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State observation for heterogeneous quasilinear traffic flow system with disturbances

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Abstract

This paper studies state observation for a heterogeneous quasilinear traffic flow system with disturbances at the inlet of a considered road section. Based on the backstepping method, an observer is designed for the quasilinear traffic flow system with only the boundary measurements at the inlet of the considered road section. The observer is constructed by duplicating the quasilinear system and adding the output injection terms to the partial differential equations and boundary conditions. Making use of the backstepping transformation, the injection gains of the observer system are derived by the computation of kernel equations, which are obtained by mapping the error system into an integral input-to-state stable target system. The applicability of the observer for the design of an output feedback controller stabilizing the quasilinear system is discussed. Finally the assumptions of the design of the observer are numerically checked on a realistic congested traffic scenario.

Keywords Heterogeneous quasilinear traffic flow system · Disturbances · Integral input-to-state stable · Observer · Backstepping

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1 Introduction

Input-to-state stability (ISS) is known as one of the central notions in the control theory of dynamical systems since the seminal paper [25]. Indeed it allows the description of the disturbance effect on the state of nonlinear finite-dimensional systems and provides some design methods of dynamical output feedback laws (see the survey [26]). A closely related notion is the notion of integral input-to-state stability (iISS) as considered, e.g., in [18]. Roughly speaking, this property estimates the impact of the integral of the disturbances to the state norm. It is very well developed for nonlinear finite-dimensional systems and networks (see e.g., [11]). The theory has been recently generalized for infinite-dimensional systems in [14].

For the infinite-dimensional system, the observers can be derived directly by using the backstepping approach which is the extension of Volterra integral transformations and was initially introduced by [15, 16] and [24] for hyperbolic partial differential equations (PDEs). Many works have been recently developed to use the backstepping method for the observer design of macroscopic traffic flow systems. As an example, consider [28] where a boundary observer is designed for a nonlinear ARZ traffic flow system.

Different macroscopic traffic flow models are possible. The first-order Lighthill-Whitham-Richards (LWR) model (see [17] and [23]) represents density-velocity relation in equilibrium and fails to model stop-and-go traffic. The second-order Payne-Whitham (PW) model (see [22] and [27]) consists of momentum equation and conservation law, and it is a nonlinear second-order deviation from density-velocity equilibrium. The second-order Aw-Rascle-Zhang (ARZ) model (see [1] and [30]) is derived from the combination of these two models (LWR model and PW model) through suitable definition and coefficients. Several equilibriums, frequent lane changes, overtaking, and platoon dispersion probably happen in congested traffic on account of the interplay between different types of vehicles and drivers [21]. Besides the homogeneous models as above, there are many macroscopic traffic flow models for heterogeneous traffic. Paper [6] studies a two-type vehicle heterogeneous traffic model to acquire overtaking and creeping traffic flows. In [19], the extended macroscopic N-type Aw-Rascle (AR) traffic model is used for heterogeneous traffic by using area occupancy. In [20], a continuum multi-type traffic model is introduced on the basis of a three-dimensional flow-concentration surface. An n-population generalization of the Lighthill-Whitham-Richards traffic flow model is presented in [3].

In this paper, we exploit the iISS notion to a quasilinear infinite-dimensional system with boundary control and perturbation. More precisely, we design an observer for a quasilinear hyperbolic system so that the estimation error system is locally asymptotic stable. Moreover, we consider a perturbed case, where the perturbation is on one boundary and we derive an ISS property. The obtained result is motivated and explicitly applied to the extended macroscopic N-type AR quasilinear traffic flow model. By doing so, the designed observer guarantees the accurate observation (no error induced by linearization) of the traffic state under the condition that the initial observation is not too far from the actual state. Exploiting the structure of the observer suggested in [29], this paper gives, for the first time, the theoretical proof of local H^2 stability of the quasilinear observer system, by the local H^2 Lyapunov analysis of the error system.

This paper is organized as follows. Section 2 contains the preliminary for the design of the quasilinear observer. The collocated observer is designed, and the theoretical proof of iISS for the target system of error system is done in Sect. 3. In Sect. 4, numerical computations are presented to check the sufficient condition for the observer design using a realistic traffic scenario. Finally, Sect. 5 contains some concluding remarks.

Notation. $\max(S)$ is the maximum value of all the elements in S, if S is a set. $\partial_t f$ and $\partial_x f$, respectively, denote the partial derivatives of a function f with respect to the variables t and x. f' denotes the first derivative of a function f with respect to the variable x, $\dot{f}$ denotes the first derivative of a function f with respect to the variable t, and $\ddot{f}$ denotes the second derivative of a function f with respect to the variable t. For a function $\varphi = [\varphi_1, \ldots, \varphi_n]^\top : [0, L] \to \mathbb{R}^n$, define $|\varphi| = \sum_{i=1}^n |\varphi_i|$, $\|\varphi\|_\infty = \operatorname{ess\,sup}_{x\in[0,L]}|\varphi|$, the L^2-norm $\|\varphi\|_{L^2} = \left(\int_0^L (\varphi_1^2(\xi,t) + \cdots + \varphi_n^2(\xi,t))\,\mathrm{d}\xi\right)^{\frac{1}{2}}$, the H^2-norm $\|\varphi\|_{H^2} = \left(\int_0^L \left(\|\varphi\|_{L^2}^2 + \|\varphi_x\|_{L^2}^2 + \|\varphi_{xx}\|_{L^2}^2\right)\mathrm{d}x\right)^{\frac{1}{2}}$, and $\|\varphi\|_{C^2} = \|\varphi\|_\infty + \|\dot{\varphi}\|_\infty + \|\ddot{\varphi}\|_\infty$. $0_{n,l}$ denotes the n, l zero matrix. I_n is a n-dimensional identity matrix. The block diagonal matrix is represented as $M = \mathrm{diag}\{M_1, M_2, \ldots, M_n\}$, where the main diagonal argument M_i $(i = 1, 2, \ldots, n)$ are matrices. M^{-1} denotes the inverse matrix of a square matrix M. $M^\top$ denotes the transpose of a matrix M. $\lambda(M)$ is the set of all the eigenvalues of a matrix M, and $|\lambda(M)|$ is the set of absolute values of all the eigenvalues if M is a square matrix. The symbol $*$ stands for a symmetric block in a matrix.

2 Preliminary

2.1 Heterogeneous traffic flow model and problem statement

Extending the results of the papers [10] and [9], we design an observer in the H^2 space for a particular quasilinear hyperbolic system, with a boundary control and a collocated perturbation. More specifically, following [19], the following heterogeneous quasilinear hyperbolic traffic flow system is considered, given a road segment with N vehicle classes, W road width and L road length, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\partial_t \rho_i(x,t) + \partial_x (\rho_i(x,t) v_i(x,t)) = 0, \tag{2.1}$$

$$\partial_t (v_i(x,t) + p_i(Ao(\rho))) + v_i(x,t)\partial_x (v_i(x,t) + p_i(Ao(\rho))) = \frac{V_{e,i}(Ao(\rho)) - v_i(x,t)}{\tau_i}, \tag{2.2}$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$\begin{aligned} &\big(\rho_1(0,t)v_1(0,t) - \rho_1^*(0)v_1^*(0), \ldots, \rho_N(0,t)v_N(0,t) - \rho_N^*(0)v_N^*(0)\big)^\top \\ &\quad = d(t) + \Theta U(t), && (2.3) \\ &\rho_i(L,t) = \rho_i^*(L), && (2.4) \end{aligned}$$

and the initial conditions

$$\rho_i(\cdot,0) = \rho_{0,i}(\cdot) \in H^2([0,L];\mathbb{R}), \tag{2.5}$$

$$v_i(\cdot,0) = v_{0,i}(\cdot) \in H^2([0,L];\mathbb{R}), \tag{2.6}$$

where the density ρ_i and velocity v_i of vehicle class i depend on the space variable $x \in [0, L]$ and the time variable $t \in [0, \infty)$, i ($i = 1, 2, \ldots, N$) is the index of vehicle class, and τ_i is the relaxation time depending on the driving behavior of vehicle class i. The area occupancy is $Ao(\rho) = \frac{a^\top \rho}{W}$, with $a = (a_1, a_2, \ldots, a_N)^\top$ (a_i is the occupied surface per vehicle for class i) and $\rho = (\rho_1, \rho_2, \ldots, \rho_N)^\top$. Area occupancy $Ao(\rho)$ describes the percentage of road space that is occupied by all the vehicle classes on the road segment, and $0 < Ao(\rho) \leq 1$. The traffic pressure function $p_i(Ao(\rho))$ of vehicle class i is (see [4]) $p_i(Ao(\rho)) = v_i^M \left(\frac{Ao(\rho)}{Ao_i^M}\right)^{\gamma_i}$, $i = 1, 2, \ldots, N$, with the free-flow velocity v_i^M ($0 < v_i \leq v_i^M$), the maximum area occupancy $0 < Ao_i^M \leq 1$, and the pressure exponent constant $\gamma_i > 1$ of class i. The free-flow velocity v_i^M stands for the desired velocity of vehicle type i, and the maximum area occupancy Ao_i^M describes the percentage of the occupied road surface for which the corresponding vehicle class i is jammed, if no other vehicle class is present. Denoting the maximal density by ρ_i^M, we assume that the inequality $0 < \rho_i \leq \rho_i^M$ holds. The equilibrium speed-Ao relationship of vehicle class i ($= 1, 2, \ldots, N$) is given by Greenshields model in [7], $V_{e,i}(Ao(\rho)) = v_i^M - p_i(Ao(\rho)) = v_i^M \left(1 - \left(\frac{Ao(\rho)}{Ao_i^M}\right)^{\gamma_i}\right)$.

The equilibrium $\rho_i^*, v_i^* \in C^2([0, L]; \mathbb{R})$ satisfies, for $i = 1, 2, \ldots, N$,

$$v_i^* \rho_i^{*\prime} + \rho_i^* v_i^{*\prime} = 0, \tag{2.7}$$

$$v_i^* v_i^{*\prime} + v_i^* p_i'(Ao(\rho^*)) = \frac{V_{e,i}(Ao(\rho^*)) - v_i^*}{\tau_i}, \tag{2.8}$$

with $\rho^* = (\rho_1^*, \rho_2^*, \ldots, \rho_N^*)^\top$. From (2.7), note that $\rho_i^* v_i^* = d_i$ with the given constant d_i and the given value for $\rho_i^*(0)$, $i = 1, 2 \ldots, N$. Assume that there exists a equilibrium $\rho_i^* > 0$, $v_i^* > 0$ defined on $[0, L]$ satisfying (2.7)–(2.8), as done in [2] for a different class of 2×2 hyperbolic systems.

As described in [8], high traffic demand is the most effective ingredient causing traffic breakdown. The disturbances caused by bottlenecks or individual drivers cannot grow and propagate on account of unconditional stability if the traffic load is low enough. In order to increase the efficiency and stability of traffic flow, we solve the

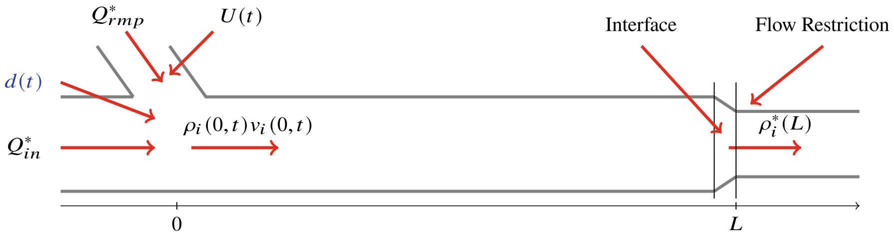

Fig. 1 Heterogeneous vehicle traffic on a road with disturbances at the inlet boundary and a velocity drop in the right boundary

control problem of high traffic demand by ramp metering in the presence of a bottleneck and disturbances on the road. The diagram is presented in Fig. 1.

The input $U \in C^1([0,\infty); \mathbb{R}^{2N-m})$ with the coefficient matrix $\Theta \in \mathbb{R}^{N,2N-m}$ acts as an on-ramp metering at the upstream boundary of the considered road segment. The referenced inflow $Q^*_{\text{in}} \in \mathbb{R}^N$ and the nominal on-ramp flux rate $Q^*_{\text{rmp}} \in \mathbb{R}^N$ at the inlet $x = 0$ satisfy

$$Q^*_{\text{in}} + Q^*_{\text{rmp}} = \left(\rho^*_1(0)v^*_1(0), \rho^*_2(0)v^*_2(0), \ldots, \rho^*_N(0)v^*_N(0)\right)^\top.$$

With the unknown disturbance input $d \in C^2([0,\infty); \mathbb{R}^N)$ at the inlet of the boundary, the total inflow at the inlet consisting of the inflow at the ramp $0 \leq \left[1\ 1\ \ldots\ 1\right](Q^*_{\text{rmp}} + \Theta U) \leq Q^{\max}_{\text{rmp}}$ ($Q^{\max}_{\text{rmp}}$ is the flux limit on the on-ramp), and the inflow at the inlet $0 \leq \left[1\ 1\ \ldots\ 1\right](Q^*_{\text{in}} + d) \leq Q^{\max}_{\text{in}}$ ($Q^{\max}_{\text{in}}$ is the flux limit of the incoming road) is limited by the maximum flow $Q_{\max} \geq \left[1\ 1\ \ldots\ 1\right](Q^*_{\text{in}} + d + Q^*_{\text{rmp}} + \Theta U) \geq 0$, and $0 < Ao(\rho(0,\cdot)) \leq \max\{Ao^M_1, Ao^M_2, \ldots, Ao^M_N\}$. As described in [10], the interface at the bottleneck is a buffer zone for velocity drop (the velocity in the interface is continuously decreasing from the left boundary of the interface to $x = L$). The value of variable speed limit $v_i(L,\cdot)$ ($0 < v_i(L,\cdot) \leq v^M_i$) is derived from the constant density $\rho^*_i(L)$ and the measurement of the flux $q_i(L,\cdot)$ (constrained by a maximum flow which is less than $Q_{\max}$) at the inlet of the bottleneck, for $i = 1, 2, \ldots, N$. From a practical perspective, an output is needed for the observer design to address the state observation problem of quasilinear traffic flow systems (2.1)–(2.6) in this paper. The boundary measurements of density $\rho_i(0,t)$ and velocity $v_i(0,t)$, $t \in [0,\infty)$ are taken at the boundary $x = 0$ collocated with a known input U; then the measured output of systems (2.1)–(2.6) is, for all $t \in [0,\infty)$,

$$y_1(t) = (\rho_1(0,t), v_1(0,t), \rho_2(0,t), v_2(0,t), \ldots, \rho_N(0,t), v_N(0,t))^\top.$$

