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Stabilization of a 1-D tank containing a fluid modeled by the shallow water equations

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Abstract

Consider a rectangular tank containing an inviscid incompressible and irrotational fluid. The tank is subject to a one-dimensional horizontal move (the control) and the motion of the fluid is described by the shallow water equations. By means of a Lyapunov approach, control laws that stabilize the state of the fluid–tank system are derived. Two classes of control are considered: full-state feedback and output feedback where the output is given by the trajectory of the tank, the level of the fluid at the boundary of the tank and the time. Although global asymptotic stability is yet to be proved, stabilization is observed through numerical simulations.

Keywords: Hyperbolic PDEs; Shallow water equations; Lyapunov approach; Boundary control; Numerical resolution

1. Introduction

Consider a tank containing an inviscid incompressible irrotational fluid. The tank is subject to a one-dimensional horizontal move. To move such a tank we need to take the motion of the fluid into account. Several recent publications deal with this question, see, e.g., [8,12,24,25]. This paper is a first attempt to study the stabilization problem with the model of the shallow water equations which are 1D-hyperbolic equations (see e.g. [3,4]).

Our main concern is the fluid state stabilization problem (level and speed relative to the tank) and the tracking problem of the tank state (position, speed and acceleration) to a prescribed trajectory (e.g. a prescribed final position of the tank) using the acceleration as the control variable.

Stabilizing feedback laws are designed using a Lyapunov approach and backstepping (see e.g. [14] for an introduction of this technique). The design process is repeated iteratively on control problems that have increasing complexity. For each control problem, an augmented Lyapunov function is built from the previous (simpler) problem and the corresponding stabilizing control laws are deduced. More specifically, the control "sub"-problems are

- fluid state stabilization (Section 3.1),
- fluid state and tank speed stabilization (Section 3.2),

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- fluid–tank state stabilization (Section 3.3) where a forward approach (see [18]) is used to design the Lyapunov function.

Two classes of stabilizing control laws are investigated: (1) time-varying full-state feedbacks and (2) output feedbacks, where the output is defined by the trajectory of the tank, the level of the fluid at the boundary of the tank and the time. Many practical and industrial motivations can be found in [11,12,19] for restricting ourselves to output feedbacks.

Some results can be found in [18] concerning the problem of the stabilization of a tank, but the input is defined as a flexible or a rigid wave generator and the equations are linearized around the equilibrium. Here, a different model of the control system is chosen. Moreover, the linearized shallow water are not stabilizable even locally (see [7]), thus a study of the non-linear equations is needed. For these non-linear equations, it is proved in [3] that one has a local controllability property.

Several other configurations are studied in [20] and it is proved that, for each configuration under consideration, the linear approximation is steady-state controllable. For our configuration, the linear approximation is not controllable, and thus we need to consider the non-linear equations to study the stabilization problem.

The paper is organized as follows. The shallow water equations are described in Section 2.1, the steady states in Section 2.2 and the stabilization problem in Section 2.3. The existence of Lyapunov functions and feedbacks are investigated in Section 3. At last, numerical simulation are used to check that asymptotic stabilization is achieved in Section 4.

2. System and control problem description

2.1. Model description

Let us consider a 1-D tank containing an inviscid incompressible irrotational fluid. The tank is subject to a one-dimensional horizontal move. Let us assume that the acceleration is small compared with the gravity constant and that the level of the fluid is small compared with the length of the tank. Hence, the dynamics of the fluid are described by the shallow water equations (see [6, Section 4.2], see also [20]):

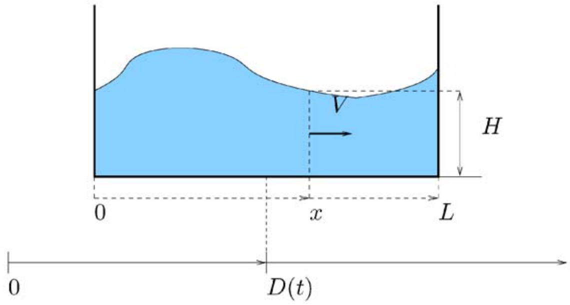

Fig. 1. A tank of length L containing a fluid.

$$\frac{\partial H}{\partial t}(x,t) + \frac{\partial}{\partial x}(HV)(x,t) = 0, \tag{1}$$

$$\frac{\partial V}{\partial t}(x,t) + \frac{\partial}{\partial x}\left(gH + \frac{V^2}{2}\right)(x,t) = -A(t), \tag{2}$$

$$\dot{S}(t) = A(t), \qquad \dot{D}(t) = S(t), \tag{3}$$

where $x \in [0,L]$ is the spatial coordinate attached to the tank of length L, $t \in [0,T]$ the time coordinate, $T > 0$, g the gravity constant, $H(x,t)$ denotes the level of the liquid, $V(x,t)$ denotes the horizontal speed of the fluid in the referential attached to the tank, and D, S and A denote, respectively, the position, the speed and the acceleration of the tank in the world coordinates. See Fig. 1.

The boundary conditions are given by, for all t in $[0,T]$,

$$V(0,t) = 0, \qquad V(L,t) = 0. \tag{4}$$

2.2. Steady states

Let us now describe the set of equilibriums: $((\bar{H},\bar{V},\bar{S},\bar{D}),\bar{A})$ in $((\mathscr{C}^1([0,L]))^2 \times \mathbb{R}^2) \times \mathbb{R}$ is said to be an equilibrium of (1)–(3) if it is a time-invariant solution of (1)–(3). This implies

$$\frac{\partial}{\partial x}(\bar{H}\bar{V}) = 0, \quad \frac{\partial}{\partial x}\left(g\bar{H} + \frac{\bar{V}^2}{2}\right) = -\bar{A}$$

which, using (1) and (2) can been easily rewritten as

$$\bar{V}(x) = 0, \qquad \bar{H}(x) = \bar{H}\left(\frac{L}{2}\right) - \left(x - \frac{L}{2}\right)\frac{\bar{A}}{g}. \tag{5}$$

By integrating (1) on $[0, L]$ and by using the boundary condition (4) together with an integration by parts, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L H(x,t)\,\mathrm{d}x\right) = 0. \tag{6}$$

This condition expresses the conservation of the mass of the fluid in the tank. Moreover, if follows from (1) and (4) that

$$\frac{\partial H}{\partial x}(0,t) = \frac{\partial H}{\partial x}(L)\left(= -\frac{u(t)}{g}\right). \tag{7}$$

Therefore, the state-space $\mathbb{X} = \{X = (H, V, S, D)\}$ is defined as the affine subspace of $(\mathscr{C}^1([0, L]))^2 \times \mathbb{R}^2$ such that the following holds:

$$\mathrm{Vol} = \int_0^L H(x)\,\mathrm{d}x, \qquad \frac{\mathrm{d}H}{\mathrm{d}x}(0) = \frac{\mathrm{d}H}{\mathrm{d}x}(L),$$

$$V(0) = V(L) = 0,$$

where the (constant) volume of the liquid in the tank is $\mathrm{Vol} := \int_0^L \bar{H}(x)\,\mathrm{d}x = L\bar{H}(L/2)$.

