MATHEMATICAL CONTROL AND
RELATED FIELDS
Volume **1**, Number **2**, June **2011**

doi:10.3934/mcrf.2011.1.129

pp. **129–147**

CONTROL OF A NETWORK OF MAGNETIC ELLIPSOIDAL SAMPLES

Shruti Agarwal

Indian Institute of Technology Madras
Department of Mathematics, Chennai - 600 036, India

Gilles Carbou

IMB, Université Bordeaux
351 cours la Libération
33405 Talence, France

Stéphane Labbé

Laboratoire Jean Kuntzmann, Université de Grenoble
Tour IRMA, 51 rue des Mathématiques, BP 53
38041 Grenoble Cedex 9, France

Christophe Prieur

Department of Automatic Control, Gipsa-lab
961 rue de la Houille Blanche, BP 46
38402 Grenoble Cedex, France

(Communicated by Emmanuel Trélat)

Abstract. In this work, we present a mathematical study of stability and controllability of one-dimensional network of ferromagnetic particles. The control is the magnetic field generated by a dipole whose position and whose amplitude can be selected. The evolution of the magnetic field in the network of particles is described by the Landau-Lifschitz equation. First, we model a network of ellipsoidal shape ferromagnetic particles. Then, we prove the stability of relevant configurations and discuss the controllability by the means of the external magnetic field induced by the magnetic dipole. Finally some numerical results illustrate the stability and the controllability results.

1. **Introduction.** The study of ferromagnetic systems is of great importance, specially for the development of modern technological devices for which there is a continuous demand of more memory. Ferromagnetic materials are used in numerous technological devices such as hard-disks, cellular phones, magnetic sensors, recording heads etc. These applications ask for the study of a system which is an assembly of magnetic domains. In order to make these objects efficient and more useful, it becomes necessary to control their magnetic behavior and to guarantee some controllability and stability properties.

For data recording applications, the first crucial problem is the correct recording of the information. In our context of an assembly of ferromagnetic materials, this is

2000 *Mathematics Subject Classification.* Primary: 93B05, 35Q93; Secondary: 93C10.
Key words and phrases. Ferromagnetic material, Landau-Lifschitz equation, Control.

a controllability issue with respect to relevant configurations. This problem is studied and solved by the control of the external magnetic field generated by a dipole. The second problem is the accurate conservation of the data. From the mathematical point of view, this latter problem corresponds to the stability of the relevant solutions (without any external magnetic field). Existence and controllability results have been already proven for ferromagnetic materials. Consider e.g. [7] where the existence of strong solutions for the Landau-Lifschitz equation is proven for finite local time, and [2] for weak solutions. See [9] for an existence result for global time of the Landau-Lisfchitz equation with a suitable control. Numerical simulations of ferromagnetic materials are studied in [4, 13, 15] among other references.

The geometry of the ferromagnetic materials considered in this paper is close to the one of [1] since in both works, the ferromagnetic domains are assumed to be ellipsoidal. In [1], only one ellipsoidal domain is considered whereas in the current paper, a network of ferromagnetic materials is considered with a coupling between them.

In this work, we deal with a one dimensional network of ferromagnetic particles. We aim to establish sufficient conditions on the cells size and on the network geometry to obtain a stability result for the relevant magnetization configurations. More precisely it is derived a sufficient condition on the volume of the ellipsoidal samples and on the distance between the samples for particular magnetic configuration to be locally asymptotically stable. This is our first main result. Our second main result is a controllability theorem when the control is defined as the amplitude of a dipole which is moving along the network with a constant speed. More precisely, given two relevant magnetization configurations (one is the initial magnetization configuration, whereas the other one is the desired final magnetization configuration), we prove a controllability result for the magnetization by the mean of an applied magnetic field generated by a magnetic dipole modeling a magnetized point. Finally we perform some simulations to illustrate both main results. We also check on a numerical simulation that the stability of relevant configurations may be violated when the geometry of the network and of the ellipsoidal samples does not satisfied our sufficient condition.

The paper is organized as follows. In Section 2, the model is derived and the problems that are solved in this paper are introduced, namely the stability of relevant solutions of the Landau-Lifschitz equation and the controllability to these magnetizations. Then both main results are given in Section 3. The proof of the stability result is given in Section 4, whereas the proof of the controllability result is given in Section 5. Some numerical simulations are performed in Section 6 to illustrate the main results and to check the accuracy of the sufficient condition for a magnetization configuration to be stable. Section 7 contains some concluding remarks and suggests further research lines.

2. Model and control problems under consideration.

2.1. **Ferromagnetism model.** The general setting of the ferromagnetism is the following (see [5], [14] and [16] and the references therein). We consider a finite homogeneous ferromagnetic medium denoted by Ω. We denote by m the magnetization:

$$m: \begin{array}{ccc} I\!R^+ \times \Omega & \longrightarrow & I\!R^3 \\ (t,x,y,z) & \mapsto & m(t,x,y,z). \end{array}$$

The magnetic moment m links the magnetic induction B and the magnetic field H by the relation $B = \bar{m} + H$, where $\bar{m}$ is the extension of m by zero outside Ω. In addition, we assume that the material is saturated so that the magnitude of m is constant. We denote by $\cdot$ the scalar product in $I\!R^3$ and $|\cdot|$ the associated norm. After renormalization we assume that

$$|m| = 1 \text{ at any point.} \tag{1}$$

The evolution of m is described by the Landau-Lifschitz equation:

$$\frac{dm}{dt} = -m \times H_{eff} - m \times (m \times H_{eff}), \tag{2}$$

where we denote by $\times$ the cross product on $I\!R^3$. See [5] for a complete description of the physical model. The existence of local weak solution of (2) was established in [18]. Global existence of weak solution is studied in [10, 11], whereas the existence and uniqueness of regular solutions for Landau-Lisfchitz equation is proven in [7] for a bounded domain, and in [8] for the domain $\mathbb{R}^3$. The effective field $H_{eff} = -\nabla\mathcal{E}$ is derived from the micromagnetism energy $\mathcal{E}$ given by

$$\mathcal{E} = \mathcal{E}_{exch} + \mathcal{E}_{dem} + \mathcal{E}_a,$$

where

- By normalizing the exchange, the exchange energy $\mathcal{E}_{exch}$ writes
$$\mathcal{E}_{exch} = \frac{1}{2}\int_\Omega |\nabla m|^2.$$
- $\mathcal{E}_{dem}$ is the demagnetizing energy:
$$\mathcal{E}_{dem} = \frac{1}{2}\int_{I\!R^3} |H_d(m)|^2.$$
The demagnetizing field $H_d(m)$ is characterized by
$$\begin{cases} \text{curl } H_d(m) = 0, \\ \text{div } (H_d(m) + \bar{m}) = 0. \end{cases} \tag{3}$$
- The applied energy $\mathcal{E}_a$ reflects the effects of an applied magnetic field H_a:
$$\mathcal{E}_a = -\int_\Omega H_a \cdot m.$$

Therefore we obtain that

$$H_{eff} = \Delta m + H_d(m) + H_a.$$

2.2. **Simplified network model.** Let us describe now the network model.

We deal with a one dimensional network of magnetic ellipsoidal shape samples. The ellipsoids are supposed to have the same geometry and to be laid on the axis $I\!R\vec{e_1}$, where $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is the canonical basis of $I\!R^3$. We denote by x_j the position of the jth cell, and we assume that $x_j = jl$, where $l > 0$ is the distance between two consecutive cells. In this paper, we consider a finite network, that is the indexes i are in the finite set $I = \{0, 1, 2, \dots, N\}$. The ith cell Ω_i is obtained from Ω_0 by a translation of vector $il\vec{e_1}$, so that

$$\Omega_i = \left\{(x, y, z) \in I\!R^3, \frac{(x - il)^2}{L_x^2} + \frac{y^2}{L_y^2} + \frac{z^2}{L_z^2} \le 1\right\},$$

where L_x, L_y and L_z are three positive values for the length of the axes of the ellipsoid in the directions $\vec{e_1}$, $\vec{e_2}$, and $\vec{e_3}$ respectively. We assume that the longest axis of the ellipsoids is in the direction $\vec{e_2}$, that is we assume that $L_y > \max\{L_x, L_z\}$. We denote by $V = \frac{4}{3}\pi L_x L_y L_z$ the volume of each cell.

