Noname manuscript No.
(will be inserted by the editor)

Boundary Control Design for Conservation Laws in the Presence of Measurement Disturbances

Francesco Ferrante · Christophe Prieur

Received: date / Accepted: date

Abstract Boundary feedback control design for systems of linear hyperbolic conservation laws in the presence of boundary measurements affected by disturbances is studied. The design of the controller is performed to achieve input-to-state stability (ISS) with respect to measurement disturbances with a minimal gain. The closed-loop system is analyzed as an abstract dynamical system with inputs. Sufficient conditions in the form of dissipation functional inequalities are given to establish an ISS bound for the closed-loop system. The control design problem is turned into an optimization problem over matrix inequality constraints. Semidefinite programming techniques are adopted to devise systematic control design algorithms reducing the effect of measurement disturbances. The effectiveness of the approach is extensively shown in several numerical examples.

Keywords hyperbolic systems, infinite-dimensional systems, input-to-state stability, Lyapunov functionals, robust control.

1 Introduction

1.1 Background

Many physical systems are accurately described by employing distributed parameters. Examples of such systems are those modeled by dynamics depending on some spatially distributed variables as temperature distribution in the heat equation, spatial density in population dynamics models, or the size of particules in crystallization process. In this paper, we focus on systems modeled by

This work has been partially supported by MIAI @Grenoble Alpes (ANR-19-P3IA-0003).

Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, Grenoble, France
E-mail: francesco.ferrante@gipsa-lab.fr

conservation laws. This type of models emerges when physical quantities (e.g. the momentum, or the mass) are conserved. See [5, Chap. 1] for many examples of such hyperbolic systems modeled by partial differential equations (*PDEs*). The control of such systems is thus relevant and useful in many real-world applications. However, due to their infinite dimensional nature, controlling these systems is generally very challenging; see [31] and the references therein. Moreover, in many practical applications pertaining to distributed parameter systems, sensors and actuators are only located at the boundaries of the spatial domain of the PDE system, i.e., boundary sensing/actuation. This leads to non-trivial controllability and stabilizability problems, even when considering linear models (see e.g. [15] for an introduction on these issues).

1.2 Contribution

This paper considers linear hyperbolic systems with boundary measurements affected by exogenous disturbances and (non-collocated) boundary controls. In particular, the control problem we solve in this paper consists of the design of boundary a feedback controller guaranteeing closed-loop input-to-state stability (*ISS*) with respect to measurement disturbances. The impact of boundary disturbances is studied by estimating the gain from the disturbances to the state. The literature on ISS infinite-dimensional systems is already very broad, as recalled in the recent textbook [29] and the recent survey [36]. General results on ISS for infinite-dimensional systems can be found in [37,35,28,18]. Depending on the considered class of infinite-dimensional systems, ISS results already exist in the literature; see e.g. [38,39,12] for delay systems, [49] for hyperbolic systems. See also [19,17,20] for ISS results on networks and [8,9] for application examples in an energy context.

In this paper, we adapt the notion of Lyapunov functionals, as presented in the survey [44] for nonlinear finite-dimensional systems, to linear hyperbolic systems with boundary inputs. More precisely, to achieve this goal, we formulate Lyapunov-like functional inequalities involving Fréchet-differentiable functionals. In addition, for the considered class of linear hyperbolic systems, we propose conditions in the form of matrix inequalities for the design of boundary controllers and for the estimation of the ISS gain with respect to measurement disturbances. These conditions can be efficiently adopted for the design of the controller by solving a set of Linear Matrix Inequalities (*LMIs*) coupled to a line search on a few scalar parameters. Other numerically tractable methods exist for systems governed by PDEs, see [48,42,4,32], to cite just a few. The conditions we propose are embedded into a convex optimization setup to allow for the design of controllers achieving optimal measurement disturbance rejection (in an appropriate sense).

In contrast to [10,46] where dynamic output feedback stabilization is solved for linear hyperbolic PDEs, here only static boundary controllers are considered. This requires the study of a particular well-posedness problem for a system written in abstract form (in a similar way as [41,33]). The well-posedness of the closed-loop system is carefully analyzed in appropriate topologies on the disturbance and state spaces.

A preliminary version of this article, yet pertaining to energy-bounded measurement disturbances, appeared in the conference paper [24]. This paper presents several additional contributions with respect to the conference version. Main additional contributions are as follows:

(i) while in [24] solutions to the closed-loop system are intended in a classical sense, in this paper we rely on a weaker notion of solution. This allows one to remove the need of any compatibility condition between the initial condition and the value of the boundary disturbance;
(ii) the control design algorithm proposed in [24] can be applied only when the plant input matrix is square and nonsingular. To overcome this limitation, in this paper, we propose a more general control design algorithm;
(iii) detailed proofs are included for all the results, proofs are omitted in [24].

The paper is organized as follows. Section 2 illustrates the problem we solve in this paper, provides some preliminary results, and discusses some technical aspects. Section 3 is dedicated to Lyapunov analysis and to the construction of an ISS Lyapunov functional via the solution to some matrix inequalities. Section 4 pertains to the control design problem and to some optimization aspects. Section 5 showcases the effectiveness of the proposed design algorithms in two numerical examples. Conclusions are drawn in Section 6. Some ancillary results and definitions are given in Appendix A.

Notation

The symbol $\mathbb{N}$ denotes the set of strictly positive integers, $\mathbb{R}_{\geq 0}$ represents the set of nonnegative real nulmbers, $\mathbb{S}^n_+$ denotes the set of real symmetric positive definite matrices of dimension n, and $\mathbb{D}^{n_p}_+$ denotes the set of real diagonal positive definite matrices of dimension n. Let $A \in \mathbb{R}^{n\times n}$, $\operatorname{He}(A) := A + A^{\mathsf{T}}$ and, when A is nonsingular, $A^{-\mathsf{T}} := (A^{-1})^{\mathsf{T}}$. In partitioned symmetric matrices, the symbol $\bullet$ stands for symmetric blocks. The symbol $\mathbf{I}$ denotes the identity matrix. Given two matrices A_1 and A_2, we denote by $A_1 \oplus A_2$ the block diagonal matrix $\begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & A_2 \end{pmatrix}$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle_{\mathbb{R}^n}$ the standard Euclidean inner product. Let X and Y be linear normed spaces, the symbol $\mathscr{L}(X, Y)$ denotes the space of all bounded linear operators from X to Y. Let $\mathscr{U} \subset \mathbb{R}$, $\mathscr{V} \subset \mathbb{R}^n$, the symbol $\mathcal{L}^2(\mathscr{U};\mathscr{V})$ denotes the set of equivalence classes of measurable functions $f\colon \mathscr{U} \to \mathscr{V}$ such that $\|f\|_{\mathcal{L}^2} := (\int_{\mathscr{U}} |f(x)|^2 dx)^{\frac{1}{2}}$ is finite. Given $f, g \in \mathcal{L}^2(\mathscr{U};\mathscr{V})$, $\langle f, g\rangle_{\mathcal{L}^2} := \int_{\mathscr{U}} \langle f(x), g(x)\rangle_{\mathbb{R}^n} dx$. Given a measurable essentially bounded function $f\colon \mathscr{U} \to \mathscr{V}$, we denote by $\|f\|_{\mathcal{L}^\infty} = \operatorname*{ess\,sup}_{x\in\mathscr{U}} |f(x)|$.

Let $\mathscr{U} \subset \mathbb{R}$ be open and $\mathscr{V}$ be a linear normed space, we define

$$\mathcal{H}^1(\mathscr{U};\mathscr{V}) := \Big\{ f \in \mathcal{L}^2(\mathscr{U};\mathscr{V}) : f \text{ is loc. absolutely continuous on } \mathscr{U}, \\ \frac{d}{dz} f \in \mathcal{L}^2(\mathscr{U};\mathscr{V}) \Big\}$$

where $\frac{d}{dz}$ stands for the weak derivative of f. Let $k \in \mathbb{N} \cup \{\infty\}$ and $\mathscr{I} \subset \mathbb{R}$, the symbol $\mathcal{C}^k(\mathscr{I};\mathscr{V})$ stands for the set of class-k functions $f\colon \mathscr{I} \to \mathscr{V}$. Let X and Y be Banach spaces, $f\colon X \to Y$, and $x \in X$, the symbol $Df(x)$ denotes (when it exists) the Fréchet derivative of f at x.

2 Problem Statement and Preliminary Results

We consider boundary feedback control of the following system of n_p linear $1 - D$ hyperbolic PDEs:

$$\begin{aligned} &\partial_t x(t,z) + \Lambda \partial_z x(t,z) = 0 \;\; \forall (t,z) \in \mathbb{R}_{\geq 0} \times (0,1) \\ &x(t,0) = Hx(t,1) + Bu(t) \; \forall t \in \mathbb{R}_{\geq 0} \end{aligned} \tag{1}$$

where $\partial_t x$ and $\partial_z x$ denote, respectively, the derivative of x with respect to "time" and the "spatial" variable z, $(z \mapsto x(\cdot,z)) \in \mathcal{L}^2((0,1);\mathbb{R}^{n_p})$ is the system state, $u \in \mathbb{R}^{n_u}$ is the control input, $\Lambda \in \mathbb{D}_+^{n_p}$, $H \in \mathbb{R}^{n_p \times n_p}$, and $B \in \mathbb{R}^{n_p \times n_u}$.

Remark 1 Assuming that Λ is positive definite implies that system (1) is characterized by positive convecting speeds. However, such an assumption does not add any loss of generality. Indeed, any system of 1-D linear hyperbolic conservation laws can be expressed as in (1) via an invertible change of variables; e.g., [11,45].

In particular, (1) takes the form of a system of 1-D boundary controlled linear hyperbolic PDEs, for which several fundamental results can be found in [5]. We further assume that the state $x(\cdot,z)$ is measurable only at the boundary point $z = 1$. Specifically, the measurable output of the system reads as $y = x(\cdot,1) + \eta$, where η is a bounded measurement distrurbance. Our goal is to design a static feedback control law u, which stabilizes system (1) in some appropriate sense and with quantifiable robustness margins with respect to the disturbance η. More precisely, in this paper we focus on static output feedback laws of the form $y \mapsto u(y) = Ky$, where $K \in \mathbb{R}^{n_u \times n_p}$ is the control gain that needs to be designed. By setting $H_{cl} := H + BK$, the closed-loop system can be formally written as:

$$\begin{aligned} &\partial_t x(t,z) + \Lambda \partial_z x(t,z) = 0 && \forall (t,z) \in \mathbb{R}_{\geq 0} \times (0,1) \\ &x(t,0) = H_{cl} x(t,1) + BK\eta(t) && \forall t \geq 0 \end{aligned} \tag{2}$$

see Fig. 1 for a pictorial representation of the closed-loop system.

