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A TRACKING PROBLEM FOR THE STATE OF CHARGE IN A ELECTROCHEMICAL LI-ION BATTERY MODEL

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Abstract. In this paper the Single Particle Model is used to describe the behavior of a Li-ion battery. The main goal is to design a feedback input current in order to regulate the State of Charge (SOC) to a prescribed reference trajectory. In order to do that, we use the boundary ion concentration as output. First, we measure it directly and then we assume the existence of an appropriate estimator, which has been established in the literature using voltage measurements. By applying backstepping and Lyapunov tools, we are able to build observers and to design output feedback controllers giving a positive answer to the SOC tracking problem. We provide convergence proofs and perform some numerical simulations to illustrate our theoretical results.

1. Introduction.

1.1. General goals.

Today batteries are being developed to power a crescent and wide range of applications as laptops, smartphones, watches, electric vehicles, medicals devices and many others. Consequently, batteries are certainly in the middle of the technological development [1]. In this direction, li-ion batteries are gaining more and more attention due to its very good properties, compared to alternative battery technologies. For example, li-ion batteries provide one of the best energy-weight ratios and have a low self discharge when is not in use [4].

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An intelligent battery control system can ensure longevity and performance of a battery, but such a type of improvements relies in an exhaustive understanding of energy storage. Thus, the modeling of li-on batteries has a key role in the design of battery management systems.

The literature on modeling of li-ion batteries is quite extensive. However, we can distinguished two groups of models. The first group is formed by equivalent circuit models (ECMs), which employ circuits elements to imitate the input-output behavior of a battery. The second group is formed by electrochemical models, which take into account electrochemical principles. Although electrochemical modeling approach has proved to have a better prediction capability compared to equivalent circuit models [4], the mathematical structure provides a huge challenge. The electro-chemicals models arise many open questions in control. For instance, in [17] the author provides, in a brief way, a survey about the main challenges in battery management in which electro-chemical models are involved. For example, the problem to estimate the State of Charge, which indicates the stored energy at certain time and its time-evolution is also useful to determinate the health of the battery. Thus, control theory of electro-chemical models needs important efforts from the control community.

This paper aims at contributing in that direction. We are interested in studying the problem of tracking of the State of Charge in a battery modeled by an electrochemical model. In other words, given a reference State of Charge profile, we want to find the appropriate input current in order to get the real State of Charge near to the given reference. The model used along this article is called the Single Particle Model. For more details about its obtention, please see [17, 4, 26].

The design of tracking controls has gained more and more attention. For instance, the growing demands on product quality and production efficiency, which require to turn away from the pure stabilization of an operating point towards tracking task as can be seen in some industrial applications, see for instance [15, 21, 5]. In that line, a potential application of the tracking of the State of Charge might be related with the dynamic pricing. For instance, for electrical vehicles, this means that the charging provider, which can be a distribution system operator or an operator/aggregator of charging stations, dynamically adapts the prices, which have to be payed by the final user for charging their electrical vehicles, see [12]. So the question of how to adapt the usage of the battery in order the operate at minimum cost could be solved via to track an optimal State of Charge profile.

1.2. **Model and properties.** As we mentioned before, in this article we use the Single Particle Model (SPM), which is a reduction of the more general model due to Doyle-Fuller-Newman (DFN). The SPM considers each electrode as a single spherical particle and neglects the electrolyte dynamics, i.e., this model considers that lithium concentration in electrolyte phase remains constant. In the following we describe the SPM.

Let us consider a spherical diffusion equation to describe the lithium concentration behavior $c^j(r,t)$ in solid phase. Thus, we obtain

$$\begin{cases} c^j_t(r,t) = D^j\left[\frac{2}{r}c^j_r(r,t) + c^j_{rr}(r,t)\right], & (r,t)\in Q_j, \\ c^j_r(0,t) = 0, \quad c^j_r(R_j,t) = -j\rho_j I(t), & t\in(0,\infty), \\ c^j(r,0) = c^j_0(r), & r\in(0,R_j), \end{cases} \tag{1}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = \phi_1(T_{amb} - T(t)) + \phi_2 I(t)V(t), \tag{2}$$

$$V(t) = \frac{RT(t)}{\alpha F} \sum_{j \in \{+,-\}} \sinh^{-1}\left(-j\omega_j(t)\right) + jU^j\left(c_s^j(t)\right) - R_f I(t), \tag{3}$$

where $j \in \{+,-\}$ indicates positive or negative electrode, D^j is the diffusivity, $Q^j = (0, R_j) \times (0, \infty)$ is the domain and R_j is the particle radius, $\rho_j = \frac{1}{D_j F a_j A L_j}$, ϕ_1, ϕ_2 are known parameters, $T(t)$ is temperature on the battery, $I(t)$ is the input current and $V(t)$ is the output voltage. The function $\omega_j(t)$ is given by

$$\omega_j(t) = \frac{I(t)}{\mu_j \sqrt{c_s^j(t)\left(c_{\max}^j - c_s^j(t)\right)}} \tag{4}$$

where $c_s^j(t) = c^j(R_j, t)$ is the surface concentration (or boundary concentration), $c_{\max}^j$ is the maximum ion concentration in the electrode j, $\mu_j = 2a^j AL^j \sqrt{c_e}$ are known parameters and U^j are equilibrium potentials of each electrode material. We detail all variables and parameters in Table 1.

TABLE 1. Model variables and electrochemical parameters

Model states, inputs and outputs	
$c^{\pm}$	Lithium concentration in solid phase $[mol/m^3]$
$c_s(t)$	Lithium concentration at solid particle surface$[mol/m^2]$
c_e	Lithium concentration in electrolyte phase $[mol/m^3]$
T	Temperature $[K]$
I	Applied current, $[A/m^2]$
V	Output Voltage $[V]$
Electrochemical model parameters	
$D^{\pm}$	Diffusivity $[m^2/s]$
$R_{\pm}$	Particle radius in solid phase $[m]$
F	Faraday Constant $[C/mol]$
R	Universal gas constant $[J/mol \cdot K]$
α	Charge transfer coefficient $[-]$
$c_{\max}^{\pm}$	Maximum concentration of solid material $[mol/m^3]$
$U^{\pm}$	Open circuit potential of solid material $[V]$
R_f	Solid interphase films resistance $[\Omega \cdot m^2]$
$L^{\pm}$	Length of region $[m]$
A	Area $[m^2]$
ϕ_1	Heat transfer coefficient $[1/s]$
ϕ_2	Inverse of heat capacity $[J/K]^{-1}$
$\varepsilon^{\pm}$	Volume fraction of solid phase $[-]$

The following definition establishes what we understand by State of Charge in a battery modeled by (1)-(3).

Definition 1.1. Let $c(r,t)$ be the Li-ion concentration in the negative electrode, then the battery State of Charge (SOC) is given by

$$SOC(t) = \frac{3}{R_-^3 c_{\max}^-} \int_0^{R_-} c(r,t) r^2 \, \mathrm{d}r. \tag{5}$$

For the sake of simplicity, we define the following adimensional variables,

$$\bar{r}_j = \frac{r}{R_j}, \quad \bar{t}_j = \frac{D_j}{R_j^2} t, \quad j \in \{+,-\}. \tag{6}$$

We take out the bars and j index on the space and time coordinates to keep the notation simple. This normalization produces the following partial differential equation (PDE)

$$\begin{cases} c_t^j = \frac{2}{r} c_r^j(r,t) + c_{rr}^j(r,t), & (r,t) \in \bar{Q}, \\ c_r^j(0,t) = 0, \quad c_r^j(1,t) = -j\tilde{\rho}_j I(t), & t \in (0,\infty), \\ c^j(r,0) = c_0^j(r), & r \in (0,1), \end{cases} \tag{7}$$

where $\bar{Q} = (0,1) \times (0,\infty)$, $j \in \{+,-\}$ and $\tilde{\rho}_j = R_j \rho_j$. In order to precise the notation used, the initial condition $c_0^j(r)$ of the system (7) is the initial condition of the system (1) scaled to the domain $r \in (0,1)$.

After this normalization the State of Charge becomes

$$SOC(t) = \frac{3}{c_{\max}^-} \int_0^1 c^-(r,t) r^2 \, \mathrm{d}r. \tag{8}$$

Henceforth we omit the index minus $-$ in the notation for concentration state or parameters related to the negative electrode.

1.3. **Problem statement and main results.** We are interested in studying the problem of tracking the SOC to a reference trajectory denoted $SOC_{ref}(t)$. This problem has already been studied in a different context, as regulation problem for Parabolic PDE. See, on the one hand, [28] and [25], where the authors deal with the regulation problem with an internal P or PI control. On the other hand, in [22, 7, 8], the authors solve a tracking problem through a control acting on the boundary.

In this work we deal with the regulation problem. To do that, we aim to apply the main idea of the certainty equivalence principle or separation principle, which refers to the fact that plug-in a convergent estimator in a stable closed-loop system does not change the stability. At this point, it is important to mention that, for the case of infinite-dimensional systems, the certainty equivalence principle may not apply as we know for the case of finite-dimensional systems. Early references recognize this difference. For example, one of the first contributions regarding the design of a finite-dimensional observer-based controller for PDEs was reported in [6]. In that paper, it is proved that under a number of suitable assumptions, a form of separation principle holds. In the same direction, in [2], referring to a Distributed Parameter System the author affirms that "There is no guarantee that a finite-dimensional controller can always produce closed-loop exponential stability with a given DPS." In that article it is proved, in the case of bounded input and bounded output, the stability of the resulting closed-loop system which was assessed for controllers with dimension large enough, but without providing an explicit criterion for the selection of the dimension parameter. Later, in [9] explicit conditions on the order of the finite-dimensional observer-based controller were reported. Another example is [24], where it is stated a certainty equivalence principle for a class of unstable-parabolic equations, more precisely

$$u_t = u_{xx} + \lambda u$$

for a large unknown parameter λ. In that paper, the authors provide a very specific update law $\dot{\hat{\lambda}}$, for the observer $\hat{\lambda}$, thus under that restrictive conditions the separation principle holds. Summarizing, in all these articles the stability of closed-loop systems were proven under very specific assumptions according to the particular cases in study. Thus, the stability of a closed-loop system must be proved.