Defining $u^* = (\rho^*_1, v^*_1, \ldots, \rho^*_N, v^*_N)^\top \in C^2([0,L]; \mathbb{R}^{2N})$, $u = (\rho_1, v_1, \ldots, \rho_N, v_N)^\top \in H^2([0,L] \times [0,\infty); \mathbb{R}^{2N})$, and $\widetilde{u} = u - u^* = (\widetilde{\rho}_1, \widetilde{v}_1, \ldots, \widetilde{\rho}_N, \widetilde{v}_N)^\top \in H^2([0,L] \times [0,\infty); \mathbb{R}^{2N})$ with $\widetilde{\rho}_i = \rho_i - \rho^*_i$, $\widetilde{v}_i = v_i - v^*_i$, for $i = 1, 2, \ldots, N$, systems (2.1)–(2.6) are rewritten as in paper [9], for all $x \in (0,L)$, $t \in [0,\infty)$,

$$\partial_t \widetilde{u}(x,t) + F(\widetilde{u}, u^*)\partial_x \widetilde{u}(x,t) = G(\widetilde{u}, u^*)\widetilde{u}(x,t), \tag{2.9}$$

with the boundary conditions, for all $t \in [0, \infty)$,

$$A_1 \widetilde{u}(0,t) = d(t) + \Theta U(t) + w_1(t) + w_2(t) - \Pi_{NL}(\widetilde{u}(0,t)), \tag{2.10}$$

$$B_1 \widetilde{u}(L,t) = 0_{2N,1}, \tag{2.11}$$

where

$$F(\widetilde{u}, u^*) = \begin{bmatrix} F_{11}(\widetilde{u}, u^*) & F_{12}(\widetilde{u}, u^*) & \cdots & F_{1N}(\widetilde{u}, u^*) \\ F_{21}(\widetilde{u}, u^*) & F_{22}(\widetilde{u}, u^*) & \cdots & F_{2N}(\widetilde{u}, u^*) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(\widetilde{u}, u^*) & F_{N2}(\widetilde{u}, u^*) & \cdots & F_{NN}(\widetilde{u}, u^*) \end{bmatrix},$$

with, for $i, j = 1, 2, \dots, N$,

$$F_{ij}(\widetilde{u}, u^*) = \begin{cases} \begin{bmatrix} \widetilde{v}_i + v_i^* & \widetilde{\rho}_i + \rho_i^* \\ 0 & \widetilde{v}_i + v_i^* - (\widetilde{\rho}_i + \rho_i^*)\delta_{ii}(\rho) \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ ((\widetilde{v}_i + v_i^*) - (\widetilde{v}_j + v_j^*))\delta_{ij}(\rho) & -(\widetilde{\rho}_j + \rho_j^*)\delta_{ij}(\rho) \end{bmatrix}, & \text{if } j \neq i, \end{cases}$$

$$G(\widetilde{u}, u^*) = \begin{bmatrix} G_{11}(\widetilde{u}, u^*) & G_{12}(\widetilde{u}, u^*) & \cdots & G_{1N}(\widetilde{u}, u^*) \\ G_{21}(\widetilde{u}, u^*) & G_{22}(\widetilde{u}, u^*) & \cdots & G_{2N}(\widetilde{u}, u^*) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}(\widetilde{u}, u^*) & G_{N2}(\widetilde{u}, u^*) & \cdots & G_{NN}(\widetilde{u}, u^*) \end{bmatrix},$$

with, for $i, j = 1, 2, \dots, N$,

$$G_{ij}(\widetilde{u}, u^*) = \begin{cases} \begin{bmatrix} v_i^{*\prime} & \rho_i^{*\prime} \\ \frac{1}{\tau_i}\delta_{ii}(\rho^*) + v_i^* \sum_{k=1}^N \sigma_{iki}(\rho^*)\rho_k^{*\prime} - \delta_{ii}(\rho) v_i^{*\prime} & \frac{1}{\tau_i} + v_i^{*\prime} + \sum_{k=1, k\neq i}^N \delta_{ik}(\rho^*)\rho_k^{*\prime} \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau_i}\delta_{ij}(\rho^*) + v_i^* \sum_{k=1}^N \sigma_{ikj}(\rho^*)\rho_k^{*\prime} - \delta_{ij}(\rho) v_j^{*\prime} & -\delta_{ij}(\rho)\rho_j^{*\prime} \end{bmatrix}, & \text{if } j \neq i, \end{cases}$$

and for $i, j, k = 1, 2, \dots, N$,

$$\delta_{ij}(\rho) = \partial_{\rho_j} p_i\,(Ao(\rho)) = \frac{v_i^M \gamma_i a_j}{Ao_i^M W} \left(\frac{Ao(\rho)}{Ao_i^M} \right)^{\gamma_i - 1},$$

$$\delta_{ij}(\rho^*) = \partial_{\rho_j} p_i\left(Ao(\rho^*)\right) = \frac{v_i^M \gamma_i a_j}{Ao_i^M W} \left(\frac{Ao(\rho^*)}{Ao_i^M} \right)^{\gamma_i - 1},$$

$$\sigma_{ikj}(\rho^*) = \partial_{\rho_k}\delta_{ij}(\rho^*) = \frac{v_i^M\gamma_i(\gamma_i-1)a_k a_j}{(Ao_i^M W)^2}\left(\frac{Ao(\rho^*)}{Ao_i^M}\right)^{\gamma_i-2}.$$

The coefficient matrices $A_1 = \mathrm{diag}\{[v_1^*(0), \rho_1^*(0)], \ldots, [v_N^*(0), \rho_N^*(0)]\} \in \mathbb{R}^{N,2N}$, $B_1 = \mathrm{diag}\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \ldots, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\} \in \mathbb{R}^{2N,2N}$, and the nonlinear term

$$\Pi_{NL}(\widetilde{u}(0,\cdot)) = \begin{bmatrix}\widetilde{\rho}_1(0,\cdot)\widetilde{v}_1(0,\cdot)\\ \widetilde{\rho}_2(0,\cdot)\widetilde{v}_2(0,\cdot)\\ \vdots\\ \widetilde{\rho}_N(0,t)\widetilde{v}_N(0,t)\end{bmatrix} \in C^1([0,\infty);\mathbb{R}^N).$$

The added terms $w_1, w_2 \in C^2([0,\infty);\mathbb{R}^N)$ are the solutions to the following system,

$$\begin{aligned}\dot{w}_1 &= -d_1 w_1,\\ \dot{w}_2 &= -d_2 w_2,\end{aligned}$$

with the constants $d_1, d_2 (d_1 \neq d_2)$, and the initial conditions $w_1(0) = -\frac{g_2(\widetilde{u}_0)+d_2 g_1(\widetilde{u}_0)}{d_1-d_2}$, $w_2(0) = \frac{d_1 g_1(\widetilde{u}_0)+g_2(\widetilde{u}_0)}{d_1-d_2}$, with

$$\begin{aligned} g_1(\widetilde{u}_0) &= A_1\widetilde{u}_0(0) - d(0) - \Theta U(0) + \Pi_{NL}(\widetilde{u}_0(0)),\\ g_2(\widetilde{u}_0) &= A_1\left(-F(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0'(0) + G(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0(0)\right) - \dot{d}(0) - \Theta\dot{U}(0)\\ &\quad + \left.\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\right|_{t=0}\left(-F(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0'(0) + G(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0(0)\right).\end{aligned}$$

The initial condition

$$\widetilde{u}(\cdot,0) = \widetilde{u}_0(\cdot) \in H^2([0,L];\mathbb{R}^{2N}), \tag{2.12}$$

satisfies the second-order compatibility conditions

$$A_1\widetilde{u}_0(0) = d(0) + \Theta U(0) + w_1(0) + w_2(0) - \Pi_{NL}(\widetilde{u}_0(0)), \tag{2.13}$$

$$B_1\widetilde{u}_0(L) = 0_{2N,1}, \tag{2.14}$$

$$\begin{aligned} &A_1\left(-F(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0'(0) + G(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0(0)\right)\\ &\quad = \dot{d}(0) + \Theta\dot{U}(0) + (-d_1 w_1(0) - d_2 w_2(0))\\ &\qquad - \left.\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\right|_{t=0}\left(-F(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0'(0) + G(\widetilde{u}_0(0), u^*(0))\widetilde{u}_0(0)\right), \end{aligned} \tag{2.15}$$

$$B_1 F(\widetilde{u}_0(L), u^*(L))\widetilde{u}_0'(L) = B_1 G(\widetilde{u}_0(L), u^*(L))\widetilde{u}_0(L). \tag{2.16}$$

The hyperbolicity of systems (2.9)–(2.16) exists around zero equilibrium on the basis of the discussion in paper [10], because for all $u^* \in C^2([0,L];\mathbb{R}^{2N})$, as $t \to \infty$, the matrix $F(\widetilde{u}, u^*) \to F(0, u^*)$, which has $2N$ real distinct nonzero eigenvalues

$\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0 > -\lambda_{m+1} > \cdots > -\lambda_{2N}$, ($\lambda_i \in C^2([0,L];\mathbb{R})$, $i = 1,\ldots,2N$, m is the number of positive eigenvalues and $0 \le m < 2N$), and $-\lambda_{m+1},\ldots,-\lambda_{2N} < 0$ means that the traffic wave moves backward in the congested regime. For $x \in [0,L]$, define $\Lambda = \mathrm{diag}\{\Lambda^+, -\Lambda^-\}$, $\Lambda^+ = \mathrm{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$, $\Lambda^- = \mathrm{diag}\{\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_{2N}\}$, $\Lambda' = \mathrm{diag}\{\lambda_1',\ldots,\lambda_m',-\lambda_{m+1}',\ldots,-\lambda_{2N}'\}$. We will study the scenarios $2N-m \ge 1$ in the H^2 sense in this paper. The corresponding right eigenvectors of $2N$ eigenvalues consist of the columns of the invertible matrix $T \in C^2([0,L];\mathbb{R}^{2N,2N})$. The density and speed gaps $\widetilde{\rho}_i(0,\cdot)$, $\widetilde{v}_i(0,\cdot)$ of vehicle class i, $i = 1,\ldots,2N$, between the measurements $\rho_i(0,\cdot)$, $v_i(0,\cdot)$ and the corresponding equilibrium $\rho_i^*(0)$, $v_i^*(0)$ at the inlet of the considered road section are involved in the output of systems (2.9)–(2.16), for $t \in [0,\infty)$,

$$y(t) = \left[0_{2N-m,m}\ \ I_{2N-m}\right] T^{-1}(0)\widetilde{u}(0,t) \in \mathbb{R}^{2N-m}. \tag{2.17}$$

2.2 State transformation

In order to simplify the analysis, by using the invertible transformation $R = \Phi\widetilde{u} \in H^2([0,L]\times[0,\infty);\mathbb{R}^{2N})$ with $\Phi \in C^\infty([0,L];\mathbb{R}^{2N,2N})$, from $\widetilde{u}$ to the new variable $R : [0,L]\times[0,\infty) \to \mathbb{R}^{2N}$, systems (2.9)–(2.16) are mapped into the following system in the form of characteristic values as in [9, 13] (see the explicit expression of Φ in [13, Equations (3.1)–(3.2)]), for all $x \in (0,L)$, $t \in [0,\infty)$,

$$\partial_t R(x,t) + \Lambda(x)\partial_x R(x,t) + \Lambda_{NL}(R,x)\partial_x R(x,t) = \Sigma(x)R(x,t) + \Sigma_{NL}(R,x)R(x,t), \tag{2.18}$$

with the boundary conditions, for all $t \in [0,\infty)$,

$$R_{\mathrm{in}}(t) = K_P R_{\mathrm{out}}(t) + \Gamma_0\Big(d(t) + \Theta U(t)\Big) + \Gamma_0(w_1(t) + w_2(t)) - \Gamma_0\Pi_{NL}(T(0)R(0,t)), \tag{2.19}$$

and the initial condition

$$R(\cdot,0) = R_0(\cdot) \in H^2([0,L];\mathbb{R}^{2N}), \tag{2.20}$$

satisfying the following second-order compatibility conditions,

$$\begin{aligned} R_{\mathrm{in}}(0) &= K_P R_{\mathrm{out}}(0) + \Gamma_0\left(d(0) + \Theta U(0)\right) + \Gamma_0(w_1(0) + w_2(0)) \\ &\quad - \Gamma_0\Pi_{NL}(T(0)R_0(0)), \end{aligned} \tag{2.21}$$

$$\begin{aligned} &([M_i^1]_{1\le i\le m}, [M_j^2]_{m+1\le j\le 2N})^\top \\ &= K_P([M_i^2]_{1\le i\le m}, [M_j^1]_{m+1\le j\le 2N})^\top + \Gamma_0\left(\dot{d}(0) + \Theta\dot{U}(0)\right) \\ &\quad + \Gamma_0(-d_1w_1(0) - d_2w_2(0)) - \Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\bigg|_{t=0} \\ &\quad T(0)\bigg(-(\Lambda(0) + \Lambda_{NL}(R_0(0),0))\,R_0'(0) + (\Sigma(0) + \Sigma_{NL}(R_0(0),0))\,R_0(0)\bigg), \end{aligned} \tag{2.22}$$

with

$$M^1 = -\left(\Lambda(0) + \Lambda_{NL}(R_0(0), 0)\right) R_0'(0) + \left(\Sigma(0) + \Sigma_{NL}(R_0(0), 0)\right) R_0(0),$$
$$M^2 = -\left(\Lambda(L) + \Lambda_{NL}(R_0(L), L)\right) R_0'(L) + \left(\Sigma(L) + \Sigma_{NL}(R_0(L), L)\right) R_0(L),$$

where $R = (R^+, R^-)^\top : [0, L] \times [0, \infty) \to \mathbb{R}^{2N}$, $R_{\text{in}} = (R^+(0, \cdot), R^-(L, \cdot))^\top \in L^\infty([0, \infty); \mathbb{R}^{2N})$, $R_{\text{out}} = (R^+(L, \cdot), R^-(0, \cdot))^\top \in L^\infty([0, \infty); \mathbb{R}^{2N})$, with $R^+ : [0, L] \times [0, \infty) \to \mathbb{R}^m$, $R^- : [0, L] \times [0, \infty) \to \mathbb{R}^{2N-m}$,

$$\Lambda_{NL} = \Phi F\left(\Phi^{-1} R, u^*\right) \Phi^{-1} - \Lambda,$$
$$\Sigma_{NL} = \Phi G\left(\Phi^{-1} R, u^*\right) \Phi^{-1} - \Phi F\left(\Phi^{-1} R, u^*\right) (\Phi^{-1})' - \Sigma,$$

