Stabilization of linear impulsive systems through a nearly-periodic reset

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Abstract

This paper deals with the class of impulsive systems constituted by a continuous-time linear dynamics for all time, except at a sequence of instants. When such a discrete-time occurs, the state undergoes a jump, or more precisely follows a discrete linear dynamics. The sequence of time instants, when a discrete dynamics occurs, is nearly-periodic only, i.e. it is distant from a periodic sequence to an uncertain error. This paper succeeds to state tractable conditions to analyze the stability, and to design reset matrices such that the hybrid system is globally exponentially stable to the origin. The approach is based on a polytopic embedding of the uncertain dynamics. Some examples illustrate the main results.

Key words: Impulsive systems. Reset laws. Stability Analysis. Stabilization. Uncertainty.

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1 Introduction

Dynamic systems subject to both continuous-time and discrete-time dynamics are a class of hybrid systems. The importance of hybrid systems in control systems analysis and design have been growing in the last decades, due mainly to their presence in practical systems and to overcome performance limitations of more classical controllers, i.e., regular linear or nonlinear controllers (see, e.g., [6, 23, 24]). Reset systems represent a particular class of hybrid systems, see [18, 5, 25, 4, 1, 28]. An interesting sub-class of hybrid systems is the class of linear systems with finite state jumps. It consists of continuous-time linear systems for which the state undergoes finite jump discontinuities at some discrete instants of time. Such systems can be regarded as a special case of reset systems in the sense that the reset rule is done through a time condition instead of a state condition. These systems can be named impulsive dynamical linear systems [12]. See also for example [10, 11, 19, 3] and [14]. Their study is motivated by the analysis of sampled-data systems (see [14]) and the design of reset controllers with upper bounded reset intervals [4]. Lyapunov theory framework provides the main tool to test the stability of reset of impulsive systems by employing an adequate Lyapunov function (or a family of Lyapunov functions). The results developed in the current paper are based on the use of adequate quasi-quadratic Lyapunov functions [21, 17, 8].

The paper deals with both the stability analysis and stabilization problems, for linear systems controlled by a reset law. Compared with the previous results, the significant difference is that the interval between two reset instants is supposed to be uncertain, namely, it is defined as the sum of a nominal reset period and an uncertain time-varying term bounded in a given interval. Preliminary results have been presented in [16]. Thus, in order to avoid Zeno phenomena, we assume that the nominal reset period is different from zero. We use our previous expertise with difference inclusions [17] and polynomials of matrices [15] to study the stability and stabilization of impulsive systems. For such systems, quasi-quadratic Lyapunov functions that decrease at reset instants are not only sufficient for exponential stability but also necessary. Sufficient conditions for the existence of such function are proposed and used for designing a reset law. The main conditions for stability and design problems are exhibited in terms of a parametric set of linear matrix inequalities (LMI). Tractable numerical solutions are proposed for these sets of parametric LMI to be expressed as a finite number of conditions using convex embeddings. A numerical construction based on Taylor series development is proposed and illustrated for design problems.

The paper is organized as follows. Section 2 describes the class of systems under consideration and states the problems of stability analysis and reset law design. In Section 3, the results to stability analysis problem are presented while Section 4 is dedicated to reset law design methods. Section 5 presents a way to solve numerically the conditions developed in the previous sections. Numerical examples are detailed in Section 6 to point out the potentiality but also the difficulty of the proposed approach. Finally some concluding remarks and forthcoming issues end the paper.

Notation. For a vector v we denote by $\|v\|$ the euclidean norm. For a matrix M, we denote by $\|M\|$ the induced Euclidean norm, $\|M\| = \max_{\|x\|=1} \|Mx\|$. By $M \succ 0$ or $M \prec 0$ we mean that the square symmetric matrix M is positive or negative definite respectively. We denote the transpose of a matrix M by M^T. By $\mathbf{I}_m$ we denote the $m \times m$ identity matrix. By $\mathbf{0}$ we denote the null matrix of appropriate dimensions. By $\lambda_{max}(M)$ we denote the maximal eigenvalue of a square symmetric matrix. For a given set $\mathcal{S}$, $co(\mathcal{S})$ denotes the convex hull of $\mathcal{S}$. For a given convex polytope $\mathcal{S}$, $vert\,(\mathcal{S})$ denotes the set of vertex of $\mathcal{S}$. We use $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ to denote the set of reals greater or equal to zero.

2 Problem formulation

Consider the following linear reset systems:

$$\dot{x}(t) = A_c x(t),\ \forall t \in \mathbb{R}^+ - \mathcal{T},\ x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

$$x(t) = A_r x(t^-),\ \forall t \in \mathcal{T} \tag{2}$$

where $x \in \mathbb{R}^n$ represents the system state and $t^- = \lim_{\tau \to 0, \tau < 0} t + \tau$. The matrices A_c and A_r are constant matrices of appropriate dimensions. The set

$$\mathcal{T} = \left\{ t_k : t_k \in \mathbb{R}^+, t_k < t_{k+1}, \forall k \in \mathbb{N}, \lim_{k \to \infty} t_k = \infty \right\} \tag{3}$$

represents the set of reset times. We assume that the reset interval has the form

$$t_{k+1} - t_k = \tau_{min} + \delta\tau_k, \tag{4}$$

where τ_{min} represents a nominal reset period and $\delta\tau_k$ an uncertain time-varying term bounded in a compact set $\Delta \subset \mathbb{R}^+$,

$$\delta\tau_k \in \Delta, \forall k \in \mathbb{N}. \tag{5}$$

with $\delta\tau_{max} = \max_{\delta\tau \in \Delta} \delta\tau$ and $\min_{\delta\tau \in \Delta} \delta\tau = 0$. In order to avoid Zeno phenomena we assume that $\tau_{min} > 0$. Note that since $t_{k+1} - t_k \geq \tau_{min}$, accumulation points are not possible. Let $\mathcal{S}(\Delta)$ denote the set of possible sequences $\{\delta\tau_k\}_{k \in \mathbb{N}}$, $\mathcal{S}(\Delta) = \left\{ \{\delta\tau_k\}_{k \in \mathbb{N}},\ \delta\tau_k \in \Delta \right\}$ and $\Theta(\Delta)$, the set of all admissible reset sequences $\mathcal{T} = \{t_k\}_{k \in \mathbb{N}}$ associated to $\mathcal{S}(\Delta)$, i.e.