We begin the controller design by dealing with the most simple case. We assume that we are able to measure the full state $c(r,t)$ of (7), then we design an output feedback control which achieves the regulation. After that, using the backstepping method, see for example [11, 27], we design an output feedback depending on the measure of the boundary concentration $c(1,t)$. The next step in our design consists of replacing the boundary measure $c(1,t)$ by a convergent observer, namely $\varphi(t)$.

As far as we know, in the literature this separation principle is used without proof. In [20] the authors propose an adaptive scheme to obtain $\varphi(t)$ based on the continuous Newton method. The proof of convergence of the scheme and of the closed-loop system is omitted. The authors of [18] state an exponential convergent scheme to obtain φ, but the proof of convergence of the system in closed loop is omitted.

In order to state the separation principle, we assume that this estimator $\varphi(t)$ satisfies the following assumption.

Assumption 1.2. *There exist a function $\varphi : [0,\infty) \to \mathbb{R}$ and positive constants L and μ such that*

$$|\varphi(t) - c(1,t)| \leq Le^{-\mu t}, \quad \forall t \geq 0,$$

where $c(r,t)$ is the solution to (7).

Let us define, for some $p_1(r,\lambda)$ and $p_0(\lambda)$ (given later by the backstepping method), the following copy of the plant

$$\begin{cases} \partial_t \widehat{c_\varphi} = \frac{2}{r}\partial_r \widehat{c_\varphi} + \partial_{rr}\widehat{c_\varphi} + p_1(r,\lambda)(\varphi(t) - \widehat{c_\varphi}(1,t)), \\ \partial_r \widehat{c_\varphi}(0,t) = 0, \quad \partial_r \widehat{c_\varphi}(1,t) = \tilde{\rho} I(t) + p_0(\lambda)(\varphi(t) - \widehat{c_\varphi}(1,t)), \\ \widehat{c_\varphi}(r,0) = \widehat{c_\varphi}_0(r). \end{cases} \tag{9}$$

Our first result consists in the exponential stability of the observer error $\widetilde{c}(r,t) = c(r,t) - \widehat{c_\varphi}(r,t)$, which is stated in Theorem 1.3. This constitutes the main contribution of our work, which presents rigorous proofs of our statements on convergence.

Theorem 1.3. *Consider $\varphi : [0,\infty) \to \mathbb{R}$ and constants $L > 0$ and $\mu > 0$ satisfying Assumption 1.2, the initial condition $\widetilde{c}_0 = c_0(r) - \widehat{c_\varphi}_0(r)$ and the gains $p_0(\lambda)$ and $p_1(r,\lambda)$ given by*

$$p_0(\lambda) = \frac{\lambda}{2} \tag{10}$$

and

$$p_1(r,\lambda) = \left(\frac{\lambda}{(r^2-1)} + \frac{\lambda}{2}\right) J_2\left(\sqrt{\lambda(r^2-1)}\right) - \frac{\lambda}{2} J_0\left(\sqrt{\lambda(r^2-1)}\right), \tag{11}$$

where J_0 and J_2 are the zero and second order Bessel functions of first kind respectively.

Therefore there exists $\lambda_{\sup} > 2 + \sqrt{6}$ such that for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$ the function $\tau(\lambda)$ defined by

$$\tau(\lambda) = \frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(\cdot,\lambda)\|^2_{L^2_r(0,1)} \tag{12}$$

is positive. Moreover, depending on μ, the L^2_r norm of the observer error $\widetilde{c}(r,t)$ satisfies one of the following cases:

1. *If* $\mu > \frac{\tau(\lambda)}{2}$ *for all* $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$*, then*

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \le \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) e^{-\tau(\lambda)t}, \quad \forall t \ge 0, \forall \lambda \in [2+\sqrt{6}, \lambda_{\text{sup}}). \tag{13}$$

2. *If* $\mu = \frac{\tau(\bar\lambda)}{2}$*, for some* $\bar\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$*, then*

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \le \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2} t\right) e^{-\tau(\bar\lambda)t}, \quad \forall t \ge 0. \tag{14}$$

Remark 1.4. In the Theorem 1.3 as well as in its proof, see Section 4.2, we consider $[2+\sqrt{6}, \lambda_{\text{sup}})$ as the biggest interval in which $\tau(\lambda) > 0$ for all $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$.

The followings results are a direct consequence of Theorem 1.3 and describe the performance of observer $\widehat{c_\varphi}(r,t)$.

Corollary 1.5. *Let* $\lambda^* = 2+\sqrt{6}$*. Depending on* μ *we have the following*

1. *if* $2\mu > \tau(\lambda^*)$*, then the highest decay rate of* $\|\widetilde{c}(\cdot,t)\|^2_{L^2_r}$ *is* $\tau(\lambda^*)$ *and the transient state is bounded. Moreover, it holds*

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \le 2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^{*3}+4\lambda^*)}{2|\tau(\lambda^*)-2\mu|}, \quad \forall t \ge 0, \tag{15}$$

2. *if* $2\mu \le \tau(\lambda^*)$*, then the decay ratio of* $\|\widetilde{c}(\cdot,t)\|^2_{L^2_r}$ *is* 2μ *and the transient state is bounded. Moreover, it holds*

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \le \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2\tau(\bar\lambda)} \exp\left\{\frac{4\|\widetilde{c}_0\|^2_{L^2_r}\tau(\bar\lambda)}{L^2(\bar\lambda^3+4\bar\lambda)} - 1\right\}, \quad \forall t \ge 0, \tag{16}$$

where $\bar\lambda$ *is solution to equation* $2\mu = \tau(\bar\lambda)$*.*

Let us define the following

$$N_1(\lambda) = 2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}.$$

Corollary 1.6. *Let* $\lambda^* = 2+\sqrt{6}$ *and* $[\lambda^*, \lambda_{\text{sup}}]$ *the interval given by the Theorem 1.3. If* $2\mu > \tau(\lambda^*)$*, then*

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \le 2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar\lambda^3+4\bar\lambda)}{2|\tau(\bar\lambda)-2\mu|}, \quad \forall t \ge 0, \tag{17}$$

where $\bar\lambda = \underset{\lambda\in[\lambda^*,\lambda_{\text{sup}}]}{\arg\min}\, N_1(\lambda)$ *and the decay ratio is given by* $\tau(\bar\lambda)$*.*

From the previous exponential stability result for $\widetilde{c}$ stated in Theorem 1.3, we are able to prove the following result.

Theorem 1.7. *Consider* $\varphi : [0,\infty) \to \mathbb{R}$ *and constants* $L > 0$ *and* $\mu > 0$ *satisfying Assumption 1.2, gains* $p_0(\lambda)$ *and* $p_1(r,\lambda)$ *given by (10) and (11) respectively. There exists* $\lambda_{\text{sup}} > 2+\sqrt{6}$ *such that for* $\lambda \in [2+\sqrt{6}, \lambda_{\text{sup}})$ *we define the input current*

$$I(t) = \frac{c_{\max}}{3\tilde\rho}\left(S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \widehat{SOC}_\varphi(t)\right)\right), \tag{18}$$

where

$$\widehat{SOC}_\varphi(t) = \frac{3}{c_{\max}} \int_0^1 \widehat{c_\varphi}(r,t) r^2 \, dr,$$

$\gamma > 0$ *is a design parameter and* $\widehat{c_\varphi}(r,t)$ *is the solution to* (9)*. This feedback control* $I(t)$ *forces the system to satisfy*

$$|SOC_{ref}(t) - SOC(t)| \to 0, \quad t \to \infty \tag{19}$$

with an exponential rate, depending on the parameters.

1.4. **Organization and notation.** The remaining part of this paper is organized as follows. In Section 2 we design an input $I(t)$ which depends on full state measurements of the concentration. In Section 3 we improve the previous design of $I(t)$ by considering partial state measurements of the concentration on the boundary. Section 4 is finally dedicated to the design of the current input $I(t)$ used in Theorem 1.7. Being precise, in Section 4.2 and Section 4.6 we provide the proof of Theorem 1.3 and Theorem 1.7 respectively. In Section 5 we illustrate the results by some numerical simulations. Section 6 collects concluding remarks. In Appendix 7, we give a proof to some intermediate results used along this article.

In order to clarify part of the notation used along this article to study the systems (7) and (9) we introduce the following notation:

$$\|f\|_{L_r^2(0,1)} = \left(\int_0^1 f^2(r) r^2 \,\mathrm{d}r\right)^{1/2}, \tag{20}$$

$$\|f\|_{H_r^1(0,1)} = \|f\|_{L_r^2(0,1)} + \|f_r\|_{L_r^2(0,1)}, \tag{21}$$

$$\|f\|_{H_r^2(0,1)} = \|f\|_{L_r^2(0,1)} + \|f_r\|_{L_r^2(0,1)} + \|f_{rr}\|_{L_r^2(0,1)}. \tag{22}$$

2. **Regulation from full state measurements.** The main goal of this work is to design an input current which allows to regulate the State of Charge of a battery. The following proposition gives a starting point for the design for the input current.

Proposition 2.1. *Consider system* (7)*, the State of Charge* $SOC(t)$ *defined by the equation* (8) *and the reference trajectory* $SOC_{ref}(t)$*. Let the input current be*

$$I(t) = \frac{c_{max}}{3\tilde{\rho}} \left(\dot{SOC}_{ref}(t) + \gamma\left(SOC_{ref}(t) - SOC(t)\right)\right), \tag{23}$$

where $\gamma > 0$ *is a constant design parameter. Then, there exists a constant* $C > 0$ *such that for all* $t > 0$

$$|SOC_{ref}(t) - SOC(t)| \le Ce^{-\gamma t}. \tag{24}$$

Proof. See Section 7.1 in the Appendix. □

Notice that this input $I(t)$ depends on full state measurements of the concentration $c(r,t)$ (see definition of SOC given by (8)). However, in most cases we have no access to the full state of the system. Thus, it is more realistic to design an output feedback which only depends on some partial measure of the state. This is the goal of next section.