$\Sigma = \begin{bmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{bmatrix}$, with $\Sigma^{++} \in C^2([0, L]; \mathbb{R}^{m,m})$, $\Sigma^{+-} \in C^2([0, L]; \mathbb{R}^{m,2N-m})$, $\Sigma^{-+} \in C^2([0, L]; \mathbb{R}^{2N-m,m})$, $\Sigma^{--} \in C^2([0, L]; \mathbb{R}^{2N-m,2N-m})$, the main diagonal elements of the matrix $\Sigma \in C^2([0, L]; \mathbb{R}^{2N,2N})$ are zeros, $K_P \in \mathbb{R}^{2N,2N}$, $\Gamma_0 \in \mathbb{R}^{2N,N}$ are given gain matrices. Since for $x \in [0, L]$, $\Lambda_{NL}(0, x) = 0_{2N,2N}$, $\Sigma_{NL}(0, x) = 0_{2N,2N}$ and $\Pi_{NL}(0) = \frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\tilde{u}}(0) = 0_{2N,1}$, then quasilinear systems (2.18)–(2.22) have zero equilibrium.

For all $t \in [0, \infty)$, the boundary measurements are taken at the boundary $x = 0$ collocated with the control input and output (2.17) that is equivalent to

$$y(t) = R^-(0, t). \tag{2.23}$$

3 Nonlinear collocated observer

3.1 Quasilinear observer design

Under the consideration of the nonobservability of the unknown disturbance input d, it is not taken into account in the design of the quasilinear observer. The following observer is designed for quasilinear systems (2.18)–(2.22) by constructing quasilinear systems (2.18)–(2.22) with the output injection terms, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\begin{aligned} &\partial_t \hat{R}(x, t) + \Lambda(x) \partial_x \hat{R}(x, t) + \Lambda_{NL}(\hat{R}, x) \partial_x \hat{R}(x, t) \\ &\quad = \Sigma(x) \hat{R}(x, t) + \Sigma_{NL}(\hat{R}, x) \hat{R}(x, t) + S(x) \left(y(t) - \hat{R}^-(0, t)\right), \end{aligned} \tag{3.1}$$

with the boundary condition, for all $t \in [0, \infty)$,

$$\begin{aligned} \hat{R}_{\text{in}}(t) &= K_P \hat{R}_{\text{out}}(t) + \Gamma_0 (w_1(t) + w_2(t)) + \Gamma_0 \left(\Theta U(t) - \Pi_{NL}\left(T(0) \hat{R}(0, t)\right)\right) \\ &\quad - \begin{bmatrix} \Gamma_1 \\ 0_{2N-m,2N-m} \end{bmatrix} \int_0^t \left(y(\tau) - \hat{R}^-(0, \tau)\right) \mathrm{d}\tau, \end{aligned} \tag{3.2}$$

and the initial condition

$$\hat{R}(\cdot,0)=\hat{R}_0(\cdot)\in H^2([0,L];\mathbb{R}^{2N}), \tag{3.3}$$

satisfying the following second-order compatibility conditions,

$$\hat{R}_{\text{in}}(0)=K_P\hat{R}_{\text{out}}(0)+\Gamma_0(w_1(0)+w_2(0))+\Gamma_0\left(\Theta U(0)-\Pi_{NL}\left(T(0)\hat{R}_0(0)\right)\right), \tag{3.4}$$

$$\begin{aligned}
&([\hat{M}_i^1]_{1\le i\le m},[\hat{M}_j^2]_{m+1\le j\le 2N})^\top\\
&\quad=K_P([\hat{M}_i^2]_{1\le i\le m},[\hat{M}_j^1]_{m+1\le j\le 2N})^\top+\Gamma_0(-d_1w_1(0)-d_2w_2(0))\\
&\qquad+\Gamma_0\Theta\dot{U}(0)-\Gamma_0\frac{\mathrm{d}\Pi_{NL}}{\mathrm{d}\widetilde{u}}\bigg|_{t=0}T(0)\Big(-\Big(\Lambda(0)+\Lambda_{NL}(\hat{R}_0(0),0)\Big)\hat{R}_0'(0)\\
&\qquad+\Big(\Sigma(0)+\Sigma_{NL}(\hat{R}_0(0),0)\Big)\hat{R}_0(0)+S(0)\left(y(0)-\hat{R}_0^-(0)\right)\Big)\\
&\qquad-\begin{bmatrix}\Gamma_1\\ 0_{2N-m,2N-m}\end{bmatrix}\left(y(0)-\hat{R}_0^-(0)\right),
\end{aligned} \tag{3.5}$$

with

$$\begin{aligned}
\hat{M}^1&=-\Big(\Lambda(0)+\Lambda_{NL}(\hat{R}_0(0),0)\Big)\hat{R}_0'(0)+\Big(\Sigma(0)+\Sigma_{NL}(\hat{R}_0(0),0)\Big)\hat{R}_0(0)\\
&\quad+S(0)\left(y(0)-\hat{R}_0^-(0)\right),\\
\hat{M}^2&=-\Big(\Lambda(L)+\Lambda_{NL}(\hat{R}_0(L),L)\Big)\hat{R}_0'(L)+\Big(\Sigma(L)+\Sigma_{NL}(\hat{R}_0(L),L)\Big)\hat{R}_0(L)\\
&\quad+S(L)\left(y(0)-\hat{R}_0^-(0)\right),
\end{aligned}$$

where $\hat{R}=(\hat{R}^+,\hat{R}^-)^\top:[0,L]\times[0,\infty)\to\mathbb{R}^{2N}$, $\hat{R}_{\text{in}}=(\hat{R}^+(0,\cdot),\hat{R}^-(L,\cdot))^\top\in L^\infty([0,\infty);\mathbb{R}^{2N})$, $\hat{R}_{\text{out}}=(\hat{R}^+(L,\cdot),\hat{R}^-(0,\cdot))^\top\in L^\infty([0,\infty);\mathbb{R}^{2N})$, $S=(S_1,S_2)^\top:(0,L)\to\mathbb{R}^{2N,2N-m}$, $\Gamma_1\in\mathbb{R}^{m,2N-m}$. In the previous equations, $\hat{R}^+:[0,L]\times[0,\infty)\to\mathbb{R}^m$ and $\hat{R}^-:[0,L]\times[0,\infty)\to\mathbb{R}^{2N-m}$ are the observation of state variables R^+ and R^-; the terms $S_1\in C^2([0,L];\mathbb{R}^{m,2N-m})$ and $S_2\in C^2([0,L];\mathbb{R}^{2N-m,2N-m})$ are output injection gains.

3.2 Local H^2 iISS of quasilinear observer system

So as to theoretically verify the applicability of state observation of the designed quasilinear observer, the following error system is obtained by subtracting observers (3.1)–(3.5) from quasilinear systems (2.18)–(2.22), for all $x\in(0,L)$, $t\in[0,\infty)$,

$$\partial_t\widetilde{R}(x,t)+\Lambda(x)\partial_x\widetilde{R}(x,t)+F_{NL}[\widetilde{R},\hat{R},\partial_x\widetilde{R},\partial_x\hat{R}]=\Sigma(x)\widetilde{R}(x,t)-S(x)\widetilde{R}^-(0,t), \tag{3.6}$$

with the boundary condition, for all $t \in [0, \infty)$,

$$\widetilde{R}_{\text{in}}(t) = K_P \widetilde{R}_{\text{out}}(t) + \begin{bmatrix} \Gamma_1 \\ 0_{2N-m,2N-m} \end{bmatrix} \int_0^t \widetilde{R}^-(0, \tau)\, \mathrm{d}\tau + \Gamma_0 d(t) \\ - G_{NL}[\widetilde{R}(0, t), \hat{R}(0, t)], \tag{3.7}$$

and the initial condition

$$\widetilde{R}(\cdot, 0) = \widetilde{R}_0(\cdot) \in H^2([0, L]; \mathbb{R}^{2N}), \tag{3.8}$$

satisfying the following second-order compatibility conditions

$$\widetilde{R}_{\text{in}}(0) = K_P \widetilde{R}_{\text{out}}(0) + \Gamma_0 d(0) - G_{NL}[\widetilde{R}_0(0), \hat{R}_0(0)], \tag{3.9}$$

$$\begin{aligned} &([\widetilde{M}_i^1]_{1\le i\le m}, [\widetilde{M}_j^2]_{m+1\le j\le 2N})^\top \\ &\quad = K_P([\widetilde{M}_i^2]_{1\le i\le m}, [\widetilde{M}_j^1]_{m+1\le j\le 2N})^\top + \begin{bmatrix} \Gamma_1 \\ 0_{2N-m,2N-m} \end{bmatrix} \widetilde{R}_0^-(0) + \Gamma_0 \dot{d}(0) \\ &\quad - \partial_{\widetilde{R}} G_{NL}\Big|_{t=0} \Big(- \Lambda(0)\widetilde{R}_0'(0) + \Sigma(0)\widetilde{R}_0(0) - F_{NL}[\widetilde{R}_0(0), \hat{R}_0(0), \widetilde{R}_0'(0), \hat{R}_0'(0)] \\ &\quad - S(0)\widetilde{R}_0^-(0)\Big) - \partial_{\hat{R}} G_{NL}\Big|_{t=0} \Big(- \Big(\Lambda(0) + \Lambda_{NL}(\hat{R}_0(0), 0)\Big) \hat{R}_0'(0) \\ &\quad + \Big(\Sigma(0) + \Sigma_{NL}(\hat{R}_0(0), 0)\Big) \hat{R}_0(0) + S(0)\widetilde{R}_0^-(0)\Big), \end{aligned} \tag{3.10}$$

with

$$\begin{aligned} \widetilde{M}^1 &= -\Lambda(0)\widetilde{R}_0'(0) + \Sigma(0)\widetilde{R}_0(0) - F_{NL}[\widetilde{R}_0(0), \hat{R}_0(0), \widetilde{R}_0'(0), \hat{R}_0'(0)] - S(0)\widetilde{R}_0^-(0), \\ \widetilde{M}^2 &= -\Lambda(L)\widetilde{R}_0'(L) + \Sigma(L)\widetilde{R}_0(L) - F_{NL}[\widetilde{R}_0(L), \hat{R}_0(L), \widetilde{R}_0'(L), \hat{R}_0'(L)] - S(L)\widetilde{R}_0^-(L), \end{aligned}$$

where $\widetilde{R} = (\widetilde{R}^+, \widetilde{R}^-)^\top : [0, L] \times [0, \infty) \to \mathbb{R}^{2N}$, $\widetilde{R}_{\text{in}} = (\widetilde{R}^+(0, \cdot), \widetilde{R}^-(L, \cdot))^\top \in L^\infty([0, \infty); \mathbb{R}^{2N})$, $\widetilde{R}_{\text{out}} = (\widetilde{R}^+(L, \cdot), \widetilde{R}^-(0, \cdot))^\top \in L^\infty([0, \infty); \mathbb{R}^{2N})$, and

$$\begin{aligned} F_{NL}[\widetilde{R}, \hat{R}, \partial_x \widetilde{R}, \partial_x \hat{R}] &= \Lambda_{NL}(\widetilde{R} + \hat{R}, x)\left(\partial_x \widetilde{R} + \partial_x \hat{R}\right) - \Sigma_{NL}(\widetilde{R} + \hat{R}, x)\left(\widetilde{R} + \hat{R}\right) \\ &\quad - \Lambda_{NL}(\hat{R}, x)\partial_x \hat{R} + \Sigma_{NL}(\hat{R}, x)\hat{R}, \\ G_{NL}[\widetilde{R}(0, \cdot), \hat{R}(0, \cdot)] &= \Gamma_0 \Pi_{NL}\left(T(0)\left(\widetilde{R}(0, \cdot) + \hat{R}(0, \cdot)\right)\right) - \Gamma_0 \Pi_{NL}\left(T(0)\hat{R}(0, \cdot)\right). \end{aligned}$$