$$\Theta(\Delta) = \Big\{ \{t_k\}_{k\in\mathbb{N}},\ t_0 = 0,\ t_{k+1} - t_k = \tau_{min} + \delta\tau_k, \\ \{\delta\tau_k\}_{k\in\mathbb{N}} \in \mathcal{S}(\Delta) \Big\}. \tag{6}$$

For any $\mathcal{T} \in \Theta(\Delta)$ the reset system (1) has a unique solution. The value of the solution at time $t \in \mathbb{R}^+$, with the initial condition $x_0 \in \mathbb{R}^n$, and the reset sequence $\mathcal{T} \in \Theta(\Delta)$ is denoted $\varphi_{\mathcal{T}}(t, x_0)$. It can be expressed as $\varphi_{\mathcal{T}}(t, x_0) = \Phi_{\mathcal{T}}(t, 0)x_0$ where the transition matrix $\Phi_{\mathcal{T}}(\cdot,\cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ is defined by the relations

$$\Phi_{\mathcal{T}}(0, 0) = \mathbf{I}_n, \tag{7}$$

$$\frac{\partial}{\partial t}\Phi_{\mathcal{T}}(t, t_k) = A_c\Phi_{\mathcal{T}}(t, t_k), \forall\ t \in [t_k, t_{k+1}), \tag{8}$$

$$\Phi_{\mathcal{T}}(t_{k+1}, t_k) = A_r\Phi_{\mathcal{T}}(t^-_{k+1}, t_k),\ \forall k \in \mathbb{N}. \tag{9}$$

In the previous equation $\Phi_{\mathcal{T}}(t^-_{k+1}, t_k)$ denotes the following limit, computed from (8), $\Phi_{\mathcal{T}}(t^-_{k+1}, t_k) = \lim_{t\to t_{k+1}, t\le t_{k+1}} \Phi_{\mathcal{T}}(t, t_k)$. See also [2] for a similar description of solutions. We may remark that in between two reset instants the flow is linear-time invariant. Therefore, the transition matrix may be computed explicitly $\Phi_{\mathcal{T}}(t, t_k) = e^{(t-t_k)A_c}$, for all $t \in [t_k, t_{k+1})$.

Definition 1 *We say that the equilibrium point $x^* = 0$ of the system* (1), (2) *is Globally Uniformly Exponentially Stable (GUES) with respect to the set of reset sequences $\Theta(\Delta)$ if there exist positive scalars c, λ such that for any $\mathcal{T} \in \Theta(\Delta)$, any $x_0 \in \mathbb{R}^n$, and any $t \geq 0$*

$$\| \varphi_{\mathcal{T}}(t, x_0) \| \leq ce^{-\lambda t} \| x_0 \|. \tag{10}$$

Two complementary problems of stability analysis and synthesis are formulated as follows:

Problem 1. *Assume that the matrices A_c and A_r are given and constant. Provide numerically tractable (LMI) conditions for verifying if the equilibrium point $x^* = 0$ of the reset system* (1), (2) *is globally uniformly exponentially stable.*

Problem 2. *Assume that the matrix A_c is given and constant. Design a reset matrix A_r to guarantee the global uniformly exponential stability of the equilibrium point $x^* = 0$.*

3 Stability analysis

In this section we present conditions for exponential stability, and we give a solution to Problem 1.

3.1 Preliminaries

Before providing LMI stability analysis conditions we present first generic conditions for the stability of the considered class of hybrid systems. Using an underlying discrete-time model at reset times, a necessary and sufficient condition for the exponential stability can be deduced. The analysis is based on the use of a class of quasi-quadratic Lyapunov functions proposed in [17, 21] for linear difference inclusions defined on compact sets (the main result is recalled in Theorem 5 in the Appendix). More precisely, we use functions of the form $V(x) = x^T \mathcal{L}_{[x]} x$, where the Lyapunov matrix, $\mathcal{L}_{[x]}$, depends on the system state in a homogeneous manner, i.e. $\mathcal{L}_{[\cdot]} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathcal{L}_{[x]} = \mathcal{L}_{[x]}^T = \mathcal{L}_{[ax]} \succ 0, \forall x \neq 0,\ a \in \mathbb{R},\ a \neq 0$. The following theorem shows that the decrease of such functions at the reset instants ensures the stability of the reset system.

Theorem 1 *Consider the reset system* (1), (2) *with* $\mathcal{T} = \{t_k\}_{k\in\mathbb{N}} \in \Theta(\Delta)$. *The equilibrium point* $x^* = 0$ *is globally uniformly exponentially stable if and only if there exists a positive definite function* $V : \mathbb{R}^n \to \mathbb{R}^+$ *strictly convex,*

$$V(x) = x^T \mathcal{L}_{[x]} x, \tag{11}$$

homogeneous (of the second order), $\mathcal{L}_{[\cdot]} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathcal{L}_{[x]} = \mathcal{L}_{[x]}^T = \mathcal{L}_{[ax]} \succ 0, \forall x \neq 0,\ a \in \mathbb{R},\ a \neq 0,\ V(0) = 0$, *such that*

$$V(x(t_k)) > V(x(t_{k+1})), \tag{12}$$

for all $x(t_k) \neq 0,\ k \in \mathbb{N}$, *and any of the possible reset sequences* $\mathcal{T} = \{t_k\}_{k\in\mathbb{N}} \in \Theta(\Delta)$.

Proof. First, note that the parameter $\delta\tau_k = t_{k+1} - t_k - \tau_{min}$ belongs to a compact set Δ and that for any sequences $\mathcal{T}$ and any $k \in \mathbb{N}$ the transition matrix $\Phi_{\mathcal{T}}(t_{k+1}, t_k) = A_r e^{A_c(\tau_{min}+\delta\tau_k)}$ is continuous with respect to $\delta\tau_k$. Therefore, at reset times, the solutions of the system (1), (2) are described by a linear difference inclusion

$$x(t_{k+1}) \in \mathcal{F}\left(x(t_k)\right),\ k \in \mathbb{N} \tag{13}$$

where

$$\mathcal{F}\left(x\right) = \left\{A_r e^{A_c(\tau_{min}+\delta\tau)} x, \delta\tau \in \Delta\right\}. \tag{14}$$

Recalling the notion of global uniform exponential stability as defined in Theorem 5 (in the Appendix) for the linear difference inclusion (13) leads to the fact that there are constants $c_d > 0$, $0 < \lambda_d < 1$ such that the relation

$$\|\varphi_{\mathcal{T}}(t_k, x_0)\| \leq c_d \lambda_d^k \|x_0\|, \ \forall k \geq 0 \tag{15}$$

holds for all initial conditions $x_0 \in \mathbb{R}^n$, all $\{t_k\}_{k\in\mathbb{N}} \in \Theta(\Delta)$ if and only if there exists a function $V(\cdot)$ as defined in (11) such that the inequality (12) is verified. We only need to show that exponential decay at reset times is equivalent to exponential decay for all $t \in \mathbb{R}^+$, that is to show that (10) $\Leftrightarrow$ (15). This statement is proven as follows.

