Noname manuscript No.
(will be inserted by the editor)

Cone-bounded feedback laws for m-dissipative operators on Hilbert spaces

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Received: date / Accepted: date

Abstract This work studies the influence of some constraints on a stabilizing feedback law. It is considered an abstract nonlinear control system for which we assume that there exists a linear feedback law that makes the origin of the closed-loop system globally asymptotically stable. This controller is then modified via a cone-bounded nonlinearity. A well-posedness and a stability theorems are stated. The first theorem is proved thanks to the Schauder fixed-point theorem, the second one with an infinite-dimensional version of LaSalle's Invariance Principle. These results are illustrated on a linear Korteweg-de Vries equation by some simulations and on a nonlinear heat equation.

Keywords Nonlinear semigroups · Stabilization · Abstract control systems

1 Introduction

The study of systems formed by a feedback interconnection of a system and a cone-bounded nonlinearity has received considerable attention in recent decades (see e.g [32], [36], or [15]). Indeed, in most of systems, the control input has a nonlinear dynamic. Nowadays, it is well known that neglecting these nonlinearities can lead to undesirable and even catastrophic behaviors for the closed-loop system. Without any assumption on the open-loop system, only a local stabilization result can be obtained. A classical research line is then to analyze the basin of attraction or to obtain a better one using anti-windup techniques in the case of saturated controls ([12] or [7]).

Tackling this kind of nonlinearities in the case of finite dimensional systems is already a difficult problem. However, nowadays, numerous techniques are available (see e.g. [32, 33, 31]) and such systems can be analyzed with different

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techniques: an appropriate Lyapunov function and a sector condition of the saturation map, as introduced in [32] or a frequency approach, leading to the so-called Popov's criterion, as it is reviewed in [15].

To the best of our knowledge, the study of this topic in the infinite dimensional case has started with [30, 28, 19]. More recently, some new results have been stated in [14, 10, 21, 25, 9]. Note that these results deal with control linear systems. The present paper aims at contributing to the study of feedback interconnection of a system (possibly nonlinear) and a cone-bounded nonlinearity in the framework of partial differential equations, more precisely for abstract control systems described with the semigroup theory ([24] and [22] are good introductions to linear semigroups and nonlinear semigroups, respectively. The Port-Hamiltonian framework, that models a lot of infinite-dimensional systems, is reviewed in [35]).

In this article, an interconnection of a system with a nonlinearity that is continuous, monotonic, linearly bounded, and vanishes at 0, is considered. Hence, these nonlinearities are more general than the saturations. When the system is linear, the feedback interconnection of a linear system and a nonlinearity can be referred to as systems of Lur'e type for which the Popov's criterion is well known (see e.g. [16]). In [14], an infinite-dimensional version of Lur'e systems is introduced. The authors derive some conditions, similar to the Popov's criterion for finite-dimensional systems, which ensure that the origin for the interconnection of a linear infinite-dimensional system and a nonlinearity satisfying a sector condition is globally exponentially stable. Let us mention also [4], where the linearized Landau-Lifshitz equation with a hysteresis is analyzed.

One of the most known functions belonging to this class of nonlinearities is the saturation. This topic has been introduced in [30] in the context of infinite-dimensional systems. In open loop, the systems considered are linear. In this article, the case of a priori bounded feedback is studied for abstract (possibly nonlinear) systems. A saturation function bounds the control input in the space where the origin is stabilized. To be more specific, for compact control operators, some conditions are derived to deduce, from a detectability assumption, the asymptotic stability when closing the loop with a saturating controller (see [30, Theorem 5.1] for a precise statement of this result). An infinite-dimensional version of the LaSalle's Invariance Principle is applied to obtain a weak convergence of the solution to the origin. This convergence becomes strong if the control space is equal to $\mathbb{R}$. This special case occurs for instance when dealing with a partial differential equation coupled with a controlled ordinary differential equation. In [28], the authors considered the same problem and obtained a better result with weaker assumptions. Indeed, they took advantage of the saturation function introduced in [30] and proved, without assuming the compactness of the control operator, but assuming only stabilizability, that saturating a stabilizable feedback law makes the origin globally asymptotically stable. Moreover, in [28], the case of unbounded control operators is tackled. A good introduction to unbounded control operators is [34].

The aim of this article is to obtain complementary results to the results of [28] and those of [30]. Moreover, in this paper, the open-loop system is nonlinear. Using a cone-bounded nonlinearity (possibly not globally Lipschitz), more general than the saturation introduced in [30], we derive some conditions to deduce the well-posedness of the closed-loop system by applying the Schauder's fixed-point theorem and the global asymptotic stability of the origin of the closed-loop system by using an infinite-dimensional version of the LaSalle's Invariance Principle. Finally, these results are applied on two specific infinite-dimensional examples, the linear Korteweg-de Vries equation and a nonlinear heat equation.

The article is organized as follows. In Section 2, we state our problem and present our main results. A subsection aims also at comparing our results to the existing results. In Section 3, the well-posedness of the Cauchy problem is tackled using the Schauder fixed-point theorem. In Section 4, the asymptotic stability of the origin for the closed-loop system is proven using an infinite-dimensional version of the LaSalle's Invariance Principle. Section 5 illustrates the main results of this paper with a Korteweg-de Vries equation with a distributed and bounded control and a nonlinear heat equation with a distributed and bounded control. Finally, Section 6 collects some concluding remarks.

Notation: Let $c \in \mathbb{C}$, $\Re(c)$ (resp. $\Im(c)$) denotes the real part (resp. the imaginary part) of c. The identity operator associated to a Hilbert space X is denoted by I_X. An operator $A : D(A) \subset X \to X$ is said *dissipative* if, for all $x, \tilde{x} \in X$, it holds that $\Re\{\langle Ax - A\tilde{x}, x - \tilde{x}\rangle_X\} \leq 0$. An operator $A : D(A) \subset X \to X$ is said to be *m-dissipative* if and only if A is dissipative and there exists $\lambda_0 > 0$ such that $\mathrm{Ran}(I - \lambda_0 A) = X$. Given a strongly continuous semigroup T over X, the *positive orbit through* $\phi \in X$ is defined by $\mathcal{O}^+ := \cup_{t\in\mathbb{R}_+} T(t)\phi$. The *strong ω-limit set of* ψ is the (possibly empty) set defined by $\omega(\psi) := \bigcap_{\tau\geq 0} \mathrm{clos}_X \left(\bigcup_{t\geq\tau} T(t)\psi\right)$. A *ball centered at* $x_0 > 0$ *of radius* r *in* X defined by $\mathcal{B}(x_0, r) = \{x \in X,\ \|x - x_0\|_X \leq r\}$. Given a Hilbert space X, a sequence $(x_n)_{n\in\mathbb{N}} \in X$ *weakly converges to* x if, for every $\tilde{x} \in X$, $\lim_{n\to+\infty}\langle x_n, \tilde{x}\rangle_X = \langle x, \tilde{x}\rangle_X$.

2 Problem statement and main results

2.1 Problem statement

Let X be a Hilbert space equipped with scalar product $\langle\cdot,\cdot\rangle_X$ and norm $\|\cdot\|_X$. Let U be another Hilbert space with scalar product $\langle\cdot,\cdot\rangle_U$ and norm $\|\cdot\|_U$. Moreover, let A be a (possibly nonlinear) dissipative operator that is the infinitesimal generator of a strongly continuous semigroup of contractions on X denoted by $(T(t))_{t\geq 0}$ with domain $D(A)$. From [8, Corollary 3.3], this implies that $D(A)$ is dense in X. Finally, let B be in $\mathcal{L}(U, X)$, the space of bounded linear operators from U to X.

We consider the stabilization problem of the origin of the following infinite-dimensional control system

$$\dot{x} = Ax + Bu \tag{1}$$

where u in U denotes the controlled input.

The aim of this paper is to study the case where the control is given by

$$u = -\sigma(B^\star x), \tag{2}$$

where $\sigma : U \to U$ is a mapping which will be characterized later on.

The system (1)-(2) is a feedback interconnection of a (possibly nonlinear) system and a nonlinearity denoted by σ. In the case of linear systems, it can be referred to a Lur'e system as in [14]. However, note that the nonlinearity σ considered all along this paper is different from the one introduced in [14].

2.2 Existing results and contributions

In [28] and [30], the authors considered the case where the control is bounded. To take into account this type of constraint, these papers introduced a saturation function, which is defined by, for all $s \in U$,

$$\mathfrak{sat}(s) = \begin{cases} s & \text{for all } \|s\|_U \leq u_s, \\ \dfrac{s}{\|s\|_U} u_s & \text{for all } \|s\|_U \geq u_s, \end{cases} \tag{3}$$

where $u_s \in (0, \infty)$ denotes the saturation level. Note that this function reduces to the identity when the U-norm of its argument is close to 0.

Such a situation arises for a large class of control systems and studying what effect can have a bounded stabilizing controller on the stability of the closed-loop system is already an open problem even for finite-dimensional systems (see e.g. [32] or [18]). In this paper, inspired by [2] and [25], we will consider nonlinearities more general than the saturations. Let us define them.

Definition 1 (Cone-bounded nonlinearities on U) Let $\sigma : U \to U$ be a continuous operator such that

1. for all u in U, $\Re\{\langle u, \sigma(u)\rangle_U\} = 0$ implies $u = 0$;
2. there exists a positive value L such that, for all $u \in U$, we have $\|\sigma(u)\|_U \leq L\|u\|_U$;
3. for all u, v in U we have $\Re\{\langle \sigma(u) - \sigma(v), u - v\rangle_U\} \geq 0$.