We assume that the characteristic lengths of the cells are small compared to l and to $V^{\frac{1}{3}}$, so that we assume that in each cell Ω_i, the magnetization is constant in the space variable (cf. [17], see also [1]) and is denoted by $m_i(t)$. We use the following notations:

- for each k in $\mathbb{N}$, $(I\!R^k)^I = \left\{u = (u_0, \ldots, u_N) \in I\!R^k \times \ldots \times I\!R^k\right\}$,
- $\|u\| = \sup_{i\in I} |u_i|$, where $|\cdot|$ is the euclidean norm in $I\!R^k$,
- $(S^2)^I = \left\{u = (u_0, \ldots, u_N) \in (I\!R^3)^I, \text{ such that } \forall\, i \in I, |u_i| = 1\right\}$.

So the unknown $m = (m_0, \ldots, m_N)$ is defined on $I\!R^+$ with values in $(S^2)^I$. Under this assumption, the exchange field vanishes and the effective field includes only the demagnetizing and the applied fields.

Let us consider a magnetization configuration $m = (m_i)_{i\in I}$. In a fixed cell Ω_{j_0}, we split the stray field induced by the distribution m in two parts: the stray field generated on Ω_{j_0} by m_{j_0} itself, denoted by $H_d^{int}(m)(j_0)$, and the field generated by the other cells, denoted by $H_d^{ext}(m)(j_0)$:

$$H_d(m)(j_0) = H_d^{int}(m)(j_0) + H_d^{ext}(m)(j_0).$$

From classical results (see [1] and [17]), the stray field generated by a uniformly magnetized ellipsoid on itself is given by

$$H_d^{int}(m)(j_0) = -\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} m_{j_0},$$

where α, β and γ depend on the ellipsoid geometry. In our case, we have $0 < \beta < \alpha$ and $0 < \beta < \gamma$.

The stray field generated by the cell i_0 on the cell j_0 is given by

$$H_{i_0,j_0}(m_{i_0})(x) = -\frac{1}{4\pi}\int_{y\in\Omega_{i_0}} \frac{m_{i_0}}{|x-y|^3}dy + \frac{3}{4\pi}\int_{y\in\Omega_{i_0}} \frac{x-y}{|x-y|^5} m_{i_0}\cdot(x-y)dy.$$

We assume that Ω_{i_0} and Ω_{j_0} are small so that we write that $H_{i_0,j_0}(m_{i_0})$ is almost constant in Ω_{j_0} and we approximate it by

$$H_{i_0,j_0}(m_{i_0}) = -\frac{V}{4\pi}\frac{m_{i_0}}{l^3|i_0-j_0|^3} + \frac{3V}{4\pi}\frac{m_{i_0}^1}{l^3|i_0-j_0|^3}\vec{e_1},$$

that is

$$H_{i_0,j_0}(m_{i_0}) = \frac{V}{4\pi l^3}\frac{1}{|i_0-j_0|^3} A m_{i_0}, \tag{4}$$

where

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The network exterior stray field at the cell j_0 is given by

$$H_d^{ext}(m)(j_0) = \sum_{i_0 \neq j_0} H_{i_0,j_0}(m(i_0)) = \begin{pmatrix} 2h_d^{ext}(m^1)(j_0) \\ \\ -h_d^{ext}(m^2)(j_0) \\ \\ -h_d^{ext}(m^3)(j_0) \end{pmatrix}, \tag{5}$$

where (m^1, m^2, m^3) are the coordinates of m, and where the operator $h_d^{ext} : (I\!R^3)^I \to (I\!R^3)^I$ is defined by, for all $u = (u_i)_{i \in I}$ in $(I\!R^3)^I$,

$$h_d^{ext}(u)(j_0) = \frac{V}{4\pi l^3} \sum_{j \neq j_0} \frac{1}{|j - j_0|^3} u(j). \tag{6}$$

In order to control the network, we apply on it a magnetic field generated by a suitable dipole of magnetic moment $M\vec{e_2}$ situated in the plane $\mathrm{Vect}(\vec{e_1}, \vec{e_2})$ at a fixed distance δl from the network. We denote by $(X, \delta l, 0)$ the coordinates of the dipole. We assume that the dipole is moving with a constant speed $\mathbf{v}$, so that $X(t) = x_0 + \mathbf{v}t$. Our control is the variable $M(t)$.

By standard results (see e.g. [3, Chap. 4.12, Eq. (4.17)] or [12, Chap. 3.4, Eq. (3.103)]), assuming that the dipole is relatively far from the network, the field induced on the cell Ω_{i_0} by this dipole is given by

$$H_{app}(t, M)(i_0) = \frac{\mu_0 M}{4\pi} \frac{1}{r^3} (2\cos(\theta) u_r + \sin(\theta)\ u_\theta), \tag{7}$$

where

- r is the distance between the dipole and the cell:
$$r = [(x_0 + \mathbf{v}t - i_0 l)^2 + \delta^2 l^2]^{\frac{1}{2}},$$
- u_r and u_θ are given by
$$u_r = \frac{1}{r} \begin{pmatrix} x_0 + \mathbf{v}t - i_0 l \\ -\delta l \\ 0 \end{pmatrix}, \quad u_\theta = \frac{1}{r} \begin{pmatrix} \delta l \\ x_0 + \mathbf{v}t - i_0 l \\ 0 \end{pmatrix}$$
- θ is the angle $(\widehat{-\vec{e_2}, u_r})$,
- μ_0 is the dielectric permitivity of the vacuum.

It yields the following model:

$$\begin{cases} \text{for } i \in I,\ m_i : I\!R^+ \longrightarrow S^2 \subset I\!R^3, \\ \\ \dfrac{dm_i}{dt} = -m_i \times H_{eff}(i) - m_i \times (m_i \times H_{eff}(i)) \text{ for } i \in I \text{ and } t \in I\!R^+, \\ \\ H_{eff}(i) = -Dm_i + H_d^{ext}(m)(i) + H_{app}(t, M)(i), \end{cases} \tag{8}$$

with

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}, \tag{9}$$

where $a = \alpha - \beta > 0$ and $b = \gamma - \beta > 0$ (for simplicity, we have replaced D by $D - \beta I_d$ without changing the model, since it only appears in the term $m \times Dm$ in the equations). Without loss of generality, we assume that $0 < a < b$.

We call relevant configurations the magnetization distributions of the form:

$$m_i^0 = \varepsilon_i \vec{e_2}, \text{ with } \varepsilon_i \in \{+1, -1\} \text{ for } i \in I. \tag{10}$$

Let us interpret these relevant configurations as a memory state in an electronic device, where $\varepsilon_i = 1$ corresponds to a bit 1, $\varepsilon_i = -1$ corresponds to a bit 0.

In order to ensure a good conservation of the memory, the key point is the stability of the relevant configurations for the system (8) without applied field.

Let us introduce the problems under consideration.

Problem 1. Exponential stability of any relevant position.
For any initial conditions sufficiently close to a given relevant configuration, the solution of the Landau-Lischitz equation (8) converges exponentially fast to the relevant configuration, without any external magnetic field.

Problem 2. Controllability under the action of a dipole.
Given any pair of relevant configurations, if the initial condition is in a neighborhood of the first relevant configuration, then, with a suitable amplitude of the magnetic field created by the dipole, the magnetic field of the network enters in finite time a neighborhood of the second relevant configuration and converges exponentially fast to it thereafter.

These problems are solved in Theorems 1 and 2 respectively under conditions on the geometry of the network.