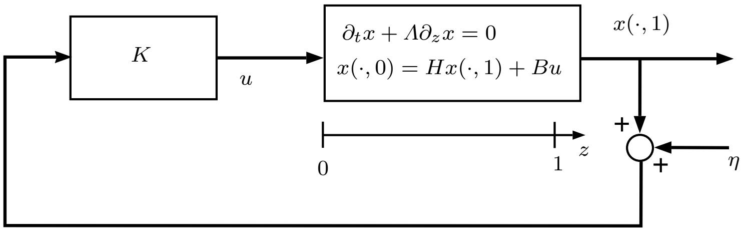

Fig. 1 Closed-loop system.

2.1 Abstract Formulation

With the objective of avoiding the use of compatibility conditions enforcing a relationship between the exogenous input η and the plant initial condition, similarly as in [33], we focus on mild solutions to (1). To this end, as in [15, 3, 25], we first reformulate the closed-loop system as an abstract differential equation. In particular, let

$$\begin{aligned} \mathfrak{A}\colon\ & \operatorname{dom}\mathfrak{A} \to \mathcal{L}^2((0,1);\mathbb{R}^{n_p}) \\ & f \mapsto -\Lambda \frac{d}{dz} f(z) \\ \mathfrak{B}\colon\ & \operatorname{dom}\mathfrak{B} \to \mathbb{R}^{n_p} \\ & f \mapsto f(0) - H_{cl} f(1) \end{aligned} \tag{3}$$

where $\operatorname{dom}\mathfrak{A} = \operatorname{dom}\mathfrak{B} = \mathcal{H}^1((0,1);\mathbb{R}^{n_p})$. Then, the closed-loop system can be formally written as:

$$\begin{aligned} \dot{x} &= \mathfrak{A}x \\ \mathfrak{B}x &= BK\eta \end{aligned} \tag{4}$$

Specifically, following the lines of [3], we consider the following notion of (mild) solution pair for (4).

Definition 1 A pair $(\phi, \eta) \in \mathcal{C}^0(\operatorname{dom}\phi; \mathcal{L}^2((0,1);\mathbb{R}^{n_p})) \times \mathcal{H}^1(\operatorname{dom}\eta;\mathbb{R}^{n_p})$ is a solution pair to (4) if $\operatorname{dom}\phi = \operatorname{dom}\eta$, $\operatorname{dom}\phi$ is an interval of $\mathbb{R}_{\geq 0}$ including zero, and for all $t \in \operatorname{dom}\phi$

$$\int_0^t \phi(s)ds \in \operatorname{dom}\mathfrak{A} \tag{5a}$$

$$\phi(t) = \phi(0) + \mathfrak{A}\int_0^t \phi(s)ds \tag{5b}$$

$$\mathfrak{B}\int_0^t \phi(s)ds = BK\int_0^t \eta(s)ds \tag{5c}$$

$\circ$

Remark 2 As a matter fact, for Definition 1 to make sense, one may select $\eta \in \mathcal{L}^1(\operatorname{dom}\eta;\mathbb{R}^{n_p})$. On the other hand, the additional level of regularity introduced in Definition 1 is exploited in the proof of Proposition 1.

2.1.1 Discussion on the notion of mild solutions

Typically, mild solutions to linear abstract dynamical systems are defined in an "explicit form" by relying on the use of the strongly continuous semigroup generated by the zero-input operator defining the system dynamics; see, e.g., [15,26,27]. In this paper, we pursue a different approach and build upon an implicit notion of solution pair, which only relies on the data of the system, i.e., the operators $\mathfrak{A}$ and $\mathfrak{B}$, and the matrices B and K. A strongly related approach, yet based on a variational formulation of the abstract dynamical system, is proposed in [33]. One interesting fact is that, as shown in the proof of Proposition 1, when the exogenous input η satisfies some regularity assumptions, the implicit characterization used in Definition 1 coincides with the more standard explicit notion of mild solution in [15, Chapter 3.3].

2.2 Control Design Problem

Now we are in a position to formally state the problem we solve in this paper.

Problem 1 Given $H \in \mathbb{R}^{n_p \times n_p}, B \in \mathbb{R}^{n_p \times n_u}$, and $\Lambda \in \mathbb{D}_+^{n_p}$. Design a control gain $K \in \mathbb{R}^{n_u \times n_p}$ such that for each solution pair (ϕ, η) to (4) the bound:

$$\|\phi(t)\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} \leq \kappa e^{-\omega t} \|\phi(0)\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} + \gamma \|\eta\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})} \qquad \forall t \in \operatorname{dom} \phi \tag{6}$$

holds for some (solution independent) $\kappa, \omega \in \mathbb{R}_{>0}$, and minimal $\gamma \in \mathbb{R}_{>0}$. $\diamond$

Relation (6) corresponds to a classical input-to-state-stability bound for the abstract closed-loop system (4). Sufficient conditions to ensure input-to-state-stability for infinite dimensional systems with respect to boundary disturbances are given in [29]; see also [36]. The main contribution of this paper is to perform an optimal design of the control gain K so to minimize the ISS gain γ. To this end, in Section 3 we provide sufficient conditions to get an explicit estimate of the gain γ.

Remark 3 Typically, fast convergence rate and robustness with respect to measurement perturbations are conflicting objectives. In this setting, optimality needs to be intended in a Pareto sense. Loosely speaking, the controller gain K needs to be designed by seeking a suitable tradeoff between performance and robustness. This can be achieved by solving Problem 1 for a range of values of ω. This aspect will be illustrated in Example 2.

2.3 Well-posedness of the closed loop system

Proposition 1 *Let $\mathcal{I} \subset \mathbb{R}_{\geq 0}$ be an interval including zero, $x_0 \in \mathcal{L}^2((0,1);\mathbb{R}^{n_p})$, and $\eta \in \mathcal{H}^1(\mathcal{I};\mathbb{R}^{n_p})$. Then, there exists a unique $\phi \in \mathcal{C}^0(\mathcal{I}, \mathcal{L}^2((0,1);\mathbb{R}^{n_p}))$ such that $\phi(0) = x_0$ and (ϕ, η) is a solution pair to (4).*

Proof. The proof largely relies on the results in [15]. As first step, define the following objects:

$$\begin{aligned} &\mathcal{A} := \mathfrak{A} \\ &\operatorname{dom}\mathcal{A} := \operatorname{dom}\mathfrak{A} \cap \ker\mathfrak{B} = \{f \in \mathcal{H}^1((0,1);\mathbb{R}^{n_p})\colon f(0) - H_{cl}f(1) = 0\} \end{aligned} \tag{7a}$$

$$\begin{aligned} &\mathit{\Pi} \in \mathscr{L}(\mathbb{R}^{n_p}; \mathcal{L}^2((0,1);\mathbb{R}^{n_p})) \\ &\mathit{\Pi}\eta := z \mapsto ((1-z)BK + zH_{cl}^\star)\,\eta \end{aligned} \tag{7b}$$

where $H_{cl}^\star \in \mathbb{R}^{n_p \times n_p}$ is any matrix such that $H_{cl}H_{cl}^\star = 0$. Now observe that the following properties hold for the above defined objects:

(a) For all $\eta \in \mathbb{R}^{n_p}$, $\mathit{\Pi}\eta \in \operatorname{dom}\mathfrak{A}$,
(b) $\mathfrak{A}\mathit{\Pi} \in \mathscr{L}(\mathbb{R}^{n_p}, \mathcal{L}^2((0,1);\mathbb{R}^{n_p}))$,
(c) for all $\eta \in \mathbb{R}^{n_p}$, $\mathfrak{B}\mathit{\Pi}\eta = BK\eta$,
(d) the operator $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup $\mathscr{T}(t)$ on the Hilbert space $\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$.

Items (a), (b), and (c) are straightforward to prove, while item (d) can be proven by following the same arguments as in the proof of [5, Theorem A.1, page 244]. Bearing in mind the above properties, we know complete the proof of the statement. Let $\mathscr{T}(t)$ be the $\mathcal{C}_0$-semigroup generated by $\mathcal{A}$. For all $t \in \mathcal{I}$, define:

$$\begin{aligned} \phi(t) :=& \mathscr{T}(t)x_0 + \mathit{\Pi}\eta(t) - \mathscr{T}(t)\mathit{\Pi}\eta(0) - \int_0^t \mathscr{T}(t-s)\mathit{\Pi}\dot{\eta}(s)ds \\ &+ \int_0^t \mathscr{T}(t-s)\mathfrak{A}\mathit{\Pi}\eta(s)ds \end{aligned} \tag{8}$$

The satisfaction of items (a), (b), and (d), ensures that $\phi \in \mathcal{C}^0(\mathcal{I}, \mathcal{L}^2((0,1);\mathbb{R}^{n_p}))$. We show that (ϕ, η) is a solution pair to (4). Let $v := \phi - \mathit{\Pi}\eta$. Then, for all $t \in \mathcal{I}$

$$v(t) = \mathscr{T}(t)v(0) + \int_0^t \mathscr{T}(t-s)\left(\mathfrak{A}\mathit{\Pi}\eta(s) - \mathit{\Pi}\dot{\eta}(s)\right)ds$$

At this stage, notice that since, from item (d), $\mathcal{A}$ is the infinitesimal generator of the $\mathcal{C}_0$-semigroup $\mathscr{T}(t)$, from [3, Proposition 3.1.16.], the above expression is the unique solution to the following initial value problem:

$$\begin{cases} \dot{v}(t) = \mathcal{A}v(t) + \mathfrak{A}\mathit{\Pi}\eta(t) - \mathit{\Pi}\dot{\eta}(t) & \forall t \in \mathcal{I} \\ v(0) = x_0 - \mathit{\Pi}\eta(0) \end{cases}$$

In particular; see [3], for all $t \in \mathcal{I}$, one has:

$$\int_0^t v(s)ds \in \operatorname{dom}\mathcal{A} \tag{9a}$$

$$v(t) = v(0) + \mathcal{A}\int_0^t v(s)ds + \int_0^t \left(\mathfrak{A}\mathit{\Pi}\eta(s) - \mathit{\Pi}\dot{\eta}(s)\right)ds \tag{9b}$$

Notice that since, from (7a), $\operatorname{dom}\mathcal{A}\subset\operatorname{dom}\mathfrak{A}$, and from item (a), $\Pi\eta\in\operatorname{dom}\mathfrak{A}$, (9a) implies that

$$\int_0^t \phi(s)ds\in\operatorname{dom}\mathfrak{A}\quad\forall t\in\mathcal{I}$$

that is (5a). Now we show that (5b) holds. From the definition of v, for all $t\in\mathcal{I}$, one has:

$$\begin{aligned}\phi(t)=v(t)+\Pi\eta(t)\overset{(9b)}{=}x_0-\Pi\eta(0)+\mathcal{A}\int_0^t v(s)ds+\int_0^t\left(\mathfrak{A}\Pi\eta(s)-\Pi\dot{\eta}(s)\right)ds\\+\Pi\eta(t)\end{aligned}$$

By integrating the term $\Pi\dot{\eta}$, the above expression yields:

$$\phi(t)=x_0+\mathcal{A}\int_0^t v(s)ds+\int_0^t\mathfrak{A}\Pi\eta(s)ds\quad\forall t\in\mathcal{I}$$

Using again the definition of v and item (b), one gets:

$$\phi(t)=x_0+\mathcal{A}\int_0^t\phi(s)ds+\int_0^t\mathfrak{A}\Pi\eta(s)ds-\mathcal{A}\int_0^t\Pi\eta(s)ds\quad\forall t\in\mathcal{I}$$

Hence, using the definition of $\mathcal{A}$ in (7a), the above expression yields:

$$\phi(t)=x_0+\mathfrak{A}\int_0^t\phi(s)ds\quad\forall t\in\mathcal{I}$$

which reads as (5b). To conclude, we show that ϕ satisfies (5c). To this end, as a first step, observe that item (c) implies

$$\mathfrak{B}\int_0^t\Pi\eta(s)ds=BK\int_0^t\eta(s)ds\quad\forall t\in\mathcal{I}\tag{10}$$

Moreover, since by construction $\operatorname{dom}\mathcal{A}\subset\ker\mathfrak{B}$ (see (7a)), from (9a) it follows that:

$$\mathfrak{B}\int_0^t v(s)ds=0\quad\forall t\in\mathcal{I}\tag{11}$$

Thus, using (10), (11), and the relationship between ϕ and v one gets (5c).