By using the backstepping transformation, for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\begin{aligned} \widetilde{R}(x, t) &= \widetilde{Z}(x, t) + \int_0^x \begin{bmatrix} 0_{m,m} & F^1(x, \xi) \\ 0_{2N-m,m} & F^2(x, \xi) \end{bmatrix} \widetilde{Z}(\xi, t)\, \mathrm{d}\xi \\ &= \mathcal{F}[\widetilde{Z}], \end{aligned} \tag{3.11}$$

where the piecewise differentiable kernels F^1 and F^2 are the solutions to the following kernel equations on the triangular domain $\mathbb{T} = \left\{(x,\xi) \in \mathbb{R}^2 | \, 0 \le \xi \le x \le L\right\}$ as described in [12],

$$\Lambda^+(x)\partial_x F^1(x,\xi) - \partial_\xi F^1(x,\xi)\Lambda^-(\xi) = F^1(x,\xi)(\Lambda^-)'(\xi) + \Sigma^{++}(x)F^1(x,\xi) + \Sigma^{+-}(x)F^2(x,\xi), \tag{3.12}$$

$$\Lambda^-(x)\partial_x F^2(x,\xi) + \partial_\xi F^2(x,\xi)\Lambda^-(\xi) = -F^2(x,\xi)(\Lambda^-)'(\xi) - \Sigma^{--}(x)F^2(x,\xi) - \Sigma^{-+}(x)F^1(x,\xi), \tag{3.13}$$

$$F^1(x,x)\Lambda^-(x) + \Lambda^+(x)F^1(x,x) = \Sigma^{+-}(x), \tag{3.14}$$

$$F^2(x,x)\Lambda^-(x) - \Lambda^-(x)F^2(x,x) = \Sigma^{--}(x), \tag{3.15}$$

$$F^2(L,\xi) - \Gamma_2 F^1(L,\xi) = K_2(L-\xi), \tag{3.16}$$

with a given coefficient matrix $\Gamma_2 \in \mathbb{R}^{2N-m,m}$. Moreover, the injection gains of the observer are, for all x in $(0, L)$,

$$S_1(x) = F^1(x,0)\Lambda^-(L), \tag{3.17}$$

$$S_2(x) = F^2(x,0)\Lambda^-(L), \tag{3.18}$$

error systems (3.6)–(3.10) are mapped into the following quasilinear target system, for all $x \in (0, L)$, $t \in [0,\infty)$,

$$\partial_t \widetilde{Z}(x,t) + \Lambda(x)\partial_x \widetilde{Z}(x,t) + Q_{NL}\left[\widetilde{Z}, \hat{R}, \partial_x \widetilde{Z}, \partial_x \hat{R}\right] = \Sigma_2(x)\widetilde{Z}(x,t) + \int_0^x D_1(x,\xi)\widetilde{Z}(\xi,t)\,\mathrm{d}\xi, \tag{3.19}$$

with the boundary condition, for all $t \in [0,\infty)$,

$$\widetilde{Z}_{\mathrm{in}}(t) = K_P \widetilde{Z}_{\mathrm{out}}(t) + \widetilde{X}(t), \tag{3.20}$$

$$\widetilde{X}(t) = K_I \int_0^t \widetilde{Z}_{\mathrm{out}}(\sigma)\,\mathrm{d}\sigma - \int_0^L k_2(x)\widetilde{Z}(x,t)\,\mathrm{d}x + \Gamma_0 d(t) - G_{NL}[\widetilde{Z}(0,t), \hat{R}(0,t)], \tag{3.21}$$

and the initial condition

$$\widetilde{Z}(\cdot,0) = \widetilde{Z}_0(\cdot) \in H^2([0,L];\mathbb{R}^{2N}), \tag{3.22}$$

$$\widetilde{X}(0) = \widetilde{X}_0 = \Gamma_0 d(0) - G_{NL}[\widetilde{Z}_0(0), \hat{R}_0(0)] \in \mathbb{R}^{2N}, \tag{3.23}$$

satisfying the second-order compatibility conditions

$$\widetilde{Z}_{\mathrm{in}}(0) = K_P \widetilde{Z}_{\mathrm{out}}(0) - \int_0^L k_2(x)\widetilde{Z}_0(x)\,\mathrm{d}x + \Gamma_0 d(0) - G_{NL}[\widetilde{Z}_0(0), \hat{R}_0(0)], \tag{3.24}$$

$$([\widetilde{N}_i^1]_{1\le i\le m}, [\widetilde{N}_j^2]_{m+1\le j\le 2N})^\top$$

$$
\begin{aligned}
&= K_P([\widetilde{N}_i^2]_{1\le i\le m}, [\widetilde{N}_j^1]_{m+1\le j\le 2N})^\top \\
&\quad - \int_0^L k_2(x)\Big(-\Lambda(x)\partial_x \widetilde{Z}_0(x) - Q_{NL}[\widetilde{Z}_0(x), \hat{R}_0(x), \partial_x \widetilde{Z}_0(x), \partial_x \hat{R}_0(x)] + \Sigma_2(x)\widetilde{Z}_0(x) \\
&\quad + \int_0^x D_1(x,\sigma)\widetilde{Z}_0(\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}x + K_I \widetilde{Z}_{\text{out}}(0) + \Gamma_0 \dot{d}(0) \\
&\quad - \partial_{\widetilde{Z}} G_{NL}\Big|_{t=0}\Big(-\Lambda(0)\widetilde{Z}_0'(0) - Q_{NL}[\widetilde{Z}_0(0), \hat{R}_0(0), \widetilde{Z}_0'(0), \hat{R}_0'(0)] + \Sigma_2(0)\widetilde{Z}_0(0)\Big) \\
&\quad - \partial_{\hat{R}} G_{NL}\Big|_{t=0}\Big(-\Big(\Lambda(0) + \Lambda_{NL}(\hat{R}_0(0), 0)\Big)\hat{R}_0'(0) \\
&\quad + \Big(\Sigma(0) + \Sigma_{NL}(\hat{R}_0(0), 0)\Big)\hat{R}_0(0) + S(0)\widetilde{R}_0^-(0)\Big),
\end{aligned}
\tag{3.25}
$$

with

$$
\begin{aligned}
\widetilde{N}^1 &= -\Lambda(0)\widetilde{Z}_0'(0) - Q_{NL}[\widetilde{Z}_0(0), \hat{R}_0(0), \widetilde{Z}_0'(0), \hat{R}_0'(0)] + \Sigma_2(0)\widetilde{Z}_0(0), \\
\widetilde{N}^2 &= -\Lambda(L)\widetilde{Z}_0'(L) - Q_{NL}[\widetilde{Z}_0(L), \hat{R}_0(L), \widetilde{Z}_0'(L), \hat{R}_0'(L)] + \Sigma_2(L)\widetilde{Z}_0(L) \\
&\quad + \int_0^L D_1(L,\xi)\widetilde{Z}_0(\xi)\,\mathrm{d}\xi,
\end{aligned}
$$

where $\widetilde{Z} = (\widetilde{Z}^+, \widetilde{Z}^-)^\top : [0,L]\times[0,\infty) \to \mathbb{R}^{2N}$, $\widetilde{Z}^+ : [0,L]\times[0,\infty) \to \mathbb{R}^m$, $\widetilde{Z}^- : [0,L]\times[0,\infty) \to \mathbb{R}^{2N-m}$, $\widetilde{Z}_{\text{in}} = (\widetilde{Z}^+(0,\cdot), \widetilde{Z}^-(L,\cdot))^\top \in L^\infty([0,\infty);\mathbb{R}^{2N})$, $\widetilde{Z}_{\text{out}} = (\widetilde{Z}^+(L,\cdot), \widetilde{Z}^-(0,\cdot))^\top \in L^\infty([0,\infty);\mathbb{R}^{2N})$,

$$
\Sigma_2 = \begin{bmatrix} \Sigma^{++} & 0_{m,2N-m} \\ \Sigma^{-+} & 0_{2N-m,2N-m} \end{bmatrix}, \qquad D_1(x,\xi) = \begin{bmatrix} D^+(x,\xi) & 0_{m,2N-m} \\ D^-(x,\xi) & 0_{2N-m,2N-m} \end{bmatrix},
$$
$$
k_2(x) = \begin{bmatrix} 0_{m,m} & 0_{m,2N-m} \\ 0_{2N-m,m} & K_2(x) \end{bmatrix} \in C^2([0,L];\mathbb{R}^{2N,2N}),
$$

for all (x,ξ) in $\mathbb{T}$, and $K_I = \begin{bmatrix} K_I^{11} & K_I^{12} \\ 0_{2N-m,m} & 0_{2N-m,2N-m} \end{bmatrix}$, $K_I^{11} \in \mathbb{R}^{m,m}$, $K_I^{12} \in \mathbb{R}^{m,2N-m}$. Here K_2 is a strictly upper triangular matrix, and D^+, D^- are given as the piecewise differentiable solutions to the Volterra integral equations

$$
\begin{aligned}
D^+(x,\xi) &= -F^1(x,\xi)\Sigma^{-+}(\xi) - \int_\xi^x F^1(x,s)D^-(s,\xi)\,\mathrm{d}s, \\
D^-(x,\xi) &= -F^2(x,\xi)\Sigma^{-+}(\xi) - \int_\xi^x F^2(x,s)D^-(s,\xi)\,\mathrm{d}s.
\end{aligned}
$$

The nonlinear terms $Q_{NL}\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right]$ and $G_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)]$ satisfy, for all $x \in (0,L)$, $t \in [0,\infty)$,

$$
\left(I_{2N} + \int_0^x \begin{bmatrix} 0_{m,m} & F^1(x,\xi) \\ 0_{2N-m,m} & F^2(x,\xi) \end{bmatrix} d\xi\right) Q_{NL}\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right]
$$

$$= \Lambda_{NL}(\mathcal{F}[\widetilde{Z}] + \hat{R}, x)\partial_x(\mathcal{F}[\widetilde{Z}] + \hat{R}) - \Sigma_{NL}(\mathcal{F}[\widetilde{Z}] + \hat{R}, x)(\mathcal{F}[\widetilde{Z}] + \hat{R})$$
$$- \Lambda_{NL}(\hat{R}, x)\partial_x(\hat{R}) + \Sigma_{NL}(\hat{R}, x)(\hat{R}), \tag{3.26}$$
$$G_{NL}[\widetilde{Z}(0,t), \hat{R}(0,t)]$$
$$= \Gamma_0 \Pi_{NL}\left(T(0)(\mathcal{F}[\widetilde{Z}](0,t) + \hat{R}(0,t))\right) - \Gamma_0 \Pi_{NL}\left(T(0)\hat{R}(0,t)\right). \tag{3.27}$$

Use the inverse backstepping transformation of (3.11), for all $x \in (0, L)$, $t \in [0, \infty)$,

$$\widetilde{Z}(x,t) = \widetilde{R}(x,t) - \int_0^x \begin{bmatrix} 0_{m,m} & K^1(x,\xi) \\ 0_{2N-m,m} & K^2(x,\xi) \end{bmatrix} \widetilde{R}(\xi,t)\,\mathrm{d}\xi, \tag{3.28}$$

where the piecewise differentiable kernels K^1 and K^2 are solutions to suitable kernel equations on the triangular domain $\mathbb{T} = \{(x,\xi) \in \mathbb{R}^2 | 0 \le \xi \le x \le L\}$, as considered in [13, Theorem A.2]. As described in [13], differentiating twice with respect to x in the invertible transformation $\mathcal{F}[\widetilde{Z}]$, it is shown that the H^2 norm of $\widetilde{Z}$ is equivalent to the H^2 norm of $\widetilde{R}$. So local iISS of error systems (3.6)–(3.10) is same as local iISS of target systems (3.19)–(3.25). Therefore, in order to prove the state observation performance of quasi-linear observer systems (3.1)–(3.5), we need to prove the iISS of quasilinear target systems (3.19)–(3.25). By using the Lyapunov method, the local iISS of target systems (3.19)–(3.25) in the H^2-norm is studied by analyzing the growth of $\|\widetilde{Z}\|_{L^2}$, $\|\partial_t\widetilde{Z}\|_{L^2}$ and $\|\partial_{tt}\widetilde{Z}\|_{L^2}$ as follows.

Theorem 3.1 *If there exist positive constants α, q_1, q_2, q_3, q_4, diagonal positive-definite matrices P_{11}, P_3, $P_4 \in \mathbb{R}^{2N,2N}$, a symmetric positive-definite matrix $P_{22} \in \mathbb{R}^{2N,2N}$, and a matrix $P_{12} \in \mathbb{R}^{2N,2N}$ such that the following matrix inequalities hold, for all $x \in [0, L]$,*

$$\Omega(x) = \begin{bmatrix} \Omega_{11}(x) & \Omega_{12} & \Omega_{13}(x) & \Omega_{14} & \Omega_{15} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ * & * & * & \Omega_{44} & \Omega_{45} \\ * & * & * & * & \Omega_{55} \end{bmatrix} \ge 0, \tag{3.29}$$

where

$$\Omega_{11}(x) = -\Lambda'(x)P_{11} - \alpha P_{11}$$
$$- \left(\Sigma_2^\top(x)P_{11} + P_{11}\Sigma_2(x) + q_1 L \nu_1^2 I_{2N} + \left(\frac{L}{q_1} + \frac{L}{q_2}\right) D_1^\top(L,x)D_1(L,x)\right),$$
$$\Omega_{12} = -P_{12}K_I,$$
$$\Omega_{13}(x) = -\Lambda'(x)P_{12} - \alpha P_{12} - \Sigma_2^\top(x)P_{12},$$
$$\Omega_{14} = \Omega_{15} = \Omega_{25} = \Omega_{34} = \Omega_{35} = 0_{2N,2N},$$
$$\Omega_{22} = \frac{1}{L}E_2P_{11} - \frac{1}{L}K_P^\top E_1 P_{11} K_P - \frac{1}{L}K_I^\top E_1 P_3 K_I,$$

$$
\begin{aligned}
\Omega_{23} &= -\frac{1}{L}K_P^\top E_1 P_{11} - \frac{1}{L}\left(K_P^\top M_1 + M_2\right) - K_I^\top P_{22},\\
\Omega_{24} &= -\frac{1}{L}K_I^\top E_1 P_3 K_P,\\
\Omega_{33} &= -\frac{1}{L}E_1 P_{11} - \frac{1}{L}\left(M_1 + M_1^\top\right) - \alpha P_{22} - q_2 L \nu_2^2 I_{2N},\\
\Omega_{44} &= \frac{1}{L}E_2 P_3 - \frac{1}{L}K_P^\top E_1 P_3 K_P - \frac{1}{L}K_I^\top E_1 P_4 K_I,\\
\Omega_{45} &= -\frac{1}{L}K_I^\top E_1 P_4 K_P,\\
\Omega_{55} &= \frac{1}{L}E_2 P_4 - \frac{1}{L}K_P^\top E_1 P_4 K_P,
\end{aligned}
$$