2.3. Control problem

Assume that the fluid level at the boundary of the tank can been measured. The control system is described as follows:

- the state at time t is $X(t) = (H(.,t), V(.,t), S(t), D(t))$,
- the control at time t is $u(t) = A(t)$,
- the output at time t is $Y(t) = (H(0,t), H(L,t), S(t), D(t), t)$.

Let $|.|$ be the usual norm of $\mathbb{R}$ and $|.|_1$ be the norm on $\mathscr{C}^1([0, L])$ defined by, for all f in $\mathscr{C}^1([0, L])$,

$$|f|_1 = \max_{x\in[0,L]} |f(x)| + \max_{x\in[0,L]} \left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\right|.$$

The set $\mathbb{X}$ is equipped with the following norm, for all $X = (H, V, S, D)$ in $\mathbb{X}$:

$$|X|_{\mathbb{X}} = |H|_1 + |V|_1 + |S| + |D|.$$

The control problem is stated as follows:

Find a *time-varying full-state feedback* (resp. an *output feedback*) u that *locally stabilizes* the system $(X(t), A(t))$ to the equilibrium $(\bar{X}, \bar{A})$ satisfying (5).

That is, find a function $u : \mathbb{X} \times [0, +\infty) \to \mathbb{R}$ *(resp.* $u : \mathbb{R}^5 \to \mathbb{R}$*) such that for all* $\varepsilon > 0$*, there exists* $C > 0$ *such that, for all initial conditions* (H_0, V_0, S_0, D_0) *in* $\mathbb{X}$ *satisfying*

$$|(H_0, V_0, S_0, D_0) - (\bar{H}, \bar{V}, \bar{S}, \bar{D})|_{\mathbb{X}} \leqslant C, \tag{8}$$

the following three properties hold on the system:

Existence and uniqueness of solutions. *There exists one and only one* $X : [0, +\infty) \to \mathbb{X}$ *such that*, (1)–(3) *hold where, for all* $t \geqslant 0$,

$$A(t) = u(X(t), t) = u(H(., t), V(., t), S(t), D(t), t)$$

$$(\text{resp. } A(t) = u(Y(t))$$

$$= u(H(0,t), H(L,t), S(t), D(t), t), \tag{9}$$

and such that the following holds:

$$H(.,0) = H_0, \quad V(.,0) = V_0,$$

$$S(0) = S_0, \quad D(0) = D_0. \tag{10}$$

Attractivity property. *Moreover, for all* $t \geqslant 0$,

$$|(H(.,t), V(.,t), S(t), D(t)) - (\bar{H}, \bar{V}, \bar{S}, \bar{D})|_{\mathbb{X}} \to_{t\to+\infty} 0.$$

Stability property. *Finally, for all* $t \geqslant 0$,

$$|(H(.,t), V(.,t), S(t), D(t)) - (\bar{H}, \bar{V}, \bar{S}, \bar{D})|_{\mathbb{X}} \leqslant \varepsilon.$$

In the next sections, a Lyapunov control law design that solves this problem is proposed. Stabilization is verified numerically in Section 4.

3. Lyapunov control design

The objective of the design is to build a Lyapunov function for the stabilization problem via a full-state feedback and an other one for the output feedback. As mentioned in the Introduction, Lyapunov functions are built for control problems with increasing complexity.

At first, a Lyapunov function for the fluid state (H, V) (i.e. a Frechet-differentiable function $R_1 : (\mathscr{C}^1([0, L]))^2 \to \mathbb{R}$ positive and null only at the point $(H,V)=(\bar{H},\bar{V})$) and a full-state feedback, making the Lyapunov function decrease, are designed. For an introduction of the Frechet-differentiation see e.g. [5, Appendix A]. This full-state feedback is used to derive an output feedback which is a good candidate to stabilize the fluid state (H,V) (see Section 3.1).

The fluid state Lyapunov function is augmented to obtain a Lyapunov function for the fluid–tank state in Sections 3.2 and 3.3.

3.1. Stabilization of the fluid state (H,V)

Let us consider first the stabilization of the fluid state.

A Lyapunov function for the variables (H,V) can be derived from the fluid entropy $E\colon [0,L]\times\mathbb{R}^2\to\mathbb{R}$ and the corresponding entropic flux $F\colon [0,L]\times\mathbb{R}^2\to\mathbb{R}$ (as in [4] for another stabilization problem). Note there is an infinite number of entropies for the shallow water equations (see [23, Vol. II, Section 9.3]).

Let us check briefly that the following functions, derived from the moments of the fluid and defined, for all $(H,V)\in\mathbb{R}^2$, by

$$E(x,H,V)=H\,\frac{V^2}{2}+g\,\frac{(H-\bar{H}(x))^2}{2}, \tag{11}$$

$$F(x,H,V)=H\,\frac{V^3}{2}+gVH(H-\bar{H}(x)) \tag{12}$$

are a couple of entropy and entropic flux respectively.

Indeed, denoting $\mathrm{D}/\mathrm{D}t$ and $\mathrm{D}/\mathrm{D}x$ the total derivative with respect to t and x, respectively, it follows that, along the solutions of (1)–(3),

$$\begin{aligned}
&\frac{\mathrm{D}}{\mathrm{D}t}E(x,H,V)+\frac{\mathrm{D}}{\mathrm{D}x}F(x,H,V)\\
&\quad=\frac{\partial E}{\partial H}\frac{\partial H}{\partial t}+\frac{\partial E}{\partial V}\frac{\partial V}{\partial t}+\frac{\partial F}{\partial H}\frac{\partial H}{\partial x}+\frac{\partial F}{\partial V}\frac{\partial V}{\partial x}+\frac{\partial F}{\partial x}\\
&\quad=-\left(\frac{V^2}{2}+g(H-\bar{H})\right)\frac{\partial}{\partial x}(HV)\\
&\qquad-HV\frac{\partial}{\partial x}\left(gH+\frac{V^2}{2}+u\right)\\
&\qquad+\left(\frac{V^3}{2}+gV(H-\bar{H})+gVH\right)\frac{\partial H}{\partial x}\\
&\qquad+\left(H\,\frac{3V^2}{2}+gH(H-\bar{H})\right)\frac{\partial V}{\partial x}-gHV\,\frac{\partial\bar{H}}{\partial x}\\
&\quad=-\frac{\partial}{\partial x}\left(HV\left(\frac{V^2}{2}+g(H-\bar{H})\right)\right)-gHV\,\frac{\partial\bar{H}}{\partial x}\\
&\qquad-HVu+\frac{\partial}{\partial x}\left(HV\left(\frac{V^2}{2}+g(H-\bar{H})\right)\right)\\
&\qquad+gHV\,\frac{\partial\bar{H}}{\partial x}+gHV\,\frac{\bar{A}}{g}\\
&\quad=-HV(u-\bar{A})
\end{aligned} \tag{13}$$

which vanishes if $u=\bar{A}$. Thus (E,F) is a couple of entropy–entropic flux.