Now we conclude the proof by showing that there does not exist any other solution pair to (4) of the form $(\bar{\phi},\eta)$ with $\bar{\phi}(0)=x_0$. Assume that there exists $\bar{\phi}$ such that $(\bar{\phi},\eta)$ is a solution pair to (4) and $\bar{\phi}(0)=x_0$. Let $e:=\phi-\bar{\phi}$, then, for all $t\in\mathcal{I}$, one has

$$\begin{aligned}&\int_0^t e(s)ds\in\operatorname{dom}\mathfrak{A}\quad e(t)=\mathfrak{A}\int_0^t e(s)ds\\&\mathfrak{B}\int_0^t e(s)ds=0\end{aligned}$$

Specifically, e is the mild solution to the following abstract initial value problem

$$\begin{aligned}\dot{e}(t) &= \mathcal{A}e(t) \quad t \in \mathcal{I}\\ e(0) &= 0\end{aligned} \tag{12}$$

where $\mathcal{A}$ is defined in (7a); see [3, Definition 3.1.1, page 108]. Therefore, as $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup on $\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$, from [3, Proposition 3.1.11, page 115] it follows that the unique solution to (12) is $e=0$, i.e., $\phi=\bar{\phi}$, thereby concluding the proof. □

Remark 4 The proof of Proposition 1 clearly shows that, under the regularity assumption considered for η, the notion of solution pair given in Definition 1 is directly related to the notion of mild solution in [15]. However, relying upon Definition 1 has two interesting advantages. The first advantage is that, as the notion of solution pair we consider only depends on the data of the system dynamics (and not on any indirectly defined object), its use is more appropriate to establish Lyapunov-like results. The second advantage is that Definition 1, by itself, does not impose any regularity assumption on the input η; see Remark 2. Therefore, the analysis carried out in this paper may potentially be extended under weaker regularity assumptions of the signal η. In this sense, an interesting aspect that deserves further investigation concerns the relationships between the notion of *weak* solution proposed in [33] and Definition 1.

3 Lyapunov-like Sufficient Conditions for Input-to-State Stability

As a first step towards the solution to Problem 1, in this section we provide Lyapunov-like sufficient conditions for input-to-state stability of the closed-loop system (4) with respect to the measurement disturbance η.

Theorem 1 *Let*

$$\mathcal{U} := \{(f,\eta) \in \operatorname{dom}\mathfrak{A} \times \mathbb{R}^{n_p} : \mathfrak{B}f = BK\eta\} \tag{13}$$

Assume that there exist a Fréchet differentiable functional $V\colon \mathcal{L}^2((0,1);\mathbb{R}^{n_p}) \to \mathbb{R}$ *and* $c_1, c_2, c_3, \chi \in \mathbb{R}_{>0}$ *such that for all* $(f,\eta)\in\mathcal{U}$

$$c_1\|f\|^2_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} \leq V(f) \leq c_2\|f\|^2_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} \tag{14}$$

$$DV(f)\mathfrak{A}f \leq -c_3 V(f) + \chi^2|\eta|^2 \tag{15}$$

Let (ϕ,η) *be a solution pair to* (4)*. Then, for all* $t\in\operatorname{dom}\phi$*, one has*

$$\|\phi(t)\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} \leq e^{-\frac{c_3}{2}t}\left(\frac{c_2}{c_1}\right)^{\frac{1}{2}}\|\phi(0)\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} + \frac{\chi}{\sqrt{c_1c_3}}\|\eta\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})} \tag{16}$$

Proof. First we show that the above result holds for all *classical* solution pairs to (4). More precisely, pick a solution pair (ϕ, η) to (4) and assume that $(\phi(0), \eta(0)) \in \mathcal{U}$ and $\eta \in \mathcal{C}^2(\operatorname{dom}\eta, \mathbb{R}^{n_p})$. Then, from [15, Theorem 3.3.3, page 123], one has that (ϕ, η) is a classical solution pair, i.e., $\phi \in \mathcal{C}^1(\operatorname{dom}\phi; \mathcal{L}^2((0,1); \mathbb{R}^{n_p}))$ and for all $t \in \operatorname{dom}\phi$

$$\begin{aligned} &(\phi(t), \eta(t)) \in \mathcal{U} \\ &\dot{\phi}(t) = \mathfrak{A}\phi(t) \end{aligned} \tag{17}$$

Now, consider the following function

$$\begin{aligned} \mathcal{W} \colon \operatorname{dom}\phi &\to \mathbb{R} \\ t &\mapsto (V \circ \phi)(t) \end{aligned} \tag{18}$$

Since $V \colon \mathcal{L}^2((0,1); \mathbb{R}^{n_p}) \to \mathbb{R}$ and $\phi \colon \operatorname{dom}\phi \to \mathcal{L}^2((0,1); \mathbb{R}^{n_p})$ are Fréchet differentiable, so is $\mathcal{W} \colon \operatorname{dom}\phi \to \mathbb{R}$ and in particular for all $t \in \operatorname{dom}\phi$

$$\dot{\mathcal{W}}(t) = DV(\phi(t))\dot{\phi}(t)$$

which, thanks to (17), yields for all $t \in \operatorname{dom}\phi$

$$\dot{\mathcal{W}}(t) = DV(\phi(t))\mathfrak{A}\phi(t)$$

Thus, since for all $t \in \operatorname{dom}\phi$, $(\phi(t), \eta(t)) \in \mathcal{U}$, using (15), one gets for all $t \in \operatorname{dom}\phi$

$$\dot{\mathcal{W}}(t) \leq -c_3\mathcal{W}(t) + \chi^2|\eta(t)|^2$$

which in turn, by the comparison lemma; see, e.g., [30, Lemma 3.4, page 103] gives, for all $t \in \operatorname{dom}\phi$

$$\mathcal{W}(t) \leq e^{-c_3 t}\mathcal{W}(0) + \chi^2 \int_0^t e^{-c_3(t-\theta)}|\eta(\theta)|^2 d\theta$$

Using

$$\int_0^t e^{-c_3(t-\theta)}|\eta(\theta)|^2 d\theta \leq \frac{1}{c_3}\|\eta\|^2_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})} \quad \forall t \in \operatorname{dom}\phi$$

allows one to conclude that, for all $t \in \operatorname{dom}\phi$

$$\mathcal{W}(t) \leq e^{-c_3 t}\mathcal{W}(0) + \frac{\chi^2}{c_3}\|\eta\|^2_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})}$$

Finally, by using (14), it follows that[1] for all $t \in \operatorname{dom}\phi$:

$$\|\phi(t)\|_{\mathcal{L}^2} \leq e^{-\frac{c_3}{2}t}\sqrt{\frac{c_2}{c_1}}\|\phi(0)\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_p})} + \frac{\chi}{\sqrt{c_1 c_3}}\|\eta\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})} \tag{19}$$

[1] We use the fact that for any $a, b \in \mathbb{R}_{\geq 0}$, $\sqrt{a^2 + b^2} \leq a + b$.

Now we conclude the proof by showing that the above bound holds also for "mild" solution pairs to (4). Let (ϕ, η) be any solution pair to (4). Then, recall that, as shown in the proof of Proposition 1, for all $t \in \operatorname{dom} \phi$:

$$\begin{aligned}\phi(t) =&\mathscr{T}(t)\phi(0) + \Pi\eta(t) - \mathscr{T}(t)\Pi\eta(0) - \int_0^t \mathscr{T}(t-s)\Pi\dot{\eta}(s)ds\\ &+ \int_0^t \mathscr{T}(t-s)\mathfrak{A}\Pi\eta(s)ds\end{aligned} \tag{20}$$

where $\mathcal{A}$ and Π are defined in (7) and $\mathbb{R}_{\geq 0} \ni t \mapsto \mathscr{T}(t)$ is the $\mathcal{C}_0$-semigroup generated by $\mathcal{A}$ on $\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$. Now we show that (ϕ, η) can be approximated (in a sense that will be clarified later) via a sequence of classical solution pairs. To this end, pick $\{\eta_k\}_{k=1}^\infty \subset \mathcal{C}^\infty(\operatorname{dom} \eta; \mathbb{R}^{n_p})$ such that

$$\eta_k \underset{k\to\infty}{\overset{\mathcal{H}^1(\operatorname{dom}\eta;\mathbb{R}^{n_p})}{\longrightarrow}} \eta \tag{21}$$

such a sequence exists by density since $\eta \in \mathcal{H}^1(\operatorname{dom} \eta; \mathbb{R}^{n_p})$. As in [33, proof of Theorem 3], for all $k \in \mathbb{N}$, define $\psi_k^0 \coloneqq \Pi\eta_k(0)$ and observe that

$$\mathfrak{B}\psi_k^0 = BK\eta_k(0) \quad \forall k \in \mathbb{N}$$

At this stage, notice that since $\operatorname{dom} \mathfrak{A} \cap \ker \mathfrak{B}$ is dense in $\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$, there exists a sequence $\{\varphi_k^0\}_{k=1}^\infty \subset \operatorname{dom} \mathfrak{A} \cap \ker \mathfrak{B}$ such that

$$\varphi_k^0 \underset{k\to\infty}{\overset{\mathcal{L}^2}{\longrightarrow}} \phi(0) - \Pi\eta(0) \tag{22}$$