Assume that the reset system (1), (2) is exponentially stable. From (10), since $t_k \in [k\tau_{min}, k(\tau_{min} + \delta\tau_{max})]$, one can check that (15) is satisfied with $c_d = c$ and $\lambda_d = e^{-\lambda\tau_{min}}$.

Consider now that the relation (15) is satisfied. Since

$$\varphi_{\mathcal{T}}(t, x_0) = \Phi_{\mathcal{T}}(t, t_k)\varphi_{\mathcal{T}}(t_k, x_0)$$

for all $t \in [t_k, t_{k+1})$, and the transition matrix $\Phi_{\mathcal{T}}(t, t_k) = e^{(t-t_k)A_c}$ is bounded for $t - t_k$ in a bounded interval, i.e. there exists a scalar $\gamma > 0$ such that

$$\|\varphi_{\mathcal{T}}(t, x_0)\| \leq \gamma \|\varphi_{\mathcal{T}}(t_k, x_0)\|, \ \forall \ t \in [t_k, t_{k+1}), \tag{16}$$

with $\gamma = e^{\epsilon(\tau_{min}+\delta\tau_{max})}$ if $\epsilon = \lambda_{max}\left(\dfrac{A_c + A_c^T}{2}\right) \geq 0$ and $\gamma = e^{\epsilon\tau_{min}}$ if $\epsilon < 0$. Therefore, $\|\varphi_{\mathcal{T}}(t, x_0)\| \leq \gamma c_d \left(e^{\ln\lambda_d}\right)^k \|x_0\|$, for all $t \in [t_k, t_{k+1})$. Furthermore, for $t \in [t_k, t_{k+1})$ and $t_{k+1} - t_k - \tau_{min} \in \Delta$, we have $k \in \left[\dfrac{t}{t_{min} + \delta\tau_{max}}, \dfrac{t}{t_{min}}\right]$, which implies that the relation (10) is satisfied with $c = c_d \times \gamma$ and $\lambda = \dfrac{\ln\lambda_d}{t_{min} + \delta\tau_{max}}$. $\square$

3.2 Parametric sets of LMI conditions

The following theorem proposes sufficient stability conditions (as a parametric set of linear matrix inequalities) for closed-loop system (1) with a reset law (2) based on a condition for the existence of quasi-quadratic Lyapunov functions.

Theorem 2 *Consider system (1), (2), with $\mathcal{T}$ in the set of admissible reset sequences $\Theta(\Delta)$ described in (6). Assume that the matrix A_r is given. If there exist matrices $P(\delta\tau)$, $P(\cdot) : \Delta \to \mathbb{R}^{n\times n}$ continuous with respect to $\delta\tau$, $P(\delta\tau) = P^T(\delta\tau) \succ 0$, $\delta\tau \in \Delta$, and G of appropriate dimensions such that the following* *set of linear matrix inequalities*

$$\begin{pmatrix} P(\delta\tau_a) & \left(GA_r e^{A_c(\tau_{min}+\delta\tau_a)}\right)^T \\ GA_r e^{A_c(\tau_{min}+\delta\tau_a)} & G+G^T-P(\delta\tau_b) \end{pmatrix} \succ 0, \tag{17}$$

is satisfied for all $\delta\tau_a, \delta\tau_b \in \Delta$, then there exists a quasi-quadratic Lyapunov function $V(x) = x^T\mathcal{L}_{[x]}x = \max_{\delta\tau\in\Delta} x^T P(\delta\tau)x$ satisfying conditions (12) *in Theorem 1, i.e. the equilibrium point $x^* = 0$ of system* (1), (2) *is globally uniformly exponentially stable.*

Proof. Let us show that if conditions (17) hold then the equilibrium point $x^* = 0$ of system (1), (2) is globally uniformly exponentially stable. Assume that there exist matrices $P(\delta\tau)$ and G satisfying the conditions (17) of Theorem 2 for all $\delta\tau_a, \delta\tau_b \in \Delta$. Since $P(\cdot)$ is continuous with respect to $\delta\tau$ and $\delta\tau$ is defined on a compact set, the image of $P(\cdot)$ with respect to Δ is a compact set. Therefore

$$\sup_{\delta\tau\in\Delta} x^T P(\delta\tau)x = \max_{\delta\tau\in\Delta} x^T P(\delta\tau)x$$

for any $x \in \mathbb{R}^n$. Consider a Lyapunov matrix defined as $\mathcal{L}_{[x]} := P\left(\delta\tau^*(x)\right)$ with $\delta\tau^*(x) = \arg\max_{\delta\tau\in\Delta} x^T P(\delta\tau)x$. Note that for any positive scalar a,

$$\arg\max_{\delta\tau\in\Delta} x^T P(\delta\tau)x = \arg\max_{\delta\tau\in\Delta} a^2 x^T P(\delta\tau)x$$

which implies that $\delta\tau^*(x) = \delta\tau^*(ax)$, that is $\mathcal{L}_{[x]} = \mathcal{L}_{[ax]}$. Furthermore, this shows that $V(x) = x^T\mathcal{L}_{[x]}x$ is an homogeneous function of the second order.

The convexity of V follows from the fact that for a set of convex functions the maximum preserves the convexity property. More explicitly, note that for each $\delta\tau \in \Delta$, the quadratic form $x^T P(\delta\tau)x$ is a convex function, i.e. for any $x, y \in \mathbb{R}^n$ and $\theta \in [0,1]$

$$\left(\theta x + (1-\theta)y\right)^T P(\delta\tau)\left(\theta x + (1-\theta)y\right) \leq \theta x^T P(\delta\tau)x + (1-\theta)y^T P(\delta\tau)y.$$

Using this property it is obtained

$$\begin{aligned} V\left(x\theta + y(1-\theta)\right) &= \max_{\delta\tau\in\Delta}\left(x\theta + y(1-\theta)\right)^T P(\delta\tau)\left(x\theta + y(1-\theta)\right) && (18)\\ &\leq \max_{\delta\tau\in\Delta}\left(\theta x^T P(\delta\tau)x + (1-\theta)y^T P(\delta\tau)y\right) && (19)\\ &\leq \theta \max_{\delta\tau\in\Delta} x^T P(\delta\tau)x + (1-\theta)\max_{\delta\tau\in\Delta} y^T P(\delta\tau)y && (20)\\ &\leq \theta V(x) + (1-\theta)V(y). && (21) \end{aligned}$$

Then V is a convex function.