Example 1 (Examples of cone-bounded nonlinearities)

1. Any linear mapping $\sigma(u) = \mu u$, where μ is a positive value, is a cone-bounded nonlinearity;
2. The saturation given by (3) is a cone-bounded nonlinearity. The fact that this function satisfies items 1 and 2 is easy to check. The last item has been checked in [28]. Indeed, in this paper, the operator given by (3) is proved to be a m-dissipative operator. Hence, in particular, the operator $\mathfrak{sat}_U$ is monotone. Therefore, it satisfies item 3;

3. For all $s \in \mathbb{R}$, the so-called *localized saturation* (as considered in e.g., [32], [16]) defined by

$$\mathfrak{sat}_{\mathtt{loc}}(s) = \begin{cases} -u_s \text{ if } s \leq -u_s, \\ s \text{ if } -u_s \leq s \leq u_s, \\ u_s \text{ if } s \geq u_s, \end{cases} \tag{4}$$

with u_s a positive value, is a cone-bounded nonlinearity;

4. For any positive value u_s, the function $s \in \mathbb{R} \mapsto u_s \tanh\left(\frac{s}{u_s}\right)$ is a cone-bounded nonlinearity;
5. The function

$$\tilde{\sigma}:\ s \in \mathbb{R} \mapsto \mathfrak{sat}_{\mathtt{loc}}(\varphi(s)), \tag{5}$$

where $u_s > 1$ and where φ is defined as follows

$$\varphi:\ s \in \mathbb{R} \mapsto \begin{cases} -\sqrt{|s|-1} - 1 \text{ if } s < -1, \\ s \text{ if } s \in [-1, 1], \\ \sqrt{s-1} + 1 \text{ if } s > 1, \end{cases} \tag{6}$$

takes values in a bounded set, but it is not globally Lipschitz because of the function $s \mapsto \sqrt{s}$ in the definition of the function φ. Figure 1 illustrates the functions $s \mapsto \tilde{\sigma}(s)$ and $s \mapsto 2s$ with $s \in [-2, 2]$ and $u_s = 1.5$. It is clear that this function is a cone-bounded nonlinearity, as introduced in Definition 1.

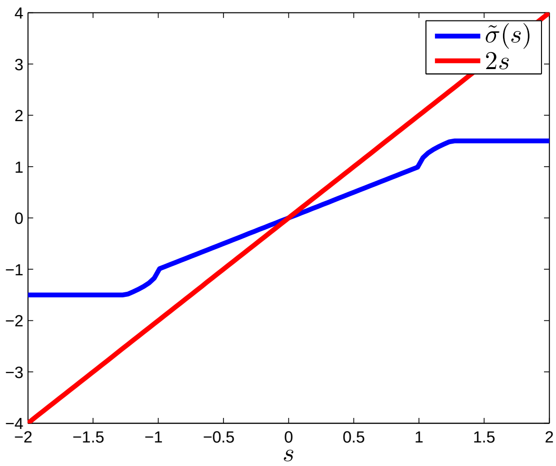

Fig. 1 Red line: $2s$; Blue line: $\tilde{\sigma}(s)$ with $u_s = 1.5$

In the following, we will consider the following closed-loop system

$$\begin{cases} \dot{x} = Ax - B\sigma(B^\star x) := A_\sigma x, \\ x(0) = x_0, \end{cases} \tag{7}$$

where $A_\sigma : D(A_\sigma) \subset X \to X$ is a nonlinear operator for which we know that

$$D(A_\sigma) = D(A). \tag{8}$$

Indeed, the domain of the operator

$$x \mapsto -B\sigma(B^\star x) \tag{9}$$

is X and from [22, Page 20]), we know that the domain of the sum of operators is the intersection of the domain of these two operators. Hence,

$$D(A_\sigma) = D(A) \cap X = D(A). \tag{10}$$

We wish to find conditions which ensure asymptotic stability of the origin of system (7).

Note that, from [22, Corollary 2.10, page 20], since A is dissipative, for all $\lambda > 0$, the operator $J_\lambda : D(J_\lambda) \to D(A)$ defined by

$$J_\lambda := (I_X - \lambda A)^{-1}$$

exists and satisfies the following inequality, for all $x, \tilde{x} \in D(J_\lambda)$

$$\|J_\lambda x - J_\lambda \tilde{x}\|_X \leq \|x - \tilde{x}\|_X. \tag{11}$$

Moreover, we have

$$D(J_\lambda) = \text{Ran}(I_X - \lambda A).$$

Moreover, since A generates a strongly continuous semigroup of contractions, from [22, Theorem 4.20, page 103], A is also a m-dissipative operator, which implies that $\text{Ran}(I_X - \lambda A) = X$.

2.2.1 Some existing results

Some existing results can be found in the literature. In this section, we will focus in particular on [30] and [28]. These papers study the particular cone-bounded nonlinearity given by (3). Hence, in this section only, we focus on the case where

$$\sigma(s) = \mathfrak{sat}_U(s), \quad \forall s \in U. \tag{12}$$

In [30], it is assumed the following properties

Assumption 1 *1. We have $\sigma(s) = \mathfrak{sat}_U(s)$;*

2. The operator A is linear and generates a strongly continuous of contractions denoted by $(W(t))_{t\geq 0}$;

3. The operator $(\lambda I_X - A)^{-1}$ is compact for all positive values λ;

4. The operator B is compact;

5. For all $\psi \in X$, the only solution to

$$B^\star T(t)\psi = 0 \tag{13}$$

is

$$\psi = 0. \tag{14}$$

Items 1., 2. and 3. allow to state the well-posedness of (7). Items 4. and 5. allow to apply a weak version of LaSalle's Invariance Principle. Note that the item 5. of these assumptions refers to a detectability property.

In [30], it is proved that, for each $x_0 \in X$, the operator A_σ generates a strongly continuous semigroup of contractions denoted by $(T_{\mathfrak{sat}_U}(t))_{t\geq 0}$ and, for each $x_0 \in X$, there exists a unique solution to (7) defined for all $t \in \mathbb{R}_{\geq 0}$ and given by $x(t) = T_{\mathfrak{sat}_U}(t)x_0$. Moreover, the following holds, for all $x_0 \in X$,

$$x(t) \rightharpoonup_X 0 \text{ as } t \to +\infty. \tag{15}$$

In his paper, the author only obtains a weak attractivity. In fact, since the paper aims at finding result for a particular partial differential equation, i.e. a beam equation, a stronger result is not necessary. The control operator for the partial differential equation belongs to the space $\mathcal{L}(\mathbb{R}, X)$. Hence, another theorem which takes into account this particular case is stated in [30]. The author of [30] proves that, under Assumption 1 and assuming moreover that $U = \mathbb{R}$, then, A_σ generates a strongly continuous semigroup of contractions denoted by $(T_{\mathfrak{sat}_U}(t))_{t\geq 0}$ and, for each $x_0 \in X$, there exists a unique mild solution to (7) denoted by $x(t) := T_{\mathfrak{sat}_U}(t)x_0$. Moreover, the following holds, for all $x_0 \in X$

$$\lim_{t\to+\infty} \|T_{\mathfrak{sat}_U}(t)x_0\|_X = 0 \tag{16}$$

Note that in the proof of these two results Slemrod does not use the particular form of $\mathfrak{sat}_U$. He only uses the fact that it is globally Lipschitz, monotone and the property 1 of Definition 1.

In [28], a better result is stated. The assumptions are weaker than Assumption 1. Let us state them

Assumption 2 *1. We have $\sigma(s) = \mathfrak{sat}_U(s)$;*

1. The operator A is linear and generates a strongly continuous of contractions denoted by $(W(t))_{t\geq 0}$;

2. The operator $A - BB^\star$ generates a strongly continuous of contractions denoted by $(T_I(t))_{t\geq 0}$ that satisfies the following, for all $z_0 \in Z$

$$\lim_{t\to+\infty} \|T_I(t)z_0\|_Z = 0. \tag{17}$$

Unlike Assumption 1 provided by [30], neither the operator B nor $(\lambda I_Z - A)^{-1}$ are assumed to be compact. Moreover, instead of assuming a detectability property as in item 3 of Assumption 1, only a stabilizability property is assumed in [28].

A stronger result than the result provided by [30] is stated in [28]. It is proved that, under Assumption 2, A_σ generates a strongly continuous semigroup of contractions denoted by $(T_{\mathfrak{sat}_U}(t))_{t\geq 0}$ and, for each $x_0 \in X$, (7) admits a unique solution denoted by $x(t) := T_{\mathfrak{sat}_U}(t)x_0$. Moreover, the following holds, for all $x_0 \in X$

$$\lim_{t\to+\infty} \|T_{\mathfrak{sat}_U}(t)x_0\|_X = 0. \tag{18}$$

Unlike the proof of the result given in [30], the proof of this latter result uses the special structure of $\mathfrak{sat}_U$. Moreover, the authors of [28] derive some conditions in order to obtain a similar result for unbounded control operators. Since this paper is devoted to the case of bounded control operators, this result will not be discussed here.

Papers [30] and [28] have inspired a lot of researchers. Among the results derived from these papers, [19] or [10] can be cited. Note that, even in the context of finite-dimensional systems, these papers have inspired some researchers (see e.g., [20]).

Remark 1 In the paper [14], the authors focus on another type of cone-bounded nonlinearity. Indeed, the nonlinearity under consideration in this paper is called a *sector condition* and is defined as follows: a nonlinearity $\Phi : U \to U$ satisfies a *sector condition* if there exist two operators $K_1, K_2 \in \mathcal{L}(X, U)$ such that

$$\Re\left\{\langle(\Phi(s) - K_1 s, \Phi(s) - K_2 s\rangle_U\right\} \leq 0, \quad \forall s \in U. \tag{19}$$

Note that the cone-bounded nonlinearity σ used all along this paper is a particular case of this nonlinearity. Indeed, if one takes $K_2 := 0$, it is easy to see that the cone-bounded nonlinearity satisfies a cone-bounded nonlinearity.