3. **Statement of the main results.** To state the stability result, we need to introduce the following notation. For $\nu > 0$ small enough, we define $\mathcal{V}_{+1}(\nu)$ and $\mathcal{V}_{-1}(\nu)$ by

$$\mathcal{V}_{+1}(\nu) = \left\{\xi = (\xi_1, \xi_2, \xi_3) \in S^2, \xi_2 > 0 \text{ and } a\xi_1^2 + b\xi_3^2 < \nu^2\right\},$$

$$\mathcal{V}_{-1}(\nu) = \left\{\xi = (\xi_1, \xi_2, \xi_3) \in S^2, \xi_2 < 0 \text{ and } a\xi_1^2 + b\xi_3^2 < \nu^2\right\}.$$

We remark that for $\xi \in S^2$, if $\xi_2 > 0$ (resp. if $\xi_2 < 0$), the quantity $a\xi_1^2 + b\xi_3^2$ measures the distance between ξ and $+\vec{e_2}$ (resp. $-\vec{e_2}$) since in this case:

$$\frac{a}{2}|\xi - \vec{e_2}|^2 \le a\xi_1^2 + b\xi_3^2 \le b|\xi - \vec{e_2}|^2.$$

For $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)$ with $\varepsilon_i \in \{-1, +1\}$, we denote for $\nu > 0$:

$$\bar{\mathcal{V}}_\varepsilon(\nu) = \left\{m \in (S^2)^I, \forall\, i \in I,\, m_i \in \mathcal{V}_{\varepsilon_i}(\nu)\right\}. \tag{11}$$

Our first main result is the following:

Theorem 1. *There exists $\gamma_0 > 0$ (depending on a and b, but independent to the size of the network) such that if*

$$\frac{V}{l^3} \le \gamma_0\ , \tag{12}$$

then there exist $\nu_0 > 0$, and $c > 0$ such that for all relevant configurations m^0 associated to ε (that is $m_i^0 = \varepsilon_i \vec{e_2}$ for all i), for all $m^{init} \in \bar{\mathcal{V}}_\varepsilon(\nu_0)$, the solution m of (8) with $M \equiv 0$ starting from the initial condition $m(0) = m^{init}$ satisfies:

$$\forall\, t \ge 0, m(t) \in \bar{\mathcal{V}}_\varepsilon(\nu_0 e^{-ct}).$$

Remark 1. This theorem means that the relevant configurations are uniformly asymptotically stable for the Landau-Lifschitz equation (8) with zero applied field ($M \equiv 0$).

The second question under consideration is the controllability of our network by the mean of a dipole generating applied field. We consider $m^\flat$ and $m^\sharp$ two relevant configurations. We assume that at $t = 0$ the magnetization of our network is close to $m^\flat$. In order to align the magnetization with $m^\sharp$, we define the control $t \mapsto M(t)$ by

$$\text{for } t \in [i\tfrac{l}{\mathbf{v}}, i\tfrac{l}{\mathbf{v}} + \delta\tfrac{l}{\mathbf{v}}[,\ M(t) = \begin{cases} +\mathcal{M} & \text{if } m_i^\flat = -m_i^\sharp = \vec{e_2}, \\ 0 & \text{if } m_i^\flat = m_i^\sharp, \\ -\mathcal{M} & \text{if } m_i^\flat = -m_i^\sharp = -\vec{e_2}. \end{cases} \tag{13}$$

$$\text{for } t \notin \bigcup_{i=0}^{N} [i\frac{l}{\mathbf{v}}, i\frac{l}{\mathbf{v}} + \delta\frac{l}{\mathbf{v}}[,\ M(t) = 0.$$

for a suitable $\mathcal{M} > 0$ and $0 \leq \delta \leq 1$. We denote $T_f = (N + \delta)l/\mathbf{v}$. For $t \geq T_f$, $M(t) = 0$, that is the dipole is switched off.

Our second main result is the following:

Theorem 2. *Let* $\mathbf{v}$ *be a fixed positive value. Let* γ_0 *and* c *be given by Theorem 1. There exists* $\gamma_1 > 0$ *with* $\gamma_1 < \gamma_0$*, there exist* $\nu_1 > 0$, $\mathcal{M} > 0$ *and* $\delta > 0$ *such that if*

$$\frac{V}{l^3} \leq \gamma_1,$$

then we have the following controllability result:
let $m^\flat$ *and* $m^\sharp$ *be two relevant configurations, and let* $t \mapsto M(t)$ *be the control given by (13). If* $m^{init} \in \bar{\mathcal{V}}_{\varepsilon^\flat}(\nu_1)$*, then the solution* m *of (8) starting from the initial condition* m^{init} *satisfies:*

$$\forall\, t \geq T_f,\ m(t) \in \bar{\mathcal{V}}_{\varepsilon^\sharp}(\nu_1 e^{-c(t-T_f)}).$$

The remaining of the paper is organized as follows. We prove Theorem 1, namely the stability of all relevant configurations, in Section 4. We remark that our stability criteria depends neither on the size of the network N nor on the considered relevant configuration $m^\flat$. Section 5 is devoted to the proof of Theorem 2, i.e. the controllability of a finite network by the mean of a magnetic dipole generating an applied field. The paper ends with some numerical simulations and concluding remarks (see respectively Sections 6 and 7).

4. **Proof of the stability for the relevant configurations.** In this section we tackle the stability of a relevant configuration for the Landau-Lifschitz equation without applied field. More precisely we prove Theorem 1 and we consider the following system with unknown $m : I\!R^+ \to (S^2)^I$:

$$\begin{aligned} \frac{dm}{dt} &= -m \times (-Dm + H_d^{ext}(m)) - m \times (m \times (-Dm + H_d^{ext}(m))) \\ &= m \times F(m) \end{aligned} \tag{14}$$

with $F(m) = -(-Dm + H_d^{ext}(m)) - m \times (-Dm + H_d^{ext}(m))$. The existence and uniqueness of a solution of (14) for any initial condition $m^0 \in (S^2)^I$ follows from the classical Cauchy-Lipschitz theorem and the constraint $m_i(t) \in S^2$ for all i in I and for all $t \geq 0$.

Let m^0 be a fixed relevant configuration: $m^0 \in (S^2)^I$ such that

$$\forall i \in I,\, m_i^0 = \varepsilon_i \vec{e_2},\ \ \varepsilon_i \in \{-1,+1\}.$$

Because of the saturation constraint (1), we only deal with perturbations m of m^0 satisfying:

$$\forall i \in I, \forall t, |m_i(t)| = 1.$$

So we describe such a perturbation writing for all $i \in I$:

$$m_i = \rho_i^1 \vec{e_1} + \rho_i^3 \vec{e_3} + \varepsilon_i \vec{e_2} + \lambda(\rho_i)\varepsilon_i \vec{e_2} \tag{15}$$

where $\rho_i = (\rho_i^1, \rho_i^3)$ and $\lambda(\rho_i) = \sqrt{1-|\rho_i|^2} - 1$.

In order to find an equivalent formulation of (14) in the variable $\rho \in \mathcal{C}^1(I\!R^+; (S^2)^I)$, we plug (15) in (14) and we project the obtained expression on both $\vec{e_1}$ and $\vec{e_3}$ axis.

We obtain that m, given by (15), satisfies (14) if and only if ρ satisfies the following system:

$$\frac{d\rho}{dt} = \begin{pmatrix} -a & \varepsilon b \\ -\varepsilon a & -b \end{pmatrix} \rho + \mathcal{L}(\rho) + \mathcal{N}(\rho) \tag{16}$$

where

$$\mathcal{L}(\rho) = \begin{pmatrix} \varepsilon h_d^{ext}(\rho^3) - \rho^3 h_d^{ext}(\varepsilon) + 2h_d^{ext}(\rho^1) + \varepsilon\rho^1 h_d^{ext}(\varepsilon) \\ \\ 2\varepsilon h_d^{ext}(\rho^1) + \rho^1 h_d^{ext}(\varepsilon) - h_d^{ext}(\rho^3) + \varepsilon\rho^3 h_d^{ext}(\varepsilon) \end{pmatrix}.$$

The nonlinear term $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_3)$ is given by

$$\begin{aligned} \mathcal{N}_1(\rho) = \quad & -\varepsilon\lambda(\rho)(-b\rho^3 - h_d^{ext}(\rho^3)) - \rho^3 h_d^{ext}(\varepsilon\lambda(\rho)) + \varepsilon\lambda(\rho)((-a\rho^1 + 2h_d^{ext}(\rho^1))\varepsilon \\ & +\rho^1 h_d^{ext}(\varepsilon)) + (-a\rho^1 + 2h_d^{ext}(\rho^1))\lambda(\rho) + \varepsilon\rho^1 h_d^{ext}(\varepsilon\lambda(\rho))\ , \end{aligned}$$

$$\begin{aligned} \mathcal{N}_3(\rho) = \quad & \varepsilon\lambda(\rho)(-a\rho^1 + 2h_d^{ext}(\rho^1)) + \rho^1 h_d^{ext}(\varepsilon\lambda(\rho)) \\ & -\varepsilon\lambda(\rho)(-h_d^{ext}(\varepsilon)\rho_3 + (b\rho^3 + h_d^{ext}(\rho^3))\varepsilon) \\ & +\lambda(\rho)(-b\rho^3 - h_d^{ext}(\rho^3)) + \varepsilon h_d^{ext}(\varepsilon\lambda(\rho))\rho^3\ . \end{aligned}$$

Remark 2. By projection, it is clear that if m satisfies (14) then ρ satisfies (16). For the converse implication, we remark that if $m \in \mathcal{C}^1(\mathbb{R}_+; (S^2)^I)$ and satisfies:

$$\forall i \in \{1,3\},\ \frac{dm}{dt} \cdot \vec{e_i} = (m \times F(m)) \cdot \vec{e_i},$$

because of the constraint $|m| = 1$, then m satisfies

$$\frac{dm}{dt} = m \times F(m).$$

This argument is used in a partial differential equation framework in [6] and in [9]. In addition, m^0 is asymptotically stable for (14) if and only if 0 is asymptotically stable for (16).