The above construction ensures that $\{(\psi_k^0 + \varphi_k^0, \eta_k(0))\}_{k=1}^\infty \subset \mathcal{U}$ and

$$\varphi_k^0 + \psi_k^0 \underset{k\to\infty}{\overset{\mathcal{L}^2}{\longrightarrow}} \phi(0) \tag{23}$$

Let, for all $k \in \mathbb{N}$, $x_k^0 \coloneqq \varphi_k^0 + \psi_k^0$. Define for all $k \in \mathbb{N}$

$$\begin{aligned}t \mapsto \varphi_k(t) \coloneqq&\mathscr{T}(t)x_k^0 + \Pi\eta_k(t) - \mathscr{T}(t)\Pi\eta_k(0) - \int_0^t \mathscr{T}(t-s)\Pi\dot{\eta}_k(s)ds\\ &+ \int_0^t \mathscr{T}(t-s)\mathfrak{A}\Pi\eta_k(s)ds \quad \forall t \in \operatorname{dom}\phi\end{aligned}$$

From [16, Theorem 3.3.4, page 124], $\{(\varphi_k, \eta_k)\}$ is a sequence of classical solution pairs to (4). Now we show that for any $t \in \operatorname{dom} \phi$

$$\lim_{k\to\infty} \|\varphi_k(t) - \phi(t)\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})} = 0$$

To this end, observe that for all $t \in \operatorname{dom} \phi$ and any $k \in \mathbb{N}$:

$$\begin{aligned}\varphi_k(t) - \phi(t) =&\mathscr{T}(t)(x_k^0 - \phi(0)) + \Pi(\eta_k(t) - \eta(t)) - \mathscr{T}(t)\Pi(\eta_k(0) - \eta(0))\\ &- \int_0^t \mathscr{T}(t-s)\Pi(\dot{\eta}_k(s) - \dot{\eta}(s))ds\\ &+ \int_0^t \mathscr{T}(t-s)\mathfrak{A}\Pi(\eta_k(s) - \eta(s))ds\end{aligned}$$

which, under (21) and (23), implies that[2]

$$\varphi_k(t) \xrightarrow[k\to\infty]{\mathcal{L}^2} \phi(t) \qquad \forall t \in \operatorname{dom}\phi \tag{24}$$

To summarize, the steps carried out so far show that, for all $t \in \operatorname{dom}\phi$:

$$(\varphi_k(t), \eta_k) \xrightarrow[k\to\infty]{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})\times\mathcal{H}^1((0,t);\mathbb{R}^{n_p})} (\phi(t), \eta) \tag{25}$$

We now conclude the proof showing that (19) holds for the mild solution pair (ϕ, η). Since, for each $k \in \mathbb{N}$, (φ_k, η_k) is a classical solution to (4), thanks to (19), one has

$$\|\varphi_k(t)\|_{\mathcal{L}^2} \leq e^{-\frac{c_3}{2}t}\sqrt{\frac{c_2}{c_1}}\|\varphi_k(0)\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_p})}+\frac{\chi}{\sqrt{c_1 c_3}}\|\eta_k\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})} \quad \forall t \in \operatorname{dom}\phi$$

Therefore, taking the limit for $k \to \infty$, by using (25), one gets[3]

$$\|\phi(t)\|_{\mathcal{L}^2} \leq e^{-\frac{c_3}{2}t}\sqrt{\frac{c_2}{c_1}}\|\phi(0)\|_{\mathcal{L}^2(0,1;\mathbb{R}^{n_p})}+\frac{\chi}{\sqrt{c_1 c_3}}\|\eta\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})} \quad \forall t \in \operatorname{dom}\phi$$

This concludes the proof. □

Remark 5 Theorem 1 provides sufficient conditions for input-to-state stability for the closed-loop system (4) in the form of a functional inequalities. This provides an elegant generalization to abstract dynamical systems of the well-known ISS dissipation inequalities for finite-dimensional nonlinear systems; see, e.g., [44]. It is interesting to observe that the gradient of V is replaced in (15) by the Fréchet derivative.

3.1 Construction of the functional V

Theorem 1 provides sufficient conditions for the solution to Problem 1. However, the applicability of such a result requires the construction of the functional V. This is a challenging task in general. To overcome this problem, next we propose a specific structure for the functional V in Theorem 1, which allows one to cast the solution to Problem 1 in the solution to some matrix inequalities.

Theorem 2 *If there exist $P \in \mathbb{D}_+^{n_p}, K \in \mathbb{R}^{n_u\times n_p}$, and $\mu \in \mathbb{R}_{>0}$ such that*

$$\begin{bmatrix}(H+BK)^{\mathsf{T}}\Lambda P(H+BK)-e^{-\mu}\Lambda P & (H+BK)^{\mathsf{T}}\Lambda PBK\\ \bullet & K^{\mathsf{T}}B^{\mathsf{T}}\Lambda PBK-\mu\lambda_{\min}(\Lambda)e^{-\mu}\mathbf{I}\end{bmatrix}\prec 0 \tag{26}$$

[2] Convergence of $\eta_k(0)$ to $\eta(0)$ follows from (21) thanks to standard Sobolev embedding arguments; see, e.g., [7, Theorem 8.8, page 212].

[3] Convergence of $\|\eta_k\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})}$ to $\|\eta\|_{\mathcal{L}^\infty((0,t);\mathbb{R}^{n_p})}$ follows from (21) thanks to [7, Theorem 8.8, page 212].

Then, K solves Problem 1 and in particular (6) *holds with:*

$$\begin{aligned}
&\omega = \frac{1}{2}\mu\lambda_{\min}(\Lambda), \quad \kappa = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}e^{\frac{\mu}{2}} \\
&\gamma = \frac{1}{\sqrt{\lambda_{\min}(P)}}
\end{aligned} \tag{27}$$

Proof. Motivated by [5], for each $f \in \mathcal{L}^2((0,1);\mathbb{R}^{n_p})$, define

$$V(f) := \int_0^1 e^{-\mu z} f(z)^\top P f(z) dz$$

Observe that for all $f \in \mathcal{L}^2((0,1);\mathbb{R}^{n_p})$, one has

$$c_1\|f\|^2_{\mathcal{L}^2} \leq V(f) \leq c_2\|f\|^2_{\mathcal{L}^2} \tag{28}$$

where $c_1 := e^{-\mu}\lambda_{\min}(P)$ and $c_2 := \lambda_{\max}(P)$ are strictly positive. Moreover, from Lemma A1 in the Appendix, one has for all $h \in \mathcal{L}^2((0,1);\mathbb{R}^{n_p})$

$$DV(f)h = \int_0^1 2f(z)^\top P e^{-\mu z} h(z) dz$$

Hence, for all $f \in \mathfrak{A}$, one gets:

$$DV(f)\mathfrak{A}f = -2\int_0^1 e^{-\mu z} f(z)^\top P\Lambda \frac{d}{dz} f(z)\, dz$$

Furthermore, due to $P, \Lambda \in \mathbb{D}^{n_p}_+$, for all $z \in [0,1]$

$$2f(z)^\top P\Lambda \frac{d}{dz} f(z) = \frac{d}{dz}\left(f(z)^\top P\Lambda f(z)\right)$$

which in turn, for all $f \in \mathfrak{A}$, gives

$$DV(f)\mathfrak{A}f = -\int_0^1 \frac{d}{dz}\left(f(z)^\top P\Lambda f(z)\right) e^{-\mu z}\, dzs$$

By integrating by parts, it follows that

$$\begin{aligned}
DV(f)\mathfrak{A}f = &- f(z)^\top P\Lambda f(z) e^{-\mu z}\Big|_0^1 \\
&- \mu\int_0^1 f(z)^\top P\Lambda f(z) e^{-\mu z} dz \quad \forall f \in \operatorname{dom}\mathfrak{A}
\end{aligned}$$

Now, pick $(f, \eta) \in \mathcal{U}$ with $\mathcal{U}$ defined as in (13). Then, it turns out that:

$$\begin{aligned}
DV(f)\mathfrak{A}f = &\begin{bmatrix} f(1) \\ \eta \end{bmatrix}^\top \Omega \begin{bmatrix} f(1) \\ \eta \end{bmatrix} \\
&- \mu\int_0^1 f(z)^\top P\Lambda f(z) e^{-\mu z} dz
\end{aligned} \tag{29}$$

where

$$\Omega := \begin{bmatrix} (H+BK)^{\mathsf{T}} \Lambda P (H+BK) - e^{-\mu}\Lambda P & (H+BK)^{\mathsf{T}} \Lambda P B K \\ \bullet & K^{\mathsf{T}} B^{\mathsf{T}} \Lambda P B K \end{bmatrix} \tag{30}$$

Let $\sigma := \mu \lambda_{\min}(\Lambda)$, then (29) implies, for all $(f, \eta) \in \mathcal{U}$

$$DV(f)\mathfrak{A}f \leq -\sigma V(f) + \begin{bmatrix} f(1) \\ \eta \end{bmatrix}^{\mathsf{T}} \Omega \begin{bmatrix} f(1) \\ \eta \end{bmatrix}$$

By setting $\chi = \sqrt{\mu \lambda_{\min}(\Lambda) e^{-\mu}}$, the latter implies that, for all $(f, \eta) \in \mathcal{U}$

$$DV(f)\mathfrak{A}f - \chi^2 \eta^{\mathsf{T}} \eta \leq -\sigma V(f) + \begin{bmatrix} f(1) \\ \eta \end{bmatrix}^{\mathsf{T}} \left(\Omega - (\mathbf{0}_{n_p \times n_p} \oplus \chi^2 \mathbf{I}) \right) \begin{bmatrix} f(1) \\ \eta \end{bmatrix} \tag{31}$$

At this stage, notice that $\Omega - (\mathbf{0}_{n_p \times n_p} \oplus \chi^2 \mathbf{I})$ reads as the matrix in the left-hand side of (26). Therefore, thanks to the satisfaction of (26), (31) implies that, for all $(f, \eta) \in \mathcal{U}$

$$DV(f)\mathfrak{A}f \leq -\sigma V(f) + \mu \lambda_{\min}(\Lambda) e^{-\mu} \eta^{\mathsf{T}} \eta$$

which reads as (15). Hence, in view of (28), by invoking Theorem 1 it follows that bound (16) holds with ω, κ, and γ defined as in (27). This concludes the proof. □

Remark 6 It is worthwhile to observe that the satisfaction of (26) ensures robustness with respect to small uncertainties in the transport velocities. This follows directly from the fact that the entries matrix in the left-hand side of (26) are continuous functions of the entries of entries of Λ. This is in accordance with the so-called Silkowski criterion; see, e.g., [5].

Remark 7 In [1], the authors propose a constructive approach based on optimization techniques for the construction of storage functionals for a wide class of PDEs. Although the results in [1] cover a large class of PDEs, their applicability to the analysis of the closed-loop system (2) does not seem straightforward due to different boundary conditions.

