Math. Control Signals Systems (2001) 14: 143–172

Mathematics of Control,
Signals, and Systems

Uniting Local and Global Controllers with Robustness to Vanishing Noise*

Christophe Prieur†

Abstract. We consider control systems for which we know two stabilizing controllers. One is globally asymptotically stabilizing, the other one is only locally asymptotically stabilizing but for some reason we insist on using it in a neighborhood of the origin. We look for a uniting control law being equal to the local feedback on a neighborhood of the origin, equal to the global one outside of a larger neighborhood and being a globally stabilizing controller. We study several solutions based on continuous, discontinuous, hybrid, time-varying controllers. One criterion of the selection of a controller is the robustness of the stability to vanishing noise. This leads us in particular to consider a kind of generalization of Krasovskii trajectories for hybrid systems.

Key words. Nonlinear stabilization, Lyapunov functions, Disturbance, Errors measurements, Generalized trajectories.

1. Problem Statement and Related Results

1.1. Introduction

In nonlinear control system theory, we now have numerous tools (backstepping, forwarding, feedback linearization, passivation, ...) to design (globally) asymptotically stabilizing feedbacks. However, if such feedbacks give a satisfactory answer to the global asymptotic stabilization problem, they are not necessarily intended to address the performance problem. As opposed to this fact, for instance via linearization, one may design controllers addressing satisfactorily both the asymptotic stabilization and the performance problems but only locally. A pratical example of such a framework is given in [TKM]. This leads us to the idea of uniting a local controller with a global controller, i.e. given (1) a controller u_l able to stabilize locally while providing better performance and (2) a controller u_g providing global asymptotic stability, we are looking for a maybe time-varying, possibly hybrid, dynamic controller providing uniform global asymptotic stability for the overall system while matching exactly the local controller u_l when the system

* Date received: November 29, 1999. Date revised: August 7, 2000.
† Laboratoire d'analyse numérique et EDP, Université Paris Sud, bâtiment 425, 91405 Orsay, France. Christophe.Prieur@math.u-psud.fr.

state component is in a neighborhood of the origin and matching the global controller u_g when this component is outside a compact set containing the origin.

1.2. Problem Statement

Let f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function such that $f(0,0) = 0$. We consider the system

$$\dot{x} = f(x,u). \tag{1}$$

We call the following the uniting problem:

Let Ω be a bounded open connected neighborhood of the origin (in $\mathbb{R}^n$) and two continuous controllers u_l: $\mathscr{D} \to \mathbb{R}^m$ and u_g: $\mathbb{R}^n \to \mathbb{R}^m$, where $\mathscr{D} \subset \mathbb{R}^n$ is an open set containing $\texttt{clos}(\Omega)$, which make the systems f_l: $\mathscr{D} \to \mathbb{R}^n$, $x \mapsto f(x, u_l(x))$ and f_g: $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f(x, u_g(x))$ admit the origin as an asymptotically stable equilibrium, globally on their respective domain of definition.

We look for:

1. A bounded closed set $A \subset \mathbb{R}^n$ (e.g. an annulus) which separates $\mathbb{R}^n$ in two connected open sets C_l and C_g (e.g. an open ball and the complement of a closed ball) such that we have
$$\Omega \subset C_l \subset \mathscr{D}.$$
2. A control law $\varphi(x)$ (at this stage we restrict to a continuous static time-invariant control law), satisfying, for all j in $\{l, g\}$,
$$\varphi(x) = u_j(x), \qquad \forall x \in C_j, \tag{2}$$
and such that the origin of the system $\dot{x} = f(x, \varphi(x))$ is globally asymptotically stable.

1.3. Related Results and the Organization of This Paper

Studies on the uniting problem have already been reported, in particular in [MMP] and [TK]. In [MMP] the solution is given in the form of a continuous static time-invariant controller φ as requested above. It assumes the existence of a continuous path of stabilizing controllers between u_l and u_g. Unfortunately we show by means of an example that this assumption can be violated. Actually, for this particular example, there is no continuous (and even discontinuous) static time-invariant controller. This shows that dynamic extension of the control φ may be necessary, this being via time variations, discrete or continuous state, i.e. the problem formulation formulated above has to be considered with a more general class of controllers. Note that in [TK] a dynamic time-invariant controller $\varphi(x, s)$ is proposed but it does not satisfy our requirement (2). Specifically, along the trajectories, the proposed control converges with time to the global one u_g.

We first return, in Section 2.1, to the static time-invariant continuous controller proposed in [MMP] but we show, in Section 2.2, a system to which it cannot be applied. In fact this system motivates us to look at obstructions for solving the problem via (dis)continuous static time-invariant controllers (Section 2.3). This negative result leads us to reformulate, in Section 3, in terms of more general controllers. A first class of such controllers is dynamic hybrid control. We show that indeed the uniting problem can be solved in terms of weak generalized trajectories (Section 4.1) but unfortunately not in terms of strong generalized trajectories (Section 4.2). So finally, in Section 5, we propose a periodic static continuous controller solving the problem in its whole generality.

2. Static Time-Invariant Controllers

2.1. A Solution to the Uniting Problem

Following the arguments and ideas of [MMP], we get:

Theorem 2.1. *Let Ω be a bounded open connected neighborhood of the origin in $\mathbb{R}^n$, and let u_l and u_g be two continuous functions on $\mathbb{R}^n$. We assume the existence of a triple (ψ, V, c) as*:

- ψ: $[0,1] \times \mathbb{R}^n \to \mathbb{R}^m$, *a continuous path connecting u_l to u_g, i.e. for all x in $\mathbb{R}^n$, we have*
$$\psi(0,x) = u_l(x), \qquad \psi(1,x) = u_g(x),$$
- V: $[0,1] \times \mathbb{R}^n \to \mathbb{R}_+$, *a $\mathscr{C}^1$ function which, for all $s \in [0,1]$, is positive definite and radially unbounded,*
- *c a positive real number such that Ω is contained in $\{x\colon V(0,x) < c\}$*

with the following properties:

1. *For each s in $[0,1]$ and each x in $\mathbb{R}^n \backslash \{0\}$, we have*
$$\frac{\partial V}{\partial x}(s,x) f(x, \psi(s,x)) < 0. \tag{3}$$
2. *For each (s,x) satisfying $V(s,x) = c$, we have*
$$\frac{\partial V}{\partial s}(s,x) < 0. \tag{4}$$

Under these conditions, we can find a bounded closed set A and a continuous function φ defined as
$$\varphi(x) = \psi(\gamma(x), x), \qquad \forall x \in \mathbb{R}^n, \tag{5}$$
where γ is a locally Lipschitz function, which solves the uniting problem.

Remark 2.2. 1. With (3) holding for $s = 1$, we impose that $\mathscr{D}$ is actually $\mathbb{R}^n$. In fact this restriction is too strong. We need only that Ω be sufficiently small inside $\mathscr{D}$. This "sufficiently" is linked to the stability properties provided by u_g. To avoid making our statement of Theorem 2.1 too complicated, we have preferred to impose $\mathscr{D} = \mathbb{R}^n$.

2. The existence of some c such that Ω is contained in $\{x, V(0,x) < c\}$ follows from the fact that V is radially unbounded. However, with (4) one asks more of such c.

Proof. Let A be the closed set $\{x\colon c \le V(0,x), V(1,x) \le c\}$. Condition (4) implies that this set is nonempty, bounded and contains the set $\{x\colon V(0,x) = c\}$. Also the sets

$$C_l = \{x\colon V(0,x) < c\}, \qquad C_g = \{x\colon c < V(1,x)\}$$

are open and connected, with an empty intersection and satisfy $\mathbb{R}^n = C_l \cup A \cup C_g$. This means that A has the required properties.

We now define the controller $\varphi(x)$. To do this, we introduce a function γ on $\mathbb{R}^n$ as follows:

- for x in C_l, we let $\gamma(x) = 0$,
- for x in C_g, we let $\gamma(x) = 1$,
- for x in A, we choose $\gamma(x)$ as the solution of

$$V(\gamma, x) = c. \tag{6}$$

Condition (4) and the Implicit Function Theorem imply that γ is well defined and Lipschitz on $\mathbb{R}^n$ and $\mathscr{C}^1$ on $\mathbb{R}^n \backslash (\partial C_l \cup \partial C_g)$ and we have

$$x \in \mathtt{int}(A) \quad \Leftrightarrow \quad \gamma(x) \in (0,1). \tag{7}$$

Then we let φ be the function defined by (5), and we readily have (2), for j in $\{l, g\}$. The global asymptotic stability of the origin for the closed-loop system is a consequence of the following three properties:

1. For each x in $\mathtt{clos}(C_g)$, we have (from (3))

   $$\frac{\partial V}{\partial x}(1,x) f(x, \varphi(x)) < 0.$$

   With continuity, it follows that there exists ε_g such that any trajectory X with initial condition x verifying $V(1,x) \ge c - \varepsilon_g$ satisfies, for x and all $t \ge t_0$, $V(1, X(t)) \le V(1,x)$, reaches in finite time, depending only on $V(1,x)$, and then remains in the set

   $$\{x\colon V(1,x) \le c - \varepsilon_g\} \subsetneq C_l \cup A.$$

2. Similarly, using $V(0,x)$, we get the existence of ε_l such that the origin is asymptotically stable with a basin of attraction containing the set

   $$C_l \subsetneq \{x\colon V(0,x) \le c + \varepsilon_l\}.$$

3. The function γ is $\mathscr{C}^1$ on $\mathtt{int}(A)$ and, from (3), (4) and (6), we get

   $$\begin{aligned}\frac{\partial \gamma}{\partial x}(x) f(x, \varphi(x)) &= -\frac{(\partial V/\partial x)(\gamma(x), x)}{(\partial V/\partial s)(\gamma(x), x)} f(x, \varphi(x)) \\ &= -\frac{1}{(\partial V/\partial s)(\gamma(x), x)} \frac{\partial V}{\partial x}(\gamma(x), x) f(x, \psi(\gamma(x), x)) \\ &< 0.\end{aligned}$$

   It follows that any trajectory with initial condition x in the set

   $$\{x\colon V(1,x) \le c - \varepsilon_g, V(0,x) \ge c + \varepsilon_l\} \subset \mathtt{int}(A)$$

satisfies, for all $t \geq 0$, $V(1, X(t)) \leq c$, converges to the set $\{x\colon \gamma(x) \leq 0\} = \mathtt{clos}(C_l)$, and therefore reaches the set $\{x\colon V(0,x) \leq c + \varepsilon_l\}$ in finite time independent of x. ■

2.2. A Topological Obstruction

Theorem 2.1 provides a solution to the uniting problem via a static time-invariant continuous controller. We show in this section that we must not restrict our attention to only such kinds of feedbacks.

Let the system be

$$\dot{x} = -y^2 x, \qquad \dot{y} = u. \tag{8}$$

The data of the uniting problem we consider are

$$u_l = -y + x, \qquad u_g = -y - x, \qquad \Omega = \{(x, y)\colon x^2 + y^2 < \tfrac{1}{2}\}, \qquad \mathscr{D} = \mathbb{R}^2.$$

The fact that u_l and u_g are global asymptotic stabilizers can be checked with LaSalle's invariance theorem and the Lyapunov function $2x^2 + y^4$.

Let A be any closed set which separates $\mathbb{R}^2$ into two connected open sets C_l and C_g with C_l containing Ω. There exists $0 < c_l < c_g$ such that

$$A \subsetneq \{(x, y)\colon c_l^2 \leq x^2 + y^2 \leq c_g^2\}.$$

Assume the existence of a static time-invariant continuous controller $\varphi(x, y)$ solving the uniting problem. Then we have

$$\varphi(x, y) = \begin{cases} -y + x & \text{if} \quad x^2 + y^2 \leq c_l^2, \\ -y - x & \text{if} \quad c_g^2 \leq x^2 + y^2, \end{cases}$$

and, in particular,

$$\varphi(c_l, 0) = c_l, \qquad \varphi(c_g, 0) = -c_g.$$

Since c_l and c_g are positive, the continuity of φ implies the existence of c, strictly positive, such that $\varphi(c, 0) = 0$. It follows that $(c, 0)$ is an equilibrium of the closed-loop system contradicting the fact that $\varphi(x, y)$ is globally asymptotically stabilizing the origin.