Let us study now the stability of 0 for (16) under a smallness condition on $\dfrac{V}{l^3}$.

The linear operator h_d^{ext} is estimated in the following way: for all $u = (u_i)_{i\in I}$ in $(I\!R^3)^I$, then

$$\|h_d^{ext}(u)\| \le \frac{V}{4\pi l^3} \sum_{j\neq 0} \frac{1}{|j|^3} \|u\|. \tag{17}$$

So, we estimate the linear operator $\mathcal{L}$ with the following lemma:

Lemma 3. *There exists a constant $K_{\mathcal{L}}$ such that we have, for all $\rho \in (\mathbb{R}^2)^I$,*

$$\|\mathcal{L}(\rho)\| \leq K_{\mathcal{L}} \frac{V}{l^3} \|\rho\|.$$

This constant K_1 depends neither on a, b, V, l nor on the size of the network.

The nonlinear right-hand side term of (16) is estimated with straightforward arguments in the following lemma.

Lemma 4. *We assume that $\frac{V}{l^3} \leq 1$. There exists a constant $K_{\mathcal{N}}$ such that, for all $\rho \in (\mathbb{R}^2)^I$ such that $\|\rho\| \leq \frac{1}{2}$, we have*

$$\|\mathcal{N}_1(\rho)\| + \|\mathcal{N}_2(\rho)\| \leq K_{\mathcal{N}} \|\rho\|^2.$$

We define $\tilde{\rho} \in \mathcal{C}^0(\mathbb{R}^+; (\mathbb{R})^I)$ by

$$\tilde{\rho}_i(t) = \left(a(\rho_i^1(t))^2 + b(\rho_i^3(t))^2\right)^{\frac{1}{2}}.$$

Recalling $0 < a < b$, we remark that

$$|(a\rho_i^1, b\rho_i^2)| \leq \sqrt{b}|\tilde{\rho}_i| \text{ and } a|\rho_i|^2 \leq |\tilde{\rho}_i|^2 \leq b|\rho_i|^2. \tag{18}$$

In addition, $m \in \bar{\mathcal{V}}_\varepsilon(\nu)$ if and only if $\tilde{\rho} < \nu$ (see (11)).

Multiplying (16) by $(a\rho_i^1, b\rho_i^3)$, we have, for all i in I,

$$\frac{1}{2}\frac{d}{dt}[a(\rho_i^1)^2 + b(\rho_i^3)^2] + a^2(\rho_i^1)^2 + b^2(\rho_i^3)^2 = (\mathcal{L}(\rho)_i + \mathcal{N}(\rho)_i) \cdot (a\rho_i^1, b\rho_i^3),$$

so using that $0 < a < b$, (18) and Lemmas 3 and 4, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(|\tilde{\rho}_i|^2\right) + a|\tilde{\rho}_i|^2 \leq \quad & K_{\mathcal{L}} \frac{V}{l^3}\|\rho\|\sqrt{b}\tilde{\rho}_i + K_{\mathcal{N}}\|\rho\|^2\sqrt{b}\tilde{\rho}_i\ , \\ \leq \quad & K_{\mathcal{L}} \frac{V}{l^3}\sqrt{\frac{b}{a}}\|\tilde{\rho}\|^2 + K_{\mathcal{N}}\frac{\sqrt{b}}{a}\|\tilde{\rho}\|^3\ . \end{aligned} \tag{19}$$

We define γ_0 by

$$\gamma_0 = \frac{a^{\frac{3}{2}}}{\sqrt{b}K_{\mathcal{L}}}. \tag{20}$$

We assume that V and l are fixed so that $\frac{V}{l^3} < \gamma_0$. We aim to show that 0 is asymptotically stable for (16).

We multiply (19) by e^{2at} and we integrate from 0 to t. We obtain that, for all i in I,

$$(\tilde{\rho}_i(t))^2 e^{2at} \leq \|\tilde{\rho}(0)\|^2 + 2K_{\mathcal{L}}\frac{V}{l^3}\sqrt{\frac{b}{a}}\int_0^t \|\tilde{\rho}(s)\|^2 e^{2as} ds + 2K_{\mathcal{N}}\frac{\sqrt{b}}{a}\int_0^t \|\tilde{\rho}(s)\|^3 e^{2as} ds.$$

So taking the supremum for i in I, we obtain that, for all $t \geq 0$,

$$\|\tilde{\rho}(t)\|^2 e^{2at} \leq \|\tilde{\rho}(0)\|^2 + 2K_{\mathcal{L}}\frac{V}{l^3}\sqrt{\frac{b}{a}}\int_0^t \|\tilde{\rho}(s)\|^2 e^{2as} ds + 2K_{\mathcal{N}}\frac{\sqrt{b}}{a}\int_0^t \|\tilde{\rho}(s)\|^3 e^{2as} ds.$$

We denote by $c = a - K_{\mathcal{L}}\frac{V}{l^3}\sqrt{\frac{b}{a}}$. From (20), since $\dfrac{V}{l^3} < \gamma_0$, c is positive. Now, while $\|\tilde{\rho}(s)\| \leq \dfrac{ca}{2K_{\mathcal{N}}\sqrt{b}}$, we have, for all $t \geq 0$,

$$\|\tilde{\rho}(t)\|^2 e^{2at} \leq \|\tilde{\rho}(0)\|^2 + (2a - c)\int_0^t \|\tilde{\rho}(s)\|^2 e^{2as} ds,$$

and by Gronwall lemma, we obtain that, while $\|\tilde{\rho}(s)\| \leq \dfrac{ca}{2K_{\mathcal{N}}\sqrt{b}}$,

$$\|\tilde{\rho}(t)\|^2 \leq \|\tilde{\rho}(0)\|^2 e^{-ct}.$$

So, if $\|\tilde{\rho}(0)\| \leq \dfrac{ca}{2K_{\mathcal{N}}\sqrt{b}}$, then for all $t \geq 0$, $\|\tilde{\rho}(t)\|^2$ remains less than $\dfrac{ca}{2K_{\mathcal{N}}\sqrt{b}}$ and

$$\forall\, t \geq 0, \|\tilde{\rho}(t)\|^2 \leq \|\tilde{\rho}(0)\|^2 e^{-ct}.$$

We set $\nu_0 = \dfrac{ca}{2K_{\mathcal{N}}\sqrt{b}}$, and it concludes the proof of Theorem 1. □

5. **Proof of the controllability result.** Let $m^\flat$ and $m^\sharp$ be two relevant configurations associated to $\varepsilon^\flat$ and $\varepsilon^\sharp$ respectively, that is $m_i^\flat = \varepsilon_i^\flat \vec{e_2}$ and $m_i^\sharp = \varepsilon_i^\sharp \vec{e_2}$ for i in I. In this section we prove Theorem 2, that is the controllability result to $m^\flat$ with initial condition in a neighborhood of $m^\sharp$. Let us introduce $\nu_0 > 0$ given by Theorem 1.