with $M_1 = \begin{bmatrix} \Lambda^+(0)P_{12}^{++} & \Lambda^+(0)P_{12}^{+-} \\ -\Lambda^-(L)P_{12}^{-+} & -\Lambda^-(L)P_{12}^{--} \end{bmatrix}$, $M_2 = \begin{bmatrix} -\Lambda^+(L)P_{12}^{++} & -\Lambda^+(L)P_{12}^{+-} \\ \Lambda^-(0)P_{12}^{-+} & \Lambda^-(0)P_{12}^{--} \end{bmatrix}$, $P_{12}^{++} \in \mathbb{R}^{m,m}$, $P_{12}^{+-} \in \mathbb{R}^{m,2N-m}$, $P_{12}^{-+} \in \mathbb{R}^{2N-m,m}$, $P_{12}^{--} \in \mathbb{R}^{2N-m,2N-m}$, $E_1 = \operatorname{diag}\{\Lambda^+(0), \Lambda^-(L)\}$, $E_2 = \operatorname{diag}\{\Lambda^+(L), \Lambda^-(0)\}$, $\nu_1 = \max(\lambda(P_{11}))$, $\nu_2 = \max(|\lambda(P_{12})|)$, *and*

$$
\begin{aligned}
M(x) = {} & \left(-\Lambda'(x) - \alpha I_{2N}\right) P_3 \\
& - \left(\Sigma_2^\top(x)P_3 + P_3\Sigma_2(x) + q_3 L \nu_3^2 I_{2N} + \frac{L}{q_3} D_1^\top(L,x) D_1(L,x)\right) \geq 0,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
K(x) = {} & \left(-\Lambda'(x) - \alpha I_{2N}\right) P_4 \\
& - \left(\Sigma_2^\top(x)P_4 + P_4\Sigma_2(x) + q_4 L \nu_4^2 I_{2N} + \frac{L}{q_4} D_1^\top(L,x) D_1(L,x)\right) \geq 0,
\end{aligned}
\tag{3.31}
$$

with $\nu_3 = \max(\lambda(P_3))$, $\nu_4 = \max(\lambda(P_4))$, *then for every* $\alpha > 0$, *there exist positive constants* δ, c, *and* b *such that, for any* d *in* $C^2([0,\infty);\mathbb{R}^N)$, $\widetilde{Z}_0 \in H^2([0,L];\mathbb{R}^{2N})$ *and* $\widetilde{X}_0 \in \mathbb{R}^{2N}$ *satisfying, for all* $t \geq 0$, $\|R(t)\|_{H^2} + \|\hat{R}(t)\|_{H^2} + \|d\|_{C^2} + |\widetilde{X}_0| + \|\widetilde{Z}_0\|_{H^2} \leq \delta$ *and compatibility conditions (3.24)–(3.25), the* H^2*-solution to Cauchy problems (3.19)–(3.23) satisfies, for all* $t \in [0,\infty)$,

$$
\begin{aligned}
& \|\widetilde{Z}(\cdot,t)\|^2_{H^2([0,L];\mathbb{R}^{2N})} + |\widetilde{X}(t)|^2 \\
& \leq c e^{-\alpha t}\left(\|\widetilde{Z}_0\|^2_{H^2([0,L];\mathbb{R}^{2N})} + |\widetilde{X}_0|^2\right) + b\int_0^t (|\dot{d}(s)|^2 + |\ddot{d}(s)|^2)\,\mathrm{d}s.
\end{aligned}
\tag{3.32}
$$

Proof The following Lyapunov function candidate is introduced for the stability analysis of systems (3.19)–(3.25), for all $x \in [0, L]$,

$$
V\left(\widetilde{Z}(x,\cdot), \widetilde{X}(\cdot), \partial_t \widetilde{Z}(x,\cdot), \partial_{tt}\widetilde{Z}(x,\cdot)\right) = V_1 + V_2 + V_3,
$$

where

$$V_1 = \int_0^L \Big(\widetilde{Z}^\top(x,\cdot)\mathcal{P}_{11}(x)\widetilde{Z}(x,\cdot) + \widetilde{X}^\top(\cdot)P_{22}\widetilde{X}(\cdot) + \widetilde{Z}^\top(x,\cdot)\mathcal{P}_{12}(x)\widetilde{X}(\cdot) + \widetilde{X}^\top(\cdot)\mathcal{P}_{12}^\top(x)\widetilde{Z}(x,\cdot)\Big)\,\mathrm{d}x, \tag{3.33}$$

$$V_2 = \int_0^L \partial_t\widetilde{Z}^\top(x,\cdot)\mathcal{P}_3(x)\partial_t\widetilde{Z}(x,\cdot)\,\mathrm{d}x, \tag{3.34}$$

$$V_3 = \int_0^L \partial_{tt}\widetilde{Z}^\top(x,\cdot)\mathcal{P}_4(x)\partial_{tt}\widetilde{Z}(x,\cdot)\,\mathrm{d}x, \tag{3.35}$$

with $\mathcal{P}_{11}(x)=P_{11}\mathrm{diag}\left\{e^{-\mu x}I_m, e^{\mu x}I_{2N-m}\right\}$, $\mathcal{P}_{12}(x)=P_{12}\mathrm{diag}\left\{e^{-\frac{\mu}{2}x}I_m, e^{\frac{\mu}{2}x}I_{2N-m}\right\}$, $\mathcal{P}_3(x) = P_3\mathrm{diag}\left\{e^{-\mu x}I_m, e^{\mu x}I_{2N-m}\right\}$, $\mathcal{P}_4(x) = P_4\mathrm{diag}\left\{e^{-\mu x}I_m, e^{\mu x}I_{2N-m}\right\}$. Under the definition of V and straightforward observations, there exists a positive real constant β such that, for every $\widetilde{Z}$, we obtain the following inequality,

$$\frac{1}{\beta}\int_0^L \left(\|\widetilde{Z}(x,\cdot)\|_{L^2}^2 + |\widetilde{X}(\cdot)|^2 + \|\partial_x\widetilde{Z}(x,\cdot)\|_{L^2}^2 + \|\partial_{xx}\widetilde{Z}(x,\cdot)\|_{L^2}^2\right)\mathrm{d}x \le V \le \beta\int_0^L \left(\|\widetilde{Z}(x,\cdot)\|_{L^2}^2 + |\widetilde{X}(\cdot)|^2 + \|\partial_x\widetilde{Z}(x,\cdot)\|_{L^2}^2 + \|\partial_{xx}\widetilde{Z}(x,\cdot)\|_{L^2}^2\right)\mathrm{d}x. \tag{3.36}$$

Taking time derivative of V_1 along the solutions to (3.19)–(3.25), using integration by parts, and defining $\dot{V}_1 = \dot{V}_{1L} + \dot{V}_{1NL}$, where $\dot{V}_{1L}$ is the time derivative of V_1 along the linear part of quasilinear target systems (3.19)–(3.25), for all $t \in [0,\infty)$, with positive constants κ_1, κ_2,

$$\begin{aligned}
\dot{V}_{1L} \le{} & \widetilde{Z}_{\mathrm{out}}^\top(\cdot)\left(K_P^\top\check{E}_1P_{11}K_P - e^{-\mu L}\check{E}_2P_{11}\right)\widetilde{Z}_{\mathrm{out}}(\cdot) + 2\widetilde{Z}_{\mathrm{out}}^\top(\cdot)K_P^\top\check{E}_1P_{11}\widetilde{X}(\cdot)\\
&+ \widetilde{X}^\top(\cdot)\check{E}_1P_{11}\widetilde{X}(\cdot) + \int_0^L \widetilde{Z}^\top(x,\cdot)\left(\Lambda'(x)\mathcal{P}_{11}(x) - \mu|\Lambda(x)|\mathcal{P}_{11}(x)\right)\widetilde{Z}(x,\cdot)\,\mathrm{d}x\\
&+ 2\int_0^L \widetilde{Z}^\top(x,\cdot)\mathcal{P}_{11}(x)\Sigma_2(x)\widetilde{Z}(x,\cdot)\,\mathrm{d}x + q_1Le^{2\mu L}\nu_1^2\int_0^L \widetilde{Z}^\top(x,\cdot)\widetilde{Z}(x,\cdot)\,\mathrm{d}x\\
&+ \left(\frac{L}{q_1} + \frac{L}{q_2}\right)\int_0^L \left(D_1(L,x)\widetilde{Z}(x,\cdot)\right)^\top\left(D_1(L,x)\widetilde{Z}(x,\cdot)\right)\mathrm{d}x\\
&+ 2\widetilde{X}^\top(\cdot)\check{M}_1\widetilde{X}(\cdot) + 2\int_0^L \widetilde{Z}^\top(x,\cdot)\Big(\Lambda'(x)\mathcal{P}_{12}(x) - \frac{\mu}{2}|\Lambda(x)|\mathcal{P}_{12}(x)\Big)\widetilde{X}(\cdot)\,\mathrm{d}x\\
&+ 2\widetilde{Z}_{\mathrm{out}}^\top(\cdot)\left(K_P^\top\check{M}_1 + \check{M}_2\right)\widetilde{X}(\cdot) + 2\int_0^L \widetilde{Z}^\top(x,\cdot)\mathcal{P}_{12}(x)K_I\widetilde{Z}_{\mathrm{out}}(\cdot)\,\mathrm{d}x\\
&+ \kappa_1\int_0^L \widetilde{Z}^\top(x,\cdot)\mathcal{P}_{12}(x)\left(\widetilde{Z}^\top(x,\cdot)\mathcal{P}_{12}(x)\right)^\top\mathrm{d}x + \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right)(\Gamma_0\dot{d}(\cdot))^\top\Gamma_0\dot{d}(\cdot)\\
&+ 2\int_0^L \widetilde{X}^\top(\cdot)\mathcal{P}_{12}^\top(x)\Sigma_2(x)\widetilde{Z}(x,\cdot)\,\mathrm{d}x + q_2Le^{\mu L}\nu_2^2\int_0^L \widetilde{X}^\top(\cdot)\widetilde{X}(\cdot)\,\mathrm{d}x
\end{aligned}$$

$$+ 2L\widetilde{Z}_{\text{out}}^{\top}(\cdot)K_I^{\top}P_{22}\widetilde{X}(\cdot) + L\kappa_2\widetilde{X}^{\top}(\cdot)P_{22}\left(\widetilde{X}^{\top}(\cdot)P_{22}\right)^{\top}, \tag{3.37}$$

where $\check{E}_1 = \text{diag}\left\{\Lambda^+(0), e^{\mu L}\Lambda^-(L)\right\}$, $\check{E}_2 = \text{diag}\left\{\Lambda^+(L), e^{\mu L}\Lambda^-(0)\right\}$,

$$\check{M}_1 = \begin{bmatrix} \Lambda^+(0)P_{12}^{++} & \Lambda^+(0)P_{12}^{+-} \\ -e^{-\frac{\mu}{2}L}\Lambda^-(L)P_{12}^{-+} & -e^{\frac{\mu}{2}L}\Lambda^-(L)P_{12}^{--} \end{bmatrix},$$

$$\check{M}_2 = \begin{bmatrix} -e^{-\frac{\mu}{2}L}\Lambda^+(L)P_{12}^{++} & -e^{\frac{\mu}{2}L}\Lambda^+(L)P_{12}^{+-} \\ \Lambda^-(0)P_{12}^{-+} & \Lambda^-(0)P_{12}^{--} \end{bmatrix},$$

and where $\dot{V}_{1NL}$ satisfies the following inequality,

$$\begin{aligned} \dot{V}_{1NL} \leq & -2\int_0^L Q_{NL}^{\top}\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right]\left(\mathcal{P}_{11}(x)\widetilde{Z}(x,\cdot) + \mathcal{P}_{12}(x)\widetilde{X}(\cdot)\right)\mathrm{d}x \\ & + \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right)\left(\Gamma_0\dot{d}(\cdot) - \dot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)] - \int_0^L k_2(\xi)\partial_t\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi\right)^{\top} \\ & \times\left(\Gamma_0\dot{d}(\cdot) - \dot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)] - \int_0^L k_2(\xi)\partial_t\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi\right) \\ & - \left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right)(\Gamma_0\dot{d}(\cdot))^{\top}\Gamma_0\dot{d}(\cdot), \end{aligned} \tag{3.38}$$

with $\dot{G}_{NL}$, the time derivative of G_{NL} along the solutions to (3.1)–(3.5) and (3.19)–(3.25).

By time differentiation of (3.19), $\partial_t\widetilde{Z}$ is shown to satisfy the following equations, for all $x \in [0, L]$, $t \in [0, \infty)$,

$$\begin{aligned} \partial_{tt}\widetilde{Z}(x,\cdot) = & -\Lambda(x)\partial_{tx}\widetilde{Z}(x,\cdot) + \Sigma_2(x)\partial_t\widetilde{Z}(x,\cdot) + \int_0^x D_1(x,\xi)\partial_t\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi \\ & - \dot{Q}_{NL}\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right], \end{aligned} \tag{3.39}$$

$$\dot{\widetilde{Z}}_{\text{in}}(\cdot) = K_P\dot{\widetilde{Z}}_{\text{out}}(\cdot) + \dot{\widetilde{X}}(\cdot), \tag{3.40}$$

$$\dot{\widetilde{X}}(\cdot) = K_I\widetilde{Z}_{\text{out}}(\cdot) - \int_0^L k_2(\xi)\partial_t\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi + \Gamma_0\dot{d}(\cdot) - \dot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)], \tag{3.41}$$

where $\dot{Q}_{NL}$ is the time derivative of Q_{NL} along the solutions to (3.1)–(3.5) and (3.19)–(3.25).