Let the function $R_1:(\mathscr{C}^1([0,L]))^2\to\mathbb{R}$, for all $(H,V)\in(\mathscr{C}^1([0,L]))^2$, be defined by

$$R_1(H,V)=\lambda_1\int_0^L E(x,H(t,x),V(t,x))\,\mathrm{d}x, \tag{14}$$

where $\lambda_1>0$ is a tuning parameter. Note that R_1 is positive and is zero only at the point $(H,V)=(\bar{H},\bar{V})$. By differentiating (14) with respect to t and by using (13) it follows that, along the solutions of (1) and (2),

$$\dot{R}_1=-(u-\bar{A})\lambda_1\left(\int_0^L HV\,\mathrm{d}x\right)-\lambda_1[F]_0^L. \tag{15}$$

Using (4) and (12), it can be seen that $[F]_0^L=0$. Hence, a natural controller is $u:(\mathscr{C}^1([0,L]))^2\to\mathbb{R}$, defined by

$$u(H,V)=\lambda_1\int_0^L HV\,\mathrm{d}x+\bar{A}. \tag{16}$$

This control law is a full-state feedback law, but one can prove that the time-derivative of $\int_0^L HV\,\mathrm{d}x$ can be expressed in terms of output variables. More precisely, using (1) and (2), it follows that

$$\frac{\partial HV}{\partial t}=-\frac{\partial}{\partial x}\left(g\,\frac{H^2}{2}+HV^2\right)-Hu \tag{17}$$

and thus, by using the boundary conditions (4) together with an integration by parts,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L HV\,\mathrm{d}x\right)\\
&\quad=-\frac{g}{2}((H(L,t))^2-(H(0,t))^2)-\mathrm{Vol}\,u
\end{aligned} \tag{18}$$

which is a function of the output $Y(t)$ only.

This allow us to use a backstepping approach which is usual in finite-dimensional control theory (see e.g. [14,21]). In this context, let us consider u as a new state variable and define the dynamics of u as

$$\dot{u} = v, \tag{19}$$

where v is the control law.

To understand how an output feedback can be derived from (15), let us prove the following.

Lemma 3.1. *Let $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{U}$ be three Banach spaces called, respectively, the state space, the output space and the control space. Let $\mathscr{F}:\mathscr{X}\times\mathscr{U}\to\mathscr{X}$ and $\mathscr{G}:\mathscr{X}\to\mathscr{Y}$ be two Frechet-differentiable functions. Let us consider the control system*

$$\dot{X} = \mathscr{F}(X,U), \quad Y = \mathscr{G}(X), \quad X(0)\in\mathscr{X}. \tag{20}$$

Let us consider an equilibrium of (20), i.e. a state 0 in $\mathscr{X}$ and a control 0 in $\mathscr{U}$ such that $\mathscr{F}(0,0)=0$ and $\mathscr{G}(0)=0$. Suppose that there exists a Lyapunov function R: $\mathscr{X}\to\mathbb{R}$ (i.e. a Frechet-differentiable function positive and zero only at the point $X=0$) such that

$$\dot{R} = (U+k(Y))l(X,t), \quad \dot{\overbrace{l(X,t)}} = \tilde{l}(Y,U), \tag{21}$$

along the solutions of (20), where $k:\mathscr{X}\to\mathbb{R}$, $l:\mathscr{Y}\times[0,+\infty)\to\mathbb{R}$ and $\tilde{l}:\mathscr{Y}\times\mathscr{U}\to\mathbb{R}$ are three Frechet-differentiable functions.

Then, for all $\lambda>0$, we can design a Lyapunov function $\tilde{R}:\mathscr{X}\times\mathscr{U}\to\mathbb{R}$, for the state variables (X,U), and an output feedback making $\tilde{R}$ decrease along the solutions of (20). This output feedback is defined by

$$\dot{U} = -\dot{\overbrace{k(Y)}} - \lambda\tilde{l}(Y,U) - \frac{1}{\lambda\tilde{\lambda}}(U+k(Y)),$$
$$U(t=0)\in\mathscr{U}. \tag{22}$$

Remark 3.2. Note that the dynamical equation of U must be a function of the output only, and, for $\tilde{R}$, a Lyapunov function for the whole state, i.e. the couple (X,U). Moreover, note that, in order to prove Lemma 3.1, only properties of the Lyapunov function R i.e. (21) and any property of the control system (20) are used. Thus the lemma can be applied to any control problem having a Lyapunov function satisfying (21).

Proof. Note first that, due to (21), with the continuous feedback $U:\mathscr{X}\to\mathscr{U}$ defined by $U(X)=-\lambda l(X,t)-k(Y)$, for all $\lambda>0$, the function R has a non-positive time derivative. Let us define the Frechet-differentiable function $\tilde{R}:\mathscr{X}\times\mathscr{U}\to\mathbb{R}$ by $\tilde{R}(X,U)=R(X)+\frac{1}{2}\tilde{\lambda}(U+k(Y)+\lambda l(X,t))^2$, where $\tilde{\lambda}>0$. By using (21), the time derivative of $\tilde{R}$ along the solution of (20) is

$$\begin{aligned}\dot{\tilde{R}} &= l(X,t)(U+k(Y))\\ &\quad+\tilde{\lambda}(U+k(Y)+\lambda l(X,t))(\dot{U}+\dot{\overbrace{k(Y)}}+\lambda\tilde{l}(Y,U))\\ &= l(X,t)(U+k(Y)+\lambda\tilde{\lambda}\dot{U}+\lambda\tilde{\lambda}\dot{\overbrace{k(Y)}}\\ &\quad+\lambda^2\tilde{\lambda}\tilde{l}(Y,U))\\ &\quad+\tilde{\lambda}(U+k(Y))(\dot{U}+\dot{\overbrace{k(Y)}}+\lambda\tilde{l}(Y,U)).\end{aligned}$$