We have established that the conclusion of Theorem 2.1 does not hold and so its assumptions are violated. Actually the same argument as above shows that there is no continuous function $\psi(s, (x, y))$ connecting u_l to u_g and providing a globally asymptotically stabilizing controller for each s in $[0, 1]$.

The obstruction observed with the system (8) leads to the following necessary condition for the solvability of the uniting problem via static time-invariant continuous feedback, where we impose $\mathscr{D} = \mathbb{R}^n$ to avoid making our statement too complicated.

Theorem 2.3. *Let (u_l, u_g, Ω) be the data of a uniting problem. If there exists a static time-invariant continuous control as a solution for this problem, then there exists $0 < c_l < c_g$ such that the functions $\tilde{u}_l$ and $\tilde{u}_g$ below are homotopic*:

$$\forall j \in \{l, g\}, \qquad \tilde{u}_j\colon \mathbb{S}^{n-1} \to \Sigma := \{(x, u), f(x, u) \neq 0\} \subset \mathbb{R}^n \times \mathbb{R}^m,$$
$$\xi \mapsto (c_j \xi, u_j(c_j \xi)).$$

Proof. Since A is a bounded closed set not containing the origin, there exists $0 < c_l < c_g$ such that A is included in $\{(x, y): c_l \le |x| \le c_g\}$. The function $H: [0, 1] \times \mathbb{S}^{n-1} \mapsto \Sigma$ defined as

$$H(s, \xi) = ([(c_g - c_l)s + c_l]\xi, u([(c_g - c_l)s + c_l]\xi))$$

provides the required homotopy. Indeed, there is no equilibrium in $\{(x, y): c_l \le |x| \le c_g\}$, so we have, for all (s, ξ) in $[0, 1] \times \mathbb{S}^{n-1}$, $f(H(s, \xi)) \ne 0$. ∎

The necessary condition given in Theorem 2.3, written in terms of homotopy, can also be expressed in terms of homology as in [C].

For the system (8), the set Σ is $\mathbb{R}^3$ without the x- and y-axis. The image of $\mathbb{S}^1$ by $\tilde{u}_l$ in $\mathbb{R}^3$ is an ellipsis, intersection of the plane $u + y - x = 0$ with the cylinder $x^2 + y^2 = c_l^2$. Similarly, the one by $\tilde{u}_g$ is an ellipsis in the plane given by $u + y + x = 0$. We can see that there is no continuous deformation allowing us to go from one ellipsis to the other without crossing the x- or y-axis. So the necessary condition is not met.

We conclude that the class of static time-invariant continuous controllers is not rich enough to address the uniting problem. Before investigating a richer class, we show that, in some cases, the class of static time-invariant discontinuous controllers is also not rich enough.

2.3. Obstruction for a Solution with a Discontinuous Static Time-Invariant Controller

In this section we want to prove a necessary condition for having a solution to the uniting problem with a discontinuous static time-invariant controller. The closed-loop system with such a controller has a discontinuous right-hand side. The notion of trajectories for such a differential equation will be the Krasovskii trajectories:

Definition 2.4. Let t_0 be in $\mathbb{R}$ and $T > t_0$. Let Φ be a locally bounded function. We say that X defined on $[t_0, T)$ is a *Krasovskii trajectory* of $\dot{x} = \Phi(x)$ if X is absolutely continuous and, for almost all t in $[t_0, T)$, we have

$$\dot{X}(t) \in K(\Phi)(X(t)),$$

where $K(\Phi)(x) = \bigcap_{\varepsilon > 0} \overline{\text{Conv}}\, \Phi(\{x\} + \varepsilon B)$ (B denotes the unit ball of $\mathbb{R}^n$ and $\overline{\text{Conv}}\, S$ the smaller closed convex set containing S).

Usually this distinction is not made because we have:

Lemma 2.5 [F, p. 50]. *When Φ is continuous each Krasovskii trajectory of $\dot{x} = \Phi(x)$ is a Peano trajectory, i.e. X is C^1 and $\dot{X} = \Phi(X(t))$, for all t.*

It follows that Theorems 2.1 and 2.3 hold for Krasovskii trajectories.

In this section we impose that f is affine in u, i.e.

$$f(x, u) = a(x) + \sum_{i=1}^{m} b_i(x) u_i,$$

where a and b are locally Lipschitz, and we have

Theorem 2.6. *Let $(u_l, u_g, \Omega, \mathscr{D})$ be the data of a uniting problem for f affine in u. If the uniting problem is solvable, in terms of Krasovskii trajectories, with a locally bounded static time-invariant controller, then, for any bounded open connected set $\tilde{\Omega}$, with a neighborhood of the origin such that*

$$\texttt{clos}(\tilde{\Omega}) \subset \Omega,$$

the uniting problem, with data $(u_l, u_g, \tilde{\Omega}, \mathscr{D})$, is also solvable in terms of Krasovskii trajectories by a static time-invariant continuous controller.

This theorem implies that, when f is affine in u, if the uniting problem cannot be solved with a continuous static time-invariant controller, then it cannot be solved with a discontinuous static time-invariant controller either. Note that this is the case for system (8).

To prove Theorem 2.6, we need the following result:

Lemma 2.7 [CLS, Theorem 1.3]. *Let Φ be a locally bounded function. If the origin is globally asymptotically stable for $\dot{x} = \Phi(x)$, in terms of the Krasovskii trajectories, then there exist a proper positive definite C^∞ function V and a positive definite continuous function W such that, for all $x \in \mathbb{R}^n$, we have*

$$\max_{v \in K(\Phi)(x)} \frac{\partial V}{\partial x}(x) v \leq -W(x).$$

Proof of Theorem 2.6. Let A and φ solve the uniting problem with φ a static time-invariant locally bounded controller. Let $\tilde{A} \subset \mathbb{R}^n$ be a compact set separating $\mathbb{R}^n$ in two connected open sets $\tilde{C}_l$ and $\tilde{C}_g$ such that we have

$$\{0\} \in \tilde{\Omega} \subset \tilde{C}_l \subset \texttt{clos}(\tilde{C}_l) \subset C_l, \qquad \texttt{clos}(C_g) \subset \tilde{C}_g. \tag{9}$$

Let V and W be given by Lemma 2.7. We have, for all x in $\mathbb{R}^n \backslash \{0\}$,

$$L_a V(x) + \varphi(x) \cdot L_b V(x) \leq -W(x),$$

where we note $\varphi \cdot L_b V(x)$ instead of $\sum_i L_{\varphi_i b_i} V$ with L_b the Lie derivative along b. Since $\tilde{A}$ is compact and does not contain the origin and W is positive definite and continuous, for all x in $\texttt{int}(\tilde{A})$, there exists $\rho_x > 0$, such that, for all y in $B(x, \rho_x) \subset \texttt{int}(\tilde{A})$, we have

$$L_a V(y) + \varphi(x) \cdot L_b V(y) \leq -\frac{W(y)}{2}, \qquad \rho_x \leq \tfrac{1}{2} d(x, \partial\tilde{A}).$$

From Theorem V.4.4 of [B], there exists a $\mathscr{C}^\infty$ partition of unity $(\Psi_i)_{i \in \mathbb{N}}$ subordinate to the open covering $\{B(x, \rho_x)\}_{x \in \texttt{int}(\tilde{A})}$ of $\texttt{int}(\tilde{A})$. For each i, let x_i be one arbitrary of the points such that

$$\texttt{supp}(\Psi_i) \subset B(x_i, \rho_{x_i}).$$

We let $\varphi_i = \varphi(x_i)$. By construction, we have, for all x in $\texttt{supp}(\Psi_i)$,

$$L_a V(x) = \varphi_i \cdot L_b V(x) \leq -\frac{W(x)}{2}. \tag{10}$$

Now we define the function $\tilde{\varphi}$ as follows:

1. For all x in $\texttt{int}(\tilde{A})$, we let $\tilde{\varphi}(x) = \sum_i \Psi_i(x)\varphi_i$.
2. For all x in $\texttt{clos}(\tilde{C}_j)$, for all j in $\{l, g\}$, we let $\tilde{\varphi}(x) = \varphi_j(x)$.

Assuming, for the time being, that $\tilde{\varphi}$ is continuous, (10) implies that the origin is a globally asymptotically stable equilibrium of $\dot{x} = a(x) + b(x)\tilde{\varphi}(x)$. So $(\tilde{A}, \tilde{\varphi})$ solves the uniting problem in terms of Peano trajectories.

We check that $\tilde{\varphi}$ is continuous on $\mathbb{R}^n$. By construction, $\tilde{\varphi}$ is continuous on $\texttt{int}(\tilde{A}) \cup \tilde{C}_l \cup \tilde{C}_g$. To prove that $\tilde{\varphi}$ is continuous on $\partial\tilde{C}_l$ (resp. $\partial\tilde{C}_g$), we fix x_0 in $\partial\tilde{C}_l$. Now let $\varepsilon > 0$ be arbitrary. From (9) and the continuity of u_l, there exists $\eta_1 > 0$ such that, for each $\eta \in (0, \eta_1]$,

$$B(x_0, \eta) \subset C_l \quad \text{and} \quad |u_l(x_0) - u_l(x)| < \varepsilon, \qquad \forall x \in B(x_0, \eta). \tag{11}$$

Now, for each x in $B(0, \frac{1}{2}\eta)$

- either x is in $\texttt{clos}(\tilde{C}_l)$ and then $|\tilde{\varphi}(x_0) - \tilde{\varphi}(x)| = |u_l(x_0) - u_l(x)| \le \varepsilon$
- or x is in $\texttt{int}(\tilde{A})$ and then

$$|\tilde{\varphi}(x) - \tilde{\varphi}(x_0)| = \left| \sum_{i \in I(x)} \Psi_i(x)[u_l(x_i) - u_l(x_0)] \right|, \tag{12}$$

  where $I(x)$ is the finite set such that, for all i in $I(x)$, $\Psi_i(x) \neq 0$. Now, for each $\eta \in (0, \eta_1]$ and each i in $I(x)$, we have

$$\begin{aligned} d(x_i, \partial\tilde{C}_l) &< |x_i - x_0| < \rho_{x_i} + \tfrac{1}{2}\eta < \tfrac{1}{2}\, d(x_i, \partial\tilde{A}) + \tfrac{1}{2}\eta \\ &< \tfrac{1}{2} \min\{d(x_i, \partial\tilde{C}_l), d(x_i, \partial\tilde{C}_g)\} + \tfrac{1}{2}\eta. \end{aligned}$$

  This implies successively

$$d(x_i, \partial\tilde{C}_l) < \eta, \qquad \rho_{x_i} < \tfrac{1}{2}\eta, \qquad |x_i - x_0| < \eta.$$

  So with (11) and (12), we get $|\tilde{\varphi}(x) - \tilde{\varphi}(x_0)| \le \varepsilon$. ■

3. A Larger Class of Controllers and Notions of Trajectories

Theorems 2.3 and 2.6 prove that we cannot restrict ourselves to static time-invariant discontinuous controller to solve the uniting problem. Therefore we now consider controllers admitting the following description (see [T]):

$$u = \varphi(x, s_d, t), \qquad s_d = k_d(x, s_d^-, t),$$

where s_d evolves in some finite set $\mathscr{F}$, the functions φ and k_d are locally bounded, and s_d^- is defined as

$$s_d^-(t) = \lim_{s \nearrow t} s_d(s).$$

For this to make sense we equip $\mathscr{F}$ with discrete topology (i.e. every set is an open set). The above controller is

- dynamic with the presence of s_d,
- time varying due to the presence of t,
- hybrid due to the presence of the discrete dynamics of s_d.

This class of hybrid controllers has received much attention in many different contexts. However, we have not seen it considered for the uniting problem. Nevertheless, it has been studied in the closely related case of achieving high performance in the presence of input saturation (see [DG] for instance).