However, let us recall that in our work the operator A may be nonlinear, which is not the case of the paper [14]. Moreover, when looking at the assumptions of [14, Page 422-423, (H1)-(H4)], imposing $K_2 = 0$ implies that the origin for the open-loop system is globally asymptotically stable. In our work, we do not need this open-loop asymptotic stability.

2.2.2 First contribution: well-posedness

Now, we are able to state our first contribution. Here is its statement.

Theorem 1 (Well-posedness and Lyapunov stability) *Assume that σ is a cone-bounded nonlinearity. Moreover, assume that one of the two conditions is fullfilled:*

1. *σ is globally Lipschitz;*
2. *There exists a Banach space X_0 such that $D(A) \subseteq X_0$ and such that*
   (a) *the canonic injection from X_0 to X is compact;*
   (b) *it holds, for all $\bar{x}$,*

$$\sup_{x\in X} \|J_1(\bar{x} - B\sigma(B^\star x))\|_{X_0} < \infty. \tag{20}$$

Then, for all x_0 in $D(A)$, there exists a unique strong solution to (7)[1] *and the operator A_σ generates a strongly continuous semigroup of contractions $(T_\sigma(t))_{t\geq 0}$ such that the two functions*

$$t \mapsto \|T_\sigma(t)x_0\|_X \ , \ t \mapsto \|A_\sigma T_\sigma(t)x_0\|_X$$

are non increasing.

Remark 2 If A is linear, the condition (20) may be reduced to the following assumption:

$$\sup_{x\in X} \|J_1(-B\sigma(B^\star x))\|_{X_0} < \infty. \tag{21}$$

Indeed, in that case, (21) implies (20).

Remark 3 Following [22, Lemma 2.13], the condition (20) may be rewritten as the following statement: there exists a positive value λ_0 such that, for all $\bar{x} \in X$,

$$\sup_{x\in X} \|J_{\lambda_0}(\bar{x} - B\sigma(B^\star x)\|_{X_0} < \infty. \tag{22}$$

In order to make easier the reading, we let $\lambda_0 = 1$ as in (20), without loss of generality.

Remark 4 The function (5) in Example 1 shows that a cone-bounded nonlinearity does not have to be globally Lipschitz to ensure the well-posedness of the closed-loop system. Therefore, Theorem 1 can be seen as an extension of the classical result stated in [29, Lemma IV 2.1. page 165], where the nonlinearity has to be globally Lipschitz.

[1] A function $x : [0,\infty) \to X$ is called a strong solution to (7) if $x(t) \in D(A)$ for all $t \geq 0$ and if it satisfies the initial value problem.

Example 2 The condition (20) imposes a global bound on the mapping σ in a specific norm. As a first illustration, consider the following linear Korteweg-de Vries (for short KdV) equation

$$\begin{cases} \partial_t x(t,z) + \partial_z x(t,z) + \partial_{zzz} x(t,z) + \mathbb{1}_\Omega(z) u(t,z) = 0, \quad (t,z) \in \mathbb{R}_+ \times (0,L), \\ x(t,0) = x(t,L) = \partial_z x(t,L) = 0, \quad t \in \mathbb{R}_+, \\ x(0,z) = x_0(z), \end{cases} \tag{23}$$

where L is a positive value, $u(t,z)$ is the control, Ω is a nonempty subset of $(0,L)$ and $\mathbb{1}_\Omega$ is defined by

$$\mathbb{1}_\Omega(z) = \begin{cases} 1 \text{ if } z \in \Omega, \\ 0 \text{ otherwise.} \end{cases} \tag{24}$$

Setting $X = L^2(0,L)$ and $U = L^2(\Omega)$, system (23) can be written as in (1) denoting

$$\begin{aligned} A : D(A) \subset L^2(0,L) &\to L^2(0,L), \\ x &\mapsto -x' - x''', \end{aligned} \tag{25}$$

where

$$D(A) = \{x \in H^3(0,L),\ x(0) = x(L) = x'(L) = 0\}. \tag{26}$$

and

$$\begin{aligned} B : L^2(\Omega) &\to L^2(\Omega), \\ u &\mapsto \mathbb{1}_\Omega(z) u. \end{aligned} \tag{27}$$

A straightforward computation, together with some integrations by parts, shows that

$$\begin{aligned} &\Re\{\langle Ax, x\rangle_X\} \leq 0, \quad x \in D(A), \\ &\Re\{\langle y, A^\star y\rangle_X\} \leq 0, \quad y \in D(A^\star). \end{aligned} \tag{28}$$

Since A is a closed linear operator and $D(A)$ is dense in X, according to [24, Corollary 4.4, Chapter 1, page 15], these latter inequalities imply that A is the infinitesimal generator of a linear strongly continuous semigroup of contractions on X.

Let $\sigma : U \to U$ be defined by

$$\sigma(u)(z) = \tilde{\sigma}(u(z)), \ \forall z \in \Omega,$$

where $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}$. The assumption given in (20) is satisfied as soon as $\tilde{\sigma}$ is bounded. Indeed, assume $\tilde{\sigma}$ is bounded by a positive value u_s, that is

$$|\tilde{\sigma}(u(z))| \leq u_s, \quad \forall z \in [0,L]. \tag{29}$$

Note that if $\tilde{\sigma}$ is bounded, it implies that σ is also bounded as follows:

$$\|\sigma(u)\|_X \leq L u_s. \tag{30}$$

To prove that (20) holds, we follow a strategy similar to the one used in [21] or [25]. First note that

$$X_0 := H_0^1(0,L) \supset D(A)$$

embeds compactly in X by the Rellich-Kondrachov theorem (see [1, Theorem 9.16, page 285]). This set satisfies item (2)(a) of Theorem 1.

The operator A has a compact resolvent (see e.g. [6]), which implies that its spectrum consists only of eigenvalues. Moreover, A generates a linear strongly continuous semigroup of contractions, hence all the eigenvalues of the operator are located in the open left half of the complex plane. In particular $1 \notin \sigma(A)$ and J_1 is invertible. Hence, there exists a unique solution x to the equation $-(I_X - A)x(z) = -B\sigma(u)$, where $u \in U$. This latter equation can be rewritten as follows

$$\begin{cases} x(z) + x'(z) + x'''(z) = -B\sigma(u), \\ x(0) = x(L) = x'(L) = 0. \end{cases} \tag{31}$$

The unique solution to this solution can be expressed compactly as follows

$$x = -J_1(B\sigma(u)). \tag{32}$$

Multiplying the first line of (31) by x and integrating between 0 and L leads to

$$\|x\|^2_{L^2(0,L)} + \int_0^L xx'dz + \int_0^L xx'''dz = -\int_0^L \sigma(Bu)xdz \tag{33}$$

Integrating by parts this latter inequality twice and using boundary condition in (23) lead to

$$\|x\|^2_{L^2(0,L)} \leq -|x'(0)|^2 - \int_0^L \sigma(Bu)xdz \tag{34}$$

Applying Young's inequality and using the fact that $\tilde{\sigma}$ is bounded, we obtain

$$\|x\|^2_{L^2(0,L)} \leq \varepsilon_1 L u_s^2 + \frac{1}{\varepsilon_1}\|x\|^2_{L^2(0,L)}, \tag{35}$$

where $\varepsilon_1 > 1$. Hence, $\|x\|^2_{L^2(0,L)} \leq \frac{Lu_s^2}{\varepsilon_2}$, with $\varepsilon_2 := 1 - \frac{1}{\varepsilon_1}$.

Now, let us multiply the first line of (31) by zx and integrate between 0 and L. After performing some integrations by parts and using the boundary conditions in (23), we obtain

$$\begin{aligned} \frac{3}{2}\|x'\|^2_{L^2(0,L)} =& \frac{1}{2}\|x\|^2_{L^2(0,L)} - \int_0^L zx^2dz - \int_0^L zx\sigma(Bu)dz \\ \leq& \frac{1}{2}\|x\|^2_{L^2(0,L)} + \frac{1}{2}\|x\|^2_{L^2(0,L)} + \frac{L^3}{2}u_s \end{aligned}$$

Therefore, we have

$$\|x'\|^2_{L^2(0,L)} \leq M, \tag{36}$$

where $M := \frac{L^3}{2}u_s + \frac{Lu_s^2}{\varepsilon_2}$. By the Poincaré inequality, there is an equivalence between the norm $\|x'\|_{L^2(0,L)}$ and $\|x\|_{H_0^1(0,L)}$. Hence, using the expression (32), we can conclude that there exists a positive value c such that

$$\|J_1(B\sigma(u))\|_{H_0^1(0,L)} \leq c. \tag{37}$$

Thus, if $\tilde{\sigma}$ is bounded, the condition (20) is satisfied (and more precisely (21) in Remark 2) for the operator A defined in (25) and (26), and the operator B defined in (27).

Example 3 As a second illustration, consider the following nonlinear heat equation

$$\begin{cases} \partial_t x(t,z) = \partial_{zz} x(t,z) + \sin(x(t,z)) + u(t,z), \quad (t,z) \in \mathbb{R}_+ \times [0,1] \\ x(t,0) = x(t,1) = 0, \quad t \in \mathbb{R}_+, \\ z(0,x) = x_0(z), \quad z \in [0,1]. \end{cases} \tag{38}$$

Setting $X := L^2(0,1)$ and $U = L^2(0,1)$, system (38) can be written as in (1) denoting

$$\begin{aligned} A : D(A) \subset L^2(0,1) &\to L^2(0,1), \\ x &\mapsto x'' + \sin(x), \end{aligned} \tag{39}$$

where

$$D(A) := \{x \in H^2(0,1),\, x(0) = x(1) = 0\}, \tag{40}$$

and

$$B := I_X. \tag{41}$$

In Appendix B, the operator (39) is proved to be m-dissipative. Therefore, it generates a strongly continuous semigroup of contractions.

