0}^{k}\frac{\lambda^j}{(M-k-j)!(k-j)!(2j)!} \le \frac{(1+\lambda)^k}{k!}, \tag{B.9}$$

$$(M-2k-1)!\sum_{j=0}^{k}\frac{\lambda^{j+1}}{(M-k-j-1)!(k-j)!(2j+1)!} \le \frac{(1+\lambda)^{k+1}}{(k+1)!}. \tag{B.10}$$

Recalling (8), we choose r sufficiently large such that the following inequalities are satisfied,

$$r \ge \max(\beta, 1), \tag{B.11}$$

$$\left[2(e^{\lambda(1+\lambda)/r^2} - 1)\left(1 + \frac{r}{\lambda}\right) + \frac{\lambda^n}{r^{2n}}\frac{(1+\lambda)^n}{n!} + \frac{1}{2nr}\right] \le 1, \quad \forall n \in \mathbb{N}, \tag{B.12}$$

$$\left[2(e^{\lambda(1+\lambda)/r^2} - 1)\left(1 + \frac{r}{\lambda}\right) + \frac{\beta\lambda^n}{r^{2n}}\frac{(1+\lambda)^n}{n!} + \frac{\beta}{2nr}\right] \le 1, \quad \forall n \in \mathbb{N}. \tag{B.13}$$

Let n be such that $2n > K$. Suppose that, for all $i = 0,\dots,2n-1$,

$$\|\mathbf{S}_N^{(i)}(T)\|_\infty \le M_K i! r^i, \quad \text{and} \quad \|(\bar{B}C)^{i+1}\mathbf{S}_N^{(-i-1)}(T)\|_\infty \le M_K i! r^i. \tag{B.14}$$

Let $\lambda = \max(\alpha, \beta)$. We have to state four estimations on $\mathbf{S}_N^{(2n)}(T)$, $\mathbf{S}_N^{(2n+1)}(T)$, $\mathbf{S}_N^{(-2n-1)}(T)$, and $\mathbf{S}_N^{(-2n-2)}(T)$.

First inequality: $\|\mathbf{S}_N^{(2n)}(T)\|_\infty \leq M_K r^{2n}(2n)!$.

Let us first note that, with (19) and (B.5), we have

$$\begin{aligned}\mathbf{S}_N^{(2n)}(T) = &-\Bigg[\sum_{k=1}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right)\\ &\times(\mathbf{S}_N^{(2n-2k)}(T)+(\bar{B}C)^{2n-2k}\mathbf{S}_N^{(-2n+2k)}(T))\\ &+\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right)\\ &\times(\mathbf{S}_N^{(2n-2k-1)}(T)+(\bar{B}C)^{2n-2k-1}\mathbf{S}_N^{(-2n+2k+1)}(T))\\ &+\left(\sum_{j=0}^{n} C_{2n}^{j+n}C_{j+n}^{n-j}\bar{A}^{2j}(\bar{B}C)^{n-j}\right)\mathbf{S}_N^{(0)}(T)+(\bar{B}C)^{2n}\mathbf{S}_N^{(-2n)}(T)\Bigg].\end{aligned} \tag{B.15}$$