3. **Regulation from partial state measurements.** We design a feedback control which depends on a partial measurement of the state given by the boundary concentration. To do that, we employ the Backstepping method (see for instance [11, 18, 19, 20]).

3.1. **State observer.** We define the anode state observer structure, which consists in a copy of (7) plus a boundary state error injection, as follows

$$\begin{cases} \widehat{c}_t(r,t) = \frac{2}{r}\widehat{c}_r + \widehat{c}_{rr} + p_1(r,\lambda)\widetilde{c}(1,t), & (r,t) \in \bar{Q}, \\ \widehat{c}_r(0,t) = 0, \quad \widehat{c}_r(1,t) = \widetilde{\rho} I(t) + p_0(\lambda)\widetilde{c}(1,t), & t \in (0,\infty), \\ \widehat{c}(r,0) = \widehat{c}_0(r), & r \in (0,1), \end{cases} \tag{25}$$

where $\widetilde{c}(1,t) = c(1,t) - \widehat{c}(1,t)$, λ is a design parameter and $p_0(\lambda)$, $p_1(r,\lambda)$ are tuning gains to be chosen later.

Remark 3.1. The observer $\widehat{c}(r,t)$ requires the measure of $c(1,t)$. The gains $p_0(\lambda)$ and $p_1(r,\lambda)$ have to be determined in the way of ensure the convergence of the observer to the real state.

The estimation error $\widetilde{c}(r,t) = c(r,t) - \hat{c}(r,t)$ follows the dynamics

$$\begin{cases} \widetilde{c}_t(r,t) = \frac{2}{r}\widetilde{c}_r + \widetilde{c}_{rr} - p_1(r,\lambda)\widetilde{c}(1,t), & (r,t) \in \bar{Q}, \\ \widetilde{c}_r(0,t) = 0, \quad \widetilde{c}_r(1,t) + p_0(\lambda)\widetilde{c}(1,t) = 0, & t \in (0,\infty), \\ \widetilde{c}(r,0) = \widetilde{c}_0(r), & r \in (0,1), \end{cases} \tag{26}$$

with $\widetilde{c}_0(r) = c_0 - \widehat{c}_0$. We search for a kernel $p(r,s)$ such that the following transformation

$$\widetilde{c}(r,t) = \widetilde{z}(r,t) - \int_r^1 p(r,s)\widetilde{z}(s)\,\mathrm{d}s \tag{27}$$

is well-defined and where $\widetilde{z}$ is the solution to the following well-posed target system

$$\begin{cases} \widetilde{z}_t = \frac{2}{r}\widetilde{z}_r + \widetilde{z}_{rr} - \lambda\widetilde{z}, & (r,t) \in \bar{Q}, \\ \widetilde{z}_r(0,t) = 0, \quad \widetilde{z}_r(1,t) = 0, & t \in (0,\infty), \\ \widetilde{z}(r,0) = \widetilde{z}_0(r), & r \in (0,1). \end{cases} \tag{28}$$

Remark 3.2. The choice of the target system is a crucial part of the Backstepping method. The main idea behind this method consists in deducing the exponential stability property of the error system from of that property for the target system.

For the sake of completeness, we include the next result for the heat equation ensuring the well-posedness and the exponential stability in $H_r^1(0,1)$ norm of the target system (28).

Proposition 3.3. *Let $\lambda > 0$. For all initial condition $\widetilde{z}_0 \in H_r^1(0,1)$, there exists an unique $\widetilde{z} \in C([0,\infty); H_r^1(0,1)) \cap C^1([0,\infty); L_r^2(0,1))$ solution to (28). Moreover, we get the estimation*

$$\|\widetilde{z}(\cdot,t)\|^2_{H_r^1(0,1)} \le e^{-2\lambda t}\|\widetilde{z}_0\|^2_{H_r^1(0,1)}, \quad \forall t \ge 0. \tag{29}$$

Proof. See Section 7.2 in the Appendix. □

Let us use the integral transformation (27) and the systems (26) and (28). After some calculations, see for instance [11, Chapter 4], we get the following system for the kernel $p(r,s)$

$$\begin{cases} p_{rr} + \frac{2}{r}p_r + 2\left(\frac{p}{s}\right)_s - p_{ss} = -\lambda p(r,s), & (r,s) \in T, \\ p(r,r) = -\frac{\lambda}{2}r, \quad p(r,0) = 0, & r \in (0,1), \end{cases} \tag{30}$$

where $T = \{(r,s) \in \mathbb{R}^2 : \ 0 \le r \le s \le 1\}$.

The gain equations for the anode observer (25) are defined by (10) and

$$p_1(r,\lambda) = 2p(r,1) - p_s(r,1), \quad \forall r \in (0,1), \tag{31}$$

where $p(r,s)$ is the solution to (30).

The well-posedness of the kernel equation (30) was studied in [27]. The following lemma explains how to solve the kernel equation and then, from the definition (31), how to get the gain observer $p_1(r,\lambda)$.

Lemma 3.4. *Let $\lambda > 0$ in the target system (28). The solution to (30) is given by*

$$p(r,s) = -\lambda s \frac{J_1\left(\sqrt{\lambda(r^2 - s^2)}\right)}{\left(\sqrt{\lambda(r^2 - s^2)}\right)}, \tag{32}$$

where J_1 is the first order Bessel function of first kind. Moreover the gain $p_1(r,\lambda)$ is given by (11).

Proof. See Section 7.3 in the Appendix. □

Now, we prove the exponential decay in $H^1_r(0,1)$ norm of the error (26) We define the following operator

$$\begin{aligned} \Lambda : H^1_r(0,1) &\longrightarrow H^1_r(0,1) \\ \widetilde{c} &\longmapsto \Lambda(\widetilde{c}) = \tilde{c} + \int_r^1 l(r,s)\widetilde{c}(s)\,\mathrm{d}s. \end{aligned}$$

The operator Λ is the inverse transformation of (27) and is well defined, linear and continuous. To see that, it is important to notice that the l-kernel associated to λ is minus the p-kernel associated to $-\lambda$, as explained in [11, Chapter 4].

The next proposition allows to infer the exponential stability property of the error system (26) from the target system (28).

Proposition 3.5. *For all $\lambda > 0$. There exists a constant $M > 1$ such that the error system (26), with the gains $p_0(\lambda)$ and $p_1(r,\lambda)$ defined by the equations (10) and (11) respectively, satisfies*

$$\|\widetilde{c}(\cdot,t)\|_{H^1_r(0,1)} \leq M e^{-\lambda t} \|\widetilde{c}_0\|_{H^1_r(0,1)}. \tag{33}$$

Proof. The map Λ is a linear continuous operator with a continuity constant greater than one. Indeed, this follows from the fact that the first term in (27) is the identity. Also Λ is invertible. Thus, the same properties hold for Λ^{-1} (thanks to the Open Map Theorem, see Corollary 2.7 in [3]). Using the exponential stability of the target system we have the following inequality, for all $t \geq 0$,

$$\|\widetilde{c}(\cdot,t)\|_{H^1_r(0,1)} \leq M e^{-\lambda t} \|\widetilde{c}_0\|_{H^1_r(0,1)}, \tag{34}$$

where $\tilde{c}_0$ is the initial condition of the system (26). □

3.2. **State of charge estimator.** From (8), an appropriate estimator for the State of Charge is

$$\widehat{SOC}(t) = \frac{3}{c_{\max}} \int_0^1 \widehat{c}(r,t) r^2\,\mathrm{d}r, \tag{35}$$

where $\widehat{c}(r,t)$ is solution to the observer system (25). This estimator of the State of Charge was used in [19, 20, 18].

Proposition 3.6. *Consider the State of Charge estimator defined by (35). If $\lambda > 0$, then*

$$|SOC(t) - \widehat{SOC}(t)| \leq \frac{\sqrt{3}M}{c_{\max}} e^{-\lambda t} \|\tilde{c}_0\|_{H_r^1}, \quad \forall t \geq 0. \tag{36}$$

Proof. We consider the estimation error for State of Charge

$$SOC(t) - \widehat{SOC}(t) = \frac{3}{c_{\max}} \int_0^1 \widetilde{c}(r,t) r^2 \, \mathrm{d}r. \tag{37}$$

Now, by Cauchy-Schwartz inequality and Proposition 3.5, we obtain the following inequality, for all $t \geq 0$,

$$|SOC(t) - \widehat{SOC}(t)| \leq \frac{\sqrt{3}M}{c_{\max}} e^{-\lambda t} \|\widetilde{c}_0\|_{H_r^1}, \tag{38}$$

that proves Proposition 3.6. □

Remark 3.7. Notice that $\widehat{SOC}(t)$ is an observer of $SOC(t)$ which depends only on a partial measurement of the full state $c(r,t)$. Being precise, $\widehat{SOC}(t)$ just depends on the boundary concentration $c(1,t)$.

3.3. **Regulation of the SOC.** In the following we prove that the feedback control (23) works even if we replace $SOC(t)$ by $\widehat{SOC}(t)$.