Let $\sigma : U \to U$ be defined by

$$\sigma(u)(z) = \tilde{\sigma}(u(z)), \quad z \in [0,L], \tag{42}$$

where $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}$. Following a similar strategy than for the KdV example and using some inequalities proved in Appendix B, the assumption given in (20) is satisfied as soon as $\tilde{\sigma}$ is bounded.

2.2.3 Second contribution: Asymptotic stability

The second result refers to the global asymptotic stability of the closed-loop system defined by (7).

Let $(T_I(t))_{t\geq 0}$ be the strongly continuous semigroup of contractions generated by $A - BB^*$.

Theorem 2 (Global asymptotic stability) *Assume that σ is a cone-bounded nonlinearity and that, for all x_0 in $D(A)$, there exists a unique strong solution to* (7)*. Suppose also that the operator A_σ generates a strongly continuous semigroup of contractions denoted by $t \mapsto T_\sigma(t)$ such that the two functions*

$$t \mapsto \|T_\sigma(t)x_0\|_X \ , \ t \mapsto \|A_\sigma T_\sigma(t)x_0\|_X$$

are non increasing, for all $x_0 \in D(A)$. Assume moreover that

1. *for all x_0 in $D(A)$,*
$$\lim_{t\to+\infty} \|T_I(t)x_0\|_X = 0;$$
2. *$D(A)$ equipped with the graph norm $\|\cdot\|_{D(A)} = \|\cdot\|_X + \|A\cdot\|_X$ is a Banach space which is compactly embedded in X.*

Then, the origin of the closed-loop system (7) *is globally asymptotically stable.*

Remark 5 Theorem 2 is a continuation of the work of [30]. The author of this latter paper assumes that the operator $(\lambda I_X - A)^{-1}$ is compact for all real $\lambda > 0$ and that the open-loop system satisfies the following observability property

$$B^\star T(t)x_0 = 0 \ , \ \forall t \geq 0 \Rightarrow x_0 = 0, \quad \forall x_0 \in X. \tag{43}$$

Our result needs only the origin to be stabilizable with the feedback law $u = -B^\star x$. In [30] it is assumed the compactness of the operator B. The latter assumption implies that the weak ω-limit set, which is defined by

$$\{\psi \in X, \text{there exists a sequence } t^n \text{ such that } T_\sigma(t^n)\psi \rightharpoonup T_\sigma(t)\phi \text{ as } t^n \to +\infty\},$$

is nonempty and invariant. In this paper, we assume an alternative property, that is $D(A)$ is compactly embedded in X, which implies that the strong ω-limit (the one we defined in the notation) is nonempty and invariant. Note that this property implies a stronger property for the open-loop system than the property assumed in [30]. However, the operator B does not require to be compact in this paper, as assumed in [30].

3 Proof of Theorem 1: well-posedness

This section aims at proving Theorem 1. A Schauder fixed-point theorem will be used. Let us recall it.

Theorem 3 (Schauder fixed-point theorem ([5], Theorem B.17, page 391)) *Let X be a Banach space and $\mathcal{C} \subseteq X$ be a convex and compact space. Therefore, every continuous mapping $f : X \to \mathcal{C}$ admits a fixed-point.*

The proof of Theorem 1 is given just below.

Proof of Theorem 1: First, note that $D(A) = D(A_\sigma)$ and A_σ is dissipative in X. Indeed, for all $x, \tilde{x} \in D(A)$

$$\begin{aligned}\Re\{\langle A_\sigma x - A_\sigma \tilde{x}, x - \tilde{x}\rangle_X\} =& \Re\{\langle Ax - A\tilde{x}, x - \tilde{x}\rangle_X\} \\ & - \Re\{\langle B\sigma(B^\star x) - B\sigma(B^\star \tilde{x}), x - \tilde{x}\rangle_X\}, \\ \leq & - \Re\{\langle \sigma(B^\star x) - \sigma(B^\star \tilde{x}), B^\star(x - \tilde{x})\rangle_U\}, \\ \leq & 0,\end{aligned} \tag{44}$$

where to obtain the last two inequality the dissipativity of A and the item 3 of Definition 1 have been used.

Now, we split our proof into two cases.

First case: item 1 holds. In this case, [29, Lemma 2.1., Part IV, page 165] implies that A_σ is a m-dissipative operator. From [22, Theorem 4.20, page 103], the operator A_σ, generates a strongly continuous semigroup of contractions on X denoted by $(T_\sigma(t))_{t\geq 0}$. Moreover, from [22, Corollary 3.7, page 53], it follows that

$$t \mapsto \|A_\sigma T_\sigma(t) x_0\|_X \tag{45}$$

is non increasing. From item 2 of Definition 1, it holds $\sigma(0) = 0$ and $T_\sigma(t)0 = 0$. Therefore, the function

$$t \mapsto \|T_\sigma(t) x_0\|_X \tag{46}$$

is a non-increasing function. This concludes the proof of Theorem 1 in the case where item 1 holds.

Second case: items 2a and 2b hold. Since A_σ is a dissipative operator, the operator $\tilde{J}_1 = (I_X - A_\sigma)^{-1}$ exists and is continuous. Moreover, from [22, Corollary 2.10, page 20], we have $D(\tilde{J}_1) = \mathrm{Ran}(I_X - A_\sigma)$.

In the second case, in order to apply [17, Theorem 4], we must show that

$$X = \mathrm{Ran}(I_X - A_\sigma).$$

The inclusion $\mathrm{Ran}(I_X - A_\sigma) \subset X$ is obvious. Let us prove that

$$X \subset \mathrm{Ran}(I_X - A_\sigma).$$

In other words, for $\bar{x}$ in X, we must show that there exists $\tilde{x}$ in $D(A)$ such that

$$(I_X - A)\tilde{x} = \bar{x} - B\sigma(B^\star \tilde{x}).$$

Let $\mathcal{T} : X \to D(A) \subseteq X_0$ be the mapping

$$\mathcal{T}(x) = J_1[\bar{x} - B\sigma(B^\star x)].$$

Let $\mathcal{C}$ be the set defined by

$$\mathcal{C} = \{x \in X_0 \,:\, \|x\|_{X_0} \leq N\},$$

where N comes from (20).

By assumption (item 2a of the statement of Theorem 1), the canonical injection from X_0 to X is compact. Thus, the set $\mathcal{C}$ is pre-compact as a subset of X and the closure in X of $\mathcal{C}$ is compact in X. It is moreover convex since it is a ball of radius N centered at 0. From item 2b in the statement of Theorem 1, we compute, for all x in $D(A)$,

$$\begin{aligned}\|\mathcal{T}(x)\|_{X_0} &= \|J_1[\bar{x} - B\sigma(B^\star x)]\|_{X_0}\,,\\ &\leq N.\end{aligned}$$

Hence, $\mathcal{T}(X) \subseteq \mathcal{C}$. Employing Schauder fixed point theorem, it implies that there exists a unique solution to $\mathcal{T}(x) = x$ and thus to (7). Therefore, from [17, Theorem 4], it implies that A_σ is a m-dissipative operator. Hence, the result is obtained similarly to the first case. It concludes the proof of Theorem 1. □

4 Proof of Theorem 2: asymptotic stability

The proof of Theorem 2 relies on the use of an infinite-dimensional version of the LaSalle's Invariance Principle stated in [13, Theorem 3]. Before proving it, let us prove the following lemma, that links the attractivity in $D(A)$ and in X.

Lemma 1 *Let $(T_\sigma(t))_{t\geq 0}$ be a semigroup of contractions on X, a Hilbert space. Let $D(A)$ be dense in X. Hence, if for all $x_0 \in D(A)$, the following holds*

$$\lim_{t\to+\infty} \|T_\sigma(t)x_0\|_X = 0. \tag{47}$$

hence, for all $x_0 \in X$,

$$\lim_{t\to+\infty} \|T_\sigma(t)x_0\|_X = 0 \tag{48}$$

Proof of Lemma 1:

Note that the proof is inspired by [19]. Pick $x_0 \in X$. Since $D(A)$ is dense in X, for all positive value ε, there exists $\tilde{x}_0 \in D(A)$ such that

$$\|x_0 - \tilde{x}_0\|_X \leq \frac{\varepsilon}{2}. \tag{49}$$

Since $(T_\sigma(t))_{\geq 0}$ is a semigroup of contractions, it holds, for all $t \geq 0$

$$\|T_\sigma(t)x_0 - T_\sigma(t)\tilde{x}_0\|_X \leq \frac{\varepsilon}{2}. \tag{50}$$

Morever, with (47), there exists $t^\star := t^\star(\varepsilon)$ such that, for all $\tilde{x}_0 \in D(A)$

$$\|T_\sigma(t)\tilde{x}_0\|_X \leq \frac{\varepsilon}{2}, \quad \forall t \geq t^\star. \tag{51}$$

Therefore, using a triangle inequality together with (50) and (51), one is able to prove that

$$\|T_\sigma(t)x_0\|_X \leq \varepsilon, \quad \forall t \geq t^\star. \tag{52}$$

This concludes the proof of Lemma 1. □

Proof of Theorem 2:

We aim at proving that, for all $x_0 \in D(A)$,

$$\lim_{t\to+\infty} \|T_\sigma(t)x_0\|_X = 0. \tag{53}$$

Indeed, once (53) holds, it is straightforward from Lemma 1 that the proof of Theorem 2 is achieved.