This controller with a switching strategy gives rise to a nonclassical ordinary differential equation describing the dynamics of the closed-loop system. In particular, this system is infinite-dimensional since to evaluate $s_d^-(t)$ at time t, we need to know s_d at time instants t_n taken in (at least) one infinite sequence converging from the left to t. As a consequence we have to make precise what we mean by trajectory. The most natural definition of trajectory is

Definition 3.1. Given (x, s_d, t_0) in $\mathbb{R}^n \times \mathscr{F} \times \mathbb{R}$ and $T > t_0$. A function (X, S_d) defined on $[t_0, T)$ is said to be a *classical trajectory* of

$$\dot{x} = f(x, \varphi(x, s_d, t)), \qquad s_d = k_d(x, s_d^-, t) \tag{13}$$

with initial condition (x, s_d) at time t_0 if:

1. X is absolutely continuous on $[t_0, T)$ and, for each t in $[t_0, T)$, there exists $\varepsilon > 0$ such that S_d is constant on $[t, t+\varepsilon)$.
2. For almost all t in $[t_0, T)$, we have
$$\dot{X}(t) = f(X(t), \varphi(X(t), S_d(t), t)),$$
and, for all t in (t_0, T), where S_d has a limit as s tends to t, $s < t$, we have
$$S_d(t) = k_d(X(t), S_d^-(t), t). \tag{14}$$
3. We have
$$(X(t_0), S_d(t_0)) = (x, s_d). \tag{15}$$

Note that by constant for S_d on $[t, t+\varepsilon)$, we mean S_d right-continuous at t because $\mathscr{F}$ is finite. In general, given (x, s_d), there may be several solutions admitting this point as the initial condition. However, there may be also none. Note that we do not ask for (14) to hold at $t = t_0$ and at all t such that $S_d^-(t)$ does not exist. In the following we denote by $\mathtt{sup}_J$ the bound of the function on J and by $\mathtt{esssup}_J$ the essential bound. Actually, we are interested in a notion of trajectories which is robust with respect to disturbances. For this reason, we introduce the notion of generalized trajectory (see also [H2], [H1], and [CR]).

Definition 3.2. Given (x, s_d, t_0) in $\mathbb{R}^n \times \mathscr{F} \times \mathbb{R}$ and $T > t_0$. A function (X, S_d) defined on $[t_0, T)$ is said to be a *weak generalized trajectory* (resp. a *strong generalized trajectory*) of (13) with initial condition (x, s_d) at time t_0, if X: $[t_0, T) \to \mathbb{R}^n$ is continuous, and with S_d: $[t_0, T) \to \mathscr{F}$, we have (15) and, for each $J = [\tau_0, \tau_1]$, a compact subinterval of $[t_0, T)$, and, for each n in $\mathbb{N}$, we can find three functions

e_n, a_n and d_n in $L^\infty_{\mathrm{loc}}([t_0, T))$, a point (x_n, s_{dn}) in $\mathbb{R}^n \times \mathscr{F}$, and a classical trajectory (X_n, S_{dn}) starting from (x_n, s_{dn}) at time τ_0 of

$$\begin{aligned} \dot{x} &= f(x, \varphi(x + e_n(t), s_d, t) + a_n) + d_n(t), \\ s_d &= k_d(x + e_n(t), s_d^-, t) \end{aligned} \tag{16}$$

defined on a right open interval containing J and satisfying

$$\sup_J (X - X_n) + \sup_J (e_n) + \operatorname*{esssup}_J (a_n) + \operatorname*{esssup}_J (d_n) \underset{n \to \infty}{\longrightarrow} 0 \tag{17}$$

$$\left(\text{resp. } \sup_J (X - X_n) + \operatorname*{esssup}_J (e_n) + \operatorname*{esssup}_J (a_n) + \operatorname*{esssup}_J (d_n) \underset{n \to \infty}{\longrightarrow} 0\right) \tag{18}$$

and such that, for all t in J, there exists N satisfying

$$S_{dn}(t) = S_d(t), \qquad \forall n \geq N. \tag{19}$$

In the above definition e_n plays the role of a measurement noise on x which disturbs the control computation, a_n is an actuator error and d_n is an external disturbance of the dynamics. A generalized trajectory is a limit, when the noise vanishes, of disturbed classical trajectories. (For other motivations for considering generalized trajectories, see pp. 164–165 of [H2].) As explained in Remark 1.4 of [LS1], with the presence of d_n we can omit any explicit reference on actuator errors because f is supposed to be locally Lipschitz. So in the following we suppose that in Definition 3.2, for all n in $\mathbb{N}$, $a_n \equiv 0$.

Of course a classical trajectory is a weak generalized trajectory and a weak generalized trajectory is a strong generalized trajectory. Moreover, the weak generalized trajectories are a generalization of Krasovskii trajectories introduced in Definition 2.4. Indeed, since f is assumed to be locally Lipschitz, by combining the technique of the proof of Proposition 1.4 of [CR] with Theorem 5.5 of [H1], we get

Lemma 3.3. *In the case without discrete dynamics (i.e. without s_d) and when φ is locally bounded, any Krasovskii trajectory of $\dot{x} = f(x, \varphi(x))$ is a weak generalized trajectory.*

Remark 3.4. By invoking Zorn's lemma exactly as in the proof of Proposition 1 of [R], we can prove that every generalized trajectory can be extended into a maximal generalized trajectory (X, S_d) defined on an interval $[t_0, T)$ with $T \leq +\infty$ (i.e. for which there exists no generalized trajectory on a $[t_0, T')$ with $T' > T$ and whose restriction is (X, S_d) on $[t_0, T)$).

In this context, if we denote a norm by $|\cdot|$, our definition of global asymptotic stability is

Definition 3.5. The origin is said to be a weakly (resp. strongly) globally asymptotically stable equilibrium of the system (13) if there exists a weak (resp. strong) generalized trajectory and the following properties hold for all t_0:

1. All the weak (resp. strong) maximal generalized trajectories are defined on $[t_0, +\infty)$.
2. There exists a function β of class $\mathcal{KL}$ such that each weak (resp. strong) generalized trajectory (X, S_d) satisfies, for all $t \geq t_0$,

$$|X(t)| \leq \beta(|X(t_0)|, t - t_0). \tag{20}$$

Remark 3.6. We observe, with Remark 2.4 and Proposition 2.5 of [LSW], that (20) is equivalent to the set of the following two properties:

1. There exists a class-$\mathcal{K}_\infty$ function α such that we have

$$|X(t)| \leq \alpha(|X(t_0)|), \qquad \forall t \geq t_0. \tag{21}$$

2. For any $r > 0$ and $\varepsilon > 0$, there exists $T > t_0$ such that

$$|X(t_0)| \leq r \quad \Rightarrow \quad |X(t)| \leq \varepsilon, \qquad \forall t \geq T. \tag{22}$$

In this paper we make the distinction of solving the uniting problem considering only the classical trajectories or by also taking into account the strong or weak generalized trajectories. Usually this distinction is not made since we have the following result proved in the Appendix.

Theorem 3.7. *In the case without discrete dynamics (i.e. without s_d) and when φ is continuous, each strong generalized trajectory is a classical trajectory.*

It follows that Theorems 2.1, 2.3 and 2.6 also give a result in terms of strong generalized trajectories.

The problem of global asymptotic stabilization for weak generalized trajectories has been considered per se in [LS1] and [LS2]. There are strong connections with the problem of uniting local and global controllers that we are considering here, both in the technicalities and the results. In particular, as we shall see, we also have in our context:

- the necessity of having a smooth Lyapunov function,
- the need for using an hybrid controller.

4. Dynamic Time-Invariant Controller with Hysteresis

4.1. A Solution to the Uniting Problem

A very natural way to overcome the difficulties encountered with static time-invariant continuous or discontinuous controllers is to introduce hysteresis, taking advantage of the existence of a region where both controllers u_l and u_g are appropriate.

Theorem 4.1. *Let $(u_l, u_g, \Omega, \mathcal{D})$ be the data of a uniting problem. There exists an appropriate bounded closed set A such that the controller below solves the uniting* *problem in terms of weak generalized trajectories*

$$u = \varphi(x, s_d), \qquad s_d = k_d(x, s_d^-), \tag{23}$$

where s_d *is in* $\{0, 1\}$ *and the functions* φ *and* k_d *satisfy*

$$\begin{aligned} \varphi(x, 0) &= u_l(x) \qquad \textit{if} \quad x \in \mathtt{clos}(\mathbb{R}^n \backslash C_g), \\ \varphi(x, 1) &= u_g(x) \qquad \textit{if} \quad x \in \mathbb{R}^n, \end{aligned} \tag{24}$$

and

$$k_d(x, s_d) = \begin{cases} 0 & \textit{if} \quad x \in \mathtt{clos}(C_l), \\ s_d & \textit{if} \quad x \in \mathtt{int}(A), \\ 1 & \textit{if} \quad x \in \mathtt{clos}(C_g). \end{cases} \tag{25}$$

The proof of Theorem 4.1 is technical and requires us to set down some machinery to handle generalized trajectories. We postpone it until Section 6.

4.2. A Problem with Strong Generalized Trajectories

By the fact that, with strong generalized trajectories, noise with very large amplitude is allowed, Theorem 4.1 is not true. Precisely, for j in $\{l, g\}$ and t_0 in $\mathbb{R}$, let X_j be the solution of

$$\forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}_{\geq t_0}, \qquad \frac{\partial X_j(x, t)}{\partial t} = f(X_j(x, t), u_j(X_j(x, t))), \tag{26}$$

$$\forall x \in \mathbb{R}^n, \qquad X_j(x, t_0) = x. \tag{27}$$

We have

Theorem 4.2. *Let* A *be the compact set and let* (φ, k_d) *be the controller given by Theorem* 4.1 *as a solution to the uniting problem in terms on weak generalized trajectories. If there exist a strictly positive real number* ε *and two compact sets* E_l *and* E_g, *subsets of* A, *such that, for all* x *in* E_l (*resp.* E_g), *there exists* $\tau_{xl} \geq \varepsilon$ (*resp.* $\tau_{xg} \geq \varepsilon$) *satisfying*

$$X_l(x, \tau_{xl}) \in E_g \qquad (\textit{resp.}\ X_g(x, \tau_{xg}) \in E_l), \tag{28}$$

$$X_l(x, t)\ (\textit{resp.}\ X_g(x, t)) \in \mathtt{int}(A), \qquad \forall t \in [0, \tau_{xl}] (\textit{resp.}\ \forall t \in [0, \tau_{xg}]), \tag{29}$$

then $(A, (\varphi, k_d))$ *does not solve the uniting problem in terms of strong generalized trajectories.*

Proof. For k in $\mathbb{N}$ we denote $i_k = l$ if k is odd and $i_k = g$ if k is even.

Now to prove Theorem 4.2 we exhibit recursively a cyclic strong generalized trajectory (X, S_d) and noise e, d. Let $d \equiv 0$ on $\mathbb{R}_{\geq 0}$.

- Let $(x, 1)$ be an initial condition in $E_g \times \{0, 1\}$. From (28), there exists a time $\tau_0 \geq \varepsilon$ such that $X_g(x, \tau_0)$ is in E_l. Let $t_1 = \tau_0$ and, for $t \in [0, t_1)$, we define

$$X(t) = X_g(x, t), \qquad S_d(t) = 1, \qquad e(t) = 0,$$

and, for $t = t_1$, we let

$$X(t_1) = X_g(x, t_1), \qquad S_d(t_1) = 0,$$

and we choose $e(t_1)$ such that $X(t_1) + e(t_1)$ is in C_l.