Theorem 3.8. *Consider the system (7), $\lambda > 0$ and $\widehat{SOC}(t)$ defined by (35). If the input current $I(t)$ is selected as following*

$$I(t) = \frac{c_{\max}}{3\tilde{\rho}} \left(S\dot{O}C_{ref}(t) + \gamma \left(SOC_{ref}(t) - \widehat{SOC}(t) \right) \right) \tag{39}$$

where $\gamma > 0$ is a design parameter. Then there exist three cases depending on γ

1. *If $\gamma < 2\lambda$, then*

$$(SOC_{ref}(t) - SOC(t))^2 \leq \left((SOC_{ref}(0) - SOC(0))^2 + \frac{3\gamma M^2 \|\widetilde{c}_0\|^2_{H_r^1}}{2c^2_{\max}|\gamma - 2\lambda|} \right) e^{-\gamma t}, \quad \forall t \geq 0. \tag{40}$$

2. *If $\gamma = 2\lambda$, then*

$$(SOC_{ref}(t) - SOC(t))^2 \leq \left((SOC_{ref}(0) - SOC(0))^2 + \frac{3\gamma M^2 \|\widetilde{c}_0\|^2_{H_r^1}}{2c^2_{\max}} t \right) e^{-\gamma t}, \quad \forall t \geq 0. \tag{41}$$

3. *If $\gamma > 2\lambda$, then*

$$(SOC_{ref}(t) - SOC(t))^2 \leq \left((SOC_{ref}(0) - SOC(0))^2 + \frac{3\gamma M^2 \|\widetilde{c}_0\|^2_{H_r^1}}{2c^2_{\max}(\gamma - 2\lambda)} \right) e^{-2\lambda t}, \quad t \geq 0. \tag{42}$$

Proof. See Section 7.4 in the Appendix. □

Remark 3.9. We have proved that the input current given by (39) achieves a regulation of the State of Charge of the system using only the partial measurement $c(1,t)$. Moreover, this regulation has an exponential convergence ratio. This is an improvement taking account the previous design proposed in Section 2. However, this design might be still being considered unrealistic, in the sense to require an online measure of the boundary concentration. In order to avoid this assumption, in the next section we propose a design which uses a convergent estimator of the boundary concentration $c(1,t)$.

4. **Regulation from a convergent estimator and proofs of the main results.** Along this section we focus on the design of a regulator which solves the problem of the tracking of SOC using a convergent estimator of the boundary concentration $c(1,t)$. We provide a proof of the main results of this work namely, Theorem 1.3, Corollaries 1.5 and 1.6 and Theorem 1.7 successively.

4.1. **Observer design for the ion concentration.** In this subsection, we do not assume that we measure $c(1,t)$ (the real surface concentration in the negative electrode). We use instead an estimator $\varphi(t)$. As we mentioned in Section 1.3, we assume the Assumption 1.2 on $\varphi(t)$.

We define a new observer equation in which we have replaced the surface concentration $c(1,t)$ by the estimation $\varphi(t)$ and we get

$$\begin{cases} \partial_t \widehat{c_\varphi}(r,t) = \frac{2}{r}\partial_r \widehat{c_\varphi} + \partial_{rr}\widehat{c_\varphi} + p_1(r,\lambda)(\varphi(t) - \widehat{c_\varphi}(1,t)), \\ \partial_r \widehat{c_\varphi}(0,t) = 0, \quad \partial_r \widehat{c_\varphi}(1,t) = \tilde{\rho} I(t) + p_0(\lambda)(\varphi(t) - \widehat{c_\varphi}(1,t)), \\ \widehat{c_\varphi}(r,0) = \widehat{c_\varphi}_0(r), \end{cases} \tag{43}$$

where the gains $p_1(r,\lambda)$ and $p_0(\lambda)$ are still defined by (11) and (10), respectively. In the following subsection we give conditions for the convergence of the observer error $\widetilde{c}(r,t) = c(r,t) - \widehat{c_\varphi}(r,t)$.

We define the surface concentration estimation error by $\eta(t) = \varphi(t) - c(1,t)$. Using the state equations (7) and the observer equations (43) we obtain the following system for the error.

$$\begin{cases} \widetilde{c}_t(r,t) - \frac{2}{r}\widetilde{c}_r - \widetilde{c}_{rr} + p_1(r,\lambda)\widetilde{c}(1,t) = -p_1(r,\lambda)\eta(t), \\ \widetilde{c}_r(0,t) = 0, \quad \widetilde{c}_r(1,t) + p_0(\lambda)\widetilde{c}(1,t) = -p_0(\lambda)\eta(t), \\ \widetilde{c}(r,0) = \widetilde{c}_0(r). \end{cases} \tag{44}$$

Remark 4.1. The system (44) can be seen as the system (26) with the perturbation terms $p_1(r,\lambda)\eta(t)$ in the domain and $p_0(\lambda)\eta(t)$ on the boundary, respectively.

4.2. **Proof of Theorem 1.3.** Here we prove Theorem 1.3 which gives the conditions for the convergence of the observer error. Consider $\widetilde{c}(r,t) = u(r,t) + v(r,t)$, where u is solution of the following system

$$\begin{cases} u_t - \frac{2}{r}u_r - u_{rr} + p_1(r,\lambda)u(1,t) = 0, \\ u_r(0,t) = 0, \quad u_r(1,t) + p_0(\lambda)u(1,t) = -p_0(\lambda)\eta(t), \\ u(r,0) = \widetilde{c}_0(r), \end{cases} \tag{45}$$

and v is solution of the following system

$$\begin{cases} v_t - \frac{2}{r}v_r - v_{rr} + p_1(r,\lambda)v(1,t) = -p_1(r,\lambda)\eta(t), \\ v_r(0,t) = 0, \quad v_r(1,t) + p_0(\lambda)v(1,t) = 0, \\ v(r,0) = 0. \end{cases} \tag{46}$$

The main idea is to prove the exponential decay of the norm for u and v. We begin by multiplying the first line of (45) by $u(r,t)r^2$ and then we integrate over $r \in (0,1)$. Using the boundary conditions in (45), we obtain, for all $t \geq 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u^2r^2\,\mathrm{d}r + \int_0^1 u_r^2r^2\,\mathrm{d}r + p_0(\lambda)u^2(1) = \\ -p_0(\lambda)u(1)\eta(t) - u(1)\int_0^1 p_1(r,\lambda)ur^2\,\mathrm{d}r. \quad (47)$$

On the righthand side of the above equality, we apply the Cauchy-Schwartz inequality and two times the Young inequality. Consequently, we obtain, for all $\delta > 0$, $\beta > 0$ and $t \geq 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \|u_r\|^2_{L^2_r} + p_0(\lambda)u^2(1) \leq \\ \left(\frac{\delta p_0(\lambda)}{2} + \frac{\beta}{2}\right)u^2(1) + \frac{1}{2\delta}p_0(\lambda)\eta^2(t) + \frac{1}{2\beta}\|p_1(r,\lambda)\|^2_{L^2_r}\|u\|^2_{L^2_r}. \quad (48)$$

Recall the following version of the Poincaré inequality for the lefthand side of (48)

$$\int_0^1 w^2r^2\,\mathrm{d}r \leq \frac{4}{\pi^2}w^2(1) + \frac{4}{\pi^2}\int_0^1 w_r^2r^2\,\mathrm{d}r. \quad (49)$$

Then, for all $\delta > 0$, $\beta > 0$ and $t \geq 0$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \frac{\pi^2}{4}\|u\|^2_{L^2_r} + (p_0(\lambda)-1)\,u^2(1) \leq \\ \left(\frac{\delta p_0(\lambda)}{2} + \frac{\beta}{2}\right)u^2(1) + \frac{1}{2\delta}p_0(\lambda)\eta^2(t) + \frac{1}{2\beta}\|p_1(r,\lambda)\|^2_{L^2_r}\|u\|^2_{L^2_r}. \quad (50)$$

Rearranging terms in previous inequality we get, for all $\delta > 0$, $\beta > 0$ and $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \left(\frac{\pi^2}{2} - \frac{1}{\beta}\|p_1(r,\lambda)\|^2_{L^2_r}\right)\|u\|^2_{L^2_r} + \\ (2p_0(\lambda) - 2 - \delta p_0(\lambda) - \beta)\,u^2(1) \leq \frac{1}{\delta}p_0(\lambda)\eta^2(t). \quad (51)$$

We recall the definition of the gain $p_0(\lambda) = \frac{\lambda}{2}$, given by (10) as we stated on Theorem 1.3. Let us set $\delta = \frac{2}{\lambda^2}$ and $\beta = \frac{\lambda}{2}$. Then, from the previous inequality, we obtain for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \left(\frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(r,\lambda)\|^2_{L^2_r}\right)\|u\|^2_{L^2_r} + \left(\frac{\lambda}{2} - 2 - \frac{1}{\lambda}\right)u^2(1) \leq \frac{\lambda^3}{4}\eta^2(t). \quad (52)$$

We define the following function which will be helpful to study (52),

$$\tau(\lambda) = \frac{\pi^2}{2} - \frac{2}{\lambda}\|p_1(\cdot,\lambda)\|^2_{L^2_r}, \quad (53)$$

where $p_1(r,\lambda)$ is given by (11).

It is possible to find an interval J such that $\tau(\lambda) > 0$ and $\frac{\lambda}{2} - 2 - \frac{1}{\lambda} \geq 0$, for all $\lambda \in J$. Indeed, on the one hand $\frac{\lambda}{2} - 2 - \frac{1}{\lambda} \geq 0$, for all $\lambda \geq 2 + \sqrt{6}$. On the other hand we check that $\tau(2+\sqrt{6}) > 0$ and since τ is a continuous function, there exists such interval.

Let us consider J as the biggest interval of the form $[2+\sqrt{6}, \lambda_{\mathrm{sup}})$ such that $\tau(\lambda) > 0$, for all $\lambda \in J$. See Figure 1 for an example.

Under those conditions over parameter λ we obtain from (52), for all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L^2_r} + \tau(\lambda)\|u\|^2_{L^2_r} + \leq \frac{\lambda^3}{4}\eta^2(t). \quad (54)$$

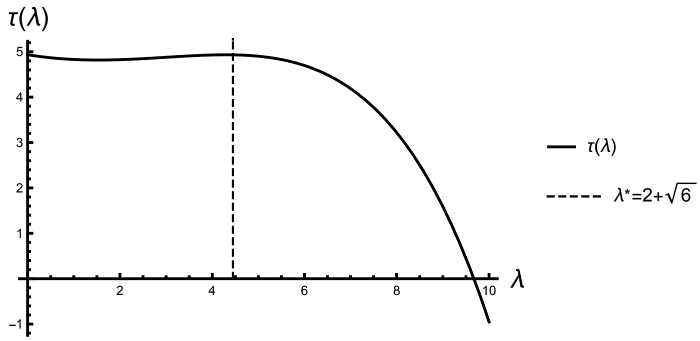


FIGURE 1. The continuous function τ is positive in $[2+\sqrt{6}, \lambda_{\sup})$.