Remark 8 It is well-known that when H is dissipative, (1) is open-loop globally exponentially stable; see, e.g., [5]. In this setting, Problem 1 can be solved by taking $K = 0$, which yields $\gamma = 0$ in (6). This feature clearly shows up in (26). Indeed, when is dissipative, (26) can be fulfilled with $K = 0$ and P with arbitrarily large eigenvalues, i.e., γ arbitrarily small. On the other hand, when the system is open-loop stable, closed-loop control may be used to improve convergence speed.

4 Control Design

Proposition 1 enables to recast the search of the parameters of a closed-loop ISS bound via the solution to some matrix inequalities. However, since condition (26) is nonlinear in the variables P, K, and μ, Proposition 1 is difficult to exploit directly for the solution to Problem 1; see [6]. While from a numerical standpoint the nonlinearities involving the variable μ can be easily addressed via a line search, further work is needed to tackle the trilinear terms involving P and K. To accomplish this task, in the result given next, we provide an equivalent condition to (26) in which no trilinear terms appear.

Lemma 1 *Let $P \in \mathbb{D}_+^{n_p}$, $\mu \in \mathbb{R}_{>0}$, and $K \in \mathbb{R}^{n_u \times n_p}$ be given. The following two statements are equivalent:*

(a) (26) *holds;*
(b) *There exists $Q \in \mathbb{R}^{n_p \times n_p}$ such that*

$$\begin{bmatrix} \mathrm{He}(Q) + \Lambda P & -Q^\mathsf{T}(H + BK) & -Q^\mathsf{T} BK \\ \bullet & -e^{-\mu}\Lambda P & 0 \\ \bullet & \bullet & -\mu\lambda_{\min}(\Lambda)e^{-\mu}\mathbf{I} \end{bmatrix} \prec 0 \tag{32}$$

Proof. Set

$$\chi = \sqrt{\mu\lambda_{\min}(\Lambda)e^{-\mu}}$$

and let

$$Z := \begin{bmatrix} P\Lambda & 0 & 0 \\ \bullet & -e^{-\mu}P\Lambda & 0 \\ \bullet & \bullet & -\chi^2\mathbf{I} \end{bmatrix} \quad N_U := \begin{bmatrix} H + BK & BK \\ I & 0 \\ 0 & I \end{bmatrix}$$

Then, one has $\Omega - (\mathbf{0}_{n_p \times n_p} \oplus \chi^2\mathbf{I}) = N_U^\mathsf{T} Z N_U$, where Ω is defined in (30). Moreover, observe that by defining

$$N_V := \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}$$

it follows that $P \succ 0$ if and only if $N_V^\mathsf{T} Z N_V \prec 0$. Therefore, by recalling that the satisfaction of (26) is equivalent to $\Omega - (\mathbf{0}_{n_p \times n_p} \oplus \chi^2\mathbf{I}) \prec 0$, the following conditions are equivalent:

(i) $-P \prec 0$ and (26) holds;
(ii) $N_U^\mathsf{T} Z N_U \prec 0$ and $N_V^\mathsf{T} Z N_V \prec 0$

Now observe that N_U and N_V are full-column rank. Thus, from the projection lemma [2,40], one has that (ii) holds if and only if there exists $Q \in \mathbb{R}^{n_p \times n_p}$ such that

$$Z + U^\mathsf{T} Q V + V^\mathsf{T} Q^\mathsf{T} U \prec 0 \tag{33}$$

where U and V are any matrices of suitable dimensions, such that

$$U N_U = 0 \quad V N_V = 0$$

In particular, by selecting

$$U = \begin{bmatrix} \mathbf{I} & -(H+BK) & -BK \end{bmatrix}$$
$$V = \begin{bmatrix} \mathbf{I} & 0 & 0 \end{bmatrix}$$

(33) specializes into (32), thereby concluding the proof. □

4.1 LMI-based Controller Design

Now we are in a position to reformulate Problem 1 as an optimization problem over matrix inequality constraints. In particular, in light of the bound on the ISS gain γ given in Theorem 2 and of the equivalence established in Lemma 1, the design of the controller gain can be turned into the following optimization problem:

$$\begin{aligned} &\underset{P,K,Q,\mu,c}{\text{maximize}} && c \\ &\text{subject to} && \begin{bmatrix} \mathrm{He}(Q)+\Lambda P & -Q^{\mathsf{T}}(H+BK) & -Q^{\mathsf{T}}BK \\ \bullet & -e^{-\mu}\Lambda P & 0 \\ \bullet & \bullet & -\mu\lambda_{\min}(\Lambda)e^{-\mu}\mathbf{I} \end{bmatrix} \prec 0 \\ &&& P - c\mathbf{I} \succeq 0, c > 0 \end{aligned} \tag{34}$$

Specifically, maximizing the value of c under the constraint $P - c\mathbf{I} \succeq 0$ is equivalent to maximize the smallest eigenvalue of P, thereby minimizing the ISS gain γ. At this stage observe that, because (32) is bilinear in the decision variables Q and K, solving (34) directly is challenging from a numerical standpoint. To overcome this problem, we propose two strategies to solve (34) via *semidefinite programming* tools.

4.1.1 Control Design Strategy I

The first strategy can be employed when when $n_p = n_u$ and B is nonsingular. This is assumption holds in several cases of practical interest as, e.g., *density-flow systems* [5, Chapter 2] or *vehicle traffic flow models* [14]. In this case, a simple invertible change of variables turns (32) into a linear matrix inequality, modulo the nonlinearity involving the variable μ. This is formalized in the result given next.

Proposition 2 *Assume that $n_u = n_p$ and that B is nonsingular. Let $P \in \mathbb{D}_+^{n_p}$, $Q \in \mathbb{R}^{n_p \times n_p}$, and $\mu \in \mathbb{R}_{>0}$. Condition* (32) *is feasible with respect to $K \in \mathbb{R}^{n_p \times n_p}$ if and only if there exists $Y \in \mathbb{R}^{n_p \times n_p}$ such that*

$$\begin{bmatrix} \mathrm{He}(Q)+\Lambda P & -(Q^{\mathsf{T}}H+Y) & -Y \\ \bullet & -e^{-\mu}\Lambda P & 0 \\ \bullet & \bullet & -\mu\lambda_{\min}(\Lambda)e^{-\mu}\mathbf{I} \end{bmatrix} \prec 0 \tag{35}$$

In particular, Q is nonsingular and (32) *holds with $K = B^{-1}Q^{-\mathsf{T}}Y$.*

Proof. Since P and Λ are positive definite, nonsingularity of Q follows directly from the satisfaction of (35). Taking $Y = Q^{\mathsf{T}} BK$ in (32) establishes the result. □

Bearing in mind the above result, when $n_p = n_u$ and B is nonsingular, Problem 1 can be cast into the following optimization problem:

$$\begin{aligned} \underset{P,Y,Q,\mu,c}{\text{maximize}} \quad & c \\ \text{subject to} \quad & (35) \\ & P - c\mathbf{I} \succeq 0, c > 0 \end{aligned} \tag{36}$$

The above optimization problem, when μ is fixed, is a genuine semidefinite program, i.e., an optimization problem over linear matrix inequality constraints with linear cost function. Therefore, by performing a line search on the scalar μ, Problem 1 can be solved in a numerically affordable fashion.

4.1.2 Control Design Strategy II

Now we propose a second strategy to solve (34), albeit more conservative, that does not rely on any assumption on the input matrix B. This strategy hinges upon the following result that provides sufficient conditions for (32) to hold.

Proposition 3 *If there exist* $W \in \mathbb{D}_{+}^{n_p}$*,* $F \in \mathbb{R}^{n_p \times n_p}$*,* $Y \in \mathbb{R}^{n_u \times n_p}$*, and* $\mu, \delta \in \mathbb{R}_{>0}$ *such that:*

$$\begin{bmatrix} \mathrm{He}(F) & -(HW+BY) & -BY & F^{\mathsf{T}} \\ \bullet & -e^{-\mu}\Lambda W & 0 & 0 \\ \bullet & \bullet & \mu e^{-\mu}\lambda_{\min}(\Lambda)(\delta^2\mathbf{I} - 2\delta W) & 0 \\ \bullet & \bullet & \bullet & -W\Lambda^{-1} \end{bmatrix} \prec 0 \tag{37}$$

Then, F *is nonsingular and* (32) *holds with:*

$$P = W^{-1}, K = YW^{-1}, Q = F^{-1}$$

Proof. Nonsingularity of F trivially follows from (37). Now we prove the second part of the above statement. By using the following elementary relationship

$$(W - \delta\mathbf{I})^{\mathsf{T}}(W - \delta\mathbf{I}) = (W - \delta\mathbf{I})^2 = W^2 + \delta^2\mathbf{I} - 2\delta W \succeq 0$$

one has that (37) implies

$$\begin{bmatrix} \mathrm{He}(F) & -(HW+BY) & -BY & F^{\mathsf{T}} \\ \bullet & -e^{-\mu}\Lambda W & 0 & 0 \\ \bullet & \bullet & -\mu e^{-\mu}\lambda_{\min}(\Lambda)W^2 & 0 \\ \bullet & \bullet & \bullet & -W\Lambda^{-1} \end{bmatrix} \prec 0 \tag{38}$$

By pre-and-post multiplying the matrix in (38), respectively, by $F^{-\mathsf{T}}\oplus\mathbf{I}\oplus\mathbf{I}\oplus\mathbf{I}$ and $F^{-1}\oplus\mathbf{I}\oplus\mathbf{I}\oplus\mathbf{I}$ yields:

$$\begin{bmatrix}\mathrm{He}(F^{-1}) & -F^{-\mathsf{T}}(HW+BY) & -F^{-\mathsf{T}}BY & \mathbf{I}\\ \bullet & -e^{-\mu}\Lambda W & 0 & 0\\ \bullet & \bullet & -\mu e^{-\mu}\lambda_{\min}(\Lambda)W^2 & 0\\ \bullet & \bullet & \bullet & -W\Lambda^{-1}\end{bmatrix}\prec 0$$

the latter, by Schur complement, gives

$$\begin{bmatrix}\mathrm{He}(F^{-1})+W^{-1}\Lambda & -F^{-\mathsf{T}}(HW+BY) & -F^{-\mathsf{T}}BY\\ \bullet & -e^{-\mu}\Lambda W & 0\\ \bullet & \bullet & -\mu e^{-\mu}\lambda_{\min}(\Lambda)W^2\end{bmatrix}\prec 0 \tag{39}$$

which, via an additional congruence transformation with $\mathbf{I}\oplus W^{-1}\oplus W^{-1}$, gives

$$\begin{bmatrix}\mathrm{He}(F^{-1})+W^{-1}\Lambda & -F^{-\mathsf{T}}(H+BYW^{-1}) & -F^{-\mathsf{T}}BYW^{-1}\\ \bullet & -e^{-\mu}\Lambda W^{-1} & 0\\ \bullet & \bullet & -\mu e^{-\mu}\lambda_{\min}(\Lambda)\mathbf{I}\end{bmatrix}\prec 0$$

By taking $Q=F^{-1}$, $P=W^{-1}$, and $K=YW^{-1}$, the latter reads as (32). This concludes the proof. □

Building upon the relationship between P in (32) and W in (37) established by the above result and the expression of γ in (27), the minimization of the ISS gain γ can be performed by minimizing the largest eigenvalue of W. Therefore, when $n_u < n_p$, Problem 1 can be cast into the following optimization problem:

$$\begin{aligned}&\underset{W,Y,F,\mu,c}{\text{minimize}} && c\\ &\text{subject to} && (37)\\ & && W-c\mathbf{I}\preceq 0, c>0\end{aligned} \tag{40}$$

Also in this case, the above optimization problem can be solved via semidefinite programming tools by performing a line search on the parameters μ and δ. Indeed, when μ and δ are fixed, (37) is a linear matrix inequality. However, notice that replacing (32) with (37) introduces additional conservatism in the solution to (34).