Therefore candidate quasi-quadratic Lyapunov functions $V(\cdot)$ of the form

$$V(x) = x^T \mathcal{L}_{[x]} x = \max_{\delta\tau \in \Delta} x^T P(\delta\tau) x \tag{22}$$

satisfy the convexity and homogeneity conditions required in Theorem 1.

Using similar arguments as in [9], one can see that multiplying the inequality (17) by

$$\mathbb{T} := [\mathbf{I}_n \quad -(A_r e^{A_c(\tau_{min}+\delta\tau_a)})^T]$$

on the left and by its transpose on the right, the following inequality holds true:

$$\begin{aligned} & P(\delta\tau_a) - (A_r e^{A_c(\tau_{min}+\delta\tau_a)})^T G A_r e^{A_c(\tau_{min}+\delta\tau_a)} \\ & \quad + (A_r e^{A_c(\tau_{min}+\delta\tau_a)})^T G A_r e^{A_c(\tau_{min}+\delta\tau_a)} \\ & \quad - (A_r e^{A_c(\tau_{min}+\delta\tau_a)})^T G^T A_r e^{A_c(\tau_{min}+\delta\tau_a)} \\ & \quad + (A_r e^{A_c(\tau_{min}+\delta\tau_a)})^T G^T A_r e^{A_c(\tau_{min}+\delta\tau_a)} \\ - & \left(e^{A_c(\tau_{min}+\delta\tau_a)}\right)^T A_r^T P(\delta\tau_b) A_r e^{A_c(\tau_{min}+\delta\tau_a)} \succ 0, \end{aligned} \tag{23}$$

which is the same as

$$\left(e^{A_c(\tau_{min}+\delta\tau_a)}\right)^T A_r^T P(\delta\tau_b) A_r e^{A_c(\tau_{min}+\delta\tau_a)} - P(\delta\tau_a) \prec 0, \tag{24}$$

$\forall\ \delta\tau_a, \delta\tau_b \in \Delta$. For any $x \neq 0$, $\delta\tau_a \in \Delta$

$$x^T P(\delta\tau_a) x \leq \max_{\delta\tau \in \Delta} x^T P(\delta\tau) x. \tag{25}$$

Then, for any $\{t_k\}_{k\in\mathbb{N}} \in \Theta(\Delta)$,

$$x(t_{k+1}) = A_r e^{(t_{k+1}-t_k)A_c} x(t_k) = A_r e^{(\tau_{min}+\delta\tau_k)A_c} x(t_k), \tag{26}$$

with $\delta\tau_k \in \Delta$. From (24), (25) and (26) the following relation holds for any $x(t_k) \neq 0$

$$x^T(t_{k+1}) P(\delta\tau_b) x(t_{k+1}) < \max_{\delta\tau \in \Delta} x^T(t_k) P(\delta\tau) x(t_k) \tag{27}$$

for any $\delta\tau_b \in \Delta$. Note that

$$\delta\tau^*(x(t_{k+1})) = \arg\max_{\delta\tau \in \Delta} x^T(t_{k+1}) P(\delta\tau) x(t_{k+1}). \tag{28}$$

The relation (27) holds for any $\delta\tau_b \in \Delta$, therefore it holds also for $\delta\tau_b = \delta\tau^*(x(t_{k+1}))$. Using the definition of $V(x)$ in (22) this leads to

$$V(x(t_{k+1})) < V(x(t_k)), \tag{29}$$

for all $x(t_k) \neq 0$, therefore there exists a function $V(x)$ satisfying the conditions in Theorem 1. $\square$

Remark 1 *Note that the presented stability conditions encompass possible solutions based on the existence of a quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}^+$, $V(x) = x^T P x$, which is strictly decreasing at reset times. This case may be obtained by setting a constant Lyapunov matrix $P(\delta\tau) = P$ in (17) for all $\delta\tau \in \Delta$.*

4 Reset Stabilization

In this section we present a solution to Problem 2. Conditions for the design of a reset matrix A_r, as defined in (2), that stabilizes system (1) are presented. The case when all variables may be reset is trivial ($A_r = \mathbf{0}$ is the trivial solution). For this reason the approach is developed for the case where the reset is constrained, in the sense that only some of the system variables may be subject to jumps. To clarify our hypothesis for the stabilization problem, we denote $x_p \in \mathbb{R}^{n_p}$ the continuous state variables and $\eta \in \mathbb{R}^{n_\eta}$ the variables that may be subject to the reset law. Obviously, the decomposition of the system state should satisfy $n_p + n_\eta = n$. Without loss of generality we may assume that the state of system (1) is decomposed in the form $x = (x_p^T \;\; \eta^T)^T$. We are interested in reset laws of the form:

$$\eta(t) = R_x C x_p(t^-) + R_\eta \eta(t^-) \tag{30}$$

where C is a known constant matrix of appropriate dimensions and R_x, R_η are to-be-designed matrices. The constraint (30) can be expressed in the form (2) using the notation

$$A_r = \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ R_x C & R_\eta \end{pmatrix}. \tag{31}$$

In the following theorem it is shown how the approach in Theorem 2 can be used in order to derive a reset matrix with the particular structure (31).

Theorem 3 *Consider system* (1), *the reset control* (30) *with the associated reset matrix* (31). *Moreover, consider that the sequence $\mathcal{T}$ takes values in the set of admissible reset sequences $\Theta(\Delta)$ described in* (6). *If there exist matrices $P(\delta\tau)$, $P(\cdot) : \Delta \to \mathbb{R}^{n\times n}$ continuous with respect to $\delta\tau$, $P(\delta\tau) = P^T(\delta\tau) \succ 0$, $\delta\tau \in \Delta$, and matrices $\bar{G}$, W_η, W_x of appropriate dimensions such that the set of linear matrix inequalities*

$$\begin{pmatrix} P(\delta\tau_a) & \left(e^{A_c(\tau_{min}+\delta\tau_a)}\right)^T W^T \\ W e^{A_c(\tau_{min}+\delta\tau_a)} & G + G^T - P(\delta\tau_b) \end{pmatrix} \succ 0, \tag{32}$$

is satisfied for all $\delta\tau_a, \delta\tau_b \in \Delta$*, with*

$$G = \begin{pmatrix} \boldsymbol{I}_{n_p} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{G} \end{pmatrix}, \; W = \begin{pmatrix} \boldsymbol{I}_{n_p} & \boldsymbol{0} \\ W_x C & W_\eta \end{pmatrix}, \tag{33}$$

then the equilibrium point $x^\star = 0$ *is globally uniformly exponentially stable for the closed-loop system* (1), (30), (31) *with* $R_x = \bar{G}^{-1} W_x$ *and* $R_\eta = \bar{G}^{-1} W_\eta$*.*