Multiplying (54) by $e^{\tau(\lambda)t}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{L^2_r}e^{\tau(\lambda)t}\right) \le \frac{\lambda^3}{4}\eta^2(t)e^{\tau(\lambda)t}$$

We recall Assumption 1.2 on the estimator $\varphi(t)$. Taking account this, and the above inequality, we get, for all $t \ge 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{L^2_r}e^{\tau(\lambda)t}\right) \le \frac{\lambda^3 L^2}{4}e^{(\tau(\lambda)-2\mu)t}. \tag{55}$$

We distinguish two cases:

1. Assume $\mu > \frac{\tau(\lambda)}{2}$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$.
   Integrating inequality (55) over $(0,t)$, we get for all $t \ge 0$

$$\begin{aligned}\|u\|^2_{L^2_r}e^{\tau(\lambda)t} &\le \|u_0\|^2_{L^2_r} + \frac{L^2\lambda^3}{4(\tau(\lambda)-2\mu)}\left(e^{(\tau(\lambda)-2\mu)t}-1\right)\\ &\le \|u_0\|^2_{L^2_r} + \frac{L^2\lambda^3}{4|\tau(\lambda)-2\mu|}.\end{aligned}$$

   This implies that, for all $t \ge 0$,

$$\|u\|^2_{L^2_r} \le \left(\|u_0\|^2_{L^2_r} + \frac{L^2\lambda^3}{4|\tau(\lambda)-2\mu|}\right)e^{-\tau(\lambda)t}. \tag{56}$$

2. Assume $\mu = \frac{\tau(\bar\lambda)}{2}$, for some $\bar\lambda \in [2+\sqrt{6}, \lambda_{\sup})$.
   From (55), integrating over $(0,t)$, it holds, for all $t \ge 0$

$$\|u\|^2_{L^2_r}e^{\tau(\bar\lambda)t} \le \|u_0\|^2_{L^2_r} + \frac{L^2\bar\lambda^3}{4}t.$$

   Therefore, for all $t \ge 0$

$$\|u\|^2_{L^2_r} \le \left(\|u_0\|^2_{L^2_r} + \frac{L^2\bar\lambda^3}{4}t\right)e^{-\tau(\bar\lambda)t}. \tag{57}$$

Now is turn to get an estimate of the L^2_r norm of v. Similar as before, let us consider the first line in (46). Multiplying by $v(r,t)r^2$ and then perform an integration by parts in $r \in (0,1)$, and using the boundary conditions in (46), we get for all $t \ge 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2_r} + \|v_r\|^2_{L^2_r} + p_0(\lambda)v^2(1) =$$

$$- v(1)\int_0^1 p_1(r,\lambda)vr^2\,\mathrm{d}r - \eta(t)\int_0^1 p_1(r,\lambda)vr^2\,\mathrm{d}r. \quad (58)$$

As before, on the righthand side of (58), we use the Cauchy-Schwartz inequality and two times the Young inequality, with the definition of $p_0(\lambda)$ given by (10), to get, for all $t \geq 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2_r} + \|v_r\|^2_{L^2_r} + \frac{\lambda}{2}v^2(1) \leq \frac{\lambda}{2}v^2(1) + \frac{\lambda}{2}\eta^2(t) + \frac{1}{\lambda}\|p_1(r,\lambda)\|^2_{L^2_r}\|v\|^2_{L^2_r}. \quad (59)$$

Applying the Poincaré inequality on the lefthand side of (59) and rearranging terms we obtain, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{L^2_r} + \tau(\lambda)\|v\|^2_{L^2_r} \leq \lambda\eta^2(t). \quad (60)$$

Multiplying the above inequality by $e^{\tau(\lambda)t}$ and taking account the Assumption 1.2 over the estimator φ it holds, for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\tau(\lambda)t}\|v\|^2_{L^2_r}\right) \leq L^2\lambda e^{(\tau(\lambda)-2\mu)t}. \quad (61)$$

As before, we distinguish two cases. We omit the computations in reason of its similarities with the computations to obtain the L^2_r norm estimation for u.

If $\mu > \frac{\tau(\lambda)}{2}$ for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$. Then it holds

$$\|v\|^2_{L^2_r} \leq \frac{L^2\lambda}{|\tau(\lambda)-2\mu|}e^{-\tau(\lambda)t}, \quad \forall t \geq 0. \quad (62)$$

If $\mu = \frac{\tau(\bar{\lambda})}{2}$, for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{\sup})$. Then we obtain

$$\|v\|^2_{L^2_r} \leq L^2\bar{\lambda}te^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \quad (63)$$

We recall in that $v(r,0) = 0$.

Finally, collecting inequalities, (56), (57), (62) and (63) and using $\|c\|^2_{L^2_r} = \|u+v\|^2_{L^2_r} \leq 2(\|u\|^2_{L^2_r} + \|v\|^2_{L^2_r})$, we obtain inequalities (13) and (14). The proof of Theorem 1.3 is completed. □

Remark 4.2. Note that, if $2\mu < \tau(\lambda)$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$ this implies that $\mu = 0$, which is a contradiction with the Assumption 1.2.

4.3. **Proof of Corollary 1.5.** Let $\lambda^* = 2+\sqrt{6}$.

Note that the function τ is a decreasing continuous function on $[\lambda^*, \lambda_{\sup})$. So its maximum is attained at $\tau(\lambda^*)$.

Let $2\mu > \tau(\lambda^*)$ then $2\mu \geq \tau(\lambda)$, for all $\lambda \in [\lambda^*, \lambda_{\sup})$, so in virtue of Theorem 1.3 it holds

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right)e^{-\tau(\lambda)t}, \quad \forall t \geq 0, \forall \lambda \in [\lambda^*, \lambda_{\sup}). \quad (64)$$

From the above inequality, is easy see that the fastest decayment for $\|\widetilde{c}(\cdot,t)\|^2_{L^2_r}$ is achieved if $\tau(\lambda) = \tau(\lambda^*)$ and since that $e^{-\tau(\lambda^*)t} < 1$, for all $t \geq 0$, we obtain an estimation of the transient state of $\|\widetilde{c}(\cdot,t)\|^2_{L^2_r}$ given by

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \leq 2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^{*3}+4\lambda^*)}{2|\tau(\lambda^*)-2\mu|}, \quad \forall t \geq 0. \quad (65)$$

If $2\mu \leq \tau(\lambda^*)$, by the monotonicity of τ on the interval $[\lambda^*, \lambda_{\sup})$ there exists $\bar{\lambda}$ such that $2\mu = \tau(\bar{\lambda})$. Now, the Theorem 1.3, it holds that

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar{\lambda}^3 + 4\bar{\lambda})}{2}t\right) e^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \tag{66}$$

The decay rate is given by $\tau(\bar{\lambda}) = 2\mu$.

Let us consider $\bar{\lambda}$ fix and we define $N_{\bar{\lambda}}(t) = \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2}t\right) e^{-\tau(\bar{\lambda})t}$. It is not difficult to see that $N_{\bar{\lambda}}(t)$ reaches its maximum at $t^* = \frac{1}{\tau(\bar{\lambda})} - \frac{4\|\widetilde{c}_0\|^2_{L^2_r}}{L^2(\bar{\lambda}^3+4\bar{\lambda})}$. It follows with $e^{-\tau(\bar{\lambda})t^*} < 1$, that

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \leq \frac{L^2(\bar{\lambda}^3 + 4\bar{\lambda})}{2\tau(\bar{\lambda})} \exp\left\{\frac{4\|\widetilde{c}_0\|^2_{L^2_r}\tau(\bar{\lambda})}{L^2(\bar{\lambda}^3 + 4\bar{\lambda})} - 1\right\}, \quad \forall t \geq 0. \tag{67}$$

The proof of Corollary 1.5 is complete. □

4.4. **Proof of Corollary 1.6.** Since that $2\mu > \tau(\lambda)$, by Theorem 1.3, it holds

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3 + 4\lambda)}{2|\tau(\lambda) - 2\mu|}\right) e^{-\tau(\lambda)t}, \quad \forall t \geq 0, \forall \lambda \in [\lambda^*, \lambda_{\sup}). \tag{68}$$

On the other hand, $\tau(\lambda) - \mu \neq 0$, for all $\lambda \in [\lambda^*, \lambda_{\sup}]$ then $N_1(\lambda)$ is a continuous function defined on a compact interval, in consequence there exists $\bar{\lambda}$ such that, $N_1(\lambda) \geq N_1(\bar{\lambda})$, for all $\lambda \in [\lambda^*, \lambda_{\sup}]$.

Let $\bar{\lambda}$ such that minimizes $N_1(\lambda)$, then it holds

$$\|\widetilde{c}(\cdot,t)\|^2_{L^2_r} \leq \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar{\lambda}^3 + 4\bar{\lambda})}{2|\tau(\bar{\lambda}) - 2\mu|}\right) e^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \tag{69}$$

From (69), we see that decay rate is $\tau(\bar{\lambda})$ and taking account that $e^{-\tau(\bar{\lambda})t} < 1$, for all $t \geq 0$ we conclude (17).

The proof of Corollary 1.6 is complete. □

4.5. **State of charge estimation.** We define a new estimator to the State of Charge by

$$\widehat{SOC}_\varphi(t) = \frac{3}{c_{\max}} \int_0^1 \widehat{c_\varphi}(r,t) r^2 \, \mathrm{d}r, \quad t \geq 0, \tag{70}$$

where $\widehat{c_\varphi}$ is the solution to (43). The following proposition gives conditions on $\widehat{SOC}_\varphi(t)$ to ensure the asymptotic convergence to the State of Charge, $SOC(t)$.