Thus, by defining U as

$$U+k(Y)+\lambda\tilde{\lambda}\dot{U}+\lambda\tilde{\lambda}\dot{\overbrace{k(Y)}}+\lambda^2\tilde{\lambda}\tilde{l}(Y,U)=0 \tag{23}$$

the first derivative of $\tilde{R}$ is $\dot{\tilde{R}}=-(1/\lambda)(U+k(Y))^2$, and thus $\tilde{R}$ is a Lyapunov function for system (20). To achieve the proof of Lemma 3.1, note that (23) is equivalent to (22), and U is indeed a dynamic output feedback. □

It is now possible to apply Lemma 3.1 to system (1)–(3) and to the Lyapunov function R_1 with $\mathscr{U}=\mathbb{R}$, $\mathscr{Y}=\mathbb{R}^4$, $\mathscr{X}=(\mathscr{C}^1([0,L]))^2$, $k(Y)=-\bar{A}$, $R=R_1$, $\lambda=1$, $l(X)=-\lambda_1\int_0^L HV$ and $\tilde{l}(Y,u)=\lambda_1(g/2)((H(L,t))^2-(H(0,t))^2)+\lambda_1\,\mathrm{Vol}\,u$. Thus let us define $\tilde{R}_1:(\mathscr{C}^1([0,L]))^2\times\mathscr{U}\to\mathbb{R}$ by, for all $(h,v,u)\in(\mathscr{C}^1([0,L]))^2\times\mathscr{U}$,

$$\tilde{R}_1(H,V,u)=R_1(H,V)+\frac{\tilde{\lambda}_1}{2}\left(u-\lambda_1\int_0^L HV\,\mathrm{d}x\right)^2, \tag{24}$$

where $\tilde{\lambda}_1>0$.

The above lemma leads to the following result.

Theorem 1.

- *For any positive gain λ_1, there exist a Lyapunov function R_1 for the variables (H,V) and a full-state feedback $u_1:\mathscr{C}^1([0,L])^2\to\mathbb{R}$, defined by, for all $(H,V)\in\mathscr{C}^1([0,L])^2$*

$$u_1(H,V):=\lambda_1\int_0^L HV\,\mathrm{d}x \tag{25}$$

such that R_1 has a non-positive time derivative.

- *For any positive gains λ_1 and $\tilde{\lambda}_1$, there exist a Lyapunov function $\tilde{R}_1$ for the variables (H,V,u) and an output feedback $\tilde{u}_1:\mathscr{C}^1([0,+\infty),\mathbb{R}^5)\times[0,+\infty)\to\mathbb{R}$, defined, for all $Y:[0,+\infty)\to\mathbb{R}^5$, $Y(t)=(H(0,t),H(L,t),S(t),D(t),t)$, by $u_1(t=0)\in\mathbb{R}$ and*

$$\overbrace{\tilde{u}_1(Y)}^{\cdot}:=-\left(\frac{1}{\tilde{\lambda}_1}+\lambda_1\,\mathrm{Vol}\right)\tilde{u}_1(Y)-\lambda_1\,\frac{g}{2}((H(L,t))^2-(H(0,t)^2)) \qquad (26)$$

such that $\tilde{R}_1$ has a non-positive time derivative.

Remark 3.3. To prove the stabilization, the LaSalle theorem must be applied. More precisely, it must be proved that the equality $(\mathrm{D}/\mathrm{D}t)(\tilde{R}_1(t))=0\ \forall t\geqslant 0$, yields $(H,V)=(\bar{H},\bar{V})$. Note, moreover, that in an infinite-dimensional space of functions, a suitable compactness property must also be proved (see e.g. [13]).

3.2. Stabilization of the fluid state (H,V) and of the tank speed S

In this section, the control problem is augmented with the stabilization of the tank speed $\dot{D}$ around $\bar{S}+\bar{A}t$. In order to achieve this, a modified "kinetic energy" of the tank is introduced in (14), i.e. the Frechet-differentiable function $R_2:(\mathscr{C}^1([0,L]))^2\times\mathbb{R}\to\mathbb{R}$ defined by

$$R_2(H,V,S)=R_1(H,V)+\frac{\lambda_2}{2}(S(t)-\bar{S}-\bar{A}t)^2, \qquad (27)$$

where R_1 is defined by (14) and λ_2 is a positive constant introduced for the tuning of the controller. Note that R_2 is positive and is zero only at the point $(H,V,S)=(\bar{H},\bar{V},\bar{S}+\bar{A}t)$. Due to (3) and (15),

$$\dot{R}_2=(u-\bar{A})\times\left(-\lambda_1\int_0^L HV\,\mathrm{d}x+\lambda_2(S-\bar{S}-\bar{A}t)\right). \qquad (28)$$

Thus a control law candidate to stabilize the variables H, V and S is $u_2:\mathscr{C}^1([0,L])\times\mathbb{R}\to\mathbb{R}$ defined, for all $(H,V,S)\in\mathscr{C}^1([0,L])\times\mathbb{R}$, by

$$u_2=\left(\lambda_1\int_0^L HV\,\mathrm{d}x-\lambda_2(S-\bar{S}-\bar{A}t)\right)+\bar{A}.$$

This is a full-state feedback. As in the previous case, it is possible to apply Lemma 3.1 with $\mathscr{X}=(\mathscr{C}^1([0,L]))^2\times\mathbb{R}$, $k(Y)=-\bar{A}$, $R=R_2$, $\lambda=1$, $l(X,t)=-\lambda_1\int_0^L HV\,\mathrm{d}x+\lambda_2(S(t)-\bar{S}-\bar{A}t)$ and $\tilde{l}(Y,u)=\lambda_1(g/2)((H(L,t)^2-H(0,t)^2)+\lambda_1\,\mathrm{Vol}\,u+\lambda_2(u-\bar{A})$. This motivates the introduction of a new Lyapunov function (see the proof of Proposition 3.1): $R_2:(\mathscr{C}^1([0,L]))^2\times\mathbb{R}^2\to\mathbb{R}$ defined by

$$\tilde{R}_2(H,V,S,u)=R_2(H,V,S)+\frac{\tilde{\lambda}_2}{2}\left(u-\lambda_1\int_0^L HV\,\mathrm{d}x+\lambda_2(S-\bar{S}-\bar{A}t)\right)^2 \qquad (29)$$

where $\tilde{\lambda}_2>0$.

The above leads to the following result.

Theorem 2.