- Now assume that (X, S_d, e) has been defined on $[0, t_k]$ with $X(t_k) \in E_{i_k}$. We extend the definition of (X, S_d, e) to $[0, t_{k+1}]$ with $t_{k+1} > t_k$ as follows:

  There exists a time $\tau_k \geq \varepsilon$ such that $X_{i_k}(X(t_k), \tau_k)$ is in $E_{i_{k+1}}$. So, we let $t_{k+1} = t_k + \tau_k$ and, for $t \in [t_k, t_{k+1})$, we define

$$X(t) = X_{i_k}(X(t_k), (t - t_k)), \qquad S_d(t) = \begin{cases} 0 & \text{if} \quad i_k = l, \\ 1 & \text{if} \quad i_k = g, \end{cases} \qquad e(t) = 0,$$

  and, for $t = t_{k+1}$, we let

$$X(t_{k+1}) = X_{i_k}(X(t_k), \tau_k), \qquad S_d(t_{k+1}) = \begin{cases} 0 & \text{if} \quad i_k = g, \\ 1 & \text{if} \quad i_k = 1, \end{cases}$$

  and we choose $e(t_{k+1})$ such that $X(t_{k+1}) + e(t_{k+1})$ is in $C_{i_{k+1}}$.
- We go on this way defining ultimately (X, S_d, e) on $\mathbb{R}_{\geq 0}$.

By construction (X, S_d) is a classical trajectory of the disturbed equation:

$$\dot{x} = f(x, \varphi(x + e(t), s_d)) + d(t), \qquad s_d = k_d(x + e(t), s_d^-). \tag{30}$$

In particular, note that $X(t)$ remains in $\mathtt{int}(A)$ so that S_d can have switches only at times t_k. Also e is zero except at a countable number of times. This implies that e is in $L^\infty(\mathbb{R}_{\geq 0})$, and

$$\underset{\mathbb{R}_{\geq 0}}{\mathtt{esssup}}\,(d) + \underset{\mathbb{R}_{\geq 0}}{\mathtt{esssup}}\,(e) = 0.$$

So (X, S_d) is a strong generalized trajectory of the system (1) with the controller given by Theorem 4.1 which does not tend to the origin. ■

We illustrate Theorem 4.2 by considering the following system in $\mathbb{R}^2$:

$$(\dot{x}, \dot{y}) = u. \tag{31}$$

The feedback $u_l(x, y) = -(x, y)$ makes the closed-loop system globally asymptotically stable. Moreover, the following trajectory defined on $[0, +\infty)$ is a trajectory of the closed-loop system (31) with $u = u_l$:

$$\forall t \geq 0, \qquad (x(t), y(t)) = (2\exp(-t), 0).$$

Let F be the closed set $F = \{(x, 0) \colon x \in [1, 2]\}$. Let s be a $\mathscr{C}^\infty$ function on $\mathbb{R}^2$ such that

$$s(x, y) = \begin{cases} 1 & \text{if} \quad (x, y) \in F, \\ 0 & \text{if} \quad d((x, y), F) \geq \frac{1}{2}. \end{cases}$$

Let θ and u_g be the functions defined, for all $(x, y) \in \mathbb{R}^2$, by

$$\theta(x, y) = s(x, y)^\pi, \qquad u_g(x, y) = \mathscr{R}((x, y), \theta(x, y))u_l(x, y),$$

where, for all (x, y) in $\mathbb{R}^2$ and for all θ in $[0, 2\pi]$, $\mathscr{R}((x, y), \theta)$ denotes the rotation with center (x, y) and angle θ. Namely, $u_g(x, y) = u_l(x, y)$ for (x, y) far enough of the segment F and $u_g(x, y) = -u_l(x, y)$ for (x, y) in the segment F.

Lemma 4.3. *u_g makes the origin of* (31) *globally asymptotically stable.*

Proof. There exist classical trajectories because u_g is continuous. Denoting B the unit ball in $\mathbb{R}^2$, for any radius $R > 3$ the ball RB is forward invariant and attractive by u_g. Moreover, in a neighborhood of the origin, u_g is equal to u_l. So we have Lyapunov stability. We now show the global attractivity of the origin. For any radius $R > 3$ the ball RB is forward invariant, so for any (x, y) in $\mathbb{R}^2$ the trajectory through (x, y) is bounded. In this case, Poincaré–Bendixon's theorem implies that any trajectory must tend either to the origin or to a periodic trajectory which must encircle the origin. However, this periodic trajectory does not exist since the set $\mathbb{R}_{\geq 0} \times \mathbb{R}$ is forward invariant by u_g. So we have attractivity. We conclude with Corollary 1 of [ABB]. ■

We now prove that the hypotheses of Theorem 4.2 are verified by taking $E_l = \{(2, 0)\}$ and $E_g = \{(1, 0)\}$. We note that, with u_g (resp. u_l), the trajectory with initial condition $(1, 0)$ (resp. $(2, 0)$) is

$$\forall t \in [0, \log(2)], \qquad (x(t), y(t)) = (\exp(t), 0) \qquad (\text{resp.} = (2\exp(-t), 0)).$$

So (28) and (29) hold. Hence, the controller given by (23) does not solve the uniting problem in terms of strong generalized trajectories for all closed sets $A \subset \mathbb{R}^2$ which separate $\mathbb{R}^2$ in two connected open sets C_l and C_g and such that we have F included in A.

5. Static Periodic Continuous Controller

Instead of enriching the class of controllers with nonsmooth components, we state here that it is sufficient to introduce time-dependence.

Theorem 5.1. *Let $(u_l, u_g, \Omega, \mathscr{D})$ be the data of the uniting problem. Suppose the existence of four bounded open connected sets Γ_j and Σ_j, for all j in $\{l, g\}$, such that*:

- $\Omega \subset \Gamma_l \subset \mathtt{clos}(\Gamma_l) \subset \Sigma_g \subset \mathtt{clos}(\Sigma_g) \subset \Sigma_l \subset \mathtt{clos}(\Sigma_l) \subset \Gamma_g \subset \mathscr{D}$.
- *For all j in $\{l, g\}$, $\mathtt{clos}(\Gamma_j)$ and $\mathtt{clos}(\Sigma_j)$ are forward invariant for the closed-loop system* (1) *obtained with $u = u_j$.*

Under these conditions, we can find an appropriate bounded closed set A and a continuous time-periodic function φ such that the controller $u = \varphi(x, t)$ solves the uniting problem in terms of strong generalized trajectories.

To prove Theorem 5.1 we use the technique of the "wave" between two controllers. This construction is quite similar to that in [SS], where a time-varying continuous controller for a one-dimensional system is exhibited.

Proof. *The set A*: Let $C_l = \Gamma_l$, $C_g = \mathtt{int}(\mathbb{R}^n \backslash \Gamma_g)$ and let finally $A = \mathbb{R}^n \backslash (C_l \cup C_g)$.

The controller: Let u_-: $\mathbb{R}^n \to \mathbb{R}^n$ be any continuous function such that

$$u_-(x) = \begin{cases} u_g(x) & \text{if} \quad x \in \mathbb{R}^n \backslash \Gamma_g, \\ u_l(x) & \text{if} \quad x \in \Sigma_l. \end{cases} \tag{32}$$

Similarly, let u_+ be any continuous function such that

$$u_+(x) = \begin{cases} u_g(x) & \text{if} \quad x \in \mathbb{R}^n \backslash \Sigma_g, \\ u_l(x) & \text{if} \quad x \in C_l. \end{cases} \tag{33}$$

Let τ_j be the real numbers defined as

$$\tau_g = \max_{\{x \in \Gamma_g \backslash \Sigma_g\}} \inf_{\{s : X_g(x,t) \in \Sigma_g, \forall t > s\}} \{s\}, \qquad \tau_l = \max_{\{x \in \Sigma_l \backslash C_l\}} \inf_{\{s : X_l(x,t) \in C_l, \forall t > s\}} \{s\},$$

where for j in $\{l, g\}$, X_j is defined by (26)–(27). These τ_j's are finite because u_l (resp. u_g) is asymptotically stabilizing the origin with the basin of attraction containing $\mathtt{clos}(\Sigma_l)$ (resp. $\mathtt{clos}(\Gamma_g)$) which is a compact set. Let $\mathscr{E}$ be the following compact set:

$$\mathscr{E} = \{(x, u) : x \in \mathtt{clos}(\Sigma_l \backslash \Sigma_g), u \in \overline{\mathrm{Conv}}(\{u_l(x), u_g(x)\})\}.$$

We define M and τ_c as follows:

$$M = \sup_{(x,u) \in \mathscr{E}} |f(x, u)|, \qquad \tau_c = \frac{\mathtt{dist}(\Sigma_g, \mathbb{R}^n \backslash \Sigma_l)}{2M}.$$

With these notations, we choose τ as any real number satisfying $\tau > (\tau_g + 2\tau_c + \tau_l)$. Let γ be a τ-periodic $\mathscr{C}^\infty$ function with value 1 on $[0, \tau_g]$ and on $[\tau_g + 2\tau_c + \tau_l, \tau]$, and 0 on $[\tau_g + \tau_c, \tau_g + \tau_c + \tau_l]$. We define the controller as

$$u = \varphi(x, t) = \gamma(t) u_+(x) + (1 - \gamma(t)) u_-(x).$$

It is continuous, τ-periodic and satisfies, for all t and for all j in $\{l, g\}$,

$$\varphi(x, t) = u_j(x), \qquad \forall x \in \mathtt{clos}(C_j). \tag{34}$$

Let $x \in \mathbb{R}^n$ and let X be a strong generalized trajectory starting from x at t_0 of

$$\dot{x} = f(x, \varphi(x, t)) \tag{35}$$

right maximally defined on $[t_0, T)$. This is a classical trajectory because of Theorem 3.7. We show that $T = +\infty$. Suppose not. Then $|X(x, t)|$ tends to $+\infty$ for t going to T. So there exists t_1 such that, for all $t \geq t_1$, $X(t) \in \mathbb{R}^n \backslash \Gamma_g$ and therefore $\varphi(X(t), t) = u_g(X(t))$. This shows that, for $t \geq t_1$, $X(t)$ is a trajectory of $\dot{x} = f(x, u_g(x))$ which tends to $+\infty$. This contradicts the hypothesis that u_g is globally stabilizing. So we must have $T = +\infty$.

Now, the fact that we have global asymptotic stability of the origin follows from Remark 3.6 and the following four points:

1. Let x be in $\mathtt{int}(C_g)$ and let X be a strong generalized trajectory of (35) starting from x. There exists t_1 in $(t_0, +\infty]$ such that X with values in $\mathtt{int}(C_g)$ is maximally defined on $[t_0, t_1)$. It follows that the restriction of X

on $[t_0, t_1]$ is a trajectory of

$$\dot{x} = f(x, u_g(x)). \tag{36}$$

However, u_g being globally asymptotically stabilizing, and $\texttt{clos}(\Gamma_g) = \mathbb{R}^n \backslash C_g$ being forward invariant and containing the origin, this set is stable and attractive. So in particular t_1 is finite and $X(t_1)$ is in $\texttt{clos}(\Gamma_g)$.