For a fixed $\zeta \in S^2$, for $\nu \in [-1, 1]$, we define $\mathcal{W}(\zeta, \nu)$ by

$$\mathcal{W}(\zeta, \nu) = \left\{x \in S^2,\, x \cdot \zeta \geq \nu\right\}.$$

We define $\vec{\xi}$ by $\vec{\xi} = \dfrac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$. We introduce ν_1 and $\mu_1 > 0$ such that $0 < \nu_1 < \nu_0$ so that

$$\mathcal{V}_{+1}(\nu_1) \subset \mathcal{W}(\vec{\xi}, -1 + \mu_1) \text{ and } \mathcal{W}(-\vec{e_2}, 1 - \mu_1) \subset \mathcal{V}_{-1}(\nu_1). \tag{21}$$

We introduce the following sequence $(t_i)_{i \in I}$ of time instants, defined by $t_i = i\frac{l}{\mathbf{V}}$, for all $i \in I$. We assume that

$$m(0) \in \mathcal{V}_{\varepsilon^\flat}(\nu_1). \tag{22}$$

We recall that our control is defined by (13) where δ and $\mathcal{M}$ are two constant values. These former values will be selected so that, for $i \in \{0, \dots, N+1\}$, the following property $(\mathcal{P}_i)$ holds:

$$(\mathcal{P}_i) \qquad \forall\, j \in \{0, \dots, N\},\; m_j(t_i) \in \begin{cases} \mathcal{V}_{\varepsilon_j^\flat}(\nu_1) \text{ for } j \geq i, \\ \\ \mathcal{V}_{\varepsilon_j^\sharp}(\nu_1) \text{ for } j < i. \end{cases}$$

Note that Theorem 2 follows from $(\mathcal{P}_{N+1})$.

The selection of δ and $\mathcal{M}$ is done by using an induction argument on the property $(\mathcal{P}_i)$.

The property $(\mathcal{P}_0)$ is true (whatever of the value of δ and $\mathcal{M}$).

Let $i_0 \in I$ such that $(\mathcal{P}_{i_0})$ holds. The conditions on $\mathcal{M}$ and δ such that $(\mathcal{P}_{i_0+1})$ are derived as follows. First, denoting $t_{i_0} = i_0\frac{l}{\mathbf{V}}$, we give a bound on the applied field during the time interval $[t_{i_0}, t_{i_0} + \delta\frac{l}{\mathbf{V}})$ in Section 5.1. Then we prove in Section

5.2 that, under suitable conditions on $\mathcal{M}$ and δ, the magnetization on the other cells is not changed. Afterwards, in Section 5.3, we give conditions on $\mathcal{M}$ and δ so that the magnetization in the cell i_0 is switched on the direction $m^\sharp_{i_0}$. In Section 5.4 the constant values $\mathcal{M}$ and δ are chosen so that the induction argument holds and we conclude the proof of Theorem 2.

5.1. **Evaluation of the applied field.** In this section some estimations on the applied field are given. For $t \in [t_{i_0}, t_{i_0} + \delta \frac{l}{\mathbf{v}})$, two cases have to be considered:

- close to the cell i_0, the applied field is given by $H_{app}(t,M)_{i_0} = -\varepsilon^\sharp_{i_0} h_a(t)\bar{\xi}(t)$, with

$$\bar{\xi}(t) = \frac{1}{\left(2+(1-\frac{t-t_{i_0}}{\tau})^2\right)^2 + (1-\frac{t-t_{i_0}}{\tau})^2} \begin{pmatrix} 2+(1-\dfrac{t-t_{i_0}}{\tau})^2 \\ \frac{t-t_{i_0}}{\tau} - 1 \\ 0 \end{pmatrix}, \tag{23}$$

  where τ is a positive value that will be chosen below (it will be selected after the proof of Lemma 6 as $\tau = \frac{\delta l}{\mathbf{v}}$) and

$$h_a(t) = \frac{\mu_0 \mathcal{M}}{4\pi} \frac{1}{(\delta l)^3} \frac{\left(2+(1-\frac{t-t_{i_0}}{\tau})^2\right)^2 + (1-\frac{t-t_{i_0}}{\tau})^2}{\left(1+(1-\frac{t-t_{i_0}}{\tau})^2\right)^{\frac{5}{2}}}. \tag{24}$$

  We remark that there exist constant values K_1 and K_2 such that:

$$\left|\bar{\xi}'(t)\right| \leq K_1 \frac{\mathbf{v}}{\delta l} \text{ and } h_a(t) \geq K_2 \frac{\mathcal{M}}{(\delta l)^{\frac{3}{2}}};$$

- On the other cells, for $j \neq i_0$, due to (7), there exists $K_3 > 0$ such that it holds

$$|H_{app}(t,M)(j)| \leq K_3 \frac{\mathcal{M}}{(1-\delta)^3 l^3}. \tag{25}$$

5.2. **Pseudo-stability under small external field.** We fix j_0 in I, $j_0 \neq i_0$. We prove in this section that, if some conditions on $\mathcal{M}$ and δ hold, the magnetization $\mathbf{m}(t) := m_{j_0}(t)$ at the cell j_0 remains almost unchanged. To do that, we denote by

$$P(t) = (P_1,P_2,P_3)(t) := \left(H_{app}(t,M)(j_0) + H_d^{ext}(m)(j_0)\right) \tag{26}$$

the contribution of the applied field and the exterior demagnetizing field at the cell j_0.

By (5) and (17), there exists C_1 such that, for all m in $(S^2)^I$,

$$\left|H_d^{ext}(m)(j_0)\right| \leq C_1 \frac{V}{l^3}. \tag{27}$$

Let h_0 be a bound of the applied field at the cell j_0, that is we assume that

$$\forall\, t \geq t_{i_0}, |H_{app}(t,M)(j_0)| \leq h_0. \tag{28}$$

From the induction assumption (we recall that we assume that $(\mathcal{P}(i_0))$ is true), we have $\mathbf{m}(t_{i_0}) \in \mathcal{V}_{+1}(\nu_1)$ or $\mathbf{m}(t_{i_0}) \in \mathcal{V}_{-1}(\nu_1)$. To ease the presentation, let us assume in this section that $\mathbf{m}(t_{i_0}) \in \mathcal{V}_{+1}(\nu_1)$ (the other case is considered in a similar way).

Due to (26), (27), and (28), we have, for all t in $[t_{i_0}, t_{i_0+1})$,

$$|P(t)| \leq C_1 \frac{V}{l^3} + h_0. \tag{29}$$

Due to (8), the magnetization $\mathbf{m}$ satisfies, for all $t \geq 0$,

$$\frac{d\mathbf{m}}{dt} = \mathbf{m} \times D\mathbf{m} + \mathbf{m} \times (\mathbf{m} \times D\mathbf{m}) - \mathbf{m} \times P(t) - \mathbf{m} \times (\mathbf{m} \times P(t)).$$

We take the scalar product of this equation with $-D\mathbf{m}$. Using (9) and (29), and writing $\mathbf{r}(t) = \big(a(\mathbf{m}_1(t))^2 + b(\mathbf{m}_3(t))^2\big)^{\frac{1}{2}}$, we obtain that:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathbf{r}^2 &= -|\mathbf{m} \times D\mathbf{m}|^2 + (\mathbf{m} \times P(t) + \mathbf{m} \times (\mathbf{m} \times P(t))) \cdot D\mathbf{m} \\ &\leq -|\mathbf{m} \times D\mathbf{m}|^2 + \sqrt{2}(C_1 \frac{V}{l^3} + h_0)|D\mathbf{m}|. \end{aligned} \tag{30}$$

We remark that $|\mathbf{m} \times D\mathbf{m}|^2 \geq a(\mathbf{m}_2)^2 \mathbf{r}^2$, and that $|D\mathbf{m}| \leq \sqrt{b}\mathbf{r}$.

From straightforward geometric arguments, if ξ is in $\mathcal{V}_{+1}(\nu_1)$ then $|\xi_2| > \left(1 - \frac{1}{a}\nu_1^2\right)^{\frac{1}{2}}$. So while $\mathbf{m}$ remains in $\mathcal{V}_{+1}(\nu_1)$, $|\mathbf{m} \times D\mathbf{m}|^2 \geq \left(1 - \frac{1}{a}\nu_1^2\right)\mathbf{r}^2$, and then, from (30),

$$\frac{1}{2}\frac{d}{dt}\mathbf{r}^2 + \left(1 - \frac{1}{a}\nu_1^2\right)\mathbf{r}^2 \leq \sqrt{2b}(C_1 \frac{V}{l^3} + h_0)\mathbf{r},$$

and so, while $\mathbf{r} \leq \nu_1$, we have

$$\frac{d\mathbf{r}}{dt} + \left(1 - \frac{1}{a}\nu_1^2\right)\mathbf{r} \leq \sqrt{2b}(C_1 \frac{V}{l^3} + h_0).$$

We assume that $\nu_1 > 0$ is sufficiently small so that $1 - \frac{1}{a}\nu_1^2 > 0$. We define γ_1 and $h_0 > 0$ by

$$\gamma_1 = \frac{1}{C_1\sqrt{2b}}(1 - \frac{1}{a}\nu_1^2)\frac{\nu_1}{2} \text{ and } h_0 = \frac{1}{\sqrt{2b}}(1 - \frac{1}{a}\nu_1^2)\frac{\nu_1}{2}. \tag{31}$$