- *For any positive gains λ_1 and λ_2, there exist a Lyapunov function R_2 for the variables (H,V,S) and a time-varying full-state feedback $u_2:\mathscr{C}^1([0,L]))^2\times\mathbb{R}\times[0,+\infty)\to\mathbb{R}$, defined, for all $(H,V,S)\in\mathscr{C}^1([0,L]))^2\times\mathbb{R}$, by*

$$u_2(H,V,S,t)=\left(\lambda_1\int_0^L HV\,\mathrm{d}x-\lambda_2(S-\bar{S}-\bar{A}t)\right)+\bar{A} \qquad (30)$$

such that R_2 has a non-positive time derivative.

- *For any positive gains λ_1, λ_2, $\tilde{\lambda}_2$, there exist a Lyapunov function $\tilde{R}_2$ for the variables (H,V,S,u) and an output feedback $\tilde{u}_2:\mathscr{C}^1([0,+\infty),\mathbb{R}^5)\times[0,+\infty)\to\mathbb{R}$, defined, for all $Y:[0,+\infty)\to\mathbb{R}^5$, $Y(t)=(H(0,t),H(L,t),S(t),D(t),t)$, by $\tilde{u}_2(t=0)\in\mathbb{R}$ and*

$$\overbrace{\tilde{u}_2(Y)}^{\cdot}=-\left(\frac{1}{\tilde{\lambda}_2}+\lambda_1\,\mathrm{Vol}+\lambda_2\right)\tilde{u}_2(Y)-\lambda_1\,\frac{g}{2}((H(L,t)^2-(H(0,t)^2)+\lambda_2\bar{A} \qquad (31)$$

such that $\tilde{R}_2$ has a non-positive time derivative.

Remark 3.4. Note that the relative degree of the dynamical system for the variable (H,V) and S is 1 (indeed the zero time-derivative of the output are independent of the control, while the first time-derivative of the output can be expressed from the control, see e.g. [15] for a precise definition of the relative degree in a finite-dimensional context).

In the following section, the system relative degree is 2, which can not be stabilized by an output feedback. See Section 3.3 for more details.

3.3. Stabilization of the fluid–tank state

In this section, the control problem is augmented with the entire function D and not only its first derivative. To stabilize also the tank position D to a prescribed trajectory $\bar{D}+\bar{S}t+\frac{1}{2}\bar{A}t^2$, one can require to stabilize the integrated trajectory $D=\int \dot{D}\,\mathrm{d}t$ around the reference trajectory $\bar{D}(t)=\int(\bar{S}+\bar{A}t)\,\mathrm{d}t$. To do this a forward approach should be used (see [17] or [21, Chapter 6]), i.e. the solvability of the following equation has to be studied:

$$\dot{\mathscr{M}}=S-\bar{S}-\bar{A}t, \tag{32}$$

where $\mathscr{M}:\mathbb{X}\to\mathbb{R}$ is a Frechet-differentiable function and: in (32) is to be understood as the time-derivative along the solutions of (1)–(3). If (32) is solvable, then $R_3:\mathbb{X}\to\mathbb{R}$ is defined as the following modification of R_2, for all (h,v,s,d) in $\mathbb{X}$:

$$R_3(H,V,S,D)$$
$$=R_2(H,V,S)+\frac{\lambda_3}{2}\left(D-\bar{D}-\bar{S}t-\tfrac{1}{2}\bar{A}t^2-\mathscr{M}\right)^2,$$

where $\lambda_3>0$. The PDE (32) is too difficult to solve (except for the trivial solution $\mathscr{M}=D-\bar{D}-\bar{S}t-\frac{1}{2}\bar{A}t^2$ which gives $R_3=R_2$). Thus a modification of the forward approach, the *Forwarding modulo L_gV* approach (see [22]), which allows us more flexibility, should be used.

More precisely instead of looking a $\mathscr{M}:\mathbb{X}\to\mathbb{R}$ which solves (32), one has to find two functions $\mathscr{M}:\mathbb{X}\to\mathbb{R}$ Frechet-differentiable and $m:\mathbb{X}\to\mathbb{R}$ continuous such that (see (28))

$$\dot{\mathscr{M}}=S-\bar{S}-\bar{A}t+m\,\frac{\dot{R}_2}{u-\bar{A}}$$
$$=(S-\bar{S}-\bar{A}t)(1+m\lambda_2)-m\lambda_1\int_0^L HV\,\mathrm{d}x.$$

This PDE is satisfied by

$$\mathscr{M}=\left(D-\bar{D}-\bar{S}t-\tfrac{1}{2}\bar{A}t^2\right)(1+m\lambda_2)$$
$$+m\lambda_1\int_0^L\left(\int_0^x(H-\bar{H})(\xi)\,\mathrm{d}\xi\right)\mathrm{d}x$$

and $m\in\mathbb{R}$. Indeed, note that, due to (1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^L\left(\int_0^x(H-\bar{H})\right)\mathrm{d}x\right)$$
$$=-\int_0^L\left(\int_0^x\frac{\partial HV}{\partial x}\right)\mathrm{d}x=-\int_0^L HV. \tag{33}$$

This allows us to introduce the Frechet-differentiable function $R_3:\mathbb{X}\to\mathbb{R}$ defined by, for all $(H,V,S,D)\in\mathbb{X}$,

$$R_3(H,V,S,D)$$
$$=R_2(H,V,S)+\frac{\lambda_3}{2}\left(\lambda_2\left(D-\bar{D}-\bar{S}t-\tfrac{1}{2}\bar{A}t^2\right)\right.$$
$$\left.+\lambda_1\int_0^L\left(\int_0^x(H-\bar{H})(\xi)\,\mathrm{d}\xi\right)\mathrm{d}x\right)^2, \tag{34}$$

where λ_1, λ_2 and λ_3 are three positive constants and R_2 is defined by (27). Note that R_3 is positive and is zero only at the point $(H,V,S,D)=(\bar{H},\bar{V},\bar{S},\bar{D})$. Due to (28) and (33), it follows that

$$\dot{R}_3=\left(-\lambda_1\int_0^L HV\,\mathrm{d}x+\lambda_2(S-\bar{S}-\bar{A}t)\right)$$
$$\times\left(u-\bar{A}+\lambda_2\lambda_3\left(D-\bar{D}-\bar{S}t-\tfrac{1}{2}\bar{A}t^2\right)\right.$$
$$\left.+\lambda_1\lambda_3\int\int(H-\bar{H})\right).$$