The proof is divided into three steps. Given $x \in D(A)$, we first prove that the ω-limit set, denoted by $\omega(x)$, is compact and invariant for the nonlinear semigroup $(T_\sigma(t))_{t\geq 0}$. Then we prove that, for all initial conditions in $\omega(x)$, the solution to (7) converges to 0 in X. Finally, it is proven that, for all initial conditions in $D(A)$, the solution to (7) converges to 0 in X.

First step: Compactness and invariance of $\omega(x)$. For all x in $D(A)$,

$$\begin{aligned}\|x\|_X + \|A_\sigma x\|_X &= \|x\|_X + \|Ax - B\sigma(B^\star x)\|_X,\\ &\geq c_1\|B\sigma(B^\star x)\|_X + \|Ax - B\sigma(B^\star x)\|_X,\\ &\geq \min\{1, c_1\}\|Ax\|_X,\end{aligned}$$

where the second inequality has been obtained from item 2 of Definition 1 and with $c_1 = \frac{1}{\|B\|_{\mathcal{L}(U,X)}\|B^\star\|_{\mathcal{L}(X,U)}L}$. This implies, for all x in $D(A)$,

$$\min\{1, c_1\}(\|x\|_X + \|Ax\|_X) \leq (1 + c_1)(\|x\|_X + \|A_\sigma x\|_X).$$

Since by sssumptions, for all x in $D(A)$, the two mappings $t \mapsto \|T_\sigma(t)x\|_X$ and $t \mapsto \|A_\sigma T_\sigma(t)x\|_X$ are nonincreasing, the former inequality implies

$$\|T_\sigma(t)x\|_{D(A)} \leq \frac{(1 + c_1)}{\min\{1, c_1\}}(\|x\|_X + \|A_\sigma x\|_X)\ ,\ \forall t \geq 0.$$

The set $D(A)$ equipped with the graph norm being compactly embedded in X, it yields that the positive orbit $\mathcal{O}^+(x)$ is precompact in X. Therefore, from [30, Theorem 3.2], for all x in $D(A)$, $\omega(x)$ is not empty, compact and invariant to the nonlinear semigroup $(T_\sigma(t))_{t\geq 0}$, i.e.,

$$T_\sigma(t)w \in \omega(x), \quad \forall (w, t) \in \omega(x) \times \mathbb{R}_+. \tag{54}$$

Second step: Asymptotic stability of the origin with initial conditions in $\omega(x)$. Let x be in $D(A)$. For all $t \geq 0$, due to the dissipativity of the operator A,

$$\frac{1}{2}\frac{d}{dt}\|T_\sigma(t)x\|_X^2 \leq -\Re\{\langle \sigma(B^\star T_\sigma(t)x), B^\star T_\sigma(t)x\rangle_U\} \leq 0. \tag{55}$$

Since $\mathcal{C} := \operatorname{clos}\{\mathcal{O}^+(x)\}$ is compact in X and σ is continuous, the function $z \in \mathcal{C} \mapsto \Re\{\langle \sigma(B^\star z), B^\star z\rangle_U\} \in \mathbb{R}$ is uniformly continuous. Let

$$\begin{aligned} W : \mathbb{R}_+ &\to \mathbb{R} \\ t &\mapsto W(t) := \Re\{\langle \sigma(B^\star T_\sigma(t)x), B^\star T_\sigma(t)x\rangle_U\}. \end{aligned} \tag{56}$$

The function $t \mapsto T_\sigma(t)x$ is continuous since $(T_\sigma(t))_{t\geq 0}$ is a strongly continuous semigroup. Moreover, by assumption, its time derivative, i.e. the function $t \mapsto A_\sigma T_\sigma(t)x$, is bounded. Therefore, the function $t \mapsto T_\sigma(t)x_0$ is uniformly continuous. Hence, W is uniformly continuous as a combination of two uniformly continuous functions.

From (55), it yields, for all $t \geq 0$,

$$\frac{1}{2}\|T_\sigma(t)x\|_X^2 - \frac{1}{2}\|x\|_X^2 \leq -\int_0^t W(s)ds. \tag{57}$$

Or, rearranging terms, it yields, for all $t \geq 0$,

$$\int_0^t W(s)ds \leq \frac{1}{2}\|x\|_X^2 - \frac{1}{2}\|T_\sigma(t)x\|_X^2 \leq \frac{1}{2}\|x\|_X^2. \tag{58}$$

Since W takes positive values, it yields

$$0 \leq \lim_{t\to+\infty} \int_0^t W(s)ds < \infty. \tag{59}$$

From Barbălat's Lemma, we get

$$\lim_{t\to+\infty} W(t) = 0. \tag{60}$$

Thus, from the definition of $\omega(x)$,

$$\Re\{\langle \sigma(B^\star w), B^\star w\rangle_U\} = 0\ ,\ \forall w \in \omega(x). \tag{61}$$

From item 1 in Definition 1 of the cone-bounded nonlinearity and (54),

$$B^\star T_\sigma(t)w = 0, \quad \forall w \in \omega(x), \quad \forall t \geq 0. \tag{62}$$

Hence, it implies that for all $w \in \omega(x)$,

$$T_\sigma(t)w = T_I(t)w\ ,\ \forall t \in \mathbb{R}_+.$$

Therefore, from Assumption 1 of Theorem 2, we have, for all $w \in \omega(x)$,

$$\lim_{t\to+\infty} \|T_\sigma(t)w\|_X = 0. \tag{63}$$

Third step: Asymptotic stability of the origin with initial conditions in $D(A)$. Let $w \in \omega(x)$. From (63), for all $w \in \omega(x)$, there exists $t(w) > 0$ such that

$$\|T_\sigma(t(w))w\|_X \leq \frac{1}{6}\|w\|_X. \tag{64}$$

Since $T_\sigma(t(w))$ is a continuous operator, there exists a positive value $\varepsilon_1(w)$ such that, for all $z \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|T_\sigma(t(w))z - T_\sigma(t(w))w\|_X \leq \frac{1}{6}\|w\|_X. \tag{65}$$

Therefore, for all $z \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|T_\sigma(t(w))z\|_X \leq \|T_\sigma(t(w))z - T_\sigma(t(w))w\|_X + \|T_\sigma(t(w))w\|_X \leq \frac{1}{3}\|w\|_X. \tag{66}$$

By reducing $\varepsilon_1(w)$ if needed, we may assume that $\varepsilon_1(w) \leq \frac{1}{3}\|w\|_X$. Hence, for all $z \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|w\|_X - \|z\|_X \leq \|z - w\|_X \leq \frac{1}{3}\|w\|_X. \tag{67}$$

Therefore, for all $z \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|w\|_X \leq \frac{3}{2}\|z\|_X, \tag{68}$$

and with (66), for all $z \in \mathcal{B}(w, \varepsilon_1(w))$,

$$\|T_\sigma(t(w))z\|_X \leq \frac{1}{2}\|z\|_X. \tag{69}$$

The family $\bigcup \{\mathcal{B}(w, \varepsilon_1(w)), w \in \omega(x)\}$ is a cover by open subsets of $\omega(x)$. Since $\omega(x)$ is a compact set, we can extract a finite cover which we index as follows

$$\omega(x) \subset \bigcup_{i=1}^{N_1} \{\mathcal{B}(w_{1i}, \varepsilon_1(w_{1i}))\}, \tag{70}$$

where (w_{1i})'s are in $\omega(x)$ and for a suitable positive integer N_1 and (69) has been used.

By considering

$$t^\star := \max_{i \in \{1, \ldots N_1\}} t(w_{1i}), \tag{71}$$

together with the fact that the function $t \mapsto \|T_\sigma(t)z\|_X$ is non increasing for any $z \in \omega(x) \subset D(A)$, we have, for all $z \in \omega(x)$,

$$\|T_\sigma(t^\star)z\|_X \leq \|T_\sigma(t(w_{1i}))z\|_X \leq \frac{1}{2}\|z\|_X, \tag{72}$$

where $i \in \{1, \ldots, N_1\}$ is selected such that $z \in \mathcal{B}(w_{1i}, \varepsilon_1(w_{1i}))$ and (72) has been used.

Since the functions $w \mapsto T_\sigma(t^\star)w$ and $V : w \to V(w) = \|w\|_X^2$ are continuous, for all $w \in \omega(x)$, there exists $\varepsilon_2(w) > 0$ such that, for all $z \in \mathcal{B}(w, \varepsilon_2(w))$,

$$\begin{aligned} |V(z) - V(w)| &\leq \frac{1}{5}V(w), \\ |V(T_\sigma(t^\star)z) - V(T_\sigma(t^\star)w)| &\leq \frac{1}{4}V(w). \end{aligned} \tag{73}$$

Therefore, with (72), for all $z \in \mathcal{B}(w, \varepsilon_2(w))$,

$$\begin{aligned} V(T_\sigma(t^\star)z) &\leq V(T_\sigma(t^\star)w) + \frac{1}{4}V(w), \\ &\leq \frac{1}{4}V(w) + \frac{1}{4}V(w), \\ &\leq \frac{1}{2}V(w). \end{aligned} \tag{74}$$

Moreover, the first inequality in (73) yields for all $z \in \mathcal{B}(w, \varepsilon_2(w))$,

$$V(w) \leq \frac{6}{5}V(z). \tag{75}$$

Finally, with (74), it follows, for all $z \in \mathcal{B}(w, \varepsilon_2(w))$,

$$V(T_\sigma(t^\star)z) \leq \frac{3}{5}V(z). \tag{76}$$

The family $\bigcup \left\{\mathcal{B}\left(w, \frac{\varepsilon_2(w)}{2}\right), w \in \omega(x)\right\}$ is a cover by open subsets of $\omega(x)$. Since $\omega(x)$ is a compact set, there exists $(w_{21}, \ldots, w_{2N_2})$ in $\omega(x)^{N_2}$ such that

$$\omega(x) \subset \bigcup_{i=1}^{N_2} \left\{\mathcal{B}\left(w_{2i}, \frac{\varepsilon_2(w_{2i})}{2}\right)\right\}. \tag{77}$$