Proposition 4.3. *Consider φ, $L > 0$ and $\mu > 0$ satisfying Assumption 1.2. Let $\tau(\lambda)$ defined by (12). Under these assumptions there exists $\lambda_{\sup} > 2 + \sqrt{6}$ such that $\tau(\lambda) > 0$, for all $\lambda \in [2 + \sqrt{6}, \lambda_{\sup})$ and there exist two cases depending on μ such that*

1. *if $\mu > \frac{\tau(\lambda)}{2}$, for all $\lambda \in [2 + \sqrt{6}, \lambda_{\sup})$, then*

$$|SOC(t) - \widehat{SOC}_\varphi(t)| \leq \frac{\sqrt{3}}{c_{\max}} \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2}{2}\frac{\lambda^3 + 4\lambda}{|\tau(\lambda) - 2\mu|}\right)^{\frac{1}{2}} e^{-\frac{\tau(\lambda)}{2}t}, \quad \forall t \geq 0, \tag{71}$$

2. *if $\mu = \frac{\tau(\bar{\lambda})}{2}$ for some $\bar{\lambda} \in [2 + \sqrt{6}, \lambda_{\sup}]$, then*

$$|SOC(t) - \widehat{SOC}_\varphi(t)| \leq \frac{\sqrt{3}}{c_{\max}} \left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2}{2}(\bar{\lambda}^3 + 4\bar{\lambda})t\right)^{\frac{1}{2}} e^{-\frac{\tau(\bar{\lambda})}{2}t}, \quad \forall t \geq 0. \tag{72}$$

Proof. Note that

$$SOC(t) - \widehat{SOC}_{\varphi}(t) = \frac{3}{c_{\max}} \int_0^1 (c(r,t) - \widehat{c_{\varphi}}(r,t)) r^2 \,\mathrm{d}r, \quad \forall t \geq 0, \tag{73}$$

where $c(r,t)$ is the real concentration in the anode and $\widehat{c_{\varphi}}$ is the solution of (43). Then, using the Cauchy-Schwartz inequality on the righthand side and Theorem 1.3, we conclude inequalities (71) and (72) respectively and prove the statement. □

4.6. **Proof of Theorem 1.7.** As in the Section 3.3, we look for the input current $I(t)$ which allows the regulation of the $SOC(t)$ to a given reference trajectory. We use here the convergence of the estimator $\widehat{SOC}_{\varphi}(t)$ depending on $\varphi(t)$ instead of the surface anode concentration $c(1,t)$.

Consider the quadratic error tracking $\kappa(t) = \frac{1}{2}\left(SOC_{ref}(t) - SOC(t)\right)^2$. Then, taking the time derivative we get for all $t \geq 0$.

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{\gamma}{2}\left(SOC(t) - SOC_{\varphi}(t)\right)^2 \tag{74}$$

On the other hand, from the proof of the Proposition 4.3 it holds for all $t \geq 0$,

$$|SOC(t) - SOC_{\varphi}(t)| \leq \frac{\sqrt{3}}{c_{\max}} \|\widetilde{c}(\cdot,t)\|_{L^2_r}. \tag{75}$$

Plugin (75) into (74), we obtain for all $t \geq 0$

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma}{2c_{\max}^2} \|\widetilde{c}(\cdot,t)\|^2_{L^2_r}. \tag{76}$$

Now, in virtue of Theorem 1.3, we have several cases.

1. Let us consider $\mu > \frac{\tau(\lambda)}{2}$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$. Then, (76) becomes,

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) e^{-\tau(\lambda)}, \quad t \geq 0. \tag{77}$$

Multiplying (77) by $e^{\gamma t}$, we obtain for all $t \geq 0$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa(t)e^{\gamma t}\right) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) e^{(\gamma-\tau(\lambda))t} \tag{78}$$

Depending on γ, it holds for all $t \geq 0$ one of the followings cases:

(a) if $\tau(\lambda) < 2\mu \leq \gamma$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2(\gamma-\tau(\lambda))}\left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right)\right) e^{-\tau(\lambda)t}. \tag{79}$$

(b) if $\tau(\lambda) < \gamma < 2\mu$, for all $\lambda \in [2+\sqrt{6}, \lambda_{\sup})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2(\gamma-\tau(\lambda))}\left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right)\right) e^{-\tau(\lambda)t}. \tag{80}$$

(c) if $\gamma = \tau(\bar{\lambda}) < 2\mu$ for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{\sup})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\lambda^3+4\lambda)}{2|\tau(\lambda)-2\mu|}\right) t\right) e^{-\gamma t}. \tag{81}$$

2. Let us consider $\mu = \frac{\tau(\bar{\lambda})}{2}$, for some $\bar{\lambda} \in [2+\sqrt{6}, \lambda_{sup})$. Then (76) becomes

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|^2_{L^2_r} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2} t\right) e^{-\tau(\bar{\lambda})t}, \quad \forall t \geq 0. \tag{82}$$

Multiplying by $e^{\gamma t}$ we obtain for all $t \geq 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa e^{\gamma t}\right) \leq \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2}t\right)e^{(\gamma-\tau(\bar{\lambda}))t}. \tag{83}$$

Depending on γ, it holds for all $t \geq 0$ one of the following cases:

(a) if $2\mu = \tau(\bar{\lambda}) < \gamma$,

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(\frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2(\gamma-\tau(\bar{\lambda}))^2} + \frac{2\|\widetilde{c}_0\|_{L_r^2}^2}{(\gamma-\tau(\bar{\lambda}))} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2(\gamma-\tau(\bar{\lambda}))}t\right)\right)e^{-\tau(\bar{\lambda})t}. \tag{84}$$

(b) if $2\mu = \tau(\bar{\lambda}) = \gamma$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(2\|\widetilde{c}_0\|_{L_r^2}^2 t + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{4}t^2\right)\right)e^{-\gamma t},\ \forall t \geq 0. \tag{85}$$

(c) if $\gamma < 2\mu = \tau(\bar{\lambda})$, then

$$\kappa(t) \leq \left(\kappa(0) + \frac{3\gamma}{2c_{\max}^2}\left(\frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2(\gamma-\tau(\bar{\lambda}))^2} + \frac{2\|\widetilde{c}_0\|_{L_r^2}^2}{|\gamma-\tau(\bar{\lambda})|} + \frac{L^2(\bar{\lambda}^3+4\bar{\lambda})}{2|\gamma-\tau(\bar{\lambda})|}\right)\right)e^{-\gamma t}. \tag{86}$$

The proof of Theorem 1.7 is completed. □

Remark 4.4. We have proved that the input current $I(t)$ given by (18), forces the system 7 to track the signal SOC_{ref} with an exponential decay rate.

Following ideas of the proof of the Corollaries 1.5 and 1.6, see sections 4.3 and 4.4 respectively, we would get similar results as Corollaries 1.5 and 1.6 in the context of Theorem 1.7.

5. **Simulations.** In this section we present some simulations to illustrate the previous results, namely Theorem 3.8 and 1.7. The model parameters used in this work have been taken from the online repository [16] (please also see the related paper [18]). We perform some simulations in two cases. Section 5.1 uses boundary measurements as output while the case where we dispose of the estimator φ is simulated in Section 5.2.

In both types of simulations the definition of $p_0(\lambda)$, $p_1(r,\lambda)$, and $\widehat{SOC}(t)$ are given by (10), (11) and (35), respectively. Moreover, we set the initial conditions of system in closed loop with an error of 50% with respect to the original value. The values for the remain parameters are shown in Table 2.

TABLE 2. Parameter simulations

Parameters	**Values**
$c(r,0)$	$1.5c_0$
$\widehat{c}(r,0)$	$1.5c_0$
$c_{\max}$	$2.5 \cdot 10^4$
λ	5
γ	70

Concerning discretization, we have used central difference method in the spatial variable to get the corresponding ODE system, which is solved with the MatLab routine *ode23tb*. Notice that the system (7) has a singularity at $r = 0$. Therefore, in order to obtain the corresponding ODE, we have done the following approximation. Consider the limit

$$c_t(0,t) = \lim_{r\to 0}\left(\frac{2}{r}c_r(r,t) + c_{rr}(r,t)\right). \tag{87}$$

Then, by the L'Hôpital's rule and the boundary condition at $r = 0$ of (7) we get that

$$\lim_{r\to 0} \frac{c_r(r,t)}{r} = c_{rr}(0,t).$$

Thus, we have that

$$c_t(0,t) \approx 3c_{rr}(0,t). \tag{88}$$

In a similar way, we get the following approximation at $r = 0$

$$\widehat{c}_t(0,t) \approx 3\widehat{c}_{rr}(0,t) + p_1(0)(c(1,t) - \widehat{c}(1,t)). \tag{89}$$

5.1. **Tracking using output.** First, we generate numerical data for the illustration of Theorem 3.8. We set a reference current input $I_{ref}(t)$ and constant initial condition $c_0 = 1.2901\cdot 10^4$. We simulate to obtain a SOC signal which is used as the reference $SOC_{ref}(t)$ in our simulations. Then, we simulate the closed-loop system (7), (25) and (39), which is

$$\begin{cases} c_t(r,t) = \frac{2}{r}c_r(r,t) + c_{rr}(r,t), \\ c_r(0,t) = 0,\ c_r(1,t) = S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}(r,t)r^2dr\right), \\ c(r,0) = c_0(r), \\ \widehat{c}_t(r,t) = \frac{2}{r}\widehat{c}_r + \widehat{c}_{rr} + p_1(r,\lambda)(c(1,t) - \widehat{c}(1,t)), \\ \widehat{c}_r(0,t) = 0, \\ \widehat{c}_r(1,t) = S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}(r,t)r^2dr\right) \\ +p_0(\lambda)(c(1,t) - \widehat{c}(1,t)), \\ \widehat{c}(r,0) = \widehat{c}_0(r). \end{cases} \tag{90}$$

We have done the previous strategy for two different situations. First we take a constant signal $I_{ref}(t) = 0.5C$ as the input current used to generate the state of charge reference $SOC_{ref}(t)$. We see in Figure 2 the input $I_{ref}(t) = 0.5C$ (on the left) and a good performance of the $SOC(t)$ tracking trajectory (on the right). Then we do the same simulations in the case of a square signal $I_{ref}(t) = 4.5square(\frac{64}{900\pi}t)C$. The results can be seen in Figure 3. These simulations illustrate an exponential rate for the tracking.