Thus, a natural expression for u_3 is

$$u_3(t)=\lambda_1\int_0^L H(x,t)V(x,t)\,\mathrm{d}x-\lambda_2(S(t)-\bar{S}-\bar{A}t)$$
$$-\lambda_2\lambda_3\left(D(t)-\bar{D}-\bar{S}t-\tfrac{1}{2}\bar{A}t^2\right)$$
$$-\lambda_1\lambda_3\int_0^L\left(\int_0^x(H(\xi,t)-\bar{H}(\xi)\,\mathrm{d}\xi)\right)\mathrm{d}x+\bar{A}.$$

This is a full-state feedback. In order to find an output feedback, R_3 has to be rewritten as

$$\dot{R}_3 = (u + \lambda_3 k(X,t))l(X,t), \quad \dot{\overbrace{k(X,t)}} = l(X,t),$$

$$\dot{\overbrace{l(X,t)}} = \tilde{l}(Y,u),$$

where $k : \mathscr{X} \times [0,+\infty) \to \mathbb{R}$, $l : \mathscr{X} \times [0,+\infty) \to \mathbb{R}$ and $\tilde{l} : \mathbb{R}^5 \times \mathbb{R} \to \mathbb{R}$ are three Frechet-differentiable functions defined, for all $(X,Y,u) \in \mathscr{X} \times \mathbb{R}^4 \times \mathbb{R}$, by

$$k(X,t) = \lambda_1 \int_0^L \left(\int_0^x (H - \bar{H})\,\mathrm{d}x \right) + \lambda_2 \left(D - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2 \right),$$

$$l(X,t) = -\lambda_1 \int_0^L HV\,\mathrm{d}x + \lambda_2 (S - \bar{S} - \bar{A}t)$$

and

$$\tilde{l}(Y,u) = \lambda_1 \frac{g}{2}(H(L,t)^2 - H(0,t)^2) + \lambda_1 \,\mathrm{Vol}\, u + \lambda_2 (u - \bar{A}).$$

Let us study Eqs. (1)–(4) and also the variables u, $\dot{u}$ as new states and $\ddot{u}$ as new control law. Note that the relative degree of this system is 2 and, for any Lyapunov function R_1 whose relative degree is 1, by using the forward approach, we need to consider Lyapunov function with relative degree 2. Therefore (see [2, Theorem 4.3] and [1, Theorem 4] in a finite-dimensional context) there does not exist an output feedback stabilizing the fluid–tank state and such that a Lyapunov function has a non-positive time derivative.

The above leads to the following result.

Theorem 3.

- *For any positive gains λ_1, λ_2 and λ_3, there exist a Lyapunov function R_3 for the variables (H,V,S,D) and a time-varying full state feedback $u_3 : \mathscr{C}^1([0,L])^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined, for all $(H,V,S,D) \in \mathscr{C}^1([0,L])^2 \times \mathbb{R}^2$, by*

$$u_3(H,V,S,D) = \lambda_1 \int_0^L HV\,\mathrm{d}x - \lambda_2(S - \bar{S} - \bar{A}t) - \lambda_2\lambda_3 \left(D - \bar{D} - \bar{S}t - \tfrac{1}{2}\bar{A}t^2 \right) - \lambda_1\lambda_3 \int_0^L \left(\int_0^x (H(\xi,t) - \bar{H}(x))\,\mathrm{d}\xi \right) \mathrm{d}x + \bar{A}. \quad (35)$$

  such that R_3 has a non-positive time derivative.
- *From the Lyapunov function R_1, there is neither a Lyapunov function $\tilde{R}_3$ for the fluid–tank state nor a dynamical output feedback $\tilde{u}_3 : \mathbb{R}^5 \to \mathbb{R}$, such that $\tilde{R}_3$ has a non-positive time-derivative.*

We are now in a position to check with numerical simulations that we have a stabilization property.

4. Numerical results

In this section we study two numerical simulations and check that the different problems of stabilization are achieved with our control law.

4.1. Observing stabilization in simulation

The shallow water equations are discretized with the 4 points semi-implicit Preissman scheme (see [16] or [10]), which is a classic finite difference scheme for this type of equations.

It is well known that the discretization of the PDEs introduces artificial damping of the solution. This numerical damping is necessary to have a stable integration of the equation, otherwise the numerical errors would not be damped and the simulation would finally blow up. The downside of this is that, in the simulations, the stabilization is due to both the control law and the numerical scheme. Therefore, it is important to take into account this numerical damping when studying the control law stabilizing performance.

The conservative character of the PDEs can be used to answer the question whether the stabilization is due to the control law and not only to the numerical scheme. In fact, if the control action is zero, $u = 0$, and if $\bar{A} = 0$, the function R_1 must be conserved for all t, according to (15). In simulation, one can observe that the entropy is actually decreasing due to the numerical damping. Hence, if the performance of a stabilizing control law is significantly better than for the zero control action, it can be deduced that the control law actually stabilizes the system (in simulation).

In the case of the Preissman scheme, this damping can be tuned by the Preissman coefficient θ and Courant number C_r. Indeed, it is possible to choose Preissman coefficient θ and Courant number C_r (namely $\theta = 0.5$ and $C_r = 1$) such that the discretization does not introduce numerical damping for the linear equations.

Other integration schemes were also investigated. For instance, the Godunov scheme from [9] leads to analogous results but introduces more numerical damping effect in the system.

For all the simulations, the following parameters have been chosen:

- The spatial and time steps of the scheme are respectively $\triangle x = 0.2$ m and $\triangle t = 0.1$ s.
- We choose the following equilibrium of the fluid–tank system: $\bar{V} = \bar{A} = \bar{S} = \bar{D} = 0$ and $\bar{H} = 1$.
- The controller gains are chosen as follows, $\lambda_1 = 0.1$, $\lambda_2 = 0.3$, $\tilde{\lambda}_2 = 2$, and $\lambda_3 = 0.15$.

Note that for these parameters, the tank length L is 10 m and the Courant number is slightly superior to 1, $C_r = 1.56$.

4.2. Stabilization with output feedback and full-state feedback

The system initial conditions are, for all $x \in [0, L]$,

$$\tilde{D} = 0, \qquad \tilde{S} = 0,$$

$$\tilde{H}(x) = 1 - 2\,\frac{(x - L/2)}{gL}, \qquad \tilde{V}(x) = 0.$$

Three control laws are compared: zero control, $\tilde{u}_2$ defined by (31), and u_3 defined by (35).

Simulation results, with $\theta = 0.53$, are presented in Figs. 2–5 where the following observations are made:

- Figs. 4 and 5 show that $\tilde{u}_2$ and u_3 control laws succeed to stabilize the fluid state in contrast with of the system without control, where oscillations of the fluid remain at the end of the simulation.
- Fig. 2 shows that $\tilde{u}_2$ stabilizes the tank velocity, while u_3 stabilizes also the tank position.