5 Numerical Examples

In this section we showcase the effectiveness of our approach in two examples. The first example is of practical interest and pertains to the vehicle traffic flow model in [14,22,23]. The second example is academic and concerns an open-loop exponentially stable system of three conservation laws[4].

[4] Numerical solutions to SDP problems are obtained in Matlab® via *SDPT3* [47] thanks to *YALMIP* [34]. Numerical integration of hyperbolic PDEs is performed via the use of the Lax-Friedrichs (Shampine's two-step variant) scheme implemented in Matlab® by Shampine [43]. Code at `https://github.com/f-ferrante/MCSS2020_HyperbolicISS`.

Example 1 Consider the example in [23] in which system (1) is defined by the following data:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, H = \begin{bmatrix} 0 & 1.1 \\ 1 & 0 \end{bmatrix}, B = \mathbf{I}$$

Notice that since $\mathrm{spec}(H) = \{1.04881, -1.04881\}$, the system in open-loop unstable; see [5]. To solve Problem 1, as $n_u = n_p$ and B is nonsingular, we rely on the control design strategy presented in Section 4.1.2. In particular, by selecting $\mu = 0.1$, the solution to (36) yields:

$$\begin{aligned} &K = \begin{bmatrix} 0 & -0.18674 \\ -0.18501 & 0 \end{bmatrix}, P = \begin{bmatrix} 0.44051 & 0 \\ 0 & 0.34583 \end{bmatrix} \\ &\gamma = 1.7, \omega = 0.05 \end{aligned} \tag{41}$$

To show the effectiveness of the proposed control design, we compare the response of the closed-loop system in the presence of measurement disturbances obtained with the gain K in (41) and with a gain K' ensuring global exponential stability with the same convergence rate. To this end, we design K' such that $\mathrm{spec}(H + BK) = \mathrm{spec}(H + BK')$. In particular, by using the standard Matlab® pole placement routine, one gets:

$$K' = \begin{bmatrix} 0.86273 & -1.1 \\ -1 & -0.86273 \end{bmatrix} \tag{42}$$

which in this specific case ensures that $H + BK'$ is dissipative. The responses of the closed-loop system obtained with the two control gains are compared in Fig. 2 and Fig. 3. The figures show that that the proposed design is effective in reducing measurement disturbance sensitivity without overly impairing transient performance.

Example 2 We consider a modified version of [50, Example 2], in which (1) is defined by the following data:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, H = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.8 & 0.2 & 0.1 \\ 0.4 & 0 & 0.2 \end{bmatrix}$$

Concerning the input matrix B, we analyze two scenarios. In the first scenario, $n_u = n_p$ and $B = \mathbf{I}$, so that the controller gain can be designed via the results in Section 4.1.1. In the second scenario, we assume that $B^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and we design the controller gain via the results in Section 4.1.2.

First scenario

It is worthwhile to notice that H is dissipative. Therefore, as pointed out in Remark 8, in this case a trivial solution to Problem 1 is given by $K = 0$. Thus, to make the design problem more interesting, we design the control gain K to

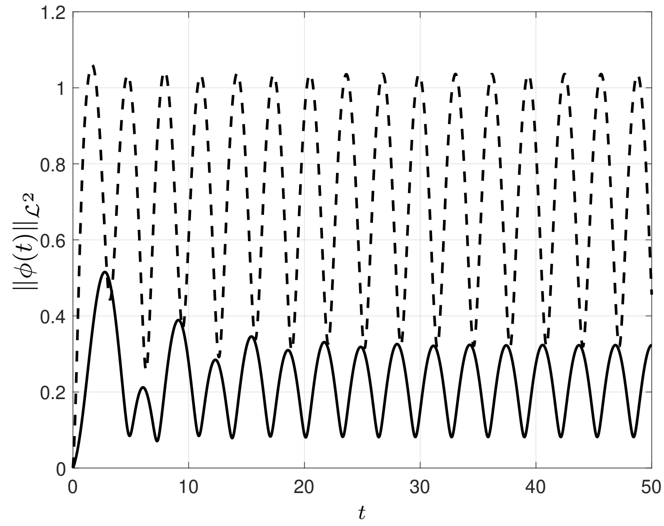

Fig. 2 Evolution of the $\mathcal{L}_2$ norm of the plant state from zero initial conditions with $\eta(t) = \sin(t)$ for different control gains. Optimal control gain (solid line) and gain K' in (42) (dashed line).

obtain a tradeoff between the ISS gain γ and the decay rate ω. To this end, as a first step, we determine an estimate of the open-loop decay rate ω_{OL}. In the light of [21] (and also Theorem 2), one can take $\omega = \frac{1}{2}\lambda_{\min}(\Lambda)\mu^\star_{OL}$ where:

$$
\begin{aligned}
\mu^\star_{OL} \coloneqq \max_{P,\mu} \quad & \mu \\
\text{subject to} \quad & H^\mathsf{T}\Lambda P H - e^{-\mu}\Lambda P \prec 0 \\
& P \in \mathbb{D}_+^{n_p}
\end{aligned}
$$

The above optimization problem can be cast as a generalized eigenvalue problem [6, Chapter 2.2.3] and can be solved via a bisection algorithm. Following this approach, one gets $\mu^\star_{OL} \approx 0.3209$, which gives the following estimate for open-loop decay rate $\omega_{OL} \approx 0.1605$. Bearing in mind, that such an estimate of the decay rate of the open-loop plant is available, we design a boundary feedback controller to improve the convergence rate of the closed-loop system, while limiting the effect of measurement disturbances. For this purpose, we solve optimization (36) by performing a line search for the variable μ over the interval $[\mu^\star_{OL}, 10 \times \mu^\star_{OL}]$. In Fig. 4 we report the tradeoff curve between γ and $\frac{\omega}{\omega_{OL}}$. From the data in Fig. 4, it is clear that convergence rate and robustness with respect to measurement disturbances are conflicting objectives. To showcase the effectiveness of the proposed control design via numerical simulations,

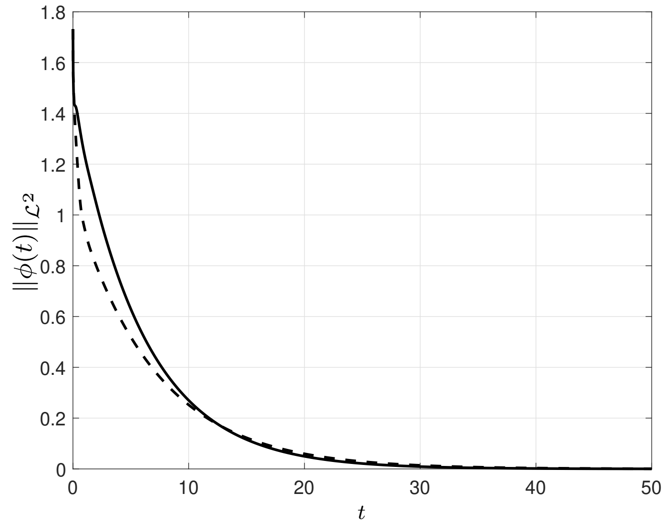

Fig. 3 Evolution of the $\mathcal{L}_2$ norm of the plant state from the initial condition $x_0 = (\cos(4\pi z) - 1, \cos(2\pi z) - 1)$ with $\eta \equiv 0$. Optimal control gain (solid line) and gain K' in (42) (dashed line).

we solve (36) with $\mu = 2 \times \mu^\star_{OL}$. This yields:

$$K = \begin{bmatrix} -0.040419 & -0.15132 & -0.048353 \\ -0.16266 & -0.076371 & -0.018732 \\ -0.080926 & -0.0017351 & -0.049936 \end{bmatrix}, P = \begin{bmatrix} 2.5228 & 0 & 0 \\ 0 & 1.0427 & 0 \\ 0 & 0 & 1.0427 \end{bmatrix}$$
$$\gamma \approx 0.9793, \omega = 0.3209 \tag{43}$$

To assess the benefit of the proposed controller in improving transient performance, in Fig. 5 we compare the evolution of the open-loop and of the closed-loop systems with $\eta = 0$ from the following initial condition:

$$x_0(z) = \begin{bmatrix} \cos(4\pi z) - 1 \\ \cos(2\pi z) - 1 \\ z(1-z) \end{bmatrix} \qquad \forall z \in [0, 1] \tag{44}$$

As foreseen, the closed-loop system manifests a faster convergence rate. To gauge the performance of the closed-loop system in the presence of measurement disturbances, in Fig. 6 we report the evolution of the closed-loop system when measurements are corrupted by a bounded disturbance η. As expected, for large times, the norm of the plant state is upper bounded by $\gamma\|\eta\|_{\mathcal{L}^\infty(\mathbb{R}_{\geq 0};\mathbb{R})}$. As a matter of fact, the bound is quite tight. To further illustrate the effectiveness of the proposed design strategy, in Fig. 7 we compare the response of the closed-loop system obtained when the control gain is designed taking different

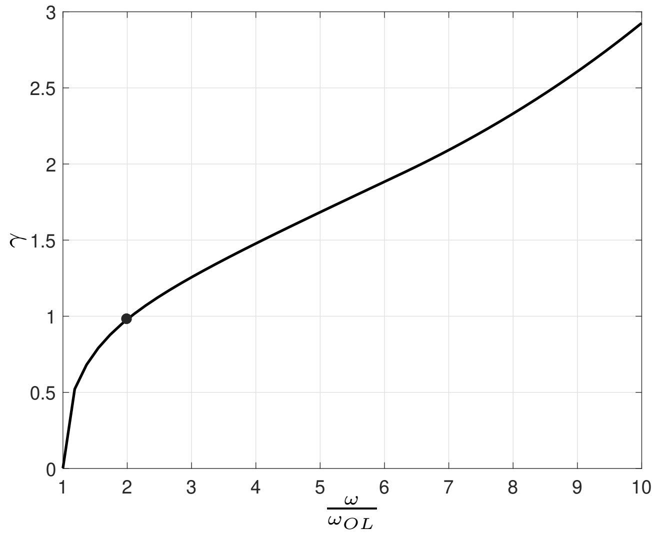

Fig. 4 Tradeoff between the ISS gain bound γ and ω/ω_{OL} in Example 2. The bullet denotes the pair $(2\mu_{OL}, \gamma_{2\mu_{OL}})$.

points on the tradeoff curve in Fig. 4. Simulations are performed from zero initial conditions and with $\eta(t) = 0.2\sin(t)$. What stands out in Fig. 7 is that more aggressive control gains increase measurement disturbance sensitivity.