Let us pick

$$\varepsilon_{2m} := \min_i \varepsilon_2(w_{2i}). \tag{78}$$

Let $x \in D(A)$. From (55), the function $t \mapsto \|T_\sigma(t)x\|_X^2$ is non-increasing and lower-bounded. Hence, there exists $V_\infty^\star(x) \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} \|T_\sigma(t)x\|_X^2 = V_\infty^\star(x) \geq 0. \tag{79}$$

Let us prove by contradiction that $V_\infty^\star(x) = 0$. We thus assume that $V_\infty^\star(x) \neq 0$. This implies that there exists $t_1 > 0$ such that, for all $t \geq t_1$,

$$\|T_\sigma(t)x\|_X^2 - V_\infty^\star(x) \leq \frac{1}{3}V_\infty^\star(x). \tag{80}$$

Moreover, there exists $w \in \omega(x)$ such that

$$\|w - T_\sigma(t_1)x\|_X \leq \frac{\varepsilon_{2m}}{2}. \tag{81}$$

Since $w \in \omega(x)$, there exists $i \in \{1, \ldots, N_2\}$ such that $w \in \mathcal{B}\left(w_{2i}, \frac{\varepsilon_2(w_{2i})}{2}\right)$. Therefore,

$$\begin{aligned} \|w_{2i} - T_\sigma(t_1)x\|_X &\leq \|w_{2i} - w\|_X + \|w - T_\sigma(t_1)x\|_X, \\ &\leq \frac{\varepsilon_2(w_{2i})}{2} + \frac{\varepsilon_{2m}}{2}, \\ &\leq \varepsilon_2(w_{2i}). \end{aligned} \tag{82}$$

Since $T_\sigma(t_1)x \in \mathcal{B}(w_{2i}, \varepsilon_2(w_{2i}))$, Equation (76) together with the fact that $T_\sigma(t_1 + t^\star)x = T_\sigma(t^\star)T_\sigma(t_1)x$ imply,

$$\|T_\sigma(t_1 + t^\star)x\|_X^2 = V(T_\sigma(t^\star)T_\sigma(t_1)x) \leq \frac{3}{5}\|T_\sigma(t_1)x\|_X^2. \tag{83}$$

Therefore, with (76) and (80), it follows, for all $t \geq t_1$

$$\begin{aligned}\|T_\sigma(t + t^*)x\|_X^2 - V_\infty^\star(x) &\leq \frac{3}{5}\|T_\sigma(t_1)x\|_X^2 - V_\infty^\star(x)\\ &\leq \frac{3}{5}\left(V_\infty^\star(x) + \frac{1}{3}V_\infty^\star(x)\right) - V_\infty^\star(x)\\ &\leq -\frac{1}{3}V_\infty^\star(x).\end{aligned} \tag{84}$$

Thus, we have

$$\|T_\sigma(t + t^\star)x\|_X^2 \leq \frac{2}{3}V_\infty^\star(x) < V_\infty^\star(x). \tag{85}$$

Since the function $t \mapsto \|T(t)x\|_X^2$ is nonincreasing, we obtain a contradiction with (79). Therefore

$$V_\infty^\star(x) = 0.$$

This concludes the proof of the global attractivity of the origin. The stability holds by assumption. Thus, using Lemma 1 , it concludes the proof of Theorem 2. □

5 Applications

5.1 Application to a linear Korteweg-de Vries equation

In this section, we illustrate Theorems 1 and 2 with the linear Korteweg-de Vries equation as considered in Example 2. In addition, we run some simulations.

Let us note that $B^\star : x \in X \mapsto x|_\Omega \in U$. Let $u(t, z) = -B^\star x(t, z) := -x(t, z)|_\Omega$, the origin for (23) is $L^2(0, L)$-globally asymptotically stable (see e.g. [3] or [27]). The stabilizability assumption of Theorem 2 is satisfied.

Now, let us tackle the case where the feedback law is bounded with the following operator defined, for all $(t, z) \in \mathbb{R}_+ \times [0, L]$

$$\sigma :\ u \in U \mapsto \sigma(u) = \tilde{\sigma}(u)(t, z), \tag{86}$$

where $\tilde{\sigma}$ is the function has been introduced in (5). Due to item 4 of Example 1, it is a cone-bounded nonliearity. This particular cone-bounded nonlinearity is illustrated by Figure 1.

The feedback law under consideration is as follows

$$u = -B\sigma(B^\star x) = -\mathbf{1}_\Omega \sigma(x|_\Omega) = -\sigma(\mathbf{1}_\Omega x). \tag{87}$$

Note that with such a feedback law the results of [30] cannot be applied since the function $u \in U \mapsto \sigma(u) \in U$ is not globally Lipschitz. Moreover, since we are considering a cone-bounded nonlinearity different from the one defined by (3), the results provided in [28] cannot be applied.

As stated in Example (2), it is known that the conditions of Theorem 1 are satisfied. Therefore, Theorem 1 applies. Thus, the operator

$$\begin{aligned} A_\sigma : D(A_\sigma) = D(A) \subset L^2(0,L) \to L^2(0,L), \\ w \mapsto -w' - w' - B\sigma(B^\star w) \end{aligned} \tag{88}$$

generates a strongly continuous semigroup of contractions.

Moreover, using the Lemma 2 given in the Appendix B.2, all the items of Theorem 2 are satisfied. Hence, Theorem 2 applies and one can conclude that the origin for (23) with $u = -\sigma(\mathbb{1}_\Omega x)$ is globally asymptotically stable.

Using a numerical scheme inspired by [23], we perfomed some numerical simulations. We note x the solution to (23) with (95) and $\tilde{x}$ the solution to

$$\begin{cases} \partial_t \tilde{x}(t,z) + \partial_z \tilde{x}(t,z) + \partial_{zzz}\tilde{x}(t,z) + \mathbb{1}_\Omega \tilde{x}(t,z) = 0, \quad (t,z) \in \mathbb{R}_+ \times (0,L) \\ \tilde{x}(t,0) = \tilde{x}(t,L) = \partial_z \tilde{x}(t,L) = 0, \quad t \in \mathbb{R}_+ \\ \tilde{x}(0,z) = \tilde{x}_0(z). \end{cases} \tag{89}$$

This latter equation refers as the Korteweg-de Vries with a linear feedback law.

We pick $x(0,z) = \tilde{x}(0,z) = 1 - \cos(z)$ and $L = 2\pi$ which is a critical case for the stability of the linear Korteweg-de Vries equation as it is reviewed in [26]. Let us choose $\Omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$. Figure 2 illustrates the solution to the system (23) with (95). We check on the simulation the origin for this equation is attractive. Figure 3 illustrates the solution to the system (89). It can be checked that the stabilizability assumption of Theorem (2) is satisfied. Figure 4 illustrates the control $u(t,z) = \sigma(\mathbb{1}_\Omega x)(t,z)$ with respect to the time and the space. We can check that the feedback law is bounded by the constant $u_s = 1.5$. Finally, Figure 5 illustrates the time-evolution of the Lyapunov functions $\|x\|^2_{L^2(0,L)}$ and $\|\tilde{x}\|^2_{L^2(0,L)}$. Note that the convergence in $L^2(0,L)$ of $\tilde{x}$ is faster than the convergence in $L^2(0,L)$ of x.

5.2 Application to a nonlinear heat equation

In this section, we illustrate Theorems 1 and 2 with the linear nonlinear heat equation as considered in Example 2.

Let us note that $B^\star : x \in X \mapsto x \in U$. Let $u(t,z) = -B^\star x(t,z) := -x(t,z)$, the origin for (38) is $L^2(0,L)$-globally asymptotically stable. Indeed, focus on the following Lyapunov function

$$V(x) = \int_0^1 x(t,z)^2 dz. \tag{90}$$

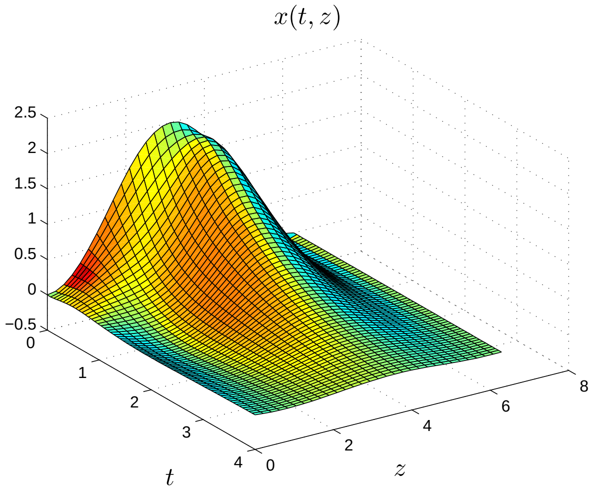

Fig. 2 Solution $x(t,z)$ with the control $u(t,z) = \sigma\left(\mathbb{1}_\Omega x\right)(t,z)$ where $u_s = 1.5$.