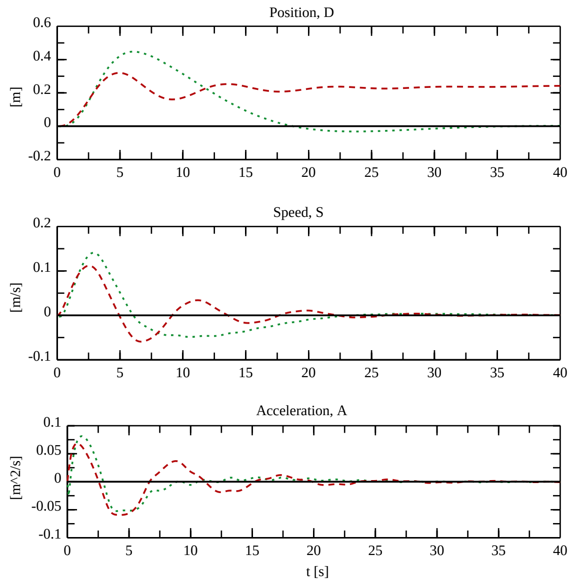

Fig. 2. Tank position D, velocity S and acceleration A. Legend: null control—black solid line, $\tilde{u}_2$—red dashed line, u_3—green dotted line.

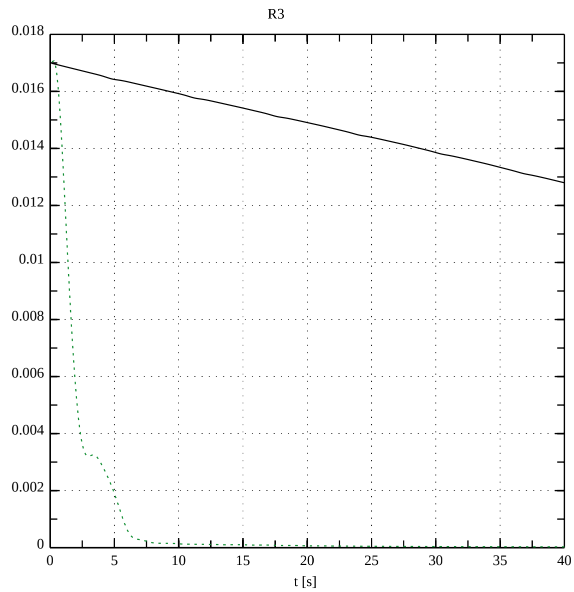

Fig. 3. Lyapunov function R_3. Legend: null control—black solid line, u_3—green dotted line.

- A slight and constant damping of the wave in the zero control simulation can be observed in Figs. 3–5. As discussed in Section 4.1, it is due to the numerical scheme since θ is set to 0.53.

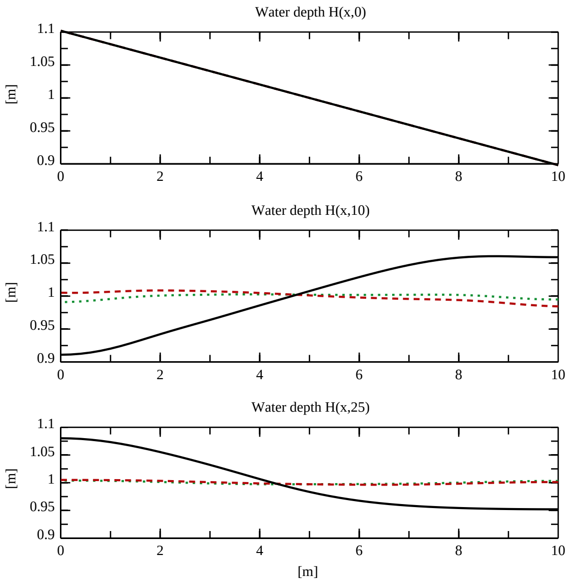

Fig. 4. Fluid level in the tank at time $t=0, 10$ and 25 s. Legend: null control—black solid line, $\tilde{u}_2$—red dashed line, u_3—green dotted line.

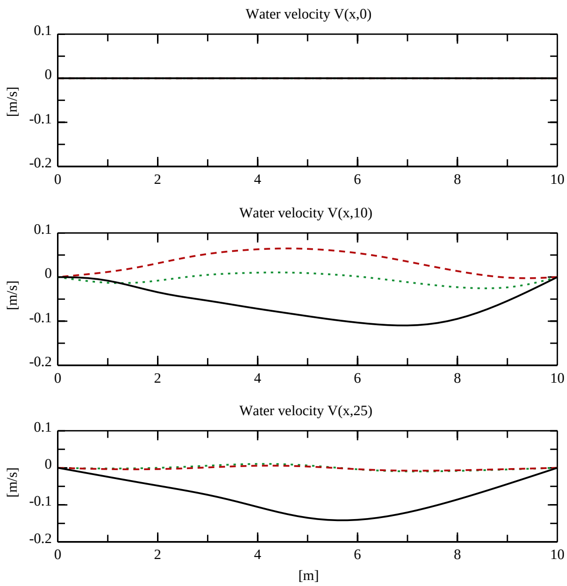

Fig. 5. Fluid velocity in the tank at time $t=0, 10$ and 25 s. Legend: null control—black solid line, $\tilde{u}_2$—red dashed line, u_3—green dotted line.

However the control law u_3 makes the Lyapunov function R_3 quite more decreasing than the numerical damping. See Fig. 3.

- Figs. 4 and 5 show that waves remain after 25 s of simulation for the null control law. Due to the conservative nature of the shallow water equation, these waves should not damp (they do because of the numerical damping) and continue to oscillate, while $\tilde{u}_2$ and u_3 succeed to stabilize in less than 10 s which is very quick in comparison with the simulation of Section 4.3.
- The controller gains were tuned in a trial and error fashion. Gains were chosen to make the fluid stabilization (λ_1) dominant over the tank acceleration, velocity (λ_2) and position (λ_3). Increasing λ_1 will not significantly improve the fluid stabilization, but will introduce a greater deviation on D and S. The λ_2 parameters behaves as one could expect from a proportional controller on the velocity, it lowers the overshoot but introduces oscillations for high gains. The simulation is quite sensitive to the λ_3 gain which introduces oscillations in the positions and velocity.