Proof. Assume that the exist symmetric positive definite matrices $P(\delta\tau_a)$, $\delta\tau_a \in \Delta$, and matrices G and W as in (33) such that conditions (32) are satisfied. Then the block $G + G^T - P(\delta\tau_b)$ is also positive definite which implies that $G + G^T \succ 0$, i.e. G is invertible. Consider the notation $A_r = G^{-1}W$. Using the change of variables $W = GA_r$, the set of inequalities (32) leads to

$$\begin{pmatrix} P(\delta\tau_a) & \left(e^{A_c(\tau_{min}+\delta\tau_a}\right)^T A_r^T G^T \\ GA_r e^{A_c(\tau_{min}+\delta\tau_a)} & G + G^T - P(\delta\tau_b) \end{pmatrix} \succ 0, \tag{34}$$

$\forall\ \delta\tau_a, \delta\tau_b \in \Delta$ which allows to prove the exponential stability of the system (1), (2) with $G^{-1}W$ as a reset matrix (as an application of Theorem 2).

Note that for G and W with the structure given in (33), the matrix A_r has the following form

$$A_r = G^{-1}W = \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ \mathbf{0} & \bar{G}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ W_x C & W_\eta \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n_p} & \mathbf{0} \\ \bar{G}^{-1} W_x C & \bar{G}^{-1} W_\eta \end{pmatrix}, \tag{35}$$

which is (31) with $R_x = \bar{G}^{-1} W_x$ and $R_\eta = \bar{G}^{-1} W_\eta$. $\square$

Remark 2 *Conditions* (17), (32) *lead to a parametric set of linear matrix inequalities, since they depend on the different values of* $\delta\tau_a, \delta\tau_b$ *in the compact set* Δ*. If the cardinal of* Δ *is finite, the set of LMIs to be solved is finite. This is not the case when* Δ *represents an interval. Tractable numerical conditions for approximating this parametric set of conditions when* Δ *is an interval are presented in the following section.*

5 Numerical evaluation

In this section it is shown how the parametric set of LMIs presented previously, namely (17), (32) can be approximated by a finite number of conditions using polytopic embeddings.

In order to obtain a finite number of LMIs from (17) or (32) we have to deal with the *exponential uncertainty* $e^{A_c\delta\tau_k}$ [15, 13, 20] that appears in the conditions. Note that the uncertain matrices $e^{A_c\delta\tau_k}$ are continuous with respect to $\delta\tau_k$ and that $\delta\tau_k$ belongs to a bounded set Δ. Then, for $\delta\tau_a \in \Delta$, the matrix $e^{A_c\delta\tau_a}$ describes a compact subset of $\mathbb{R}^{n\times n}$:

$$\mathcal{E} = \left\{X \in \mathbb{R}^{n\times n} : X = e^{A_c\delta\tau_a}, \delta\tau_a \in \Delta\right\}. \tag{36}$$

The basic idea is to embed the set of matrices $\mathcal{E}$ into a polytopic set $\mathcal{Z}$, i.e. to find a set of N matrices Z_i such that

$$\mathcal{E} \subset \mathcal{Z} = co\{Z_1, Z_2, \ldots, Z_N\}. \tag{37}$$

This implies that for all $\delta\tau_a \in \Delta$ there exists a set of scalars $\mu_i(\delta\tau_a) \in [0,1],\ i = 1,\ldots,N$, such that

$$e^{A_c\delta\tau_a} = \sum_{i=1}^{N}\mu_i(\delta\tau_a)Z_i,\ \sum_{i=1}^{N}\mu_i(\delta\tau_a) = 1. \tag{38}$$

In order to provide some insights about how a polytopic embedding can be approximated numerically, we recall briefly the method proposed in [15] for the case when Δ is a bounded interval $[0, \delta\tau_{max}]$.

Consider an h-order Taylor expansion of the matrix exponential $e^{A_c\delta\tau}$:

$$e^{\delta\tau_a A_c} \approx \left(\sum_{l=0}^{h}\frac{{A_c}^l}{l!}\delta\tau_a^l\right). \tag{39}$$

Then we may construct a convex polytope by considering the terms $\delta\tau^l, l = 1,2,\ldots,h$, as independent parameters. The h-order Taylor approximation can be embedded in a matrix hypercube with 2^h vertices. However, one can exploit the relation between the different parameters to construct a convex polytope inside the hypercube. This approach is mathematically formalized in Lemma 6 (given in the Appendix). Applying Lemma 6 to the polynomial form (39) for $\delta\tau_a \in \Delta$, leads to a polytope with $N = h+1$ vertices given by:

$$
\begin{aligned}
Z_1 &= \mathbf{I}_n, \\
Z_2 &= \delta\tau_{max} A_c + \mathbf{I}_n, \\
Z_3 &= \delta\tau_{max}^2 \frac{{A_c}^2}{2!} + \delta\tau_{max} A_c + \mathbf{I}_n, \\
&\vdots \\
Z_{h+1} &= \delta\tau_{max}^h \frac{{A_c}^h}{h!} + \delta\tau_{max}^{h-1} \frac{{A_c}^{h-1}}{(h-1)!} + \ldots + \delta\tau_{max}^2 \frac{{A_c}^2}{2!} + \delta\tau_{max} A_c + \mathbf{I}_n
\end{aligned}
$$

which embeds the polynomial approximation (39). For evaluating the approximation error induced by the use of Taylor series we mention the survey paper of Moler and Loan [22]. If ε is some prescribed error tolerance, the simplest method consists in choosing h large enough so that $\|A_c\|\,\delta\tau_{max}/(h+2) < 1$ and

$$
\left\| e^{\delta\tau_a A_c} - \left(\sum_{l=0}^{h} \frac{{A_c}^l}{l!} \delta\tau_a^l \right) \right\| \leq \frac{\|A_c\|^{h+1}\,\delta\tau_{max}^{h+1}}{(h+1)!} \frac{1}{1 - \|A_c\|\,\delta\tau_{max}/(h+2)} \leq \varepsilon. \quad (40)
$$