5.2. **Tracking using output estimator.** As above, we generate a synthetic state of charge $SOC_{ref}(t)$ from a known $I_{ref}(t)$ and then we simulate the controlled system (7), (18) and (43), that is

$$\begin{cases} c_t(r,t) = \frac{2}{r}c_r(r,t) + c_{rr}(r,t), \\ c_r(0,t) = 0,\ c_r(1,t) = S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c}_\varphi(r,t)r^2dr\right), \\ c(r,0) = c_0(r), \\ \partial_t\widehat{c_\varphi}(r,t) = \frac{2}{r}\partial_r\widehat{c_\varphi} + \partial_{rr}\widehat{c_\varphi} + p_1(r,\lambda)(\varphi(t) - \widehat{c_\varphi}(1,t)), \\ \widehat{c_\varphi}(0,t) = 0,\ \partial_r\widehat{c_\varphi}(1,t) = S\dot{O}C_{ref}(t) + \gamma\left(SOC_{ref}(t) - \frac{3}{c_{\max}}\int_0^1 \widehat{c_\varphi}(r,t)r^2dr\right) \\ +p_0(\lambda)(\varphi(t) - \widehat{c_\varphi}(1,t)), \\ \widehat{c_\varphi}(r,0) = \widehat{c_\varphi}_0(r). \end{cases} \tag{91}$$

Note that instead of $c(1,t)$, in this simulation, we have used an artificial estimator $\varphi(t)$ of the boundary concentration $c(1,t)$, namely $\varphi(t) = c(1,t)+Me^{-\mu t}$, with $M > 0$ and $\mu > 0$. This $\varphi(t)$ satisfies the Assumption 1.2. To run out the simulations we have used the parameters values given by the Table 2 and set up $M = c_{\max}$ and $\mu = 70$ to characterize the estimator $\varphi(t)$.

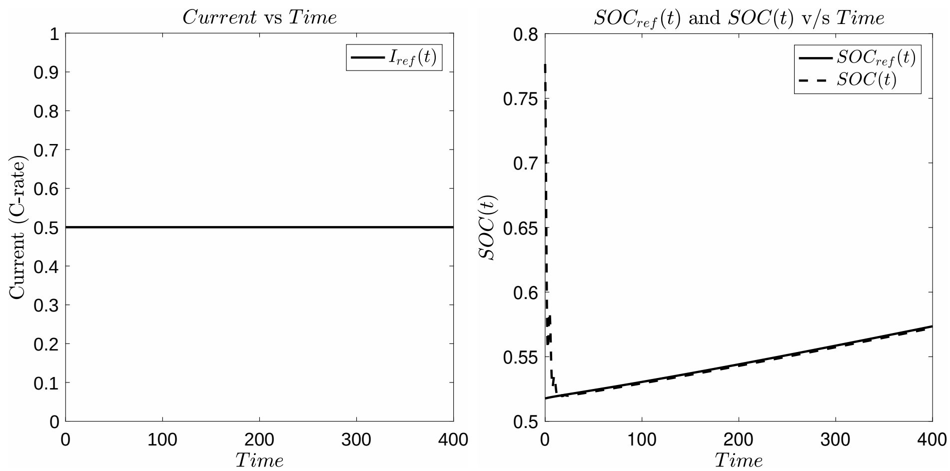


FIGURE 2. (*Left*) The input $I_{ref}(t) = 0.5C$. (*Right*) We compare $SOC(t)$ for the controlled system with the reference $SOC_{ref}(t)$, generated by $I_{ref}(t)$.

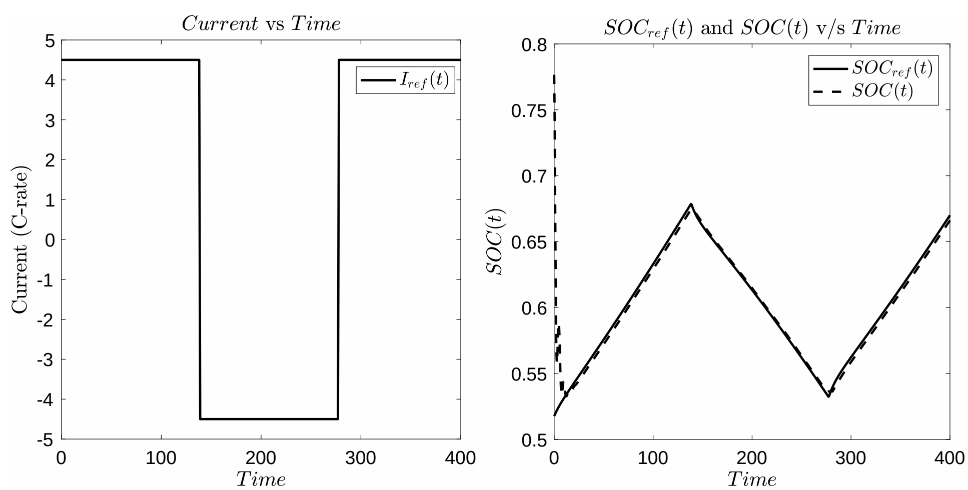
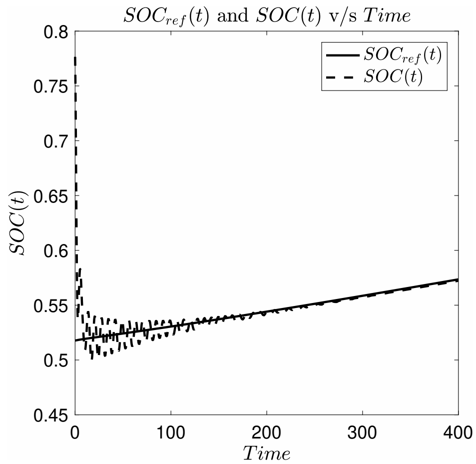


FIGURE 3. (*Left*) The input $I_{ref}(t) = 4.5\, square(\frac{64}{900\pi}t)C$. (*Right*) We compare $SOC(t)$ for the controlled system with the reference $SOC_{ref}(t)$, generated by $I_{ref}(t)$.

As in Section 5.1 we run simulations in two cases. First for $I_{ref}(t) = 0.5C$ and then for $I_{ref}(t) = 4.5 square(\frac{64}{900\pi}t)C$. The results of the tracking of $SOC(t)$ to the reference $SOC_{ref}(t)$ are presented in Figure 4 confirming the good performance of our controllers. As predicted by Theorem 1.7, in simulations the convergence seems to be of exponential type.

6. **Conclusions.** In this paper the tracking problem of the State of Charge (SOC) to a given reference trajectory has been solved. The Single Particle Model ([4, 23, 17]), which belongs to the class of Electrochemical models describing the dynamic



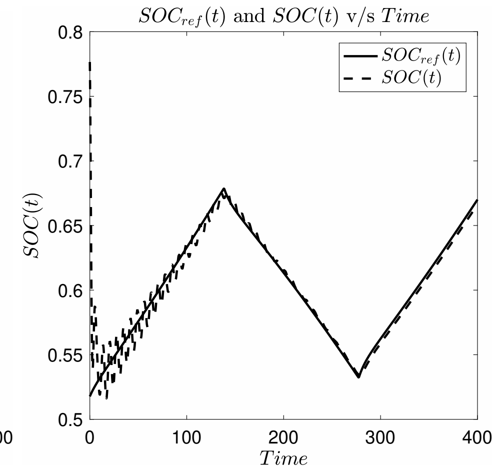


FIGURE 4. (a) We compare $SOC(t)$ for the controlled system with the reference $SOC_{ref}(t)$ generated by $I_{ref}(t) = 0.5C$ (Left) and $I_{ref}(t) = 4.5square(\frac{64}{900\pi}t)C$ (Right).

of lithium ions concentration has been used. This tracking problem consisted in designing a current input for the battery such that the SOC converges to a prescribed trajectory as time goes to infinity.

The approach to solve the tracking problem consisted, in a first stage, in designing an input feedback $I(t)$ which depends on the full ion concentration in the anode. The exponential convergence to zero of the tracking error has been proven. Moreover, an observer has been designed to avoid the online measurement of the full anode concentration. The proof was based on the backstepping method, yielding an exponential decay rate of the reference reference error for the SOC.

An implicit difficult of this approach is that the ion concentration observer depends on an online boundary measurement of the lithium ion concentration. Even if only the boundary is measured, it is very difficult to get proper measurements. To avoid this difficult an observer has been designed for the ion concentration depending on an estimator of surface concentration satisfying Assumption 1.2. Some numerical simulations illustrated the obtained results.

Possible future extensions naturally appear. We could consider models including the dynamics of the ions in electrolyte phase or a distributed temperature. Concerning the controller, a nice extension would be to consider saturated inputs.