4.3. Importance of the non-linear terms of the shallow water equations

Note that the shallow water equations linearized are uncontrollable, even locally (see [7] and also the Introduction section). Indeed the functions $H, V : [0, L] \times [0, +\infty) \to \mathbb{R}$ and $D : [0, +\infty) \to \mathbb{R}$ defined by, for all $t \geqslant 0$ and for all x in $[0, L]$,

$$D(t) = 0, \qquad H(x,t) = 0.5 + \sin^2\left(\frac{\pi x}{L}\right),$$

$$V(x,t) = -2\,\frac{\pi}{gL}\,t\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right) \tag{36}$$

are solutions of the linearized equations around $((\bar{H}, \bar{V}, \bar{S}, \bar{D}), \bar{A})$ with $u = 0$. However the nonlinear shallow water equations are locally controllable (see [3]), we check numerically that the nonlinear equations are stabilizable.

To do this, let us consider as initial condition the value of functions (36) at $t=0$ and the feedback (35). In this simulation, θ is chosen as 0.5001, which is very close to the critical value (namely 0.5), in order to minimize the numerical damping. Therefore non-smooth numerical solutions (see Fig. 8) are obtained. Fig. 6 shows that the tank stays very close to the initial

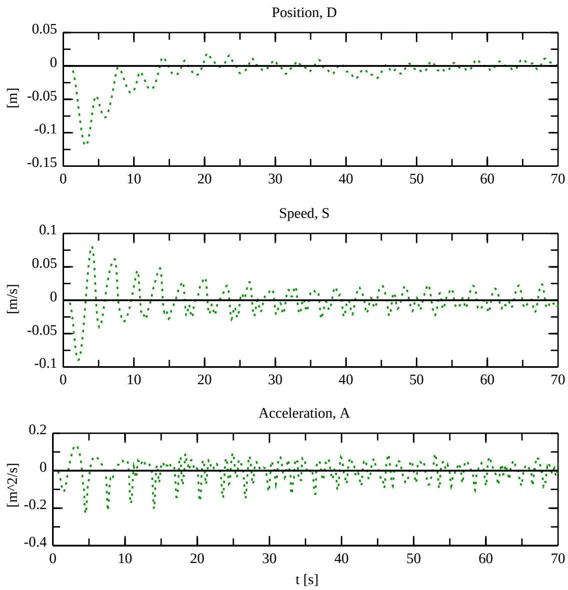

Fig. 6. Tank trajectory. Legend: null control—black solid line, u_3—green dashed line.

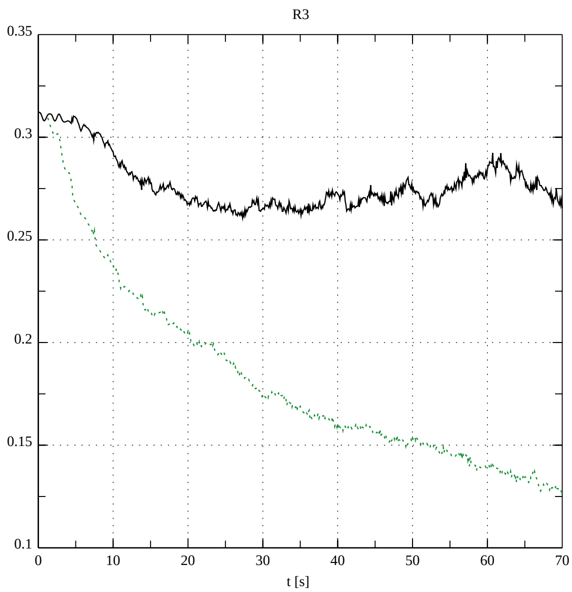

Fig. 7. Lyapunov function R_3. Legend: null control—black solid line, u_3—green dashed line.

position but succeed in stabilizing the fluid speed (see Fig. 9) and the fluid level (see Fig. 8). In Fig. 7, one can see that the numerical damping of the Lyapunov function R_3 which is measured by the simulation with

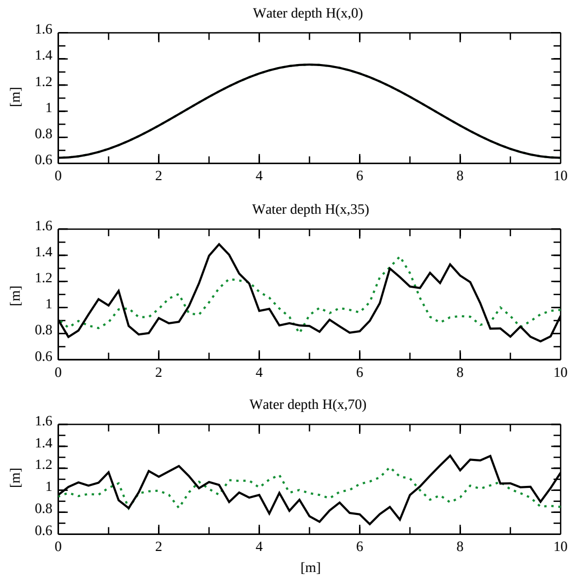

Fig. 8. Fluid depth at 0, 30 and 70 s. Legend: null control—black solid line, u_3—green dashed line.

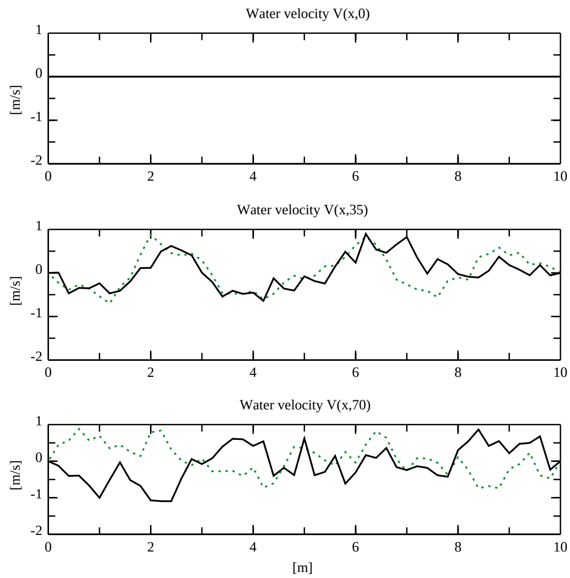

Fig. 9. Fluid velocity at 0, 30, 30 and 70 s. Legend: null control—black solid line, u_3—green dashed line.

the null control, may been neglected in the decreasing rate of the R_3 in the simulation with feedback (35).

Note finally that the stabilization is very slow in comparison with the simulation of Section 4.2 and some waves remain after 70 s. This is due to the fact that only the nonlinear part of the shallow water equations stabilized the system (since the linear approximation is uncontrollable).

5. Conclusion

In this paper we study the problems of the stabilization of a tank containing a fluid by an full-state feedback and by an output feedback. We use a Lyapunov approach to do this. We check numerically this stabilization problems are achieved with our control laws.

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