Thus with $r > 1$ (recall (B.11)) we have

$$\begin{aligned}\|\mathbf{S}_N^{(2n)}(T)\|_\infty \leq M_K &\Bigg[\sum_{k=1}^{n-1}\big((2n-2k)!r^{2n-2k}\\ &+(2n-2k-1)!r^{2n-2k-1}\big)\left(\sum_{j=0}^{k} C_{2n}^{j+k}C_{j+k}^{k-j}\lambda^{k+j}\right)\\ &+\sum_{k=0}^{n-1}\big((2n-2k-1)!r^{2n-2k-1}+(2n-2k-2)!r^{2n-2k-2}\big)\\ &\times\left(\sum_{j=0}^{k} C_{2n}^{j+k+1}C_{j+k+1}^{k-j}\lambda^{k+j+1}\right)\\ &+\sum_{j=0}^{n} C_{2n}^{j+n}C_{j+n}^{n-j}\lambda^{n+j}+(2n-1)!r^{(2n-1)}\Bigg],\\ \leq M_K(2n)!r^{2n}&\Bigg[2\sum_{k=1}^{n-1}(2n-2k)!r^{-2k}\lambda^k\\ &\times\sum_{j=0}^{k}\frac{\lambda^j}{(2n-k-j)!(k-j)!(2j)!}\\ &+2\sum_{k=0}^{n-1}(2n-2k-1)!r^{-2k-1}\lambda^k\\ &\times\sum_{j=0}^{k}\frac{\lambda^{j+1}}{(2n-k-j-1)!(k-j)!(2j+1)!}\\ &+\frac{\lambda^n}{r^{2n}}\sum_{j=0}^{n}\frac{\lambda^j}{(n-j)!(2j)!(n-j)!}+\frac{1}{2nr}\Bigg].\end{aligned}$$

Now, with Claim 11, we deduce

$$\begin{aligned}\|\mathbf{S}_N^{(2n)}(T)\|_\infty \leq M_K(2n)!r^{2n}&\Bigg[2\sum_{k=1}^{n-1}\frac{r^{-2k}\lambda^k(1+\lambda)^k}{k!}\\ &+2\sum_{k=0}^{n-1}\frac{r^{-2k-1}\lambda^k(1+\lambda)^{k+1}}{(k+1)!}+\frac{\lambda^n}{r^{2n}}\frac{(1+\lambda)^n}{n!}+\frac{1}{2nr}\Bigg],\\ \leq M_K(2n)!r^{2n}&\Bigg[2(e^{\lambda(1+\lambda)/r^2}-1)\left(1+\frac{r}{\lambda}\right)\\ &+\frac{\lambda^n}{r^{2n}}\frac{(1+\lambda)^n}{n!}+\frac{1}{2nr}\Bigg].\end{aligned}$$

The estimation $\|\mathbf{S}_N^{(2n)}(T)\|_\infty \leq M_K r^{2n}(2n)!$ follows from (B.12).

Second inequality: $\|\mathbf{S}_N^{(2n+1)}(T)\|_\infty \leq M_K r^{2n+1}(2n+1)!$.

We now prove the result for $2n+1$ in a similar way. First note that, with (19) and (B.6), we have

$$\begin{aligned}\mathbf{S}_N^{(2n+1)}(T) = \boldsymbol{\gamma}_{2n+1}^T &-\Bigg[\sum_{k=1}^{n}\left(\sum_{j=0}^{k} C_{2n+1}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right)\\ &\times(\mathbf{S}_N^{(2n+1-2k)}(T)+(\bar{B}C)^{2n-2k+1}\mathbf{S}_N^{(-(2n+1-2k))}(T))\\ &+\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n+1}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right)\\ &\times(\mathbf{S}_N^{(2n-2k)}(T)+(\bar{B}C)^{2n-2k}\mathbf{S}_N^{(-(2n-2k))}(T))\\ &+\left(\sum_{j=0}^{n} C_{2n+1}^{j+n+1}C_{j+n+1}^{n-j}\bar{A}^{2j+1}(\bar{B}C)^{n-j}\right)\\ &\times\mathbf{S}_N^{(0)}(T)+(\bar{B}C)^{2n+1}\mathbf{S}_N^{(-(2n+1))}(T)\Bigg].\end{aligned}$$

Recalling the definition of λ, we obtain

$$\begin{aligned}\|\mathbf{S}_N^{(2n+1)}(T)\|_\infty \leq M_K&(2n+1)!r^{2n+1}\\ \times&\Bigg[\frac{\lambda^{n+1}}{r^{2n+1}}\sum_{j=0}^{n}\frac{\lambda^j}{(n-j)!(n-j)!(2j+1)!}+\frac{1}{r(2n+1)}\\ &+2\sum_{k=0}^{n-1}(2n-2k)!r^{-2k-1}\lambda^k\\ &\times\sum_{j=0}^{k}\frac{\lambda^{j+1}}{(2n+1-k-j-1)!(k-j)!(2j+1)!}\\ &+2\sum_{k=1}^{n}(2n+1-2k)!r^{-2k}\lambda^k\\ &\times\sum_{j=0}^{k}\frac{\lambda^j}{(2n+1-k-j)!(k-j)!(2j)!}\Bigg].\end{aligned}$$