Note that the accuracy of the analysis may also be improved by dividing the interval of analysis in several subintervals and applying the embedding procedure locally. Assuming that we use a division into $m-1$ subintervals of the form $\mathcal{I}_j = [\delta\tau_{max} j/m, \delta\tau_{max}(j+1)/m],\ j = 0, \ldots, m-1$, we only need to compute the vertex $\{Z_i\}_{i=1}^{h+1}$ for $\delta\tau \in [0, \delta\tau_{max}/m]$. Next, we use the fact that $\forall \delta\tau \in \mathcal{I}_j,\ e^{A_c \delta\tau} = e^{A_c \delta\tau_{max} j/m} e^{A_c \rho}$ with $\rho = \delta\tau - \delta\tau_{max} j/m \in [0, \delta\tau_{max}/m]$, to derive a local polytopic approximation where the set of vertices is given by $\mathcal{Z}_j^{loc} = \left\{ e^{A_c \delta\tau_{max} j/m} Z_i \right\}_{i=1}^{h+1}$. Then, the extreme points of the global polytope can be determined among the vertices of each local embedding $\mathcal{Z} = vert\left(\cup_{j=1}^{m-1} co\left(\mathcal{Z}_j^{loc} \right) \right)$ using classical convex hull algorithms.

In the following subsection we illustrate the use of polytopic embedding methods for reset matrices design.

5.2 Stabilization based on polytopic sets

Here we show how the parametric set of linear matrix inequalities, such as (17), (32), can be reduced to a finite number of linear matrix inequalities. The approach is illustrated for the condition (32). The conditions (17) proposed in Theorem 2 can be treated in a similar manner. The reset matrix design procedure based on polytopic set (37) is formulated in the following result.

Theorem 4 *Consider system* (1) *with* $\mathcal{T} \in \Theta(\Delta)$. *Moreover, consider the polytopic set* (37) *with* N *vertices. If there exist symmetric positive definite matrices* $P_i,\ i = 1, \ldots, N$, *and matrices* G, W *as described in* (33) *such that* *the following set of linear matrix inequalities*

$$\begin{pmatrix} P_i & Z_i^T \left(e^{A_c \tau_{min}}\right)^T W^T \\ W e^{A_c \tau_{min}} Z_i & G + G^T - P_j \end{pmatrix} \succ 0, \tag{41}$$

is satisfied for all $i, j = 1, \ldots, N$, then the equilibrium point $x^\star = 0$ is globally uniformly exponentially stable for the closed-loop system (1), (2), (31) *with $A_r = G^{-1}W$.*

Proof. Assume that there exist a set of matrices $P_i,\ i = 1, \ldots, N$, and matrices G and W such that the set of inequalities (41) hold true. Then the condition

$$\begin{pmatrix} \sum_{i=1}^N \mu_i P_i & \sum_{i=1}^N \mu_i Z_i^T \left(e^{A_c \tau_{min}}\right) W^T \\ \sum_{i=1}^N \mu_i W e^{A_c \tau_{min}} Z_i & G + G^T - \sum_{i=j}^N \mu_j P_j \end{pmatrix} \succ 0, \tag{42}$$

is satisfied for any set of scalars $\mu_i, \mu_j \in [0, 1],\ i, j = 1, \ldots, N$, such that $\sum_{i=1}^N \mu_i = \sum_{j=1}^N \mu_j = 1$. Note that $e^{A_c \delta\tau_a} \in \mathcal{Z}$ for all $\delta\tau_a \in [0, \delta\tau_{max}]$. Then the previous condition implies the existence of matrices $P(\delta\tau_a)$ such that the conditions (32) hold with

$$P(\delta\tau_a) = \sum_{i=1}^N \mu_i(\delta\tau_a) P_i,\ P(\delta\tau_b) = \sum_{j=1}^N \mu_j(\delta\tau_b) P_j, \tag{43}$$

where $\mu_i(\delta\tau_a), \mu_j(\delta\tau_b) \in [0, 1],\ i, j = 1, \ldots, N$, represent the barycentric coordinates of $e^{A_c \delta\tau_a}$ and $e^{A_c \delta\tau_b}$ in the polytope $\mathcal{Z}$. $\square$

Remark 3 *When polytopic embeddings are used, the underlying quasi-quadratic Lyapunov function* (22) *takes the form $V(x) = \max_{P \in \mathcal{P}} x^T P x$, with $\mathcal{P} = co\{P_1, P_2, \ldots, P_N\}$. Using classical arguments for convex functions one can prove that in this case $V(x) = \max_{i=1,\ldots,N} x^T P_i x$.*

Remark 4 *The result in Theorem 4 is generic and holds for any polytopic embedding* (37) *of the matrix exponential. Alternatively, for numerical implementations, we could use a polytopic development based on Jordan Forms, such as in [7], or on the Cayley-Hamilton Theorem [13]. Note that to deal with the possible numerical approximations errors it is of interest to extend the proposed stabilization conditions to robust stabilization. This issue may be considered in the future from the L_2 or ISS stabilization frameworks. Further extensions are possible for systems that have both switching dynamics and resets. This may be possible for the case when such hybrid events have a synchronous occurrence, by addressing the problem from the point of view of switched polytopic systems (see for example [27]).*

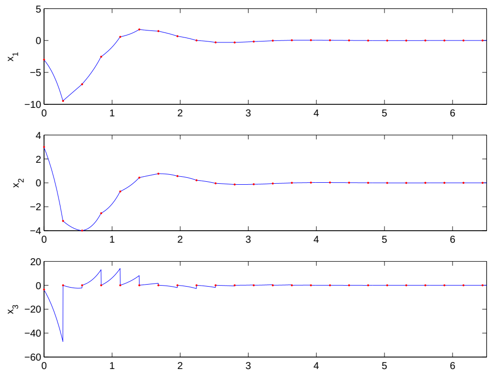

Fig. 1. Illustration of system evolution with a periodic reset interval $\tau^a = 0.28$.

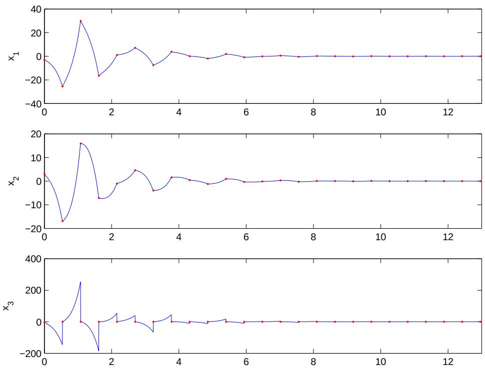

Fig. 2. Illustration of system evolution with a periodic reset interval $\tau^b = 0.54$.