2. Let x be in $\texttt{clos}(\Gamma_g)$ and let X be a strong generalized trajectory of (35) starting from x. Then X cannot leave $\texttt{clos}(\Gamma_g)$ since this set is supposed to be forward invariant for $u = u_g$ and we have (34).
3. Let k be in $\mathbb{Z}$, let t_1 be in $[k\tau, (k+1)\tau)$ and let X be a strong generalized trajectory of (35) such that $x_1 = X(t_1)$ is in $\texttt{clos}(\Gamma_g)$. Let $t_2 = (k+1)\tau$. We have $X(t_2)$ in $\texttt{clos}(\Gamma_g)$, and:
   (a) For t in $[t_2, t_2 + \tau_g]$, γ is 1 so $\varphi(x,t) = u_+(x)$. It follows that there exists t_3 in $[t_2, t_2 + \tau_g]$ such that $X(t_3)$ is in $\texttt{clos}(\Sigma_g)$. Indeed suppose not. Then for t in $[t_2, t_2 + \tau_g]$, $\varphi(x,t) = u_g(x)$ and the restriction of X on $[t_2, t_2 + \tau_g]$ is a trajectory of (36). The definition of τ_g implies that $X(t_2 + \tau_g)$ is in $\texttt{clos}(\Sigma_g)$ which is a contradiction. So there exists t_3 in $[t_2, t_2 + \tau_g]$ such that $X(t_3)$ is in $\texttt{clos}(\Sigma_g)$. Then X cannot leave $\texttt{clos}(\Sigma_g)$ since this set is supposed to be forward invariant for $u = u_g$ and we have (33). So $X(t_2 + \tau_g)$ is in $\texttt{clos}(\Sigma_g)$.
   (b) For t in $[t_2 + \tau_g, t_2 + \tau_g + \tau_c]$, we have no specific properties given by the controller except that it is in $\overline{\text{Conv}}(\{u_l, u_g\})$. The definition of τ_c implies that $X(t_2 + \tau_g + \tau_c)$ is in Σ_l.
   (c) For t in $[t_2 + \tau_g + \tau_c, t_2 + \tau_g + \tau_c + \tau_l]$, γ is 0 so $\varphi(x,t) = u_-(x)$. Then X cannot leave $\texttt{clos}(\Sigma_l)$ between $t_2 + \tau_g + \tau_c$ and $t_2 + \tau_g + \tau_c + \tau_l$ because this set is supposed to be forward invariant for $u = u_l$ and we have (32). So for t in $[t_2 + \tau_g + \tau_c, t_2 + \tau_g + \tau_c + \tau_l]$, we have $\varphi(x,t) = u_l(x)$ and X is a trajectory of

$$\dot{x} = f(x, u_l(x)). \tag{37}$$

   The definition of τ_l implies that $X(t_2 + \tau_g + \tau_c + \tau_l)$ is in $\texttt{clos}(\Sigma_l)$.
4. Let x be in $\texttt{clos}(C_l)$ and let X be a generalized trajectory starting from x. Then X cannot leave $\texttt{clos}(C_l)$ since this set is supposed to be forward invariant for $u = u_l$ and we have (34). So X is a trajectory of (37). ■

6. Basic Properties of Weak Generalized Trajectories and Proof of Theorem 4.1

6.1. *Basic Properties of Weak Generalized Trajectories Given by a Controller with Hysteresis*

Understanding the behavior of weak generalized trajectories is of interest on its own. In this section we study some of the properties of such trajectories for the system

$$\dot{x} = f(x, \varphi(x, s_d)), \qquad s_d = k_d(x, s_d^-),$$

with φ and k_d satisfying (24) and (25) but without relying on our specific problem of uniting a local and a global controller. In particular, in this paragraph, we do not suppose that u_l and u_g are asymptotically stabilizing. We impose only that u_l and u_g are two continuous functions $\mathbb{R}^n \to \mathbb{R}^m$.

Definition 6.1. A function (X, S_d) defined on $[t_0, T)$ is said to have a switch at time $t \in [t_0, T)$ if S_d is not continuous at t.

We start by locating the points where a weak generalized trajectory may have a switch:

Theorem 6.2. *Let (X, S_d) be a weak generalized trajectory defined on $[t_0, T)$ with a switch at time $t \in [t_0, T)$. Consider the sets*

$$S_d^p(t) = \{s\colon \exists t_n \in [t, T), t_n \xrightarrow[n\to\infty]{} t, S_d(t_n) \xrightarrow[n\to\infty]{} s\}, \tag{38}$$

$$S_d^m(t) = \{s\colon \exists t_n \in [t_0, t), t_n \xrightarrow[n\to\infty]{} t, S_d(t_n) \xrightarrow[n\to\infty]{} s\}. \tag{39}$$

- *If the switch is such that $1 \in S_d^m(t)$ and $0 \in S_d^p(t)$, then $X(t)$ is in ∂C_l,*
- *or if the switch is such that $0 \in S_d^m(t)$ and $1 \in S_d^p(t)$, then $X(t)$ is in ∂C_g.*

Proof. Suppose that, for $t \in [t_0, T)$, we have $1 \in S_d^m(t)$ and $0 \in S_d^p(t)$, i.e. for each $\varepsilon \in (t_0, T - t)$ there exists t' and $t'' \in [t', t' + \varepsilon]$ such that $S_d(t') = 1$, $S_d(t'') = 0$ and $t \in [t', t' + \varepsilon]$. Let J_ε be the compact interval $[t', t' + \varepsilon]$. There exists a sequence (X_n, S_{dn}) of classical trajectories such that, from (19), for n sufficiently large, we have $S_{dn}(t') = S_d(t')$ and $S_{dn}(t'') = S_d(t'')$. So, for n sufficiently large, each classical trajectory must have a switch at some time $t_n \in [t', t'']$ with $1 \in S_{dn}^m(t_n)$ and $0 \in S_{dn}^p(t_n)$. Due to (39), there exists a sequence (t_n^p), $t_n^p < t_n$, which tends to t_n as p tends to infinity and such that $S_{dn}(t_n^p) = 1$. Therefore, due to (38), there exists a maximal ε_n^p such that, for all s in $[t_n^p, t_n^p + \varepsilon_n^p)$, we have $S_{dn}(s) = 1$ and such that $S_d(\varepsilon_n^p + t_n^p) = 0$. This implies with (14) that

$$0 = S_{dn}(\varepsilon_n^p + t_n^p) = k_d(X_n(t_n^p + \varepsilon_n^p) + e_n(t_n^p + \varepsilon_n^p), 1).$$

So, from the definition of k_d, we have $X_n(t_n^p + \varepsilon_n^p) + e_n(t_n^p + \varepsilon_n^p)$ in $\mathtt{clos}(C_l)$ and $X_n(s) + e_n(s)$ in $\mathtt{int}(A \cup C_g)$, for all s in $[t_n^p, t_n^p + \varepsilon_n^p)$. By taking the limit as p and n tend to the infinity, we get $X(t) \in \partial C_l$.

The other case is established in exactly the same way. ■

The switches cannot happen too often:

Theorem 6.3. *There exists a strictly positive number η such that, for every weak generalized trajectory and for every time t, there exists at most one switch in the interval $(t, t + \eta)$.*

Before proving Theorem 6.3, we establish a similar result for disturbed classical trajectories:

Lemma 6.4. *For any $D > 0$, there exist strictly positive numbers ε and η such that, for any disturbed classical trajectory with $\mathtt{sup}(e) \le \varepsilon$ and $\mathtt{esssup}(d) \le D$, there is at most one switch in $(t, t+\eta)$.*

Proof. Let Σ_l and Σ_g be two open sets such that we have $\partial C_l \subset \Sigma_l$, $\partial C_g \subset \Sigma_g$ and $d(\Sigma_l, \Sigma_g) > 0$. Let

$$M = \sup_{x \in \Sigma_g \cup A \cup \Sigma_l, s \in \{l,g\}, d \in D.\mathtt{clos}(B)} |f(x, u_s(x)) + d| < +\infty. \tag{40}$$

Let $\varepsilon > 0$ be such that, for all j in $\{l, g\}$,

$$\{x \in \partial C_j, |e| \le \varepsilon\} \Rightarrow x + e \in \Sigma_j. \tag{41}$$

Let (X, S_d) be a disturbed classical trajectory, defined on $[t_0, T)$, of

$$\dot{x} = f(x, \varphi(x + e(t), s_d)) + d(t), \qquad s_d = k_d(x + e(t), s_d^-),$$

with $\mathtt{sup}_{[t_0,T)}(e) \le \varepsilon$ and $\mathtt{esssup}_{[t_0,T)}(d) \le D$. Since S_d is by definition constant on $[t, t+\varepsilon)$ for some $\varepsilon > 0$, we have $S_d^p(t) = \{S_d(t)\}$ and, in each compact subinterval of $[t_0, T)$, we can have at most a finite number of switches. So, suppose this trajectory has two consecutive switches, at times $t_1 < t_2$ in $[t_0, T)$. We must have

$$\begin{gathered}
0 \in S_d^m(t_1), \qquad S_d^p(t_1) = \{1\}, \qquad X(t_1) + e(t_1) \in \partial C_g \\
(\text{resp.} \quad 1 \in S_d^m(t_1), \qquad S_d^p(t_1) = \{0\}, \qquad X(t_1) + e(t_1) \in \partial C_l), \\
1 \in S_d^m(t_2), \qquad S_d^p(t_2) = \{0\}, \qquad X(t_2) + e(t_2) \in \partial C_l \\
(\text{resp.} \quad 0 \in S_d(t_2), \qquad S_d^p(t_2) = \{1\}, \qquad X(t_2) + e(t_2) \in \partial C_g).
\end{gathered}$$

Note that, with (41), we also have $X(t_1) \in \Sigma_g$ (resp. $X(t_1) \in \Sigma_l$) and $X(t_2) \in \Sigma_l$ (resp. $X(t_2) \in \Sigma_g$). The continuity of X implies the existence of $\tilde{t}_1$ and $\tilde{t}_2$ such that we have the following three properties (resp. the other cases):

$$t_1 \le \tilde{t}_1 < \tilde{t}_2 \le t_2, \qquad X(\tilde{t}_1) \in \Sigma_g, \qquad X(\tilde{t}_2) \in \Sigma_l, \tag{42}$$

$$\forall t \in [\tilde{t}_1, \tilde{t}_2], \qquad (X(t), X(t) + e(t)) \in (\Sigma_l \cup A \cup \Sigma_g) \times (\Sigma_l \cup A \cup \Sigma_g).$$

Then the definition of M and φ implies

$$d(\Sigma_g, \Sigma_l) \le |X(\tilde{t}_1) - X(\tilde{t}_2)| \le M(\tilde{t}_2 - \tilde{t}_1) \le M(t_2 - t_1).$$

So, by letting $\eta = (d(\Sigma_g, \Sigma_l))/M$, we get that any two consecutive times of switch t_1 and t_2 must satisfy $t_2 - t_1 \ge \eta$. ■

Proof of Theorem 6.3. Let $D = 1$ and let ε and η be given by Lemma 6.4. Let (X, S_d) be a weak generalized trajectory defined on $[t_0, T)$. For the sake of getting a contradiction, suppose it has two switches at times t_1 and $t_2 < t_3$, with $t_3 = \min\{t_1 + \eta/4, T\}$. Let t_4 in $[\max(t_0, t_1 - \eta/4), t_1]$ be such that S_d has two switches on $[t_4, t_3]$. Let $J = [t_4, t_3]$. Let (X_n, S_{dn}) be a sequence of classical disturbed trajectories which tends to (X, S_d) on J, with $\mathtt{sup}_J(e_n) \le \varepsilon$ and with $\mathtt{esssup}_J(d_n) \le D$. So, as in the proof of Theorem 6.2, for n sufficiently large, the classical disturbed trajectories must also have a first switch at some time larger than or equal to t_4 and another one in the neighborhood of t_2. This contradicts Lemma 6.4. ■

Between two consecutive switches, a weak generalized trajectory is a classical trajectory. Precisely, we have the following:

Theorem 6.5. *Let t_0 be in $\mathbb{R}$ and let (X, S_d) be a weak generalized trajectory such that $S_d \equiv 0$ (resp. 1) on $[t_0, T)$ with $T > t_0$. Then X is an undisturbed classical trajectory of $\dot{X} = f(X, u_l(X))$ (resp. $\dot{X} = f(X, u_g(X))$) on $[t_0, T)$.*

Proof. Let $T' \in (t_0, T)$. Let (X_n, S_{dn}) be a sequence of disturbed classical trajectories which tends to (X, S_d) on $[t_0, T']$. Let $t_n \in [t_0, T']$ be defined by

$$t_n = \mathtt{sup}\{t \leq T' \colon S_{dn}(s) \equiv 0, \forall s \in [t_0, t]\}.$$

Since $S_d(t_0) = 0$, (19) implies, for n sufficiently large, $S_{dn}(t_0) = 0$ and therefore that t_n exists. Since S_{dn} is constant on $[t, t + \varepsilon)$, for all t in $[t_0, T)$, we have either $t_n = T'$ or $S_{dn}(t_n) = 1$. We can find a subsequence such that $t_n \to \tilde{t} \in [t_0, T']$. We show that $\tilde{t} = T'$. Suppose the contrary, i.e. $\tilde{t} < T'$. Then for n sufficiently large we have $t_n < T'$, so $S_{dn}(t_n) = 1$. Then, due to Lemma 6.4, $S_{dn} \equiv 1$ on $[t_n, \min(t_n + \frac{1}{2}\eta, T'))$. By taking the limit when n goes to infinity, $S_d \equiv 1$ on $(\tilde{t}, \min(\tilde{t} + \frac{1}{2}\eta, T'))$, which is a nonempty set. This contradicts the assumption of Theorem 6.5. So we have $S_{dn} \equiv 0$ on $[t_0, t_n)$ with $t_n \to T'$.