If $\frac{V}{l^3} \leq \gamma_1$, then while $\mathbf{r} \leq \nu_1$,

$$\frac{d\mathbf{r}}{dt} + \left(1 - \frac{1}{a}\nu_1^2\right)\mathbf{r} \leq \sqrt{2b}(C_1\gamma_1 + h_0) = \left(1 - \frac{1}{a}\nu_1^2\right)\nu_1.$$

So, $\mathbf{r}$ is a sub-solution of the following ordinary differential equation:

$$\frac{dz}{dt} + \left(1 - \frac{1}{a}\nu_1^2\right)z = \left(1 - \frac{1}{a}\nu_1^2\right)\nu_1$$

for which ν_1 is a constant solution. So, γ_1 and h_0 being given by (31), assuming

$$\frac{V}{l^3} \leq \gamma_1 \text{ and } |H_{app}(t, M)_{(j_0)}| \leq h_0 \; \forall \, t \geq t_{i_0},$$

if $\mathbf{m}(t_{i_0})$ is in $\mathcal{V}_{+1}(\nu_1)$, then, for all $t \geq t_{i_0}$, $\mathbf{m}(t)$ is in $\mathcal{V}_{+1}(\nu_1)$. Therefore, with the bound (25), we have proved

Lemma 5. *If the conditions*

$$\frac{V}{l^3} \leq \gamma_1 \textit{ and } K_3 \frac{\mathcal{M}}{(1-\delta)^3 l^3} \leq \frac{1}{\sqrt{2b}}(1 - \frac{1}{a}\nu_1^2)\frac{\nu_1}{2} \tag{32}$$

hold, then for $j_0 \neq i_0$, if $m_{j_0}(t_{i_0})$ is in $\mathcal{V}_{+1}(\nu_1)$, then $m_{j_0}(t)$ remains in the same neighborhood $\mathcal{V}_{+1}(\nu_1)$ of $\vec{e_2}$, for all $t \geq t_{i_0}$.
Moreover, the same conclusion holds replacing +1 by -1 and $\vec{e_2}$ by $-\vec{e_2}$.

5.3. **Reversal for a fixed cell.** We describe the reversal of one cell with the following lemma:

Lemma 6. *Let $\xi \in \mathcal{C}^1(\mathbb{R}^+; S^2)$ and $A \in \mathcal{C}^0(\mathbb{R}^+; \mathbb{R}^3)$. We assume that*

$$\forall\, t, |A(t)| \leq \kappa_1 \text{ and } |\xi'(t)| \leq \kappa_2. \tag{33}$$

For a fixed $\zeta \in S^2$, for $\mu \in [-1,1]$, we define $\mathcal{W}(\zeta,\nu)$ by

$$\mathcal{W}(\zeta,\nu) = \left\{x \in S^2,\, x\cdot\zeta \geq \nu\right\}.$$

Let $h_a \in \mathcal{C}^0(\mathbb{R}^+; \mathbb{R})$. We consider the following ordinary differential equation

$$\frac{du}{dt} = -u \times (A(t) + h_a(t)\xi(t)) - u \times (u \times (A(t) + h_a(t)\xi(t)). \tag{34}$$

Let $\mu_1 > 0$, let $\tau > 0$. There exists $h_{inf} \in \mathbb{R}^+$ such that for all A and ξ satisfying (33), for all h_a such that $h_a(t) \geq h_{inf}$ for $t \geq 0$, then, if $u(0)$ is in $\mathcal{W}(\xi(0), -1+\mu_1)$, then the solution u of (34) with initial data $u(0)$ at $t = 0$ satifies $u(\tau) \in \mathcal{W}(\xi(\tau), 1-\mu_1)$.

Proof. We set $\beta(t) = u(t)\cdot\xi(t)$. We have

$$\beta'(t) = h_a(t)(1-(\beta(t))^2) + \bar{H}(t) \tag{35}$$

with $\bar{H}(t) = u(t)\cdot\xi'(t) - u(t)\times A(t)\cdot\xi(t) - u(t)\times(u(t)\times A(t))\cdot\xi(t)$. We remark that from (33), for all $t \geq 0$, $|\bar{H}(t)| \leq \kappa_2 + \sqrt{2}\kappa_1$. We define h^1_{inf} by

$$h^1_{inf} = \frac{\kappa_2+\sqrt{2}\kappa_1}{\mu_1(2-\mu_1)}.$$

We assume that

$$\forall\, t \geq 0, h_a(t) \geq h^1_{inf}. \tag{36}$$

Then the constant maps $-1+\mu_1$ and $1-\mu_1$ are subsolutions of (35) since under assumption (36),

$$\begin{array}{rl} h_a(t)(1-(1-\mu_1)^2) + \bar{H}(t) \geq & h_a(t)(1-(1-\mu_1)^2) - (\kappa_2+\sqrt{2}\kappa_1)\ , \\ \geq & h^1_{inf}(1-(1-\mu_1)^2) - (\kappa_2+\sqrt{2}\kappa_1)\ , \\ \geq & 0\ . \end{array}$$

Thus we get

Claim 7. *Under Assumption (36), if at the initial time, $u(0) \in \mathcal{W}(\xi(0), -1+\mu_1)$, then for all $t \geq 0$, $u(t)$ remains in $\mathcal{W}(\xi(0), -1+\mu_1)$. In addition, if at a time t_0, $u(t_0)$ is in $\mathcal{W}(\xi(t_0), 1-\mu_1)$, then for all $t \geq t_0$, $u(t)$ remains in $\mathcal{W}(\xi(t), 1-\mu_1)$.*

Furthermore, while $u(t)$ remains in $\mathcal{W}(\xi(t), -1+\mu_1) \setminus \mathcal{W}(\xi(t), 1-\mu_1)$, we have: $1-(\beta(t))^2 \geq 1-(1-\mu_1)^2 \geq \mu_1(2-\mu_1)$ that is

$$\beta'(t) \geq \mu_1(2-\mu_1)h_a(t) - (\kappa_2+\sqrt{2}\kappa_1).$$

We define h_{inf} by

$$h_{inf} = \frac{1}{\mu_1(2-\mu_1)}\left(\kappa_2+\sqrt{2}\kappa_1+\frac{2(1-\mu_1)}{\tau}\right). \tag{37}$$

If $h_a(t) \geq h_{inf}$, then, while $|\beta(t)| \leq 1-\mu_1$, then $\beta'(t) \geq \dfrac{2(1-\mu_1)}{\tau}$. Thus, with $\beta(0) \geq -1+\mu_1$, we deduce

$$\forall t \text{ such that } |\beta(t)| \leq 1-\mu_1, \text{ it holds } \beta(t) \geq (1-\mu_1)\left(\frac{2t}{\tau}-1\right).$$

First case: Assume that there exists $t_0 < \tau$ such that $u(t_0)$ is in $\mathcal{W}(\xi(t_0), 1-\mu_1)$. Due to Claim 7, for all $t \geq t_0$, $u(t)$ remains in $\mathcal{W}(\xi(t), 1-\mu_1)$.

Second case: if for all $t < \tau$, $u(t)$ remains out of $\mathcal{W}(\xi(t), 1-\mu_1)$, then

$$\forall\, t < \tau, \beta(t) \geq (1-\mu_1)\left(\frac{2t}{\tau}-1\right)$$

and in particular, $\beta(\tau) \geq 1-\mu_1$, that is

$$u(\tau) \in \mathcal{W}(\xi(t), 1-\mu_1).$$

It concludes the proof of Lemma 6. □

To prove the switch, namely that at time $t_{i_0}+\tau$ the magnetization m_{i_0} at the cell i_0 enters a neighborhood of $\varepsilon^\sharp_{i_0}$, and remains in this neighborhood thereafter, two cases may occur:

First case: if $\varepsilon^\sharp_{i_0} = -1$, that is $m^\sharp_{i_0} = -\vec{e_2}$, for $t \in [t_{i_0}, t_{i_0+1})$, we set:

$$M(t) = \begin{cases} \mathcal{M} & \text{for} \quad t_{i_0} \leq t < t_{i_0} + \frac{\delta l}{\mathbf{V}} \\ 0 & \text{for} \quad t_{i_0} + \frac{\delta l}{\mathbf{V}} \leq t < t_{i_0+1} \end{cases}.$$

We apply Lemma 6 at this cell, with $\bar{\xi}(t)$ given by (23), $\tau = \frac{\delta l}{\mathbf{V}}$, $A(t) = -Dm_{i_0}(t) + H_d^{ext}(m(t))_{(i_0)}$. We remark that

- $\xi(t_{i_0}) = \vec{\xi}$ so by Assumption $(\mathcal{P}(i_0))$ and with (21), $m(t_{i_0})$ is in $\mathcal{W}(\xi(t_{i_0}), \mu_1)$,
- $|A(t)| \leq K_2$ (since m takes its values in the sphere, and since V/l^3 is supposed to be small),
- $|\xi'(t)| \leq K_1 \dfrac{\mathbf{V}}{\delta l}$ and $h_a \geq K_2 \frac{\mathcal{M}}{(\delta l)^{\frac{3}{2}}}$.