Now, with Claim 11, we easily get

$$\begin{aligned}\|\mathbf{S}_N^{(2n+1)}(T)\|_\infty \leq M_K(2n+1)!r^{2n+1}&\Bigg[2(e^{\lambda(1+\lambda)/r^2}-1)\left(\frac{\lambda+r}{\lambda}\right)\\ &+\frac{\lambda^{n+1}}{r^{2n+1}}\frac{(1+\lambda)^n}{n!}+\frac{1}{(2n+1)r}\Bigg]\end{aligned}$$

and the estimation on $\|\mathbf{S}_N^{(2n+1)}(T)\|_\infty$ follows from (B.12).

Third inequality: $\|(\bar{B}C)^{2n+1}\mathbf{S}_N^{(-2n-1)}(T)\| \leq M_K r^{2n}(2n)!$.

We now prove the result for $-(2n+1)$. With (19) and (B.7), we have

$$\begin{aligned}(\bar{B}C)^{2n+1}\mathbf{S}_N^{(-2n-1)}(T) = -\bar{B}C&\Bigg[\mathbf{S}_N^{(2n-1)}(T)\\ &+\left(\sum_{j=0}^{n} C_{2n}^{j+n}C_{j+n}^{n-j}\bar{A}^{2j}(\bar{B}C)^{n-j}\right)\mathbf{S}_N^{(-1)}(T)\\ &+\sum_{k=1}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right)\\ &\times(\mathbf{S}_N^{(2n-2k-1)}(T)+(\bar{B}C)^{2n-2k}\mathbf{S}_N^{(-2n+2k-1)}(T))\\ &+\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k} C_{2n}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right)\\ &\times(\mathbf{S}_N^{(2n-2k-2)}(T)+(\bar{B}C)^{2n-2k-1}\mathbf{S}_N^{(-2n+2k)}(T))\Bigg].\end{aligned}$$

Due to (B.11), we have $1/r \le 1/\beta$ and thus we get

$$\|(\bar{B}C)^{2n+1}\mathbf{S}_N^{-2n-1}(T)\|_\infty \le M_K r^{2n}(2n)!\Bigg[\frac{\beta}{2nr}+\frac{\beta\lambda^n}{r^{2n}} \times \sum_{j=0}^{n}\frac{\lambda^j}{(n-j)!(n-j)!(2j)!}+2\sum_{k=1}^{n-1}(2n-2k)!r^{-2k}\lambda^k \times \sum_{j=0}^{k}\frac{\lambda^j}{(2n-k-j)!(k-j)!(2j)!} +2\sum_{k=0}^{n-1}(2n-2k-1)!r^{-2k-1}\lambda^k \times \left(\sum_{j=0}^{k}\frac{\lambda^{j+1}}{(2n-k-j-1)!(k-j)!(2j+1)!}\right)\Bigg],$$

and then, with Claim 11,

$$\|(\bar{B}C)^{2n+1}\mathbf{S}_N^{(-(2n+1))}(T)\|_\infty \le M_K(2n)!r^{2n} \times \left[2(e^{\lambda(1+\lambda)/r^2}-1)\left(1+\frac{r}{\lambda}\right)+\frac{\beta\lambda^n}{r^{2n}}\frac{(1+\lambda)^n}{n!}+\frac{\beta}{2nr}\right].$$

Thus the estimation on $\|(\bar{B}C)^{2n+1}\mathbf{S}_N^{(-2n-1)}(T)\|_\infty$ follows from (B.13).

Fourth inequality: $\|(\bar{B}C)^{2n+2}\mathbf{S}_N^{(-2n-2)}(T)\|_\infty \le M_K r^{2n+1}(2n+1)!$.