Let $s < t$ in $[t_0, T')$. With a similar argument of the proof of Theorem 3.7, we prove that

$$X(t) - X(s) = \int_s^t f(X(\tau), u_l(X(\tau)))\, d\tau.$$

This proves that X is a classical trajectory of (37) on $[t_0, T']$, for all $T' \in (t_0, T)$. So X is a classical trajectory of (37) on $[t_0, T)$. ■

With the properties we know now for the weak generalized trajectories, we can write exactly when such trajectories exist.

Theorem 6.6. *Let t_0 be in $\mathbb{R}$. There exists a weak generalized trajectory starting from (x_0, s_0) in $\mathbb{R}^n \times \{0, 1\}$ and defined on $[t_0, T)$ for some $T > t_0$ if and only if we have one of the following configurations*:

- *x_0 in C_l and $s_0 = 0$.*
- *x_0 in A and s_0 in $\{0, 1\}$.*
- *x_0 in C_g and $s_0 = 1$.*

Proof.

- Let x_0 be in C_l and $s_0 = 1$. There is no classical (slightly) disturbed trajectory starting from $(x_0, 1)$. Indeed, for such a trajectory, for all sufficiently small nonnegative times t, we must have S_d constant and X in C_l (neighborhood of x_0). However, for all x in a neighborhood of x_0 and all e sufficiently small,

(25) yields $s_d = k_d(x+e, s_d^-) = 0$. So S_d must be 0. As a consequence, we cannot find $\tau > t_0$ and a sequence of classical disturbed trajectories (X_n, S_{dn}) defined at least on $[t_0, \tau]$ and with noise e_n, d_n such that $\mathtt{sup}_{[t_0,\tau]}(e_n) + \mathtt{esssup}_{[t_0,\tau]}(d_n)$ goes to 0, $X_n(t_0)$ goes to x_0 and $S_{dn}(t_0)$ goes to 1. So there is no weak generalized trajectory starting from $(x_0, 1)$ when x_0 is in C_l.

- Let x_0 be in $\mathtt{clos}(C_l)$ and $s_0 = 0$. Then, due to the continuity of f and $\varphi(\cdot\,, 0)$, there exists a C^1 function Y: $[t_0, T) \to \mathtt{int}(\mathbb{R}^n \backslash C_g)$ with $T > t_0$, satisfying

$$\dot{Y} = f(Y(t), \varphi(Y(t), 0)), \qquad Y(t_0) = x_0.$$

By letting

$$X(t) = Y(t), \qquad S_d(t) = 0,$$

for $t \in [t_0, T)$, we get an (undisturbed) classical trajectory starting from $(x_0, 0)$. Such a trajectory is also a weak generalized trajectory.

- Let x_0 in ∂C_l and $s_0 = 1$. There exists a sequence $\{x_n\}$ of points in $\mathtt{int}(A)$ which tends to x_0 and satisfying

$$d(x_n, \partial C_g) \geq \Delta > 0.$$

With M defined in (40) and η given by Lemma 6.4, let τ be equal to $\min\{\frac{1}{2}\eta, \Delta/2M\}$. Now, for each n, there exists a C^1 function Y_n with values in $\mathtt{int}(A)$, right maximally defined on $[0, T_n)$ and satisfying

$$\dot{Y}_n = f(Y_n(t), \varphi(Y_n(t), 1)), \qquad Y_n(0) = x_n.$$

Then for n sufficiently large, we can build an undisturbed classical trajectory (X_n, S_{dn}) defined on $[t_0, t_0 + \tau)$ with initial condition $(x_n, 1)$. Indeed

  - either $T_n > \tau$, then by letting

$$X_n(t) = Y(t - t_0), \qquad S_{dn}(t) = 1,$$

  for $t \in [t_0, T_n + t_0)$, we get an undisturbed classical trajectory, defined on $[t_0, t_0 + \tau)$ starting from $(x_n, 1)$,

  - or $T_n \leq \tau$, then, when t tends to T_n, $Y_n(t)$ tends to z_n in ∂A and, more precisely in ∂C_l, since $T_n < \Delta/M$. In this case, there exists a C^1 function Z_n with values in $\mathtt{int}(\mathbb{R}^n \backslash C_g)$, right maximally defined on $[0, T_n')$ and satisfying

$$\dot{Z}_n = f(Z_n(t), \varphi(Z_n(t), 0)), \qquad Z_n(0) = z_n.$$

  Since $\mathbb{R}^n \backslash C_g$ is a compact set and its boundary is ∂C_g, the argument used in the proof of Lemma 6.4 shows that we must have $T_n' \geq \frac{1}{2}\eta \geq \tau$. Then by letting

$$X_n(t) = Y_n(t - t_0), \qquad S_{dn}(t) = 1,$$

  for $t \in [t_0, t_0 + T_n)$, and

$$X_n(t) = Z_n(t - T_n - t_0), \qquad S_{dn}(t) = 0,$$

  for $t \in [t_0 + T_n, t_0 + T_n' + T_n)$, we get an undisturbed classical trajectory, defined on $[t_0, t_0 + \tau)$ and starting from $(x_n, 1)$.

The sequence X_n takes values in the compact set $\mathbb{R}^n \backslash C_g$ and is equicontinuous. So due to Ascoli's theorem, we can extract a subsequence converging as n goes to ∞ to a continuous function Y, uniformly on $[t_0, (t_0 + \tau)/2]$. Then, in the subsequence, either one or the other of the following occurs:

– There is subsubsequence such that $S_{dn}(t) = 1$ for all t in $[t_0, (t_0 + \tau)/2]$. In this case, by letting

$$X(t) = Y(t), \qquad S_d(t) = 1,$$

for $t \in [t_0, (t_0 + \tau)/2)$, we get a weak generalized trajectory starting from $(x_0, 1)$,

– From the construction of S_{dn}, for all n sufficiently large, there exists $\varepsilon_n \in (t_0, (t_0 + \tau)/2]$ such that $S_{dn}(t) = 1$ for t in $[t_0, \varepsilon_n)$ and 0 for t in $[\varepsilon_n, (t_0 + \tau)/2]$. There exists a subsubsequence such that ε_n tends monotonically to $\varepsilon \in [t_0, (t_0 + \tau)/2]$. In this case, by letting

$$X(t) = Y(t), \qquad S_d(t) = \begin{cases} 1 & \text{if} \quad t < \varepsilon, \\ 1 & \text{if} \quad t = \varepsilon \quad \text{and} \quad \varepsilon_n > \varepsilon, \qquad \forall n, \\ 0 & \text{if} \quad t = \varepsilon \quad \text{and} \quad \varepsilon \geq \varepsilon_n, \qquad \forall n, \\ 0 & \text{if} \quad \varepsilon < t, \end{cases}$$

for $t \in [t_0, (t_0 + \tau)/2)$, we get a weak generalized trajectory starting from $(x_0, 1)$. Indeed, in particular, note that we have

$$\varepsilon_n > \varepsilon \quad \Rightarrow \quad S_{dn}(\varepsilon) = 1; \qquad \varepsilon \geq \varepsilon_n \quad \Rightarrow \quad S_{dn}(\varepsilon) = 0.$$

• Let x_0 be in $\mathtt{int}(A)$, and s_0 in $\{0, 1\}$. Then, as in the previous cases, we can construct an undisturbed classical trajectory. So there exists a weak generalized trajectory.

• The other cases:

– x_0 in C_g and $s_0 = 0$ is studied similarly to the case x_0 in C_l and $s_0 = 1$,

– x_0 in $\mathtt{clos}(C_g)$ and $s_0 = 1$ is studied similarly to the case x_0 in $\mathtt{clos}(C_l)$ and $s_0 = 0$,

– x_0 in ∂C_g and $s_0 = 0$ is studied similarly to the case x_0 in ∂C_l and $s_0 = 1$. ■

With this theorem of existence at hand, we remark:

Remark 6.7. If (X, S_d) is a weak generalized trajectory defined on $[t_0, T)$ starting from (x_0, s_0), then, for any s in $[t_0, T)$, $(\tilde{X} = X(\cdot + s), \tilde{S}_d = S_d(\cdot + s))$ is a weak generalized trajectory defined on $[0, T - s)$, starting from $(X(s), S_d(s))$.

We conclude our general study of weak generalized trajectories by noting that, as classical trajectories, weak generalized trajectories can be maximally extended (see Remark 3.4) and such an extension must blow up if its domain of definition is bounded:

Theorem 6.8. *Let (X, S_d) be a weak generalized trajectory defined on $[t_0, T)$ with $T < +\infty$. Then we have $\lim_{t \to T} |X(t)| = +\infty$.*

Proof. We start with a claim:

Claim 6.9. *If, for a weak generalized trajectory* (X, S_d), *we have that* $X(t)$ *is in a given compact set, for all* t *in* $[t_2, t_2]$, *then there exists* ξ *such that, for all* (s, t) *in* $[t_1, t_2]$, *we have*

$$|X(s) - X(t)| \leq \xi|s - t|. \tag{43}$$

Claim 6.9 is a direct consequence of the fact that this holds for the disturbed classical trajectories tending to (X, S_d) on $[t_1, t_2]$, since $f(x, \varphi(x + e, s_d))$ is bounded in this case.

Suppose the conclusion of Theorem 6.8 does not hold, i.e. there exists K a compact set of $\mathbb{R}^n$ and times t_n in $[t_0, T)$ tending monotonically to $T < +\infty$ such that $(X(t_n), S_d(t_n))$ is in $K \times \{0, 1\}$ for all n. We show the following:

Claim 6.10. *If* $X(t_n)$ *is in* K *for all* n, *with* t_n *converging to* T, *then* $(X(t_n), S_d(t_n))$ *has a limit* (x_0, s_0^-) *when* t *tends to* T.

Proof of Claim 6.10. We show first that, for n sufficiently large, $X(t)$ is in the bounded open set $K + B$ for all $t \in [t_n, T)$. Indeed if this is not true, the continuity of X implies the existence of $s_n \in (t_n, T)$ such that

$$|X(t_n) - X(s_n)| = 1 \quad \text{and} \quad |X(t_n) - X(t)| < 1, \qquad \forall t \in [t_n, s_n).$$

It follows that $X(t)$ is in the compact set $K + \mathtt{clos}(B)$, for all t in $[t_n, s_n]$. So, from Claim 6.9, there exists ξ such that

$$1 = |X(t_n) - X(s_n)| \leq \xi|s_n - t_n| \leq \xi|T - t_n|.$$

This cannot hold since t_n converges to t. So, for n sufficiently large, $X(t)$ in $K + B$ for all t in $[t_n, T)$. From Claim 6.9 this implies that there exists ξ such that, for all (s, t) in $[t_n, T)$, we have (43). It follows (by invoking the Cauchy criterion) that $X(t)$ has a limit x_0 when t tends to T. Finally to conclude the proof of Claim 6.10 we invoke Theorem 6.3. It guarantees the existence of $\sigma < \eta$ such that there is no switch in $[T - \sigma, T)$. This implies that S_d is constant on $[T - \sigma, T)$ and so $S_d(t)$ has a limit, denoted s_0^- when t tends to T. ■

From this point we want to show that the weak generalized trajectory can be extended beyond T. We do this by extending the approximating disturbed classical generalized trajectories:

Extension of the Approximating Disturbed Classical Generalized Trajectories to $[t_0, T + \rho/(4M))$. Let $X(T) = x_0$, $K = X([t_0, T])$ and

$$M = \sup_{x \in K + \mathtt{clos}(B), s \in \{l, g\}} |f(x, u_s(x))| < +\infty. \tag{44}$$

Note that we have, for all t in $[t_0, T]$, $|X(t) - x_0| \leq M(T - t)$. Let s_0 be equal to $k_d(x_0, s_0^-)$. Note that, from (25), this definition of s_0 implies the existence of