By Lemma 6, the switch is obtained if

$$K_2 \frac{\mathcal{M}}{(\delta l)^{\frac{3}{2}}} \geq \frac{1}{\mu_1(2-\mu_1)}\left(\kappa_2 + \sqrt{2}\kappa_1 + \frac{2(1-\mu_1)}{\tau}\right). \tag{38}$$

Under this condition, $m_{i_0}(t_{i_0}+\tau)$ is in $\mathcal{W}(\xi(t_{i_0}+\tau), \mu_1) = \mathcal{W}(-\vec{e_2}, \mu_1)$. Since, by (21), $\mathcal{W}(-\vec{e_2}, \mu_1) \subset \mathcal{V}_{-1}(\nu_1)$, then $m_{i_0}(t_{i_0}+\tau)$ is in $\mathcal{V}_{-1}(\nu_1)$ and remains in this neighborhood of $-\vec{e_2}$ for $t \in [t_{i_0}+\tau, t_{i_0+1})$ since the applied field vanishes in this time interval, and applying Theorem 1.

Second case: if $\varepsilon^\sharp_{i_0} = 1$, that is $m^\sharp_{i_0} = \vec{e_2}$, for $t \in [t_{i_0}, t_{i_0+1})$, we set:

$$M(t) = \begin{cases} -\mathcal{M} & \text{for} \quad t_{i_0} \leq t < t_{i_0} + \frac{\delta l}{\mathbf{V}} \\ 0 & \text{for} \quad t_{i_0} + \frac{\delta l}{\mathbf{V}} \leq t < t_{i_0=1} \end{cases}.$$

We conclude that in both cases, the following result holds:

Lemma 8. *If Condition (38) is satisfied, then $m_{i_0}(t_{i_0}+\tau)$ is in $\mathcal{V}_{-1}(\nu_1)$ and remains in this neighborhood of $-\vec{e_2}$ for $t \in [t_{i_0}+\tau, t_{i_0+1})$*

5.4. **Conditions on $\mathcal{M}$ and δ for the induction argument.** If both Conditions (32) and (38) are satisfied, then if $\mathcal{P}(i_0)$ is true, then $\mathcal{P}(i_0+1)$ holds. The existence of $\mathcal{M}$ satisfying both (32) and (38) is ensured assuming that

$$\frac{\delta^{\frac{3}{2}}}{l^{\frac{3}{2}} K_2 \mu_1 (2-\mu_1)} \left(\kappa_2 + \sqrt{2}\kappa_1 + \frac{2(1-\mu_1)}{\tau} \right) \leq \frac{\mathcal{M}}{l^3} \leq \frac{(1-\delta)^3}{K_3\sqrt{2b}} (1 - \frac{1}{a}\nu_1^2) \frac{\nu_1}{2}. \tag{39}$$

The $\mathcal{M}$ satisfying this condition exists for a sufficiently small δ. With this selection of $\mathcal{M}$ and δ, the induction argument applies and $(\mathcal{P}_{N+1})$ holds (recall $I = \{0, \dots, N\}$). It concludes the proof of Theorem 2.

Remark 3. We remark that in (39), if $\mathbf{v}$ is large, then we must take δ small, so that the model is not yet valid, since it is based on the fact that the cells' diameter is small compared to the other lengths. In addition, the model for the field generated by the dipole is valid far from the dipole. Moreover some constants values in (39) are defined only implicitly in function of the geometry of the network. Therefore, the optimization of the controllability time and thus on the speed on the writing of the network is still an open question, to the best to our knowledge.

6. **Numerical simulations.** In order to illustrate the previous results, we perform a set of simulations. Then, through these simulations, we point out the stability of the relevant configurations and the efficiency of the developed controls.

6.1. **Numerical scheme.** The numerical scheme used in these simulations is an explicit Euler scheme with re-normalization. For small set of particles, the problem is not steep and the basic explicit scheme is highly adapted. Simulation has been performed using Matlab.

6.2. **Illustration of the stability result.** Let us consider a set of ten particles (see Figure 1) with the parameters and shape of Table 1.

l	10^{-6}
V	1.8963×10^{-17}
a	0.25
b	0.5

Table 1. A first set of physical parameters

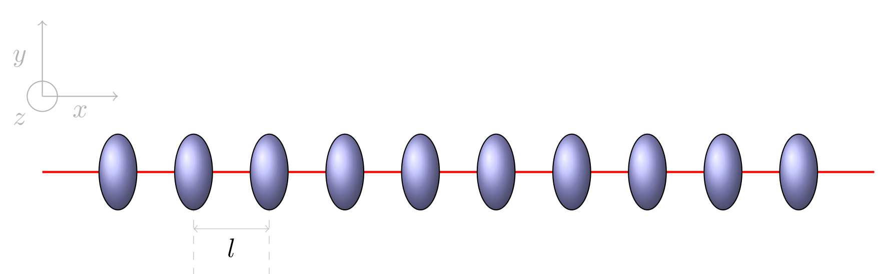

Figure 1. Set of particles.

The ratio $\frac{V}{l^3}$ appears in the sufficient condition of Theorem 1. With the physical parameters of Table 1, we compute $\frac{V}{l^3} = 0.1435$. Let us illustrate Theorem 1 by showing that modifying the ratio $\frac{V}{l^3}$ we may have either unstable relevant configurations or stable ones.

To do that we first give an example of a randomly modified relevant configuration whose, with no external field, goes to the relevant configuration when times goes to infinity. In the following computations we set, for m^0 a relevant configuration,

$$\forall i \in \{1, \cdots, 10\},\ m_i = m_i^0 + \eta_i,$$

where η is a random vector whose components follows the uniform law on $(-1, 1)$.

Let us consider again the set of ten particles of Figure 1 and let us modify the physical parameters of Table 1 so that $\frac{V}{l^3} = 4.7038$. The matrix D defined by (9) and which is a parameter of the model (8) is computed with $a = 0.4$ and $b = 1$. The initial magnetization configuration is illustrated in Figure 2 by the red arrows on the associated set of particles.

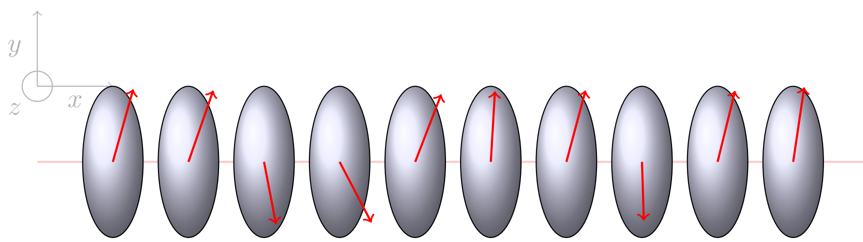

FIGURE 2. Initial magnetization configuration: random perturbation of a given relevant configuration with a dense set of particles.

Figure 3 gives the final configuration ($T = 10$ and 2000 time discretization points).

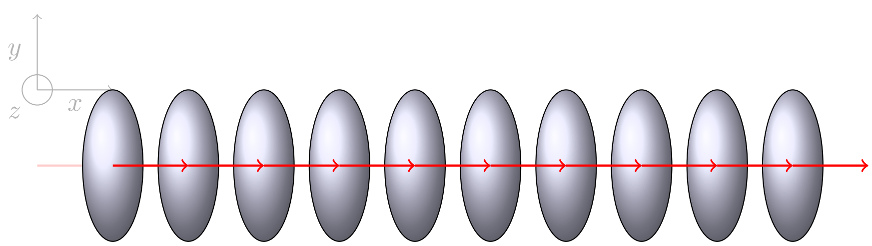

FIGURE 3. Final configuration (for $T = 10$) with a dense set of particles.