We prove the last case, $-(2n+2)$, in a similar way. First, with (19) and (B.8), we compute

$$(\bar{B}C)^{2n+2}\mathbf{S}_N^{(-2n-2)}(T) = -\bar{B}C\Bigg[\mathbf{S}_N^{(2n)}(T) + \left(\sum_{j=0}^{n}C_{2n+1}^{j+n+1}C_{j+n+1}^{n-j}\bar{A}^{2j+1}(\bar{B}C)^{n-j}\right)\mathbf{S}_N^{(-1)}(T) \times \sum_{k=1}^{n}\left(\sum_{j=0}^{k}C_{2n+1}^{j+k}C_{j+k}^{k-j}\bar{A}^{2j}(\bar{B}C)^{k-j}\right) \times (\mathbf{S}_N^{(2n-2k)}(T)+(\bar{B}C)^{2n-2k+1}\mathbf{S}_N^{(-2n+2k-2)}(T)) + \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}C_{2n+1}^{j+k+1}C_{j+k+1}^{k-j}\bar{A}^{2j+1}(\bar{B}C)^{k-j}\right) \times (\mathbf{S}_N^{(2n-2k-1)}(T)+(\bar{B}C)^{2n-2k}\mathbf{S}_N^{(-2n+2k-1)}(T))\Bigg]. \quad (B.16)$$

Thus we have

$$\|(\bar{B}C)^{2n+2}\mathbf{s}_N^{(-2n-2)}(T)\|_\infty \le M_K(2n+1)!r^{2n+1} \times \Bigg\|\Bigg[\frac{\beta}{(2n+1)r}+2\sum_{k=1}^{n}(2n+1-2k)!r^{-2k}\lambda^k \times \sum_{j=0}^{k}\frac{\lambda^j}{(2n+1-k-j)!(k-j)!(2j)!} +2\sum_{k=0}^{n-1}(2n+1-2k-1)r^{-2k-1}\lambda^k \times \left(\sum_{j=0}^{k}\frac{\lambda^{j+1}}{(2n+1-k-j-1)!(k-j)!(2j+1)!}\right) +\frac{\beta\lambda^{n+1}}{r^{2n+1}}\sum_{j=0}^{n}\frac{\lambda^j}{(n-j)!(n-j)!(2j+1)!}\Bigg]\Bigg\|.$$

With Claim 11, we get the inequality

$$\|(\bar{B}C)^{2n+2}\mathbf{s}_N^{(-(2n+2))}(T)\|_\infty \le M_K(2n+1)!r^{2n+1} \times \left[2(e^{\lambda(1+\lambda)/r^2}-1)\left(1+\frac{r}{\lambda}\right) +\frac{\beta\lambda^{n+1}}{r^{2n+1}}\frac{(1+\lambda)^n}{n!}+\frac{\beta}{(2n+1)r}\right]$$

and the estimation on $\|(\bar{B}C)^{2n+2}\mathbf{S}_N^{(-2n-2)}(T)\|_\infty$ follows from (B.13). This concludes the proof of Lemma 10. ■

For larger index n, the estimations are obtained more easily.

Lemma 12. *There exists $M_K > 0$ and $r > 0$ such that, for all $n \in \mathbb{N}$, $n \ge N+1$,*

$$\mathbf{S}_N^{(n)}(0) = \mathbf{S}_N^{(-n)}(0) = \mathbf{0}, \quad (B.17)$$

$$\mathbf{S}_N^{(n)}(T) = \mathbf{0}, \quad (B.18)$$

$$\left\|\mathbf{S}_N^{(-n)}(T)\right\|_\infty \le M_K r^N N!. \quad (B.19)$$

Proof. With (19), we get that for all $i \in \mathbb{N}$, $n \ge N+1$, we have $\mathbf{S}_N^{(n)}(0) = \mathbf{S}_N^{(-n)}(0) = 0$ and thus (B.17). Furthermore, using (19) again, we get that for all $n > N$, $\mathbf{S}_N^{(n)}(T) = 0$, which is (B.18). It remains to state (B.19). Using (19) and Lemma 10, we compute