$\rho \in (0, 1]$ such that we have

$$s_0 = k_d(x, s_0), \qquad \forall x \in \{x_0\} + \rho B.$$

From Theorem 6.3, we know that the weak generalized trajectory has a finite number of switches occurring at times s_i in $[t_0, T - \sigma]$ with σ introduced in the proof of Claim 6.10. Then let

$$\mathcal{T}_n = \{s_i\} \cup \left\{ t_0, \max\left(t_0, \frac{1}{n}\left(T - \frac{1}{n}\right)\right), \max\left(t_0, \frac{2}{n}\left(T - \frac{1}{n}\right)\right), \ldots, \max\left(t_0, \left(T - \frac{1}{n}\right)\right)\right\}. \tag{45}$$

Now from the definition of a weak generalized trajectory, (19), Lemma 6.4 and the convergence of $X(t)$ to x_0, for n sufficiently large, there exists a disturbed classical trajectory (X_n, S_{dn}) defined at least on $[t_0, T - 1/n]$ and such that

$$\left| X_n\left(T - \frac{1}{n}\right) - x_0 \right| \le \frac{\rho}{2}, \tag{46}$$

$$\sup_{[t_0, T-1/n]} (X - X_n) + \sup_{[t_0, T-1/n]} (e_n) + \operatorname{esssup}_{[t_0, T-1/n]}(d_n) \le \frac{1}{n}, \tag{47}$$

$$S_{dn}(\tau) = S_d(\tau), \qquad \forall \tau \in \mathcal{T}_n, \tag{48}$$

$$S_{dn}(t) = s_0^-, \qquad \forall t \in \left[T - \sigma, T - \frac{1}{n}\right]. \tag{49}$$

The latter following from the fact that we cannot have two switches in $[T - \sigma, T - 1/n]$. Then let Y_n with values in $\{x_0\} + \rho B$ be a trajectory of

$$\dot{Y}_n = f(Y_n, \varphi(x_0, s_0)), \qquad Y_n(0) = X_n\left(T - \frac{1}{n}\right),$$

right maximally defined on $[0, T_n)$. So $|Y_n(T_n) - x_0| = \rho$. Using (44), (46) and (47) we have

$$\rho \le |Y_n(T_n) - Y_n(0)| + \left|X_n\left(T - \frac{1}{n}\right) - X\left(T - \frac{1}{n}\right)\right| + \left|X\left(T - \frac{1}{n}\right) - x_0\right)\right|$$
$$\le MT_n + \frac{1}{n} + \frac{\rho}{2}.$$

Therefore, with n large enough, $T_n > \rho/(2M)$. This proves that Y_n is defined on $[0, \rho/(2M)]$. Now, we define a function (X_n', S_{dn}', e_n') on $[t_0, T + \rho/(2M) - 1/n]$ by letting, for t in $[t_0, T - 1/n)$,

$$X_n'(t) = X_n(t), \qquad S_{dn}'(t) = S_{dn}(t), \qquad e_n'(t) = e_n(t), \qquad d_n'(t) = d_n(t),$$

and, for t in $[T - 1/n, T + \rho/(2M) - 1/n]$,

$$X_n'(t) = Y_n\left(t - T + \frac{1}{n}\right), \qquad S_{dn}'(t) = s_0, \qquad e_n'(t) = 0, \qquad d_n'(t) = 0.$$

This function gives rise to a disturbed classical trajectory on $[t_0, T+\rho/(2M)-1/n]$. This is clear for the interval $[t_0, T-1/n)$. For the remainder of the interval, this follows from the equalities:

$$f(X_n'(t), \varphi(X_n'(t)+e_n'(t), S_{dn}'(t))) = f\left(X_n'(t), \varphi\left(Y_n\left(t-T+\frac{1}{n}\right), s_0\right)\right) = f(X_n'(t), s_0),$$

$$S_{dn}'(t) = s_0 = k_d(x_0, s_0) = k_d(X_n'(t) + e_n'(t), S_{dn}'^{-}(t)),$$

for t in $(T-1/n, T+\rho/(2M)-1/n]$ and, with (49),

$$S_{dn}'\left(T-\frac{1}{n}\right) = s_0 = k_d(x_0, s_0^-) = k_d\left(X_n'\left(T-\frac{1}{n}\right) + e_n'\left(T-\frac{1}{n}\right), S_{dn}'^{-}\left(T-\frac{1}{n}\right)\right).$$

Now, on the interval $[T, T+\rho/(4M)]$, the sequence of functions $X_n'(t)$ takes values on $\{x_0\}+\rho B$ so is bounded and equicontinuous. Then there exists a subsequence such that X_n' converges uniformly to a continuous function Y. We define a function (X', S_d') on $[t_0, T+\rho/(4M)]$ by letting, for t in $[t_0, T)$,

$$X'(t) = X(t), \qquad S_d'(t) = S_d(t),$$

and, for t in $[T, T+\rho/(4M)]$,

$$X'(t) = Y(t), \qquad S_d'(t) = s_0.$$

We prove that the above subsequences provide an appropriate approximating sequence of disturbed classical trajectories converging to (X', S_d') in the sense of Definition 3.2:

1. We have that X_n' converge uniformly to X on $[t_0, T+\rho/(4M)]$. Indeed:
   - On $[T, T+\rho/4M]$ this follows from the convergence of X_n' to Y.
   - On $[t_0, T-1/n]$ this follows from the convergence of X_n' to X.
   - Finally, for t in $[T-1/n, T]$, we have

$$X_n'(t) - X'(t) = X_n'\left(T-\frac{1}{n}\right) - X\left(T-\frac{1}{n}\right) + X\left(T-\frac{1}{n}\right) - X(t) + \int_{T-1/n}^{t} f(X_n'(s), \varphi(x_0, s_0)). \tag{50}$$

   This implies, for t in $[T-1/n, T]$,

$$|X_n'(t) - X(t)| \leq \sup_{[t_0, T-1/n]} (X_n' - X) + M\left(t-T+\frac{1}{n}\right) + M\left(t-T+\frac{1}{n}\right).$$

2. On the interval $[0, \rho/(4M)]$, the sequence $S_{dn}'(t+T)$ is constant and equal to s_0 as the function $S_d'(t+T)$. Also pick t in $[t_0, T]$.
   - Either t is not a time s_i of a switch of the weak generalized trajectory, i.e. S_d is continuous at t, and is constant on some neighborhood of t. This implies that for all n sufficiently large, we can find m such that
     (1) t is in the interval $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$,
     (2) S_d is constant on $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$,

(3) using Lemma 6.4, S_{dn} has at most one switch in $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$,

(4) and, using (48),

$$S_{dn}\left(\frac{m}{n}\left(T-\frac{1}{n}\right)\right)=S_d\left(\frac{m}{n}\left(T-\frac{1}{n}\right)\right),$$

$$S_{dn}\left(\frac{m+1}{n}\left(T-\frac{1}{n}\right)\right)=S_d\left(\frac{m+1}{n}\left(T-\frac{1}{n}\right)\right).$$

This implies that S_{dn} is constant on $[m/n(T-1/n), ((m+1)/n)(T-1/n)]$ and therefore $S_{dn}(t)=S_d(t)$.

- Or $t=s_i$, for some i. Then, from (48) and the definition of $\mathcal{T}_n$ in (45), we have, for n sufficiently large, $S_{dn}(t)=S_d(t)$.

3. We have that (47) implies that e'_n and d'_n converge uniformly to zero on the interval $[t_0, T+\rho/(4M)]$.

So (X', S'_d) is a weak generalized trajectory on $[t_0, T+\rho/(4M))$ and its restriction to $[t_0, T)$ is (X, S_d). So this contradicts the fact that (X, S_d) is a maximal weak generalized trajectory. ■

6.2. *Proof of Theorem 4.1*

The Set A. Since the origin is asymptotically stable for (37) with domain of attraction $\mathcal{D}$ and $f(\cdot, u_l(\cdot))$ is continuous, there exists a C^∞ function V which is positive definite and proper on $\mathcal{D}$ such that, for all x in $\mathcal{D}\backslash\{0\}$, we have

$$\frac{\partial V}{\partial x}(x)f(x, u_l(x))<0. \tag{51}$$

Then, since $\mathtt{clos}(\Omega)$ is a subset of $\mathcal{D}$, there exists c_l such that $\Omega \subsetneq C_l := \{x\in\mathcal{D}\colon V(x)<c_l\}$. Let c_g, C_g and A be defined by

$$c_g=2c_l, \qquad C_g=\{x\colon V(x)>c_g \text{ or } x\notin\mathcal{D}\}, \qquad A=\mathbb{R}^n\backslash(C_l\cup C_g). \tag{52}$$

Maximality of Trajectories. Let $(x, s_d)\in\mathbb{R}^n\times\{0,1\}$ be any point so that there exists a weak generalized trajectory, maximally defined on $[t_0, T)$ (see Theorem 6.6). We show that, for each such trajectory, we have $T=+\infty$. Suppose not. Then, with Theorem 6.8, we know that $|X(t)|$ tends to $+\infty$ when t goes to T. So there exists t_1 such that, for all $t\geq t_1$, $X(t)\in C_g$ and in particular $X(t)\notin\partial C_g\cup\partial C_l$. With Theorem 6.2, we know that, in this case, this trajectory cannot have switches in $[t_1, T)$. So S_d is constant on $[t_1, T)$ and, by using an approximating sequence of disturbed classical trajectories, with (25), we can claim that its value is 1. So, with Theorem 6.5, X restricted to $[t_1, T)$ is a classical trajectory of the closed-loop system with the global controller $u=u_g$ which tends to $+\infty$. This is a contradiction of the fact that u_g is a global asymptotic stabilizer. So we must have $T=+\infty$.

Now in order to prove the global asymptotic stability of the origin, we first establish

Claim 6.11. *For any weak generalized trajectory, only two cases are possible*:

1. *There exists no switch and the generalized trajectory is a classical trajectory of* (37) *on* $[t_0, +\infty)$ *and remains in the set* $A \cup C_l$.
2. *There exists at least one switch at a nonnegative time* σ *and the generalized trajectory is such that*:
   - $X(\sigma)$ *is in* ∂C_l.
   - *For all* t *in* (t_0, σ), X *is a classical trajectory of* (36) *and it is not in* C_l.
   - *For all* t *in* $[\sigma, +\infty)$, X *is a classical trajectory of* (37).