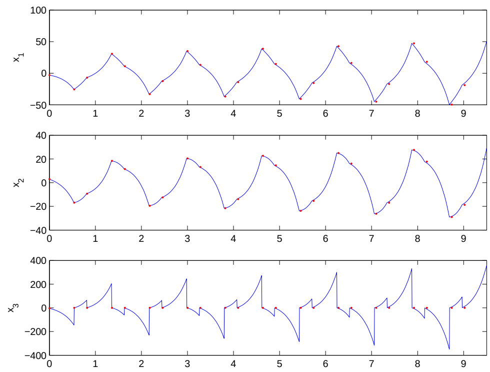

Fig. 3. Illustration of unstable system behavior when the reset interval switches among two values $\tau_{min} \in \{0.28, 0.54\}$.

6 Illustrative Examples

Example 1. Consider a reset system (1), (2) with

$$A_c = \begin{pmatrix} 0 & -3 & 1 \\ 1.4 & -2.6 & 0.6 \\ 8.4 & -18.6 & 4.6 \end{pmatrix}, \quad A_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{44}$$

The matrix A_c has the eigenvalues

$$\lambda_1 = -1.44, \lambda_2 = 3.44, \lambda_3 = 0. \tag{45}$$

For this system the matrix $A_r e^{\tau_{min} A_c}$ is Schur for $\tau_{min} \in [0, 0.58]$. This implies that the system is stable if the reset occurs periodically, with constant reset interval τ_{min} in $[0, 0.58]$ and $\delta\tau_k = 0$. However, variation of the reset interval may induce instability. While the system is stable for constant reset intervals $\tau^a = 0.28$ and $\tau^b = 0.54$ (see Fig. 1 and Fig. 2, respectively), the matrix $A_r e^{\tau^a A_c} A_r e^{\tau^b A_c}$ has eigenvalues outside the unit circle. Therefore, resetting the system with the pattern $\tau^a \to \tau^b \to \tau^a \to \tau^b \ldots$, leads to an unstable behavior. This phenomena is illustrated in Fig. 3.

Consider now $\tau_{min} = 0.1$ and $\delta\tau_{max} = 0.2$. In order to illustrate graphically the construction of the polytopic embedding $\mathcal{Z}$ in (37), consider the Jordan normal form of A_c:

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3. \end{pmatrix} \tag{46}$$

with $\lambda_i = 1, 2, 3$, given in (45). For this particular case, the exponential uncertainty $e^{\delta\tau_k J}$ has the form

$$e^{\delta\tau_k J} = T \begin{pmatrix} \phi_1(\delta\tau_k) & 0 & 0 \\ 0 & \phi_2(\delta\tau_k) & 0 \\ 0 & 0 & 1 \end{pmatrix} T^{-1} \tag{47}$$

with two uncertain scalar parameters $\phi_i(\delta\tau_k) = e^{\delta\tau_k \lambda_i}$, $i = 1, 2$, and $\delta\tau_k \in [0, 0.2]$ and T an invertible matrix. In Fig. 4, we illustrate the curve described by the exponential uncertainty in the $\phi_1(\delta\tau_k) - \phi_2(\delta\tau_k)$ plane as well as the obtained polytopic embedding based on a 8th order Taylor series expansion (in dark gray). Note that the Jordan form is used here only for providing a graphical illustration of the polytopic embedding methodology. The LMI procedure can be applied by using directly the Taylor series expansion in the original coordinates, without using the transformation matrix T.

The polytopic embedding procedure can be refined by dividing the interval $[0, \delta\tau_{max}]$ in several subintervals, and locally applying the Taylor method.

An illustration based on 5 subintervals is given in Fig. 4 (light gray). For the considered example this leads to a Taylor series approximation error less than $\varepsilon = 10^{-10}$. Using such an embedding and adapting the conditions in Theorem 2 similarly to Theorem 4, we can provide an estimate of the domain Δ for which the system is stable in spite of variations of the reset interval. For

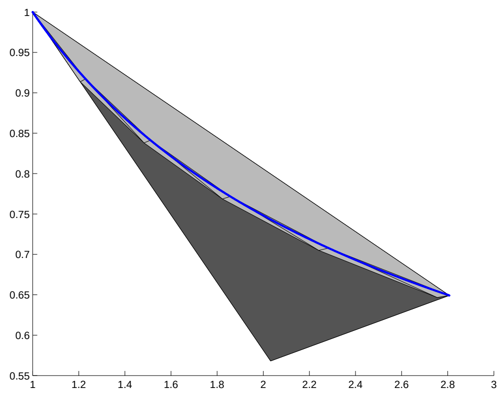

Fig. 4. Representation of exponential uncertainty and of construction of convex polytopes in the $\phi_1(\delta\tau_a) - \phi_2(\delta\tau_b)$ plane, for $\tau_{min} = 0.1$, $\delta\tau_a \in [0, 0.2]$. The polytopic embedding in dark gray is obtained based on a $8th$ order Taylor approximation. In light gray we represent a polytopic embedding obtained based on a $8th$ order Taylor approximation applied on 5 subintervals.

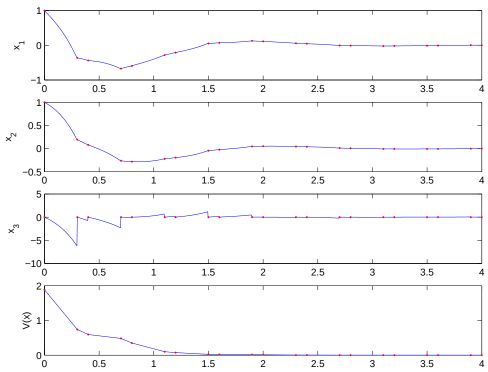

Fig. 5. Example of system evolution for the Example 1 with $\tau_{min} = 0.1$, $\delta\tau \in [0, 0.2]$, based on an arbitrary sequence of reset intervals.

$\tau_{min} = 0.1$, Theorem 4 indicates stability with variations of the reset interval of a maximum amplitude $\delta\tau_{max} = 0.2$. An example of system evolution according to a random variation of the reset interval is given in Fig. 5. The first three graphics correspond to the evolution of state variables (with discontinuities in the $3rd$ system state) while the last one illustrates the convergence of the obtained Lyapunov function (illustrating stability).