Taking time derivative of V_2 along the solutions to (3.19)–(3.25), using integration by parts, and defining $\dot{V}_2 = \dot{V}_{2L} + \dot{V}_{2NL}$, where $\dot{V}_{2L}$ is the time derivative of V_2 along the linear part of quasilinear target systems (3.19)–(3.25), with positive constants κ_3 and κ_4,

$$\dot{V}_{2L} \leq \dot{\widetilde{Z}}_{\text{out}}^{\top}(\cdot)\left(K_P^{\top}\check{E}_1P_3K_P - e^{-\mu L}\check{E}_2P_3\right)\dot{\widetilde{Z}}_{\text{out}}(\cdot) + 2\dot{\widetilde{Z}}_{\text{out}}^{\top}(\cdot)K_P^{\top}P_3\check{E}_1K_I\widetilde{Z}_{\text{out}}(\cdot)$$

$$+\widetilde{Z}_{\text{out}}^{\top}(\cdot)K_I^{\top}\check{E}_1P_3K_I\widetilde{Z}_{\text{out}}(\cdot)+\kappa_3\dot{\widetilde{Z}}_{\text{out}}^{\top}(\cdot)K_P^{\top}\check{E}_1P_3\left(K_P^{\top}\check{E}_1P_3\right)^{\top}\dot{\widetilde{Z}}_{\text{out}}(\cdot)$$
$$+\kappa_4\widetilde{Z}_{\text{out}}^{\top}(\cdot)K_I^{\top}\check{E}_1P_3\left(K_I^{\top}\check{E}_1P_3\right)^{\top}\widetilde{Z}_{\text{out}}(\cdot)$$
$$+\int_0^L\partial_t\widetilde{Z}^{\top}(x,\cdot)\left(\Lambda'(x)\mathcal{P}_3(x)-\mu|\Lambda(x)|\mathcal{P}_3(x)\right)\partial_t\widetilde{Z}(x,\cdot)\,\mathrm{d}x$$
$$+2\int_0^L\partial_t\widetilde{Z}^{\top}(x,\cdot)\mathcal{P}_3(x)\Sigma_2(x)\partial_t\widetilde{Z}(x,\cdot)\,\mathrm{d}x$$
$$+\dot{d}(\cdot)^{\top}\Gamma_0^{\top}\left(\left(\frac{1}{\kappa_3}+\frac{1}{\kappa_4}\right)I_{2N}+\check{E}_1P_3\right)\Gamma_0\dot{d}(\cdot)$$
$$+q_3Le^{2\mu L}\nu_3^2\int_0^L\partial_t\widetilde{Z}^{\top}(x,\cdot)\partial_t\widetilde{Z}(x,\cdot)\,\mathrm{d}x$$
$$+\frac{L}{q_3}\int_0^L\left(D_1(L,x)\partial_t\widetilde{Z}(x,\cdot)\right)^{\top}\left(D_1(L,x)\partial_t\widetilde{Z}(x,\cdot)\right)\,\mathrm{d}x,\tag{3.42}$$

and where $\dot{V}_{2NL}$ satisfies the following inequality,

$$\dot{V}_{2NL}\leq\left(\Gamma_0\dot{d}(\cdot)-\dot{G}_{NL}[\widetilde{Z}(0,\cdot),\hat{R}(0,\cdot)]-\int_0^Lk_2(\xi)\partial_t\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi\right)^{\top}$$
$$\times\left(\left(\frac{1}{\kappa_3}+\frac{1}{\kappa_4}\right)I_{2N}+\check{E}_1P_3\right)$$
$$\times\left(\Gamma_0\dot{d}(\cdot)-\dot{G}_{NL}[\widetilde{Z}(0,\cdot),\hat{R}(0,\cdot)]-\int_0^Lk_2(\xi)\partial_t\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi\right)$$
$$-2\int_0^L\dot{Q}_{NL}^{\top}\left[\widetilde{Z},\hat{R},\partial_x\widetilde{Z},\partial_x\hat{R}\right]\mathcal{P}_3(x)\partial_t\widetilde{Z}(x,\cdot)\,\mathrm{d}x$$
$$-\dot{d}(\cdot)^{\top}\Gamma_0^{\top}\left(\left(\frac{1}{\kappa_3}+\frac{1}{\kappa_4}\right)I_{2N}+\check{E}_1P_3\right)\Gamma_0\dot{d}(\cdot).\tag{3.43}$$

By second time differentiation of (3.19), $\partial_{tt}\widetilde{Z}$ is shown to satisfy the following equations, for all $x\in[0,L]$, $t\in[0,\infty)$,

$$\partial_{ttt}\widetilde{Z}(x,\cdot)=-\Lambda(x)\partial_{ttx}\widetilde{Z}(x,\cdot)+\Sigma_2(x)\partial_{tt}\widetilde{Z}(x,\cdot)+\int_0^xD_1(x,\xi)\partial_{tt}\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi$$
$$-\ddot{Q}_{NL}\left[\widetilde{Z},\hat{R},\partial_x\widetilde{Z},\partial_x\hat{R}\right],\tag{3.44}$$
$$\ddot{\widetilde{Z}}_{\text{in}}(\cdot)=K_P\ddot{\widetilde{Z}}_{\text{out}}(\cdot)+\ddot{\widetilde{X}}(\cdot),\tag{3.45}$$
$$\ddot{\widetilde{X}}(\cdot)=K_I\dot{\widetilde{Z}}_{\text{out}}(\cdot)-\int_0^Lk_2(\xi)\partial_{tt}\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi+\Gamma_0\ddot{d}(\cdot)-\ddot{G}_{NL}[\widetilde{Z}(0,\cdot),\hat{R}(0,\cdot)],\tag{3.46}$$

where $\ddot{Q}_{NL}$ and $\ddot{G}_{NL}$ are, respectively, the second-order time derivative of Q_{NL} and G_{NL} along the solutions to (3.1)–(3.5) and (3.19)–(3.25).

Taking time derivative of V_3 along the solutions to (3.19)–(3.25), using integration by parts, and defining $\dot{V}_3 = \dot{V}_{3L} + \dot{V}_{3NL}$, where $\dot{V}_{3L}$ is the time derivative of V_3 along the linear part of quasilinear target systems (3.19)–(3.25), with positive constants κ_5 and κ_6,

$$\begin{aligned}\dot{V}_{3L} \leq & \ddot{\check{Z}}_{\text{out}}^{\top}(\cdot)\left(K_P^{\top}\check{E}_1P_4K_P - e^{-\mu L}\check{E}_2P_4\right)\ddot{\check{Z}}_{\text{out}}(\cdot) + 2\ddot{\check{Z}}_{\text{out}}^{\top}(\cdot)K_P^{\top}P_4\check{E}_1K_I\dot{\check{Z}}_{\text{out}}(\cdot)\\ &+ \dot{\check{Z}}_{\text{out}}^{\top}(\cdot)K_I^{\top}\check{E}_1P_4K_I\dot{\check{Z}}_{\text{out}}(\cdot) + \kappa_5\ddot{\check{Z}}_{\text{out}}^{\top}(\cdot)K_P^{\top}\check{E}_1P_4\left(K_P^{\top}\check{E}_1P_4\right)^{\top}\ddot{\check{Z}}_{\text{out}}(\cdot)\\ &+ \kappa_6\dot{\check{Z}}_{\text{out}}^{\top}(\cdot)K_I^{\top}\check{E}_1P_4\left(K_I^{\top}\check{E}_1P_4\right)^{\top}\dot{\check{Z}}_{\text{out}}(\cdot)\\ &+ \int_0^L \partial_{tt}\widetilde{Z}^{\top}(x,\cdot)\left(\Lambda'(x)\mathcal{P}_4(x) - \mu|\Lambda(x)|\mathcal{P}_4(x)\right)\partial_{tt}\widetilde{Z}(x,\cdot)\,\mathrm{d}x\\ &+ 2\int_0^L \partial_{tt}\widetilde{Z}^{\top}(x,\cdot)\mathcal{P}_4(x)\Sigma_2(x)\partial_{tt}\widetilde{Z}(x,\cdot)\,\mathrm{d}x\\ &+ q_4Le^{2\mu L}\nu_4^2\int_0^L \partial_{tt}\widetilde{Z}^{\top}(x,\cdot)\partial_{tt}\widetilde{Z}(x,\cdot)\,\mathrm{d}x + \ddot{d}^{\top}(\cdot)\Gamma_0^{\top}\\ &\times\left(\left(\frac{1}{\kappa_5}+\frac{1}{\kappa_6}\right)I_{2N} + P_4\check{E}_1\right)\Gamma_0\ddot{d}(\cdot)\\ &+ \frac{L}{q_4}\int_0^L \left(D_1(L,x)\partial_{tt}\widetilde{Z}(x,\cdot)\right)^{\top}\left(D_1(L,x)\partial_{tt}\widetilde{Z}(x,\cdot)\right)\,\mathrm{d}x, \end{aligned}\tag{3.47}$$

and where $\dot{V}_{3NL}$ satisfies the following inequality,

$$\begin{aligned}\dot{V}_{3NL} \leq & \left(\Gamma_0\ddot{d}(\cdot) - \ddot{G}_{NL}[\widetilde{Z}(0,\cdot),\hat{R}(0,\cdot)] - \int_0^L k_2(\xi)\partial_{tt}\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi\right)^{\top}\\ &\left(\left(\frac{1}{\kappa_5}+\frac{1}{\kappa_6}\right)I_{2N} + P_4\check{E}_1\right)\\ &\times\left(\Gamma_0\ddot{d}(\cdot) - \ddot{G}_{NL}[\widetilde{Z}(0,\cdot),\hat{R}(0,\cdot)] - \int_0^L k_2(\xi)\partial_{tt}\widetilde{Z}(\xi,\cdot)\,\mathrm{d}\xi\right)\\ &- 2\int_0^L \ddot{Q}_{NL}^{\top}\left[\widetilde{Z},\hat{R},\partial_x\widetilde{Z},\partial_x\hat{R}\right]\mathcal{P}_4(x)\partial_{tt}\widetilde{Z}(x,\cdot)\,\mathrm{d}x\\ &- \ddot{d}^{\top}(\cdot)\Gamma_0^{\top}\left(\left(\frac{1}{\kappa_5}+\frac{1}{\kappa_6}\right)I_{2N} + P_4\check{E}_1\right)\Gamma_0\ddot{d}(\cdot). \end{aligned}\tag{3.48}$$

For the linear term $\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L}$, by using (3.37), (3.42), and (3.47), there exists a constant $\alpha > 0$ such that

$$\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L}$$

$$
\begin{aligned}
&\leq -\alpha V - \int_0^L \begin{bmatrix} \widetilde{Z}(x,\cdot) \\ \widetilde{Z}_{\mathrm{out}}(\cdot) \\ \widetilde{X}(\cdot) \\ \dot{\widetilde{Z}}_{\mathrm{out}}(\cdot) \\ \ddot{\widetilde{Z}}_{\mathrm{out}}(\cdot) \end{bmatrix}^\top \check{\Omega}(x) \begin{bmatrix} \widetilde{Z}(x,\cdot) \\ \widetilde{Z}_{\mathrm{out}}(\cdot) \\ \widetilde{X}(\cdot) \\ \dot{\widetilde{Z}}_{\mathrm{out}}(\cdot) \\ \ddot{\widetilde{Z}}_{\mathrm{out}}(\cdot) \end{bmatrix} \mathrm{d}x \\
&\quad - \int_0^L \partial_t \widetilde{Z}^\top(x,\cdot)\check{M}(x)\partial_t \widetilde{Z}(x,\cdot)\,\mathrm{d}x - \int_0^L \partial_{tt}\widetilde{Z}^\top(x,\cdot)\check{K}(x)\partial_{tt}\widetilde{Z}(x,\cdot)\,\mathrm{d}x \\
&\quad + \dot{d}^\top(\cdot)\Gamma_0^\top\left(\left(\frac{L}{\kappa_1}+\frac{L}{\kappa_2}+\frac{1}{\kappa_3}+\frac{1}{\kappa_4}\right)I_{2N} + P_3\check{E}_1\right)\Gamma_0\dot{d}(\cdot) \\
&\quad + \ddot{d}^\top(\cdot)\Gamma_0^\top\left(\left(\frac{1}{\kappa_5}+\frac{1}{\kappa_6}\right)I_{2N} + P_4\check{E}_1\right)\Gamma_0\ddot{d}(\cdot),
\end{aligned}
\tag{3.49}
$$

where, for $x \in [0, L]$, $\check{\Omega}(x) = \begin{bmatrix} \check{\Omega}_{11}(x) & \check{\Omega}_{12}(x) & \check{\Omega}_{13}(x) & \check{\Omega}_{14} & \check{\Omega}_{15} \\ * & \check{\Omega}_{22} & \check{\Omega}_{23} & \check{\Omega}_{24} & \check{\Omega}_{25} \\ * & * & \check{\Omega}_{33} & \check{\Omega}_{34} & \check{\Omega}_{35} \\ * & * & * & \check{\Omega}_{44} & \check{\Omega}_{45} \\ * & * & * & * & \check{\Omega}_{55} \end{bmatrix}$, with