Second Scenario

We know assume that

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and design the controller gain K by relying on the approach outlined in Section 4.1.2. In particular, by selecting again $\mu = 2 \times \mu^\star_{OL}$ and solving (40) by performing a line search on δ, one gets:

$$K = \begin{bmatrix} -0.2277 & -0.05109 & -0.07146 \\ -0.1637 & -0.2154 & -0.07010 \end{bmatrix}, P = \begin{bmatrix} 1.413 & 0 & 0 \\ 0 & 0.7826 & 0 \\ 0 & 0 & 0.7826 \end{bmatrix}$$
$$\delta = 1, \gamma = 1.1303 \tag{45}$$

From this data, we can see that the design considered in the first scenario results in the lowest value of γ. The responses of the closed-loop system for the control gains in (43) and (45) are compared in Fig. 8. As Fig. 8 shows, a larger value of γ reflects into a significant deterioration of the response in terms of measurement disturbance sensitivity.

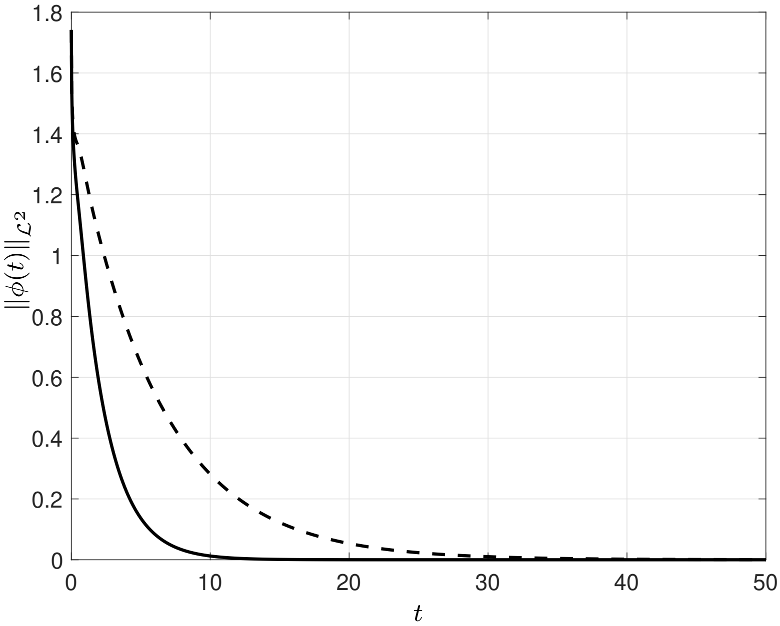

Fig. 5 Evolution of the $\mathcal{L}^2$-norm of the plant state from the initial condition (44) with $\eta = 0$: open-loop system (dashed line) and closed-loop system (solid line).

6 Conclusion

Boundary control design for hyperbolic systems of conservation laws has been considered in this paper. The proposed boundary controller depends only on non-collocated boundary output and is designed to ensure input-to-state stability of the closed-loop system with respect to measurement disturbances with a minimal ISS gain. The control design problem is recast into the feasibility problem of some matrix inequalities. Semidefinite programming tools are used to design the controller while minimizing the gain between the $\mathcal{L}^\infty$ of the measurement disturbance and the $\mathcal{L}^2$ of the plant state. Numerical experiments underlined the significance of the proposed suboptimal design.

The extension of the methodology we presented in this paper to specific physical applications, as for example the Shallow-Water equations is currently part of our ongoing research. Furthermore, the analysis of the conservatism induced by the class of Lyapunov functionals considered herein offers an interesting direction for future research.

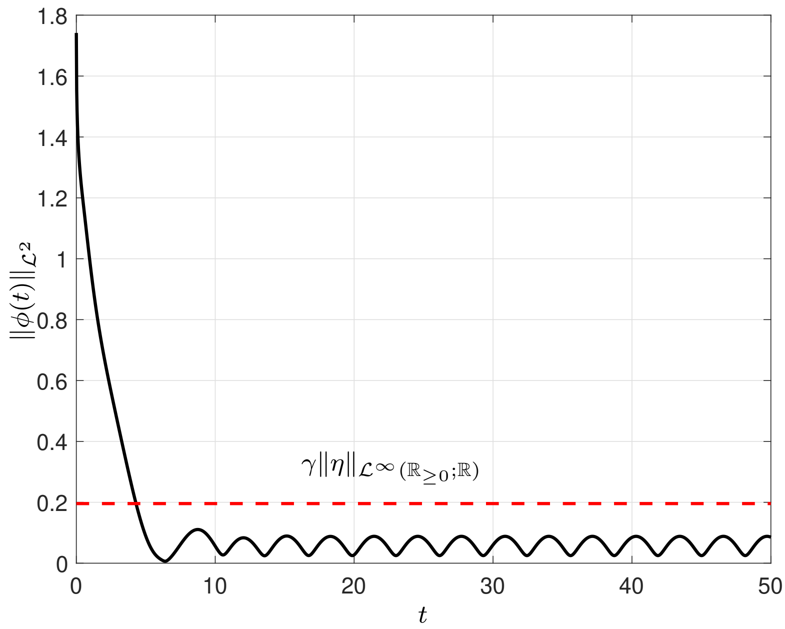

Fig. 6 Evolution of the $\mathcal{L}^2$-norm of the plant state from the initial condition (44) with $\eta(t) = 0.2\sin(t)$.

References

1. M. Ahmadi, G. Valmorbida, and A. Papachristodoulou. Dissipation inequalities for the analysis of a class of PDEs. *Automatica*, 66:163–171, 2016.
2. P. Apkarian, Hoang D. Tuan, and J. Bernussou. Continuous-time analysis, eigenstructure assignment, and h_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations. *IEEE Transactions on Automatic Control*, 46(12):1941–1946, 2001.
3. W. Arendt, C. J.K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96. Springer Science & Business Media, 2011.
4. M. Barreau, A. Seuret, F. Gouaisbaut, and L. Baudouin. Lyapunov stability analysis of a string equation coupled with an ordinary differential system. *IEEE Transactions on Automatic Control*, 63(11):3850–3857, 2018.
5. G. Bastin and J.-M. Coron. *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*, volume 88 of *Progress in Nonlinear Differential Equations and Their Applications*. Springer, 2016.
6. S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics, June 1997.
7. H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.
8. F. Bribiesca Argomedo, E. Witrant, and C. Prieur. D1-input-to-state stability of a time-varying nonhomogeneous diffusive equation subject to boundary disturbances. In *Proceedings of the American Control Conference*, pages 2978–2983, 2012.
9. F. Bribiesca Argomedo, E. Witrant, and C. Prieur. *Safety factor profile control in a Tokamak*. Springer International Publishing, 2013.

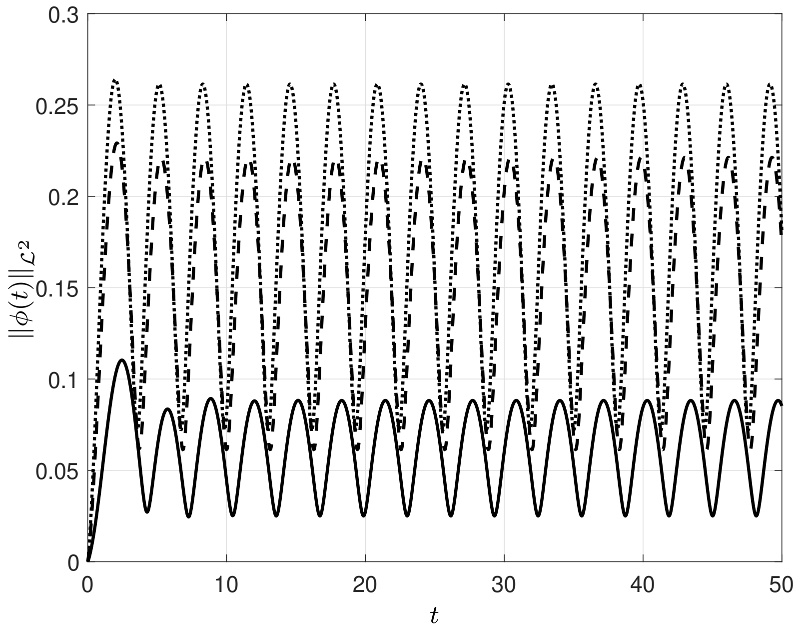

Fig. 7 Evolution of the $\mathcal{L}^2$-norm of the plant state from zero initial conditions in the presence of measurement disturbances for different control design tradeoff pairs $\left(\frac{\omega}{\omega_{OL}}, \gamma\right)$. Solid line $(2, 0.9793)$, dashed line $(5, 1.6831)$, and dotted line $(10, 2.9250)$.