Example 2. Next, a comparison between the use of a common quadratic and of a quasi-quadratic Lyapunov function is given. Consider a reset system (1), (2) with

$$A_c = \begin{pmatrix} -1.5 & -2.5 & 5 \\ -0.4 & 2.5 & 2 \\ -0.5 & 0.75 & 2 \end{pmatrix}, \; A_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.4 & -2.31 & 0 \end{pmatrix} \tag{48}$$

with $\tau_{min} = 0.1$ and $\Delta = \{0, 0.29\}$. For this example it is not possible to show

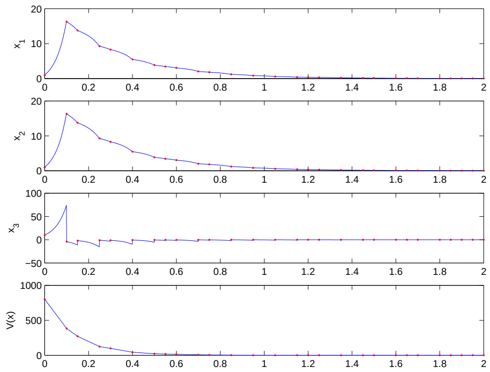

Fig. 6. Example of system evolution for the Example 3, based on an arbitrary sequence of reset intervals.

stability using LMIs based on a common quadratic Lyapunov function $V(x) = x^T Px$. However, using quasi-quadratic Lyapunov functions (11) and the set of LMIs (17) it is possible to prove stability. This example shows the interest of using more advanced Lyapunov functions for reducing the conservatism in stability analysis.

Example 3. Consider a continuous-time system (1) with a partial reset (30), where

$$A_c = \begin{pmatrix} -2 & 0 & 5 \\ 0 & -1 & 5 \\ 0 & -1/5 & 20 \end{pmatrix}, \tag{49}$$

$C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $n_p = 2$, $n_\eta = 1$, $\tau_{min} = 0.05$ and $\delta\tau_k \in [0, 0.5]$. Note that the base matrix A_c is unstable. We use a polytopic embedding similar to the one in Example 1, based on a 8th order Taylor series expansion applied on 5 subintervals. The Taylor series approximation error is less than $\varepsilon = 10^{-10}$. Using the set of LMIs in Theorem 4 with structured matrices G and W as in (33), the reset gains $R_x = -0.16$, $R_\eta = -0.02$ are obtained. An example of system evolution is presented in Figure 6. The first three plots are the state components, whereas the last one is the time-evolution of the Lyapunov function.

7 Conclusion

Focusing on linear impulsive systems, we stated tractable conditions to analyze the exponential stability. It is assumed that the jump instants are not periodic but only nearly-periodic, i.e. with an uncertain distance to a periodic sequence. It is shown that quasi-quadratic Lyapunov functions that decrease at reset instants are not only sufficient for exponential stability but also necessary. Sufficient LMI conditions are derived to compute reset matrices such that the hybrid system is global exponentially stable to the origin based on the existence of such quasi-quadratic Lyapunov functions. Numerical comparisons of conservatism reduction with respect to common quadratic Lyapunov functions are presented.

This paper lets many questions open. The first one may be the generalization of this work to nonlinear systems. Also the performance issues may be considered either for linear impulsive systems or for nonlinear ones [26]. One criterion to be optimized may be the rejection of perturbations and the gain perturbations/output to be made as low as possible. The connection with the Input-to-State Stability (ISS) of impulsive systems as in [14] may be fruitful.

Appendix

The following results will be useful for the proofs of our theorems:

Theorem 5 *(Adapted from Theorem 1 in [17]) Consider $\mathcal{A}$, a compact set, $\Lambda(\cdot) : \mathcal{A} \to \mathbb{R}^{n\times n}$, a continuous function over $\mathcal{A}$ and the following Linear Difference Inclusion*

$$z_{k+1} \in \mathcal{H}(z_k), \mathcal{H}(z) = \{\Lambda(\alpha)z, \alpha \in \mathcal{A}\}, \tag{50}$$

$k \in \mathbb{N},\ z_0 \in \mathbb{R}^n$. *The following statements are equivalent:*

1) The equilibrium point $z^* = 0$ of (50) is globally uniformly exponentially stable, i.e. there are constants $c_d > 0$, $0 < \lambda_d < 1$ such that $\|z_k\| \leq c_d \lambda_d^k \|z_0\|$, $\forall k \geq 0$, holds for all initial conditions $z_0 \in \mathbb{R}^n$, all $k \in \mathbb{N}$.

2) There exists a positive definite quasi-quadratic function $V : \mathbb{R}^n \to \mathbb{R}^+$ strictly convex, homogeneous (of the second order), $V(z) = z^T \mathcal{L}_{[z]} z$, $\mathcal{L}_{[\cdot]} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\mathcal{L}_{[z]} = \mathcal{L}_{[z]}^T = \mathcal{L}_{[az]} \succ 0$, $\forall z \neq 0, a \in \mathbb{R}, a \neq 0$, $V(0) = 0$ such that the following relation is satisfied : $V(z) - \max_{\alpha\in\mathcal{A}} V(\Lambda(\alpha)z) > 0, \forall z \neq 0$.

Lemma 6 *[15] Consider a polynomial matrix*

$$L(\rho) = L_0 + \rho L_1 + \rho^2 L_2 + \ldots + \rho^p L_p, \tag{51}$$

where the parameter $\rho \in \mathbb{R}$ and $L_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, p$. For each upper bound $\bar{\rho} \geq 0$ on ρ there exist matrices $U_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, p+1$, such that the following property holds:
For all $\rho \in [0, \bar{\rho}]$ there exist parameters $\mu_i(\rho)$, $i = 1, \ldots, p+1$, with $\sum_{i=1}^{p+1} \mu_i(\rho) = 1$, and $\mu_i(\rho) \geq 0$, $i = 1, \ldots, p+1$, such that $L(\rho) = \sum_{i=1}^{p+1} \mu_i(\rho) U_i$. In particular, U_i, $i = 1, \ldots, p$, can be chosen as $U_1 = L_0$, $U_2 = \overline{\rho} L_1 + L_0$, $U_3 = \overline{\rho}^2 L_2 + \overline{\rho} L_1 + L_0, \ldots, U_{p+1} = \overline{\rho}^p L_p + \overline{\rho}^{p-1} L_{p-1} + \ldots + \overline{\rho}^2 L_2 + \overline{\rho} L_1 + L_0$.

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