$$
\begin{aligned}
\check{\Omega}_{11}(x) &= \mu|\Lambda(x)|\mathcal{P}_{11}(x) - \Lambda'(x)\mathcal{P}_{11}(x) - \alpha\mathcal{P}_{11}(x) - \kappa_1\mathcal{P}_{12}(x)\left(\mathcal{P}_{12}(x)\right)^\top \\
&\quad - \Big(\Sigma_2^\top(x)\mathcal{P}_{11}(x) + \mathcal{P}_{11}(x)\Sigma_2(x) + q_1 L e^{2\mu L}\nu_1^2 I_{2N} \\
&\quad + \left(\frac{L}{q_1}+\frac{L}{q_2}\right)D_1^\top(L,x)D_1(L,x)\Big), \\
\check{\Omega}_{12}(x) &= -\mathcal{P}_{12}(x)K_I, \\
\check{\Omega}_{13}(x) &= \frac{\mu}{2}|\Lambda(x)|\mathcal{P}_{12}(x) - \Lambda'(x)\mathcal{P}_{12}(x) - \alpha\mathcal{P}_{12}(x) - \Sigma_2^\top(x)\mathcal{P}_{12}(x), \\
\check{\Omega}_{14} &= \check{\Omega}_{15} = \check{\Omega}_{25} = \check{\Omega}_{34} = \check{\Omega}_{35} = 0_{2N,2N}, \\
\check{\Omega}_{22} &= \frac{e^{-\mu L}}{L}\check{E}_2 P_{11} - \frac{1}{L}K_P^\top\check{E}_1 P_{11}K_P - \frac{1}{L}K_I^\top\check{E}_1 P_3 K_I - \frac{\kappa_4}{L}K_I^\top\check{E}_1 P_3\left(K_I^\top\check{E}_1 P_3\right)^\top, \\
\check{\Omega}_{23} &= -\frac{1}{L}K_P^\top\check{E}_1 P_{11} - \frac{1}{L}\left(K_P^\top\check{M}_1 + \check{M}_2\right) - K_I^\top P_{22}, \\
\check{\Omega}_{24} &= -\frac{1}{L}K_I^\top\check{E}_1 P_3 K_P, \\
\check{\Omega}_{33} &= -\frac{1}{L}\check{E}_1 P_{11} - \frac{1}{L}\left(\check{M}_1 + \check{M}_1^\top\right) - \kappa_2 P_{22}\left(P_{22}\right)^\top - \alpha P_{22} - q_2 L e^{\mu L}\nu_2^2 I_{2N}, \\
\check{\Omega}_{44} &= \frac{e^{-\mu L}}{L}\check{E}_2 P_3 - \frac{1}{L}K_P^\top\check{E}_1 P_3 K_P - \frac{\kappa_3}{L}K_P^\top\check{E}_1 P_3\left(K_P^\top\check{E}_1 P_3\right)^\top - \frac{1}{L}K_I^\top\check{E}_1 P_4 K_I \\
&\quad - \frac{\kappa_6}{L}K_I^\top\check{E}_1 P_4\left(K_I^\top\check{E}_1 P_4\right)^\top, \\
\check{\Omega}_{45} &= -\frac{1}{L}K_I^\top\check{E}_1 P_4 K_P, \\
\check{\Omega}_{55} &= \frac{e^{-\mu L}}{L}\check{E}_2 P_4 - \frac{1}{L}K_P^\top\check{E}_1 P_4 K_P - \frac{\kappa_5}{L}K_P^\top\check{E}_1 P_4\left(K_P^\top\check{E}_1 P_4\right)^\top,
\end{aligned}
$$

$$\check{M}(x) = \left(-\Lambda'(x) + \mu|\Lambda(x)| - \alpha I_{2N}\right)\mathcal{P}_3(x) - \left(\Sigma_2^\top(x)\mathcal{P}_3(x) + \mathcal{P}_3(x)\Sigma_2(x) + q_3 L e^{2\mu L}\nu_3^2 I_{2N} + \frac{L}{q_3}D_1^\top(L,x)D_1(L,x)\right),$$
$$\check{K}(x) = \left(-\Lambda'(x) + \mu|\Lambda(x)| - \alpha I_{2N}\right)\mathcal{P}_4(x) - \left(\Sigma_2^\top(x)\mathcal{P}_4(x) + \mathcal{P}_4(x)\Sigma_2(x) + q_4 L e^{2\mu L}\nu_4^2 I_{2N} + \frac{L}{q_4}D_1^\top(L,x)D_1(L,x)\right).$$

Under conditions (3.29), (3.30), (3.31), there exist constants μ,κ_1, κ_2, κ_3, κ_4, κ_5, $\kappa_6 > 0$ small enough, such that $\check{\Omega} \geq 0$ and $\check{M}$, $\check{K} \geq 0$; thus, there exist positive constants α_1, α_2 such that, with $\alpha_1 = \max\left\{\lambda\left(\Gamma_0^\top\left(\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4}\right)I_{2N} + P_3\check{E}_1\right)\Gamma_0\right)\right\}$ and

$$\alpha_2 = \max\left\{\lambda\left(\Gamma_0^\top\left(\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6}\right)I_{2N} + P_4\check{E}_1\right)\Gamma_0\right)\right\},$$

the linear term $\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L}$ satisfies the following inequality,

$$\dot{V}_{1L} + \dot{V}_{2L} + \dot{V}_{3L} \leq -\alpha V + \alpha_1 \dot{d}^\top \dot{d} + \alpha_2 \ddot{d}^\top \ddot{d}. \tag{3.50}$$

Now we analyze the nonlinear term $\dot{V}_{1NL} + \dot{V}_{2NL} + \dot{V}_{3NL}$. From (3.38), (3.43), and (3.48), there exist positive constants h_1, h_2, h_3, h_4, h_5, and h_6 such that

$$\dot{V}_{1NL} \leq 2h_1\int_0^L \left|Q_{NL}^\top\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right]\right| \left(|\mathcal{P}_{11}(x)|\left|\widetilde{Z}(x,\cdot)\right| + |\mathcal{P}_{12}(x)|\left|\widetilde{X}(\cdot)\right|\right)\mathrm{d}x + h_2\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right)\left(2\left|\dot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)]\right|^2 + 2V_2 + \dot{d}^\top(\cdot)\Gamma_0^\top\Gamma_0\dot{d}(\cdot)\right), \tag{3.51}$$

$$\dot{V}_{2NL} \leq h_3\left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + \left|\check{E}_1 P_3\right|\right)\left(2\left|\dot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)]\right|^2 + 2V_2 + \dot{d}^\top(\cdot)\Gamma_0^\top\Gamma_0\dot{d}(\cdot)\right) + 2h_4\int_0^L \left|\dot{Q}_{NL}^\top\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right]\right| |\mathcal{P}_3(x)| \left|\partial_t\widetilde{Z}(x,\cdot)\right| \mathrm{d}x, \tag{3.52}$$

$$\dot{V}_{3NL} \leq h_5\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \left|P_4\check{E}_1\right|\right)\left(2\left|\ddot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)]\right|^2 + 2V_3 + \ddot{d}^\top(\cdot)\Gamma_0^\top\Gamma_0\ddot{d}(\cdot)\right) + 2h_6\int_0^L \left|\ddot{Q}_{NL}^\top\left[\widetilde{Z}, \hat{R}, \partial_x\widetilde{Z}, \partial_x\hat{R}\right]\right| |\mathcal{P}_4(x)| \left|\partial_{tt}\widetilde{Z}(x,\cdot)\right| \mathrm{d}x. \tag{3.53}$$

As derived in paper [9], since $\Lambda_{NL}(\widetilde{Z},\cdot)$ is twice differentiable with respect to $\widetilde{Z}$ and x, and $\Lambda_{NL}(0,\cdot) = 0_{2N,2N}$, there exist positive constants δ_Λ and s_1, s_2, s_3 such that for any $w_1, \nu_1 \in \mathbb{R}^{2N}$, if $\|\widetilde{Z}\|_\infty \leq \delta_\Lambda$, it holds that

$$\left\|\Lambda_{NL}(\widetilde{Z},\cdot)\right\|_\infty \leq s_1\|\widetilde{Z}\|_\infty, \tag{3.54}$$
$$\left\|\partial_{\widetilde{Z}}\Lambda_{NL}(\widetilde{Z},\cdot)w_1\right\|_\infty + \left\|\partial_x\Lambda_{NL}(\widetilde{Z},\cdot)\right\|_\infty \leq s_2\|w_1\|_\infty, \tag{3.55}$$
$$\left\|\partial^2_{\widetilde{Z}}\Lambda_{NL}(\widetilde{Z},\cdot)\nu_1\right\|_\infty \leq s_3\|\nu_1\|_\infty. \tag{3.56}$$

Similarly, since $\Sigma_{NL}(\widetilde{Z},\cdot)$ is twice differentiable with respect to $\widetilde{Z}$ and x, and $\Sigma_{NL}(0,\cdot) = 0_{2N,2N}$, there exist positive constants δ_Σ and s_4, s_5, s_6 such that for any $w_2, \nu_2 \in \mathbb{R}^{2N}$, if $\|\widetilde{Z}\|_\infty \le \delta_\Sigma$, it holds that

$$\left\|\Sigma_{NL}(\widetilde{Z},\cdot)\right\|_\infty \le s_4\|\widetilde{Z}\|_\infty, \tag{3.57}$$

$$\left\|\partial_{\widetilde{Z}}\Sigma_{NL}(\widetilde{Z},\cdot)w_2\right\|_\infty + \left\|\partial_x\Sigma_{NL}(\widetilde{Z},\cdot)\right\|_\infty \le s_5\|w_2\|_\infty, \tag{3.58}$$

$$\left\|\partial^2_{\widetilde{Z}}\Sigma_{NL}(\widetilde{Z},\cdot)\nu_2\right\|_\infty \le s_6\|\nu_2\|_\infty. \tag{3.59}$$

Because Π_{NL} is once differentiable with respect to $\widetilde{Z}$, and $\Pi_{NL}(0) = 0_{2N,1}$, there exist positive constants δ_Π and s_7, s_8 such that for any $w_3 \in \mathbb{R}^{2N}$, if $\|\widetilde{Z}\|_\infty \le \delta_\Pi$,

$$\left|\Gamma_0\Pi_{NL}(\widetilde{Z}(0,\cdot))\right| \le s_7\|\widetilde{Z}(0,\cdot)\|_\infty, \tag{3.60}$$

$$\left|\Gamma_0\partial_{\widetilde{Z}}\Pi_{NL}(\widetilde{Z}(0,\cdot))w_3\right| \le s_8\|w_3\|_\infty. \tag{3.61}$$

Note that the nonlinear terms Q_{NL} and G_{NL} in (3.26)–(3.27) are dependent on the variables $\widetilde{Z}$, $\partial_x\widetilde{Z}$, $\hat{R}$, and $\partial_x\hat{R}$. For $\hat{R} \in H^2([0,L]\times[0,\infty);\mathbb{R}^{2N})$, recall the well-known inequalities: $||\hat{R}||_{L^1} \le C_1||\hat{R}||_{L^2} \le C_2||\hat{R}||_\infty$, $||\hat{R}||_\infty \le C_3(||\hat{R}||_{L^2} + ||\partial_x\hat{R}||_{L^2}) \le C_4||\hat{R}||_{H^1}$, $||\partial_x\hat{R}||_\infty \le C_5(||\partial_x\hat{R}||_{L^2} + ||\partial_{xx}\hat{R}||_{L^2}) \le C_6||\hat{R}||_{H^2}$ with the positive constants $C_1, C_2, C_3, C_4, C_5, C_6$. By using these inequalities with inequalities (3.54)–(3.61), relations (3.11) and (3.26)–(3.27), recalling assumptions on $R(t)$, $\hat{R}(t)$, and $d(t)$ in Theorem 3.1, with (3.26)–(3.27), there exist positive constants δ_1, h_7, h_8, h_9, h_{10}, h_{11} such that for all $\widetilde{Z}_0$ satisfying $\|\widetilde{Z}_0\|_\infty \le \delta_1 \le \min\{\delta_\Lambda, \delta_\Sigma, \delta_\Pi\}$, it holds $\left\|\mathcal{F}[\widetilde{Z}]\right\|_{L^2} \le h_7\|\widetilde{Z}\|_{L^2}$, $\|\widetilde{Z}\|^2_{L^2} + |\widetilde{X}|^2 \le h_8V_1$, $\|\partial_t\widetilde{Z}\|^2_{L^2} \le h_9V_2$, and

$$\left|Q_{NL}\left[\widetilde{Z},\hat{R},\partial_x\widetilde{Z},\partial_x\hat{R}\right]\right| \le h_{10}\left(\|\partial_t\widetilde{Z}\|^2_\infty + \|\widetilde{Z}\|^2_\infty\right),$$

$$\left|\dot{G}_{NL}[\widetilde{Z}(0,\cdot),\hat{R}(0,\cdot)]\right| \le h_{12}|\partial_t\widetilde{Z}(0,\cdot)|.$$

For all $\widetilde{Z}$ satisfying $\|\widetilde{Z}\|_\infty \le \delta_1$, the following inequality is deduced from (3.51),

$$\begin{aligned}\dot{V}_{1NL} \le& 2h_1h_{10}\left(\|\partial_t\widetilde{Z}\|_\infty + \delta_1\right)(V_1+V_2)\\ &+ h_2\left(\frac{L}{\kappa_1}+\frac{L}{\kappa_2}\right)\left(2h_{11}^2\|\partial_t\widetilde{Z}\|_\infty V_2 + 2V_2 + \dot{d}^\top(\cdot)\Gamma_0^\top\Gamma_0\dot{d}(\cdot)\right).\end{aligned} \tag{3.62}$$

From (3.26)–(3.27) and inequalities (3.54)–(3.61), there exist positive constants $\delta_2 \le \delta_1$, h_{12} such that for all $\widetilde{Z}$ satisfying $\|\widetilde{Z}\|_\infty + \|\partial_t\widetilde{Z}\|_\infty \le \delta_2$, it holds $\left|\dot{Q}_{NL}\left[\widetilde{Z},\hat{R},\partial_x\widetilde{Z},\partial_x\hat{R}\right]\right| \le h_{12}\left(\|\partial_{tt}\widetilde{Z}\|^2_\infty + \|\partial_t\widetilde{Z}\|^2_\infty + \|\widetilde{Z}\|^2_\infty\right)$, so from (3.52),

$$\begin{aligned}\dot{V}_{2NL} \le& h_3\left(\frac{1}{\kappa_3}+\frac{1}{\kappa_4}+\max\left\{\lambda\left(P_3\check{E}_1\right)\right\}\right)\left(2h_{11}^2\delta_2V_2 + 2V_2 + \dot{d}^\top(\cdot)\Gamma_0^\top\Gamma_0\dot{d}(\cdot)\right)\\ &+ 2h_4h_{12}\delta_2(V_1+V_2+V_3).\end{aligned} \tag{3.63}$$