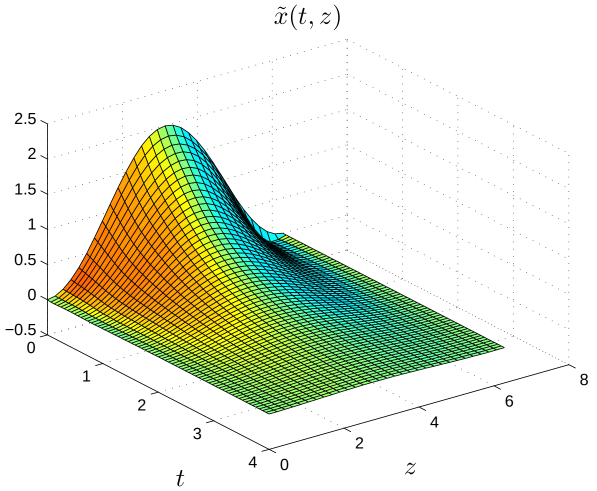

Fig. 3 Solution $\tilde{x}(t,z)$ with the control $u(t,z) = \mathbb{1}_\Omega \tilde{x}(t,z)$.

Its derivative along (38) yields

$$\frac{d}{dt}V(x) = \int_0^1 x(t,z)\partial_{zz}x(t,z)dz + \int_0^1 sin(x)(t,z)x(t,z)dz - \int_0^1 x(t,z)^2 \quad (91)$$

Performing some integrations by parts and using a Poincaré inequality leads to

$$\frac{d}{dt}V(x) \leq -\int_0^1 \partial_z x(t,z)^2dz + \frac{4}{\pi^2}\int_0^1 \partial_z x(t,z)^2dz - \int_0^1 x(t,z)^2 \quad (92)$$

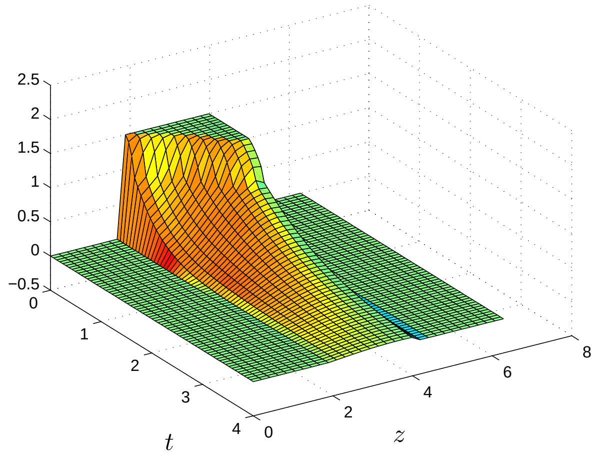

Fig. 4 Control $u(t,z) = \sigma\left(\mathbb{1}_{\Omega} x)\right)(t,z)$ where $u_s = 1.5$.

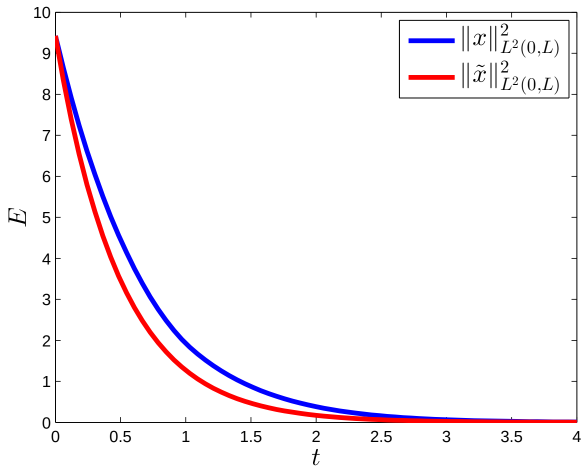

Fig. 5 Time-evolution of the Lyapunov functions $\|x\|^2_{L^2(0,L)}$ and $\|\tilde{x}\|^2_{L^2(0,L)}$

Hence, we have

$$\frac{d}{dt}V(x) \leq -V(x). \tag{93}$$

Therefore, the stabilizability assumption of Theorem 2 is satisfied.

Now, let us tackle the case where the feedback law is bounded with the following operator defined, for all $(t,z) \in \mathbb{R}_+ \times [0,L]$

$$\sigma : \; u \in U \mapsto \sigma(u) = \tilde{\sigma}(u)(t,z), \tag{94}$$

where $\tilde{\sigma}$ is the function has been introduced in (5). Due to item 4 of Example 1, it is a cone-bounded nonlinearity. This particular cone-bounded nonlinearity is illustrated by Figure 1.

The feedback law under consideration is as follows

$$u = -B\sigma(B^\star x) = -\sigma(x). \tag{95}$$

Note that with such a feedback law neither the results of [30], nor the ones of [28] cannot be applied since we are considering a nonlinear operator A.

As stated in Example 3, it is known that the conditions of Theorem 1 are satisfied. Therefore, Theorem 1 applies. Thus, the operator

$$\begin{aligned} A_\sigma : D(A_\sigma) = D(A) \subset L^2(0,L) \to L^2(0,L), \\ w \mapsto w'' + \sin(w) - B\sigma(B^\star w) \end{aligned} \tag{96}$$

generates a strongly continuous semigroup of contractions.

Moreover, using the Lemma 3 given in the Appendix B.2, all the items of Theorem 2 are satisfied. Hence, Theorem 2 applies and one can conclude that the origin for (38) with $u = -\sigma(x)$ is globally asymptotically stable.

6 Conclusion

In this paper, the analysis of a stabilizing feedback law modified via a cone-bounded nonlinearity has been tackled with various techniques. The well-posedness and the Lyapunov stability are proved using a Schauder fixed-point theorem and some nonlinear semigroups results. Finally, assuming a stabilizability property and precompactness of the trajectories of the solution, an infinite-dimensional version of the LaSalle's Invariance Principle has been used to conclude on the asymptotic stability of the origin. These results have been illustrated on a linear Korteweg-de Vries equation and on a nonlinear heat equation.

A possible future research line could be the study of unbounded control operators. Assuming the existence of a stabilizing feedback law for an unbounded control operator, is the origin still asymptotically stable when saturating the controller? In [28], the question has been tackled assuming that the semigroup associated to the closed-loop system with a saturated controller generates a strongly continuous semigroup of contractions. A natural question is: without assuming this latter property, is the Cauchy Problem well-posed? Is the origin of the closed-loop system still globally asymptotically stable ?

A Precompacity of the KdV equation with a cone-bounded nonlinearity

This section is devoted to the proof of the precompactness of the canonical embedding from $D(A_\sigma) = D(A)$, defined in (26), into $X := L^2(0,L)$. Let us state the lemma and prove it.

Lemma 2 *The canonical embedding from $D(A_\sigma)$, equipped with the graph norm, into $X := L^2(0,L)$ is compact.*

Proof of Lemma 2: We follow the strategy of [21], [25] and [11]. Let us recall the definition of the graph norm

$$\begin{aligned}\|x\|^2_{D(A_\sigma)} :=&\|x\|^2_{L^2(0,L)} + \|A_\sigma x\|^2_{L^2(0,L)}\\ =&\int_0^L \left(|x(z)|^2 + |-x'''(z) - x'(z) - \sigma\left(\mathbb{1}_\Omega x\right)(z)|^2\right) dz\\ =&\int_0^L \left(|x(z)|^2 + |x'''(z) + x'(z) + \sigma\left(\mathbb{1}_\Omega x\right)(z)|^2\right) dz.\end{aligned} \tag{97}$$

Note that

$$\|\sigma\left(\mathbb{1}_\Omega x\right)\|_{L^2(0,L)} \leq 2\|x\|_{L^2(0,L)}. \tag{98}$$

From the definition of the graph norm, we get the following two inequalities

$$\|x\|^2_{D(A_\sigma)} \geq \|x\|^2_{L^2(0,L)} \tag{99}$$

and, since, for all $(s,\tilde{s}) \in \mathbb{C}^2$, it holds $|s+\tilde{s}|^2 \leq 2|s|^2 + 2|\tilde{s}|^2$, we have

$$\begin{aligned}\|x\|^2_{D(A_\sigma)} \geq&\frac{1}{2}\int_0^L |-\sigma\left(\mathbb{1}_\Omega x\right)(z)|^2 dz\\ &+\frac{1}{2}\int_0^L |x'''(z) + x'(z) + \sigma\left(\mathbb{1}_\Omega x\right)(z)|^2 dz\\ \geq&\frac{1}{4}\int_0^L |x'''(z) + x'(z)|^2 dz.\end{aligned} \tag{100}$$

Noticing that $\|x'''\|^2_{L^2(0,L)} = \|x''' + x' - x'\|^2_{L^2(0,L)}$, we have

$$\|x'''\|^2_{L^2(0,L)} \leq 2\|x''' + x'\|^2_{L^2(0,L)} + 2\|x'\|^2_{L^2(0,L)}, \tag{101}$$

and using that $\|x'\|^2_{L^2(0,L)} = \|x' + x''' - x''' + zx - zx\|^2_{L^2(0,L)}$, we obtain

$$\begin{aligned}\|x'\|^2_{L^2(0,L)} \leq&2\|x' + x'''\|^2_{L^2(0,L)} + 2\|x''' - zx + zx\|^2_{L^2(0,L)}\\ \leq&2\|x' + x'''\|^2_{L^2(0,L)} + 4\|x''' - zx\|^2_{L^2(0,L)} + 4\|zx\|^2_{L^2(0,L)}\\ \leq&2\|z' + x'''\|^2_{L^2(0,L)} + 4\|x'''\|^2_{L^2(0,L)} - 8\int_0^L zx'''(z)x(z)dz + 8\|zx\|^2_{L^2(0,L)}.\end{aligned}$$