7. Appendix.

7.1. **Proof of Proposition 2.1.** Consider a given reference trajectory $SOC_{ref}(t)$. We look for an input $I(t)$ to regulate the $SOC(t)$ of the system (7) to $SOC_{ref}(t)$. To do that we define

$$\kappa(t) = \frac{1}{2}\left(SOC_{ref}(t) - SOC(t)\right)^2 . \tag{92}$$

Now, taking the time derivative over $\kappa(t)$ and using the system (7) we get

$$\dot{\kappa}(t) = \left(SOC_{ref}(t) - SOC(t)\right)\left(S\dot{O}C_{ref}(t) - \int_0^1 c_t r^2 \,\mathrm{d}r\right), \tag{93}$$

$$= (SOC_{ref}(t) - SOC(t)) \left(S\dot{O}C_{ref}(t) - \frac{3\tilde{\rho}}{c_{\max}} I(t) \right). \tag{94}$$

Then, if we select the current as

$$I(t) = \frac{c_{\max}}{3\tilde{\rho}} \left(S\dot{O}C_{ref}(t) + \gamma \left(SOC_{ref}(t) - SOC(t) \right) \right), \tag{95}$$

where $\gamma > 0$ is a constant design parameter, then we obtain

$$\dot{\kappa}(t) = -\gamma \left(SOC_{ref}(t) - SOC(t) \right)^2, \tag{96}$$

$$= -2\gamma\kappa(t). \tag{97}$$

Last equation implies that $\kappa(t) = \kappa(0)e^{-2\gamma t}$ and in particular

$$\lim_{t\to\infty} \kappa(t) = 0. \tag{98}$$

In conclusion we have that $|SOC_{ref}(t) - SOC(t)| \to 0$ when $t \to \infty$. $\square$

7.2. **Proof of Proposition 3.3.** Consider the linear operator $A : D(A) \subset L^2_r(0,1) \to L^2_r(0,1)$ defined by $A\tilde{z} := -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tilde{z}_r)$ and $D(A) = \{\tilde{z} \in H^2_r(0,1) : \tilde{z}_r(0) = \tilde{z}_r(1) = 0\}$. It is easy to check that A is maximal monotone. Thus, by the Hille-Yosida theorem (see Theorem 7.4 in [3, Chapter 7]), if $z_0 \in D(A)$, then equation (28) has a unique solution $\tilde{z} \in C([0,\infty); D(A)) \cap C^1([0,\infty); L^2_r(0,1))$.

Now, we perform some energy estimations. For the moment we assume $z_0 \in D(A)$ and then we easily obtain that the solutions to (28) satisfy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}\|^2_{L^2_r(0,1)} = -\lambda \int_0^1 \tilde{z}^2 r^2 \,\mathrm{d}r - \int_0^1 \tilde{z}_r^2 r^2 \,\mathrm{d}r \le -\lambda \|\tilde{z}\|^2_{L^2_r(0,1)} \tag{99}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{z}_r\|^2_{L^2_r(0,1)} = -\lambda \int_0^1 \tilde{z}_r^2 r^2 dr - \int_0^1 (\tilde{z}_{rr} r + 2\tilde{z}_r)^2 \,\mathrm{d}r. \tag{100}$$

Consequently, we get the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{z}\|^2_{L^2_r(0,1)} + \|\tilde{z}_r\|^2_{L^2_r(0,1)} \right) \le -2\lambda \left(\|\tilde{z}\|^2_{L^2_r(0,1)} + \|\tilde{z}_r\|^2_{L^2_r(0,1)} \right) - 2\|\tilde{z}_{rr} r + 2\tilde{z}_r\|^2_{L^2(0,1)}. \tag{101}$$

and applying the Gronwall's lemma we get

$$\|\tilde{z}(\cdot,t)\|^2_{L^2_r(0,1)} + \|\tilde{z}_r(\cdot,t)\|^2_{L^2_r(0,1)} \le e^{-2\lambda t} \left(\|\tilde{z}(\cdot,0)\|^2_{L^2_r(0,1)} + \|\tilde{z}_r(\cdot,0)\|^2_{L^2_r(0,1)} \right) \tag{102}$$

given (29). This inequality also allows to use a density argument to conclude that (28) has a unique solution $\tilde{z} \in C([0,\infty); H^1_r(0,1)) \cap C^1([0,\infty); L^2_r(0,1))$ if $\tilde{z}_0 \in H^1_r(0,1)$. $\square$

7.3. **Proof of Lemma 3.4.** Consider the following function

$$\check{p}(r,s) = \frac{r}{s} p(r,s). \tag{103}$$

After some calculations we get that

$$p_r(r,s) = -\frac{s}{r^2}\check{p} + \frac{s}{r^2}\check{p}_r, \tag{104}$$

$$p_{rr}(r,s) = -\frac{2s}{r^3}\check{p} - \frac{s}{r^2}\check{p}_r - \frac{s}{r^2}\check{p}_r + \frac{s}{r}\check{p}_{rr}, \tag{105}$$

$$p_s(r,s) = \frac{1}{r}\check{p} + \frac{s}{r}\check{p}_s, \tag{106}$$

$$p_{ss}(r,s) = \frac{1}{r}\check{p}_s + \frac{1}{r}\check{p}_s + \frac{s}{r}\check{p}_s, \tag{107}$$

then, using (30) and equations (104)-(107), we get the following equation and boundary conditions for $\check{p}(r,s)$

$$\begin{cases} \check{p}_{rr}(r,s) - \check{p}_{ss}(r,s) = -\lambda \check{p}(r,s), & (r,s) \in T, \\ \check{p}(r,0) = 0, \quad \check{p}(r,r) = -\frac{\lambda}{2} r, & r \in (0,1). \end{cases} \tag{108}$$

Using the Successive Approximations Method we solve the equation (108), see [11, Chapter 4], and we obtain that

$$\check{p}(r,s) = -\lambda r \frac{J_1\left(\sqrt{\lambda(r^2-s^2)}\right)}{\left(\sqrt{\lambda(r^2-s^2)}\right)}, \tag{109}$$

where J_1 is the first order Bessel function of first kind. Then the kernel $p(r,s)$ is given by (32). Equation (11) follows from (32). This concludes the proof of Lemma 3.4. □

Remark 7.1. From the kernel transformation (103) and boundary condition of (30) we observe that the boundary condition $\check{p}(r,0)$ remains free. The selection of $\check{p}(r,0) = 0$ ensures a well-posed equation (108) and an explicit solution.

7.4. **Proof of Theorem 3.8.** Consider the quadratic tracking error

$$\kappa(t) = \frac{1}{2}\left(SOC_{ref}(t) - SOC(t)\right)^2,$$

and take the time derivative. Thus, we obtain

$$\begin{aligned} \dot{\kappa}(t) &= \left(SOC_{ref}(t) - SOC(t)\right)\left(\dot{SOC}_{ref}(t) - \frac{3\tilde{\rho}}{c_{\max}} I(t)\right) \\ &= -\gamma\left(SOC_{ref}(t) - SOC(t)\right)\left(SOC_{ref}(t) - \widehat{SOC}(t)\right) \\ &= -\gamma\left(SOC_{ref}(t) - SOC(t)\right)\left(SOC_{ref}(t) - SOC(t) + SOC(t) - \widehat{SOC}(t)\right) \\ &= -\gamma\left(SOC_{ref}(t) - SOC(t)\right)^2 - \gamma\left(SOC_{ref}(t) - SOC(t)\right)\left(SOC(t) - \widehat{SOC}(t)\right). \end{aligned}$$

Moreover, by the Young inequality, for all $t \geq 0$,

$$\begin{aligned} &\left|-\gamma\left(SOC_{ref}(t) - SOC(t)\right)\left(SOC(t) - \widehat{SOC}(t)\right)\right| \leq \\ &\qquad \frac{\gamma}{2}\left(SOC_{ref}(t) - SOC(t)\right)^2 + \frac{\gamma}{2}\left(SOC(t) - \widehat{SOC}(t)\right)^2. \end{aligned} \tag{110}$$

Then we obtain that for all $t \geq 0$,

$$\dot{\kappa}(t) \leq -\gamma\kappa(t) + \frac{\gamma}{2}\left(SOC(t) - \widehat{SOC}(t)\right)^2. \tag{111}$$

By Proposition 3.6, we obtain that for all $t \geq 0$,

$$\dot{\kappa}(t) + \gamma\kappa(t) \leq \frac{3\gamma M^2}{2c_{\max}^2}\|\widetilde{c}_0\|_{H_r^1}^2 e^{-2\lambda t}. \tag{112}$$

Multiplying by $e^{\gamma t}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\kappa(t)e^{\gamma t}\right) \leq \frac{3\gamma M^2}{2c_{\max}^2}\|\widetilde{c}_0\|_{H_r^1}^2 e^{(\gamma-2\lambda)t} \tag{113}$$

From the above inequality we distinguish three cases depending on the value of γ.

1. Let $\gamma < 2\lambda$. Integrating (113) over $(0,t)$ we get

$$\kappa(t)e^{\gamma t} - \kappa(0) \leq \frac{3\gamma M^2}{2c_{\max}^2} \frac{\|\widetilde{c}_0\|^2_{H^1_r}}{(\lambda - 2\gamma)} (e^{(\gamma-2\lambda)t} - 1). \tag{114}$$

We ignore the negative terms in the righthand side of the previous inequality to get, for all $t \geq 0$

$$\kappa(t) \leq \left(\kappa^2(0) + \frac{3\gamma M^2}{2c_{\max}^2} \frac{\|\widetilde{c}_0\|^2_{H^1_r}}{|\gamma - 2\lambda|} \right) e^{-\gamma t}. \tag{115}$$

2. Let $\gamma = 2\lambda$. We integrate (113) over $(0,t)$, we get for all $t \geq 0$

$$\kappa(t) \leq \left(\kappa^2(0) + \frac{3\gamma M^2}{2c_{\max}^2} \|\widetilde{c}_0\|^2_{H^1_r} t \right) e^{-\gamma t}. \tag{116}$$

3. Let $\gamma > 2\lambda$. Similar as before, we integrate (113) over $(0,t)$ to get for all $t \geq 0$

$$\kappa(t) \leq \kappa^2(0)e^{-\gamma t} + \frac{3\gamma M^2}{2c_{\max}^2} \frac{\|\tilde{c}_0\|^2_{H^1_r}}{(\gamma - 2\lambda)} e^{-2\lambda t}. \tag{117}$$

Finally we collect the inequalities (115), (116) and (117) to conclude. The proof of Theorem 3.8 is complete. □

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