Proof of Claim 6.11. We consider five cases depending on the location of the initial condition:

1. Let $x \in \mathtt{int}(A) \cup \mathtt{clos}(C_l)$ and $s_d = 0$. From Theorem 6.6 and the above, there exists (X, S_d), a weak generalized trajectory right maximally defined on $[t_0, +\infty)$ and with initial condition (x, s_d). We show that it has no switch. Indeed, if not, since the possible number of switches is finite on each compact interval of time (see Theorem 6.3), there exists a time τ for a first switch. At this time, S_d must change from 0 to 1. However, since $x_0 \notin \partial C_g$ and $s_d = 0$, Theorem 6.2 implies that τ cannot be 0 and that $X(\tau)$ is in ∂C_g. On the other hand, Theorem 6.5 says that X restricted to $[t_0, \tau)$ is a classical trajectory of (37). Then, from (51), we have, for all t in $[t_0, \tau)$,
$$V(X(t)) \le V(x) < c_g. \tag{53}$$
So, for t going to τ, we get $V(X(\tau)) < c_g$ which contradicts the fact that $X(\tau)$ is in ∂C_g, from the definition of C_g (52). Consequently, there is no switch, $S_d \equiv 0$ on $[t_0, +\infty)$ and X is a classical trajectory of (37) on $[t_0, +\infty)$ and remains in $A \cup C_l$.
2. Let $x \in C_l$ and $s_d = 1$. There is no weak generalized trajectory (see Theorem 6.6).
3. Let $x \in C_g$ and $s_d = 0$. There is no weak generalized trajectory (see Theorem 6.6).
4. Let $x \in \partial C_g$ and $s_d = 0$. From Theorem 6.6 and the above, there exists (X, S_d), a weak generalized trajectory right maximally defined on $[t_0, +\infty)$ and with initial condition (x, s_d).
   (a) If there is no switch at $t = t_0$, there exists $t_1 > t_0$ such that $S_d(t) = 0$ for all t in $[t_0, t_1)$. It follows that the restriction of X to $[t_0, t_1)$ is a classical trajectory of (37) and satisfies, for t in (t_0, t_1), (53). So, by Remark 6.7, the restriction of (X, S_d) to $[(t_0 + t_1)/2, +\infty)$ is one of the weak generalized trajectories studied in Case 1 above.
   (b) If there is a switch at $t = t_0$, then there is another one. Indeed, if not, for all $t_1 > t_0$, the restriction of X to $[t_1, +\infty)$ is a classical trajectory of (36) and the restriction of S_d to $[t_1, +\infty)$ is the constant 1. However, in this case, since u_g is globally asymptotically stabilizing, there exists s such that $X(t)$ is in C_l for all $t \ge s$. So, by Remark 6.7, the restriction of

(X, S_d) to $[s, +\infty)$ is a weak generalized trajectory with initial condition $X(s)$ in C_l and $S_d(s) = 1$, which is impossible in view of Theorem 6.6. So there is another switch. Let σ be the first one. We must have $1 \in S_d^m(\sigma)$ and $0 \in S_d^p(\sigma)$, and, from Theorem 6.2, $X(\sigma)$ is in ∂C_l. Then:

- Since, for each $t_1 \in (t_0, \sigma)$, the restriction of S_d to $[t_1, \sigma)$ is the constant 1, it follows from Case 2 and the continuity of X that, for all t in $[t_1, \sigma)$, X is not in C_l.
- By the continuity of X and the definition of S_d^p, there exists t_2, close to σ, such that $X(t_2)$ is in $\mathtt{int}(A) \cup \mathtt{clos}(C_l)$ and $S_d(t_2) = 0$. So the restriction of (X, S_d) to $[t_2, +\infty)$ is one of the weak generalized trajectories studied in Case 1 above.

5. Let $x \notin C_l$ and $s_d = 1$. As in Case 4(b) above, there is a weak generalized trajectory right maximally defined on $[t_0, +\infty)$ and with initial condition (x, s_d). It has one switch at $\sigma > 0$ and satisfies the property stated in Claim 6.11. ■

With the help of Claim 6.11 and Remark 3.6, we can now prove the global asymptotic stability:

1. We have proved that all the trajectories are defined on $[t_0, +\infty)$.
2. *Establishing* (21): Let α_l be a class-$\mathcal{K}_\infty$ function such that

$$\alpha_l(s) \geq s \quad \text{and} \quad (V(x) \leq c_g \Rightarrow |x| \leq \alpha_l(V(x))).$$

Let also v be the positive definite function defined on $[0, c_g]$ as

$$v(s) = \inf_{\{x\colon V(x) \geq s \text{ or } x \notin \mathcal{D}\}} |x|, \qquad \forall s \in [0, c_g].$$

Finally, because for the system (36) the origin is globally asymptotically stable, there exists a class-$\mathcal{K}_\infty$ function α_g such that, for all x, we have

$$|X_g(x, t)| \leq \alpha_g(|x|), \qquad \forall t \geq t_0,$$

where $X_g(x, t)$ is defined by (26)–(27). Without loss of generality, we can impose

$$\alpha_g(v(s)) \geq s, \qquad \forall s \in [0, c_g].$$

We show that, given (x, s_d) an initial condition, a generalized trajectory (X, S_d) satisfies

$$|X(t)| \leq \alpha_l(\alpha_g(|x|)), \qquad \forall t \geq t_0. \tag{54}$$

Indeed:

- If X has no switch, then X is a classical trajectory of (37) and is in $A \cup C_l$. So, from (51), we have

$$V(X(t)) \leq V(x) \leq c_g, \qquad \forall t \geq 0,$$

and therefore, for all $t \geq 0$,

$$|X(t)| \leq \alpha_l(\alpha_g(v(V(x)))) \leq \alpha_l(\alpha_g(|x|)).$$

- If X has one switch at time $\sigma \geq 0$. Then the restriction of X to $[t_0, \sigma)$ is a classical trajectory of (36) not in C_l. So we have

$$|X(t)| \leq \alpha_g(|x|) \leq \alpha_l(\alpha_g(|x|)) \quad \text{and} \quad \{V(X(t)) \geq c_l \text{ or } X(t) \notin \mathscr{D}\}.$$

The restriction of X to $[\sigma, +\infty)$ is a classical trajectory of (37). So we have

$$V(X(t)) \leq V(X(\sigma)) = c_l \leq \alpha_g(v(c_l)) \leq \alpha_g(|x|),$$

and therefore we have (54).

3. *Establishing* (22): Let $r > 0$ and $\varepsilon > 0$. Let r_l and r_g be defined by

$$r_l = \sup\{|x|, x \in C_l \cup A\} < +\infty, \qquad r_g = \tfrac{1}{2} \inf\{|x|, x \in \partial C_l\} > 0.$$

Due to the global asymptotic stability of the systems (37) and (36), there exists $T_g, T_l < +\infty$ such that

$$|x| \leq r \quad \Rightarrow \quad |X_g(x,t)| \leq r_g, \qquad \forall t \geq T_g, \tag{55}$$

and such that

$$|x| \leq r_l \quad \Rightarrow \quad |X_l(x,t)| \leq \varepsilon, \qquad \forall t \geq T_l. \tag{56}$$

Let $T = T_l + T_g$. We show that

$$|x| \leq r \quad \Rightarrow \quad |X(t)| \leq \varepsilon, \qquad \forall t \geq T, \tag{57}$$

where (X, S_d) is any generalized trajectory starting from (x, s_d). Indeed:

- If X has no switch, then X is a classical trajectory of (37) and is contained in $A \cup C_l$. So (57) is a consequence of the definition of r_l and (56).
- If X has one switch at time σ, then $X(\sigma)$ is in ∂C_l, and X is a classical trajectory of (36) on (t_0, σ) and a classical trajectory of (37) on $[\sigma, +\infty)$. So, for all t in $[T_l + \sigma, +\infty)$, we have

$$|X(t)| \leq \varepsilon. \tag{58}$$

Also we have $\sigma \leq T_g$. Suppose not. Then (55) implies $0 < |X(\sigma)| \leq \frac{1}{2} \inf\{|x|, x \in \partial C_l\}$, which is a contradiction to $X(\sigma)$ in ∂C_l. So we have (58) for all t in $[T, +\infty)$.

This achieves the proof of Theorem 4.1. ■

Acknowlegements. The author is deeply grateful to Laurent Praly and Jean-Michel Coron for many constructive suggestions.

Appendix. Proof of Theorem 3.7

Proof. Let X, defined on $[t_0, T)$, be a strong generalized trajectory of

$$\dot{x} = f(x, \varphi(x,t)), \tag{59}$$

where the functions f and φ are continuous. Let $J = [\tau_0, \tau_1]$ be a compact sub-interval of $[t_0, T)$. Let (X_n, S_n) be disturbed classical trajectories of

$$\dot{x} = f(x, \varphi(x + e_n, t)) + d_n,$$

with

$$\sup_J (X - X_n) + \texttt{esssup}(e_n) + \texttt{esssup}(d_n) \xrightarrow[n\to\infty]{} 0. \tag{60}$$

Let $\sigma < t$ be in J. We have, for all n in $\mathbb{N}$,

$$\begin{aligned} X(t) - X(\sigma) - \int_\sigma^t f(X(\tau), \varphi(X(\tau), \tau))\, d\tau \\ = X(t) - X_n(t) + X_n(t) - X_n(\sigma) + X_n(\sigma) \\ - X(\sigma) - \int_\sigma^t f(X(\tau), \varphi(X(\tau), \tau))\, d\tau. \end{aligned} \tag{61}$$

From (16), for all n in $\mathbb{N}$, we have

$$X_n(t) - X_n(\sigma) = \int_\sigma^t (f(X(\tau), \varphi(X(\tau) + e_n(\tau), \tau)) + d_n(\tau))\, d\tau.$$

With (60) and the continuity of f and φ, we have, for almost all τ in $[\sigma, t]$,

$$\lim_{n\to\infty} f(X_n(\tau), \varphi(X_n(\tau) + e_n(\tau), \tau)) + d_n(\tau) = f(X(\tau), \varphi(X(\tau), \tau)),$$

and the sequence $(f(X_n(\tau), \varphi(X_n(\tau) + e_n(\tau), \tau)))$ is essentially bounded on $[\sigma, t]$. So with the Dominated Convergence Theorem, we obtain with (61)

$$\lim_{n\to\infty} \left(X(t) - X(\sigma) - \int_\sigma^t f(X(\tau), \varphi(X(\tau), \tau))\, d\tau \right) = 0.$$

Since X is independent of n, and since f is continuous, we conclude that X is in $\mathscr{C}^1$ and is a classical trajectory of (59) on all J compact subintervals of $[t_0, T)$. So X is a trajectory of (59) on $[t_0, T)$. ■

References

[ABB] V. Andriano, A. Bacciotti and G. Beccari, Global stability and external stability of dynamical systems, *Nonlinear Anal. TMA*, **28**(7) (1997), 1167–1185.

[B] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, New York, 1975.

[CLS] F. H. Clarke, Yu. S. Ledyaev and R. J. Stern, Asymptotic stability and smooth Lyapunov functions, *J. Differential Equations*, **149** (1998), 69–114.

[C] J.-M. Coron, A necessary condition for feedback stabilization, *Systems Control Lett.*, **14** (1990), 98–105.

[CR] J.-M. Coron and L. Rosier, A relation between continuous time-varying and discontinuous feedback stabilization, *J. Math. Systems Est. Control*, **4** (1994), 67–84.

[DG] J. A. De Doná and G. G. Goodwin, A general scheme that combines piecewise-linear control, low-high-gain control and over-saturation to achieve high performance in the presence of input saturation, *Proc. Amer. Cont. Conf.*, 2000.

[F] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Side*, Kluwer Dordrecht, 1988.

[H1] O. Hájek, Discontinuous differential equations, part I, *J. Differential Equations*, **32** (1979), 149–170.

[H2] H. Hermes, Discontinuous vector fields and feedback control, in *Differential Equations and Dynamic Systems*, J. K. Hale and J. P. La Salle, eds., Academic Press, New York, 1967, pp. 155–165.

[LS1] Y. S. Ledyaev and E. D. Sontag, A remark on robust stabilization of general asymptotically controllable systems, *Proc. Conf. on Information Sciences and Systems* (*CISS* 97), Johns Hopkins University, Baltimore, MD, 1997, pp. 246–251.

[LS2] Y. S. Ledyaev and E. D. Sontag, A Lyapunov characterization of robust stabilization, *Nonlinear Anal.*, **37** (1999), 813–840.

[LSW] Y. Lin, E. Sontag and Y. Wang, A smooth converse Lyapunov theorem for robust stability, *SIAM J. Control Optim.*, **34**(1) (1996), 124–160.

[MMP] P. Morin, R. M. Murray and L. Praly, *Nonlinear Rescaling of Control Laws with Application to Stabilization in the Presence of Magnitude Saturation, Proc. Conf. on NOLCOS '98*, 1998, pp. 690–695.

[R] E. P. Ryan, Discontinuous feedback and universal adaptive stabilization, in *Control of Uncertain Systems*, D. Hinrichsen and B. Mårtensson, eds., Birkhäuser, Boston 1990, pp. 245–258.

[SS] E. D. Sontag and H. J. Sussmann, Remarks on continuous feedback, *Proc. IEEE Conf. on Decision and Control*, Albuquerque, NM, Dec. 1980, pp. 916–921.

[T] L. Tavernini, Differential automata and their discrete simulators, *Nonlinear Anal. TMA*, **11**(6) (1997), 665–683.

[TKM] A. R. Teel, O. L. Kaiser and R. M. Murray, Uniting local and global controllers for the Caltech ducted fan, *Proc. Amer. Cont. Conf.*, 1997.

[TK] A. R. Teel and N. Kapoor, Uniting local and global controllers, *Proc. Euro. Cont. Conf.*, 1997.