Deriving some integrations by parts, we get

$$\int_0^L zx'''(z)x(z)dz = \frac{3}{2}\|x'\|^2_{L^2(0,L)},$$

and therefore

$$\|x'\|^2_{L^2(0,L)} \leq 2\|x' + x'''\|^2_{L^2(0,L)} + 4\|x'''\|^2_{L^2(0,L)} - 12\|x'\|^2_{L^2(0,L)} + 8\|zx\|^2_{L^2(0,L)}. \tag{102}$$

Hence,

$$13\|x'\|^2_{L^2(0,L)} \leq 2\|x' + x'''\|^2_{L^2(0,L)} + 4\|x'''\|^2_{L^2(0,L)} + 8L^2\|x\|^2_{L^2(0,L)}. \tag{103}$$

Plugging inequality (101) in (103), we have

$$\begin{aligned}13\|x'\|^2_{L^2(0,L)} \leq & 2\|x'+x'''\|^2_{L^2(0,L)} + 4\left(2\|x'''+x'\|^2_{L^2(0,L)} + 2\|x'\|^2_{L^2(0,L)}\right)\\ & + 8L^2\|x\|^2_{L^2(0,L)}\\ \leq & 10\|x'+x'''\|^2_{L^2(0,L)} + 8\|x'\|^2_{L^2(0,L)} + 8L^2\|x\|^2_{L^2(0,L)}.\end{aligned}$$

Therefore,

$$\|x'\|^2_{L^2(0,L)} \leq 2\|x'+x'''\|^2_{L^2(0,L)} + \frac{8L^2}{5}\|x\|^2_{L^2(0,L)}. \tag{104}$$

Considering Equations (99) and (100), it leads us to the following inequality, for all $x \in D(A)$,

$$\|x'\|^2_{L^2(0,L)} \leq \Delta\|x\|^2_{D(A_\sigma)} \tag{105}$$

where Δ is a term which depends only on L.

Thus, if we consider now a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $D(A_\sigma)$ bounded for the graph norm of $D(A_\sigma)$, we have from (105) that this sequence is bounded in $H^1_0(0,L)$. Since the canonical embedding from $H^1_0(0,L)$ to $L^2(0,L)$ is compact, there exists a subsequence still denoted $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x$ in $L^2(0,L)$. Thus x belongs to $L^2(0,L)$ which allows us to state that $D(A_\sigma)$ embedds compactly in X. It concludes the proof of Lemma 2. □

B Nonlinear heat equation

B.1 Proof of the m-dissipativity of the nonlinear heat equation

This subsection is devoted to the proof of the following theorem.

Theorem 4 *The operator defined by* (39) *is m-dissipative*

Proof of Theorem 4:

The proof of Theorem 4 is divided in two steps. First, the operator A is proved to be dissipative. Secondly, we prove that, for all $f \in L^2(0,L)$, there exist $x \in D(A)$ such that

$$x - Ax = f. \tag{106}$$

Let us recall that the dissipativity and the existence of $x \in D(A)$ such that (106) holds imply that A is a m-dissipative operator.

First step: Dissipativity of the operator A.

Note that we have

$$\langle Ax - A\tilde{x}, x-\tilde{x}\rangle_{L^2(0,1)} = \int_0^1 (x-\tilde{x})(x''-\tilde{x}'')dz + \int_0^L (x-\tilde{x})(\sin(x)-\sin(\tilde{x}))dz. \tag{107}$$

Performing some integrations by parts leads to

$$\int_0^L (x-\tilde{x})(x''-\tilde{x}'')dz = -\int_0^1 (x'-\tilde{x}')^2 dz. \tag{108}$$

Moreover, using the fact that sin is Lipschitz together with a Poincaré inequality, one has

$$\int_0^L (x-\tilde{x})(\sin(x)-\sin(\tilde{x}))dz \leq \int_0^L (x-\tilde{x})^2 dz \leq \frac{4}{\pi^2}\int_0^1 (x'-\tilde{x}')^2 dz. \tag{109}$$

Hence, it is easy to see that

$$\langle Ax - A\tilde{x}, x-\tilde{x}\rangle_{L^2(0,1)} \leq 0. \tag{110}$$

Second step: Existence of $x \in D(A)$ such that (106) holds

To prove the existence of $x \in D(A)$ such that (106) holds, on has to prove that there exists a solution to the following nonlinear ODE

$$\begin{cases} x - x'' + \sin(x) = f, \\ x(0) = x(1) = 0. \end{cases} \tag{111}$$

We aim at applying the Schauder fixed-point theorem to the following nonhomogeneous linear ODE

$$\begin{cases} x - x'' = -\sin(y) + f, \\ x(0) = x(1) = 0, \end{cases} \tag{112}$$

where $y \in L^2(0,1)$. It is easy to see that there exists a unique solution to (112).

Focus on the map

$$\begin{aligned} \mathcal{T} : L^2(0,1) &\to L^2(0,1) \\ y &\mapsto x = \mathcal{T}(y) \end{aligned} \tag{113}$$

where $x = \mathcal{T}(y)$ is the unique solution to (112).

We define

$$C := \{x \in H_0^1(0,1) \mid \|x\|_{H_0^1(0,1)} \leq M\}. \tag{114}$$

From the theorem of Rellich, the injection of $H_0^1(0,1)$ in $L^2(0,1)$ is compact, then C is bounded in $H_0^1(0,1)$ and is relatively compact in $L^2(0,1)$. Moreover, it is a closed subset of $L^2(0,1)$. Thus C is a compact subset of $L^2(0,1)$. In order to apply the Schauder theorem, we have to prove that $\mathcal{T}(L^2(0,1)) \subset C$ for a suitable choice of M and λ. Let us multiply the first line of (112) by z and then integrate between 0 and 1. After some integrations by parts, one has

$$\|x\|_{L^2(0,1)}^2 + \|x'\|_{L^2(0,1)}^2 = \int_0^1 fx dz - \int_0^1 \sin(y) x dz. \tag{115}$$

Hence, applying Cauchy Schwarz inequality leads to

$$\|x'\|_{L^2(0,1)}^2 \leq \frac{1}{2}\|f\|_{L^2(0,1)}^2 + \frac{1}{2} - \|x\|_{L^2(0,1)}^2 + \|x\|_{L^2(0,1)}^2. \tag{116}$$

Therefore, since $\|x'\|_{L^2(0,1)}^2$ and $\|x\|_{H_0^1(0,1)}$ are equivalent by the Poincaré inequality, one has

$$\|x\|_{H_0^1(0,1)} \leq M, \tag{117}$$

where

$$M := \sqrt{\frac{1}{2}\left(\|f\|_{L^2(0,1)}^2 + 1\right)}.$$

Hence, applying Theorem 3, it concludes the proof of Theorem 4. □

B.2 Precompacity of the nonlinear heat equation with a cone-bounded nonlinearity

This subsection is devoted to the proof of the following lemma.

Lemma 3 *The canonical embedding from $D(A_\sigma)$, equipped with the graph norm, into $X := L^2(0,1)$ is compact.*

Proof of Lemma 3:

We follow the strategy of [21], [25] and [11]. Let us recall the definition of the graph norm

$$\begin{aligned}\|x\|^2_{D(A_\sigma)} :=&\|x\|^2_{L^2(0,1)} + \|A_\sigma x\|^2_{L^2(0,1)} \\ =&\int_0^1 \left(|x(z)|^2 + |x''(z) + \sin(x)(z) - \sigma\,(x)\,(z)|^2\right) dz\end{aligned} \tag{118}$$

Note that we have

$$\|x\|^2_{D(A_\sigma)} \geq \|x\|^2_{L^2(0,1)} \tag{119}$$

and

$$\begin{aligned}\|x\|^2_{D(A_\sigma)} \geq& \frac{1}{4}\int_0^1 |\sigma(x)(z)|^2 dz + \frac{1}{4}\int_0^1 |-\sin(x)(z)|^2 dz \\ &+ \frac{1}{4}\int_0^1 |x''(z) + \sin(x)(z) - \sigma(x)(z)|^2 dx \\ \geq& \frac{1}{8}\int_0^1 |x''(z)|^2 dz.\end{aligned} \tag{120}$$

Hence,

$$\|x\|^2_{D(A_\sigma)} \geq \frac{1}{8}\|x''\|^2_{L^2(0,1)}. \tag{121}$$

Noticing that $\|x\|^2_{L^2(0,1)} = \|x - x'' + x''\|^2_{L^2(0,1)}$, we have

$$\begin{aligned}\|x\|^2_{L^2(0,1)} =&\|x + x''\|^2_{L^2(0,1)} + \|x''\|_{L^2(0,1)^2} \\ =&\|x\|^2_{L^2(0,1)} + \|x''\|_{L^2(0,1)} + 2\int_0^1 x(z)x''(z)dz + \|x''\|_{L^2(0,1)}.\end{aligned} \tag{122}$$

Therefore, we have

$$-\int_0^1 x(z)x''(z)dz = \|x''(z)\|^2_{L^2(0,1)}. \tag{123}$$

Performing an integration by parts, we obtain

$$\int_0^1 x(z)x''(z)dz = -\|x'(z)\|^2_{L^2(0,1)}. \tag{124}$$

Hence, using (121), the following inequality holds

$$\|x'\|^2_{L^2(0,1)} \leq 8\|x\|^2_{D(A_\sigma)}. \tag{125}$$

Thus, if we consider now a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $D(A_\sigma)$ bounded for the graph norm of $D(A_\sigma)$, we have from (105) that this sequence is bounded in $H^1_0(0,L)$. Since the canonical embedding from $H^1_0(0,L)$ to $L^2(0,L)$ is compact, there exists a subsequence still denoted $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to x$ in $L^2(0,L)$. Thus x belongs to $L^2(0,L)$ which allows us to state that $D(A_\sigma)$ embedds compactly in X. It concludes the proof of Lemma 3. □

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