From (3.26)–(3.27) and inequalities (3.54)–(3.61), there exist positive constants $\delta_3 \le \delta_2$, h_{13}, h_{14} such that for all $\widetilde{Z}$ satisfying $\|\widetilde{Z}\|_\infty + \|\partial_t \widetilde{Z}\|_\infty + \|\partial_{tt}\widetilde{Z}\|_\infty \le \delta_3$, it holds $\left|\ddot{Q}_{NL}\left[\widetilde{Z}, \hat{R}, \partial_t \widetilde{Z}, \partial_x \hat{R}\right]\right| \le h_{13}\left(\|\partial_{tt}\widetilde{Z}\|_\infty^2 + \|\partial_t \widetilde{Z}\|_\infty^2 + \|\widetilde{Z}\|_\infty^2\right)$, $\left|\ddot{G}_{NL}[\widetilde{Z}(0,\cdot), \hat{R}(0,\cdot)]\right| \le h_{14}\|\partial_{tt}\widetilde{Z}(0,\cdot)\|_\infty$ and we deduce from (3.53),

$$\begin{aligned}\dot{V}_{3NL} \le & h_5\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \max\left\{\lambda\left(P_4\check{E}_1\right)\right\}\right)\left(2h_{14}^2\delta_2 V_2 + 2V_3 + \ddot{d}^\top(\cdot)\Gamma_0^\top\Gamma_0\ddot{d}(\cdot)\right)\\ & + 2h_6h_{13}\delta_3(V_1 + V_2 + V_3).\end{aligned} \tag{3.64}$$

Therefore, the nonlinear term $\dot{V}_{1NL} + \dot{V}_{2NL} + \dot{V}_{3NL}$, by using (3.62), (3.63), and (3.64) with $\alpha_3 = \max\left\{\lambda\left(\Gamma_0^\top\left(h_2\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right)I_{2N} + h_3\left(\frac{1}{\kappa_3}I_{2N} + \frac{1}{\kappa_4}I_{2N} + P_3\check{E}_1\right)\right)\Gamma_0\right)\right\}$ and

$$\alpha_4 = \max\left\{\lambda\left(\Gamma_0^\top\left(h_5\left(\frac{1}{\kappa_5}I_{2N} + \frac{1}{\kappa_6}I_{2N} + P_4\check{E}_1\right)\right)\Gamma_0\right)\right\},$$

satisfies, for $\|\widetilde{Z}\|_\infty + \|\partial_t\widetilde{Z}\|_\infty + \|\partial_{tt}\widetilde{Z}\|_\infty \le \delta_3$,

$$\begin{aligned}&\dot{V}_{1NL} + \dot{V}_{2NL} + \dot{V}_{3NL}\\ &\quad\le 2h_1h_{10}(\delta_2 + \delta_1)(V_1 + V_2) + (2h_4h_{12}\delta_2 + 2h_6h_{13}\delta_3)(V_1 + V_2 + V_3)\\ &\qquad + \left(h_2\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right) + h_3\left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + \max\left\{\lambda\left(P_3\check{E}_1\right)\right\}\right)\right)(2h_{14}^2\delta_2 V_2 + 2V_2)\\ &\qquad + h_5\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \max\left\{\lambda\left(P_4\check{E}_1\right)\right\}\right)(2h_{14}^2\delta_2V_2 + 2V_3) + \alpha_3\dot{d}^\top(\cdot)\dot{d}(\cdot) + \alpha_4\ddot{d}^\top(\cdot)\ddot{d}(\cdot).\end{aligned} \tag{3.65}$$

Thus combining inequalities (3.50) with (3.65), along the solutions to systems (3.19)–(3.25), for all $t \in [0, \infty)$, we get the existence of a positive constant $\delta_4 \le \delta_3$ such that, for all $\widetilde{Z}$ satisfying $\|\widetilde{Z}\|_\infty + \|\partial_t\widetilde{Z}\|_\infty + \|\partial_{tt}\widetilde{Z}\|_\infty \le \delta_4$,

$$\begin{aligned}V &\le V(0)e^{-\alpha t/2} + \alpha_5 e^{-\alpha t/2}\int_0^t (|\dot{d}(s)|^2 + |\ddot{d}(s)|^2)e^{\alpha s/2}\,\mathrm{d}s\\ &\le V(0)e^{-\alpha t/2} + \alpha_5\int_0^t (|\dot{d}(s)|^2 + |\ddot{d}(s)|^2)\,\mathrm{d}s,\end{aligned} \tag{3.66}$$

with $\alpha_5 = \max\{\alpha_1 + \alpha_3, \alpha_2 + \alpha_4\}$ and such that

$$\begin{aligned}&4h_1h_{10}\delta_4 + 2h_4h_{12}\delta_4 + 2h_6h_{13}\delta_4\\ &\quad + \left(h_2\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2}\right) + h_3\left(\frac{1}{\kappa_3} + \frac{1}{\kappa_4} + \max\left\{\lambda\left(P_3\check{E}_1\right)\right\}\right)\right)(2h_{11}^2\delta_4 + 2)\\ &\quad + h_5\left(\frac{1}{\kappa_5} + \frac{1}{\kappa_6} + \max\left\{\lambda\left(P_4\check{E}_1\right)\right\}\right)(2h_{14}^2\delta_4 + 2) < \alpha/2.\end{aligned}$$

Combining this relation with (3.36), there exist positive constants

$$c = \beta^2, \qquad b = \beta\alpha_5,$$

such that, for all $t \geq 0$,

$$\begin{aligned}
&\int_0^L \left(|\widetilde{Z}(x,t)|^2 + |\widetilde{X}(x,t)|^2 + |\partial_x \widetilde{Z}(x,t)|^2 + |\partial_{xx}\widetilde{Z}(x,t)|^2\right) \mathrm{d}x \\
&\leq \beta V(\cdot) \\
&\leq \beta \left(\widetilde{V}(0)e^{-\alpha t/2} + \alpha_5 \int_0^t (|\dot{d}(s)|^2 + |\ddot{d}(s)|^2)\, \mathrm{d}s\right) \\
&\leq \beta^2 e^{-\alpha t/2}\left(\int_0^L (|\widetilde{Z}_0(x)|^2 + |\widetilde{X}_0|^2 + |\partial_x \widetilde{Z}(x,0)|^2 + |\partial_{xx}\widetilde{Z}(x,0)|^2)\, \mathrm{d}x\right) \\
&\quad + \beta\left(\alpha_5 \int_0^t (|\dot{d}(s)|^2 + |\ddot{d}(s)|^2)\, \mathrm{d}s\right) \\
&= ce^{-\alpha t/2}\left(\int_0^L (|\widetilde{Z}_0(x)|^2 + |\widetilde{X}_0|^2 + |\partial_x \widetilde{Z}(x,0)|^2 + |\partial_{xx}\widetilde{Z}(x,0)|^2)\, \mathrm{d}x\right) \\
&\quad + b\int_0^t (|\dot{d}(s)|^2 + |\ddot{d}(s)|^2)\, \mathrm{d}s, \qquad (3.67)
\end{aligned}$$

completing the proof of Theorem 3.1. □

Remark 3.1 Based on the reversibility of backstepping transformation, it is straightforward to deduce the iISS of error systems (3.6)–(3.10) in the H^2 sense by studying the stability of target systems (3.19)–(3.25) under the assumptions of Theorem 3.1. The iISS of error systems (3.6)–(3.10) implies that the state observation goes to the real values as time goes on. This observer-based input is obtained by applying (formally) the separation principle between the control in [9] and observation problems (3.1)–(3.5).

4 Numerical computation

In order to validate the observer design for the heterogeneous congested traffic with high traffic demand and a velocity drop, respectively, at the inlet and outlet boundaries of the considered road segment, the traffic parameters of two vehicle classes on a road section of 1 km length and 6.5m width are chosen as in papers [10] and [9], see Table 1.

The relationships $a_1 < a_2, \tau_1 < \tau_2$, and $\rho_1^*(0) > \rho_2^*(0), v_1^*(0) > v_2^*(0)$, class 1 represents small and fast vehicles, and class 2 describes big and slow vehicles. Given $\rho_1^*(0)$ on the domain [90, 120] with a step length 2 and $\rho_2^*(0)$ on the domain [60, 80] with a step length 2, we search a discrete quantity of $(\rho_1^*(0), \rho_2^*(0))$ such that the linearized system of (2.1)–(2.2) is stabilized, and the value of $\|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})}$ is minimal. The used function "optimize" is common for solving optimization problems on MATLAB including the chosen solver "sdpt3" and the objective $\|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})}$. We

Table 1 Selected values of parameters

Name	Symbol	Value	Unit
Relaxation time	τ_1	30	s
	τ_2	60	s
Pressure exponent	γ_1	2.5	1
	γ_2	2	1
Free-flow velocity	v_1^M	80	$\frac{\text{km}}{\text{h}}$
	v_2^M	60	$\frac{\text{km}}{\text{h}}$
Maximum $Ao(\rho)$	Ao_1^M	0.9	1
	Ao_2^M	0.85	1
Occupied surface per vehicle	a_1	10	m^2
	a_2	42	m^2
equilibrium density at the inlet	$\rho_1^*(0)$	110	$\frac{\text{veh}}{\text{km}}$
	$\rho_2^*(0)$	70	$\frac{\text{veh}}{\text{km}}$
equilibrium velocity at the inlet	$v_1^*(0)$	50	$\frac{\text{km}}{\text{h}}$
	$v_2^*(0)$	25	$\frac{\text{km}}{\text{h}}$

obtain the optimal values of $\rho_1^*(0)$, $\rho_2^*(0)$ in Table 1 and see Fig. 2as in papers [10] and [9].

The values of parameters K_P, K_I, Γ_0, Γ_2 derived from seeking the optimal tuning known input U to minimize the likelihood of congested traffic in paper [8], and the coefficient matrices Θ of the known input U are given as in paper [9],

$$K_P = \begin{bmatrix} 0 & 0 & 0 & -7.85 \\ 0 & 0 & 0 & 10.47 \\ 0 & 0 & 0 & -42.04 \\ -5.68 & 5.08 & -7.16 & 0 \end{bmatrix} \times 10^{-5},$$

$$K_I = \begin{bmatrix} -20 & 30 & 30 & 60 \\ -24 & -7 & 26 & 30 \\ -10 & 20 & -30 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^{-5}, \qquad \Gamma_0 = \begin{bmatrix} 0 & 0.0469 \\ 0 & -0.0625 \\ 0.0332 & 0.2051 \\ 0 & 0 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} -5.677 & 5.085 & -7.162 \end{bmatrix} \times 10^{-5}, \qquad \Theta = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} \times 10^{-5}.$$

By solving the linear matrix inequalities (LMIs) conditions, we derive the values of the variables P_{11}, P_{12}, P_{22}, P_3, P_4,

$$P_{11} = \text{diag}\{1.7179, 2.2212, 4.3163, 2.4493\} \times 10^3,$$

$$P_{12} = \begin{bmatrix} -10.3574 & -0.0253 & -0.0142 & -0.0073 \\ 0.0289 & -13.1242 & -0.0041 & -0.0853 \\ 0.0284 & 0.0073 & -25.1181 & 0.6772 \\ -0.0080 & -0.0819 & 0.3714 & 14.1855 \end{bmatrix},$$

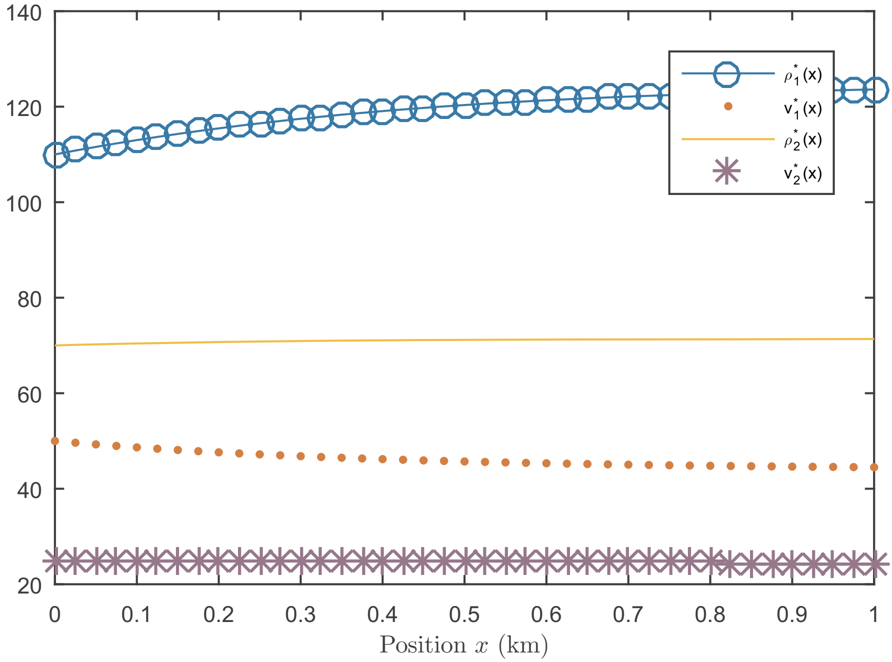

Fig. 2 Relation between spacial variable x and the nonuniform equilibrium $u^* = (\rho_1^*(x), v_1^*(x), \rho_2^*(x), v_2^*(x))^\top$

$$
\begin{aligned}
P_{22} &= \text{diag}\,\{5.1230, 5.1231, 5.1236, 5.1231\} \times 10^3,\\
P_3 &= \text{diag}\,\{2.4187, 2.8217, 4.4091, 2.8180\} \times 10^3,\\
P_4 &= \text{diag}\,\{2.4187, 2.8217, 4.4091, 2.8180\} \times 10^3,
\end{aligned}
$$

for which the conditions of Theorem 3.1 are satisfied. Therefore, Theorem 3.1 applies and the iISS of the quasilinear observer dynamics is proven. As a final remark, let us explain how to simulate systems (3.1)–(3.5) in a closed loop with the control U and these control gains. It asks in particular to discretize the piecewise continuously differentiable kernel functions F^1 and F^2 by following the approach of [12]. See also [5] where discontinuous kernel functions are numerically computed for a different control problem. After computing these kernel functions, running simulations for the observer dynamics and the closed-loop system could be done as in, e.g., [10].

5 Conclusion

This paper has developed a backstepping PDE method for the design of an observer for a heterogeneous quasilinear traffic model. Some sufficient conditions are derived in the main result for the computation of the injection gains and of the observer dynamics. These conditions are checked on numerical simulations using a realistic scenario of a congested traffic with high demand traffic flow and a velocity drop. It would be of interest to use the obtained results for the stability proof of the closed-loop system when combining the derived observer with a state feedback-stabilizing controller. The study of regulation problem is another open question.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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