10. F. Castillo, E. Witrant, C. Prieur, and L. Dugard. Dynamic boundary stabilization of linear and quasi-linear hyperbolic systems. In *Proceedings of the IEEE 51st Conference on Decision and Control*, pages 2952–2957, Maui (HI), USA, 2012.
11. F. Castillo, E. Witrant, C. Prieur, and L. Dugard. Boundary observers for linear and quasi-linear hyperbolic systems with application to flow control. *Automatica*, 49(11):3180–3188, 2013.
12. A. Chaillet and P. Pepe. Integral input-to-state stability of delay systems based on Lyapunov-Krasovskii functionals with point-wise dissipation rate. In *Proc. of the 2018 IEEE Conference on Decision and Control (CDC)*, pages 5451–5456, 2018.
13. W. Cheney. *Analysis for applied mathematics*, volume 208. Springer Science & Business Media, 2013.
14. G. M. Coclite, M. Garavello, and B. Piccoli. Traffic flow on a road network. *SIAM journal on mathematical analysis*, 36(6):1862–1886, 2005.
15. R. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*. Springer-Verlag, 1995.
16. R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21. Springer Science & Business Media, 2012.
17. S. Dashkovskiy and M. Kosmykov. Input-to-state stability of interconnected hybrid systems. *Automatica*, 49(4):1068–1074, 2013.
18. S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, 2013.
19. S. Dashkovskiy, B. Rüffer, and F. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

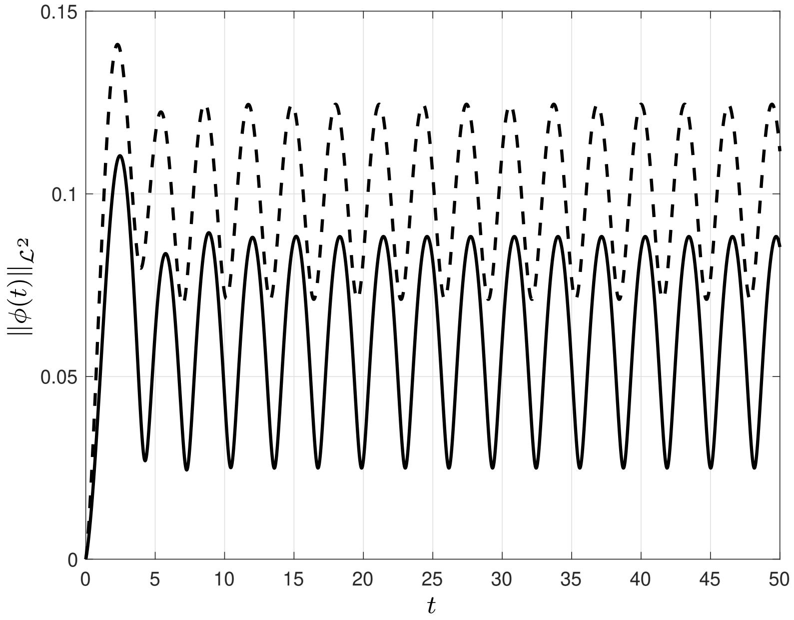

Fig. 8 Evolution of the $\mathcal{L}^2$-norm of the closed-loop system from zero initial conditions with $\eta(t) = \sin(t)$ for different control gains. Gain in (43) (solid line) and gain in (45) (dashed line).

20. S. N. Dashkovskiy, B. S. Ruffer, and F. R. Wirth. Applications of the general Lyapunov ISS small-gain theorem for networks. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 25–30, 2008.
21. A. Diagne, G. Bastin, and J.-M. Coron. Lyapunov exponential stability of 1-d linear hyperbolic systems of balance laws. *Automatica*, 48(1):109–114, 2012.
22. N. Espitia, A. Girard, N. Marchand, and C. Prieur. Event-based control of linear hyperbolic systems of conservation laws. *Automatica*, 70:275–287, 2016.
23. N. Espitia, A. Tanwani, and S. Tarbouriech. Stabilization of boundary controlled hyperbolic PDEs via Lyapunov-based event triggered sampling and quantization. In *Proceedings of the IEEE 56th Conference on Decision and Control*, pages 1266–1271, 2017.
24. F. Ferrante and C. Prieur. Boundary control design for linear conservation laws in the presence of energy-bounded measurement noise. In *Proceedings of the 57th IEEE Conference on Decision and Control*, pages 6550–6555, 2018.
25. B. Jacob, K. Morris, and H. Zwart. C_0-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain. *Journal of evolution equations*, 15(2):493–502, 2015.
26. B. Jacob, R. Nabiullin, J. Partington, and F. Schwenninger. On input-to-state-stability and integral input-to-state-stability for parabolic boundary control systems. In *Proceddings of the IEEE 55th Conference on Decision and Control*, pages 2265–2269, 2016.
27. B. Jacob, R. Nabiullin, J. R. Partington, and F. L. Schwenninger. Infinite-dimensional input-to-state stability and Orlicz spaces. *SIAM Journal on Control and Optimization*, 56(2):868–889, 2018.

28. B. Jacob, F.L. Schwenninger, and H. Zwart. On continuity of solutions for parabolic control systems and input-to-state stability. *Journal of differential equations*, 266(10):6284–6306, 2019.
29. I. Karafyllis and M. Krstic. *Input-to-State Stability for PDEs*. Communications and Control Engineering. Springer, 2019.
30. H. Khalil. *Nonlinear systems*. Prentice Hall, 3, 2002.
31. M. Krstic and A. Smyshlyaev. *Boundary Control of PDEs: A Course on Backstepping Designs*. SIAM, Philadelphia, PA, USA, 2008.
32. P.-O. Lamare, A. Girard, and C. Prieur. An optimisation approach for stability analysis and controller synthesis of linear hyperbolic systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 22(4):1236–1263, 2016.
33. H. Lhachemi and R. Shorten. ISS property with respect to boundary disturbances for a class of riesz-spectral boundary control systems. *Automatica*, 109:108504, 2019.
34. J. Lofberg. Yalmip: A toolbox for modeling and optimization in matlab. In *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pages 284–289. IEEE, 2004.
35. A. Mironchenko and H. Ito. Characterizations of integral input-to-state stability for bilinear systems in infinite dimensions. *Mathematical Control and Related Fields*, 6(3):447–466, 2016.
36. A. Mironchenko and C. Prieur. ISS of infinite-dimensional systems: recent results and open questions. *Under submission*, 2019.
37. A. Mironchenko and F. Wirth. Lyapunov characterization of input-to-state stability for semilinear control systems over Banach spaces. *Systems & Control Letters*, 119:64–70, 2018.
38. P. Pepe. Input-to-state stabilization of stabilizable, time-delay, control-affine, nonlinear systems. *IEEE Transactions on Automatic Control*, 54(7):1688–1693, jul. 2009.
39. P. Pepe and Zhong-Ping Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, 55(12):1006–1014, 2006.
40. G. Pipeleers, B. Demeulenaere, J. Swevers, and L. Vandenberghe. Extended LMI characterizations for stability and performance of linear systems. *Systems & Control Letters*, 58(7):510–518, 2009.
41. C. Prieur and J. Winkin. Boundary feedback control of linear hyperbolic systems: Application to the Saint-Venant–Exner equations. *Automatica*, 89:44–51, 2018.
42. M. Safi, L. Baudouin, and A. Seuret. Tractable sufficient stability conditions for a system coupling linear transport and differential equations. *Systems & Control Letters*, 110:1–8, 2017.
43. L. F. Shampine. Solving hyperbolic PDEs in MATLAB. *Applied Numerical Analysis & Computational Mathematics*, 2(3):346–358, 2005.
44. E.D. Sontag. Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, pages 163–220. Springer-Verlag, Berlin, 2007.
45. Y. Tang, C. Prieur, and A. Girard. Tikhonov theorem for linear hyperbolic systems. *Automatica*, 57:1–10, 2015.
46. N.-T. Trinh, V. Andrieu, and C.-Z. Xu. Design of integral controllers for nonlinear systems governed by scalar hyperbolic partial differential equations. *IEEE Transactions on Automatic Control*, 62(9):4527–4536, 2017.
47. R. H. Tütüncü, K.-C. Toh, and M. J. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical programming*, 95(2):189–217, 2003.
48. G. Valmorbida, M. Ahmadi, and A. Papachristodoulou. Stability analysis for a class of partial differential equations via semidefinite programming. *IEEE Transactions on Automatic Control*, 61(6):1649–1654, 2016.
49. G. Y. Weldegiyorgis and Mapundi K. Banda. An analysis of the input-to-state-stabilisation of linear hyperbolic systems of balance laws with boundary disturbances. *arXiv preprint arXiv:2006.02492*, 2020.
50. L. Zhang and C. Prieur. Necessary and sufficient conditions on the exponential stability of positive hyperbolic systems. *IEEE Transactions on Automatic Control*, 62(7):3610–3617, 2017.

A Ancillary Results and Definitions

Definition A1. *[15] Let $\mathcal{Z}$ be a Hilbert space. The function $\mathscr{T}\colon \mathbb{R}_{\geq 0} \to \mathscr{L}(\mathcal{Z},\mathcal{Z})$ is a $\mathcal{C}_0$-semigroup on $\mathcal{Z}$ if it satisfies the following properties:*

(a) *For all $t,s\in\mathbb{R}_{\geq 0}$, $\mathscr{T}(t+s)=\mathscr{T}(t)\mathscr{T}(s)$*
(b) $\mathscr{T}(0)=\mathbf{I}$
(c) *For all $z_0\in\mathcal{Z}$, $\lim\limits_{t\to 0^+}\|\mathscr{T}(t)z_0-z_0\|=0$*

$\circ$

Definition A2. *[15] Let $\mathcal{Z}$ be a Hilbert space and $\mathcal{A}\colon \operatorname{dom}\mathcal{A}\to\mathcal{Z}$. We say that $\mathcal{A}$ generates a $\mathcal{C}_0$-semigroup $\mathscr{T}$ on $\mathcal{Z}$ if for all $z\in\operatorname{dom}\mathcal{A}$*

$$\mathcal{A}z=\lim_{t\to 0^+}\frac{1}{t}\left(\mathscr{T}(t)-\mathbf{I}\right)z$$

$\circ$

Definition A3 ([13])**.** *Let X and Y be linear normed spaces, U be an open subset of X, $f\colon U\to Y$, and $x\in U$. We say that f is Fréchet differentiable at x if there exists $L\in\mathscr{L}(X,Y)$ such that*

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-Lh\|_Y}{\|h\|_X}=0$$

In particular L is the Fréchet derivative of f at x and is denoted by $Df(x)$. When $X=\mathbb{R}$, we denote

$$\dot f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$\circ$

Lemma A1. *Let $\psi\in\mathcal{C}^0([0,1];\mathbb{R}), G\in\mathbb{S}^n_+$ and $\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$ be endowed with its standard inner product. Consider the following functional*

$$\begin{aligned}V\colon \mathcal{L}^2((0,1);\mathbb{R}^{n_p})&\to\mathbb{R}\\ X\mapsto V(X)&\coloneqq\int_0^1\psi(z)\langle GX(z),X(z)\rangle_{\mathbb{R}^{n_p}}\,dz\end{aligned}$$

Then, V is Fréchet differentiable on $\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$ and in particular for each $X,h\in\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$

$$DV(X)h=2\langle\psi GX,h\rangle_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})}$$

Proof. For any $X,h\in\mathcal{L}^2((0,1);\mathbb{R}^{n_p})$, one has

$$\begin{aligned}V(X+h)-V(X)&=\int_0^1\psi(z)\left(\langle h(z),Gh(z)\rangle_{\mathbb{R}^{n_p}}\right.\\&\quad\left.+\,2\langle X(z),Gh(z)\rangle_{\mathbb{R}^{n_p}}\right)dz\\&\leq\lambda_{\max}(G)\|\psi\|_{\mathcal{L}^\infty(0,1;\mathbb{R})}\|h\|^2_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})}\\&\quad+2\langle\psi GX,h\rangle_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})}\end{aligned}$$

Thus, it follows that

$$\lim_{\|h\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})}\to 0}\frac{|V(X+h)-V(X)-2\langle X,\psi Gh\rangle_{\mathcal{L}^2(0,1;\mathbb{R}^{n_p})}|}{\|h\|_{\mathcal{L}^2((0,1);\mathbb{R}^{n_p})}}=0$$

This concludes the proof. $\square$