$$\sup_{t\in[0,T]}\left\|\mathbf{S}_N^{(-N-1)}(t)\right\|_\infty = \left\|\int_0^t\sum_{i=0}^{2N}\mathbf{S}_N^{(i-N)}(T)\frac{(\tau-T)^i}{i!}\Phi_\sigma\left(\frac{\tau}{T}\right)d\tau\right\|_\infty \le M_K r^N N!\int_0^t\sum_{i=0}^{2N}\frac{(T-\tau)^i}{i!}d\tau \le M_K r^N N! e^T,$$

and we can deduce by recurrence that, for all $n \ge N$, $\sup_{t\in[0,T]}\left\|\mathbf{S}_N^{(-n)}(t)\right\|_\infty \le M_K e^{2T} r^N N!$. By redefining M_K, inequality (B.19) follows. This concludes the proof of Lemma 12. ■

We deduce Proposition 6 from Lemmas 10 and 12. ■

B.2. Proof of Theorem 7

We first prove some existence and unicity results. First, we transform boundary control problem (4) in a system with a distributed control. Suppose first that $\bar{\mathbf{u}} \in H^1(0,T)$. Let $f \in C^\infty(0,1)$ such that $f'(0)-\frac{\mu}{2}f(0) = f'(1)-\frac{\mu}{2}f(1) = 0$ and $f(1) = 1$. Let $(\tilde{\mathbf{y}},\tilde{z}) = (\bar{\mathbf{y}}-f(x)\bar{\mathbf{u}},\bar{z})$. So we get the following problem,

$$\begin{cases}\tilde{\mathbf{y}}_t-(\tilde{\mathbf{y}}_{xx}-\bar{A}\tilde{\mathbf{y}}-\bar{B}\tilde{z}) = -f\bar{\mathbf{u}}_t-(\bar{A}f-f'')\bar{\mathbf{u}},\\ \tilde{z}_t-C\tilde{\mathbf{y}} = Cf\bar{\mathbf{u}},\\ \tilde{\mathbf{y}}_x(0,t)-\dfrac{\mu}{2}\tilde{\mathbf{y}}(0,t) = 0,\\ G\tilde{\mathbf{y}}_x(1,t)+(I_p-G\mu/2)\tilde{\mathbf{y}}(1,t) = 0,\\ \tilde{\mathbf{y}}(.,0) = -f(.)\bar{\mathbf{u}}(0), \qquad \tilde{z}(.,0) = 0.\end{cases} \quad (B.20)$$

Let $\mathcal{A}$ be the operator defined on $\mathcal{D}(\mathcal{A}) = \{(g,h) \in L^2(0,1)^{p+1}, g_{xx} \in L^2(0,1)^p, g_x(0)-\mu/2g(0) = 0, Gg_x(1)+(I_p-G\mu/2)g(1) = 0\}$ by $\mathcal{A}(g,h) = (-g_{xx}+\bar{A}g+\bar{B}h,-Cg)$. The operator $\mathcal{A}$ is maximal monotone symmetric, thus, thanks to Hille–Yosida theorem, it is closed, densely defined and self-adjoint. Furthermore, thanks to the Hille–Yosida theorem, for $\bar{\mathbf{u}} \in H^1(0,T)$, problem (B.20) has a unique solution

$$(\tilde{\mathbf{y}},\tilde{z}) \in C([0,T],L^2(0,1)^{p+1})\cap C^1((0,T),L^2(0,1)^{p+1}) \cap C((0,T),\mathcal{D}(\mathcal{A}))$$

such that, $(\tilde{\mathbf{y}},\tilde{z}) = S_{\mathcal{A}}(-f\bar{\mathbf{u}}(0),0)-\int_0^t S_{\mathcal{A}}(t-\tau)(f\bar{\mathbf{u}}_t+(\bar{A}f-f'')\bar{\mathbf{u}},Cf\bar{\mathbf{u}})d\tau$, where $S_{\mathcal{A}}$ denotes the strongly continuous