Remark 4. In this example we see that a too dense set of particles has not relevant configurations as stable configurations and can not be considered for magnetic recording purposes.

Now the physical parameters are chosen such that we have $a = 0.25$, $b = 0.5$ and $\frac{V}{l^3} = 0.1435$. In this case, for the following random perturbation of a relevant configuration chose as initial configuration, Theorem 1 stands (see Figure 4 for initial configuration and Figure 5 for final configuration for $T = 10$ and 2000 time discretization points).

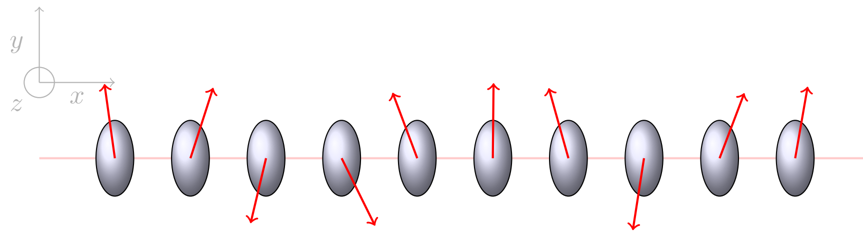

FIGURE 4. Initial configuration: random perturbation of a given relevant configuration with a sparse set of particles.

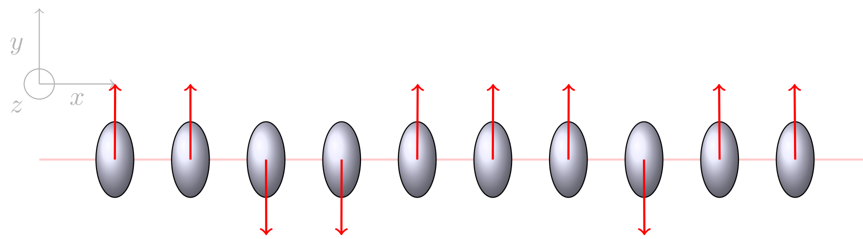

FIGURE 5. Final configuration (for $T = 10$) with a sparse set of particles.

6.3. **Illustration of the controllability result.** Let us illustrate the controllability result when the ratio $\frac{V}{l^3}$ is small (when this ratio is too large, then, as remarked in the previous sub-section with Figure 3, the relevant magnetizations may be not stable, and thus could not be used neither for applications nor to save memory on electronic devices). The controllability part is more time consuming than the relaxation part seen in the previous sub-section. To illustrate the efficiency of the developed control strategy suggested by Theorem 2, we apply the control $M(t)$ to the previous set of particles. The initial and final configurations are relevant configurations (see Figure 6 for the initial configuration and Figure 7 for the final configuration).

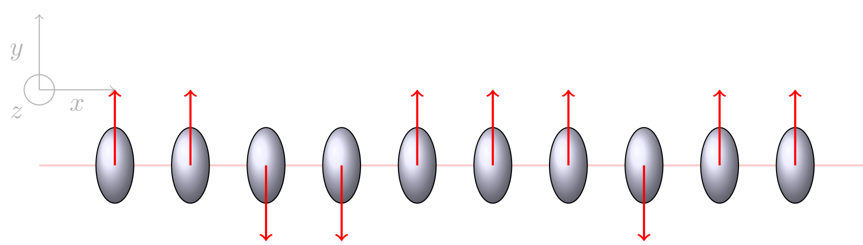

FIGURE 6. Initial configuration.

Choosing $\mathbf{v}= l$, $\mathcal{M} = 10^9 \times l^3$, $x_0 = 0$ and $\delta = 0.3$, we may check on numerical simulations that the equilibrium is reached at time $T = 40$, choosing 40 000 time steps for the discretization.

7. **Conclusion.** In this paper, we modeled and studied an one-dimensional finite network of ferromagnetic particles in respect of its stability and controllability. It is

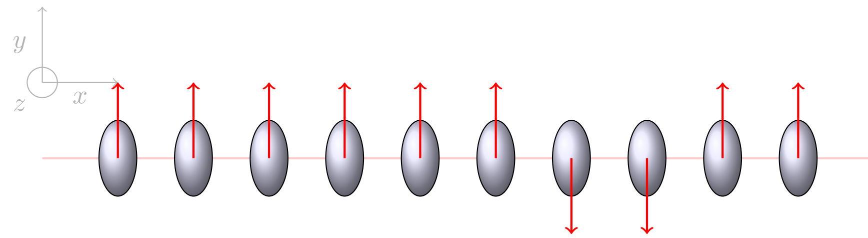

FIGURE 7. Goal configuration.

of great importance because of the practical applications of such network. To establish the stability of relevant configurations of a network, a stability criteria is given. For controllability, the most crucial issue is the coupled behavior of the particles in a network, that is, on the use of control, the change of the magnetization of one particle should perturb only a few the magnetization of other particles. This issue is carefully considered and under some conditions, it is proved that the magnetization of the particles in a network enters in a neighborhood of the relevant configuration in finite time and converges exponentially fast to it after this time. Also, numerical simulations are done to illustrate the obtained analytic results. In a future work, we will consider the speed of convergence and the optimal controllability problem under a constraint on the amplitude of the dipole and in presence of a limitation of the speed of its movement.

REFERENCES

[1] François Alouges and Karine Beauchard, *Magnetization switching on small ferromagnetic ellipsoidal samples*, ESAIM Control Optim. Calc. Var., **15** (2009), 676–711.

[2] François Alouges and Alain Soyeur, *On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness*, Nonlinear Anal., **18** (1992), 1071–1084.

[3] S. W. Anwane, "Fundamentals of Electromagnetic Fields," Infinity Science Press, Hingham, Masschusetts, 2007.

[4] L'ubomír Baňas, Sören Bartels and Andreas Prohl, *A convergent implicit finite element discretization of the Maxwell-Landau-Lifshitz-Gilbert equation*, SIAM J. Numer. Anal., **46** (2008), 1399–1422.

[5] William F. Brown, "Micromagnetics," Wiley, New York, 1963.

[6] Gilles Carbou, *Stability of static walls for a three-dimensional model of ferromagnetic material,* J. Math. Pures Appl., **93** (2010), 183–203.

[7] Gilles Carbou and Pierre Fabrie, *Regular solutions for Landau-Lifschitz equation in a bounded domain,* Differential Integral Equations, **14** (2001), 213–229.

[8] Gilles Carbou and Pierre Fabrie, *Regular solutions for Landau-Lifschitz equation in* $I\!R^3$, Commun. Appl. Anal., **5** (2001), 17–30.

[9] Gilles Carbou, Stéphane Labbé and Emmanuel Trélat, *Control of travelling walls in a ferromagnetic nanowire*, Discrete Contin. Dyn. Syst., **1** (2008), 51–59.

[10] Shijin Ding, Boling Guo, Junyu Lin and Ming Zeng, *Global existence of weak solutions for Landau-Lifshitz-Maxwell equations,* Discrete Contin. Dyn. Syst., **17** (2007), 867–890.

[11] Boling Guo and Fengqiu Su, *Global weak solution for the Landau-Lifshitz-Maxwell equation in three space dimensions*, J. Math. Anal. Appl., **211** (1997), 326–346.

[12] D. J. Griffiths, "Introduction to Electrodynamics," 3^rd edition, PearsonBenjamin Cummings, San Francisco, CA, 2008.

[13] Stéphane Labbé, "Simulation Numérique du Comportement Hyperfréquence des Matériaux Ferromagnétiques," Editions Universitaires Européennes, 2010.

[14] Stéphane Labbé, *Fast computation for large magnetostatic systems adapted for micromagnetism,* SIAM J. Sci. Comp., **26** (2005), 2160–2175.

[15] Stéphane Labbé and Pierre-Yves Bertin, *Microwave polarisability of ferrite particles with non-uniform magnetization*, Journal of Magnetism and Magnetic Materials, **206** (1999), 93–105.

[16] L. Landau and E. Lifschitz, "Electrodynamique des Milieux Continus, Cours de Physique Théorique," (French) [Electrodynamic of Continuous Media, Theoretical Physics Course], **VIII**, Mir, Moscou, 1969.

[17] J. A. Osborn, *Demagnetizing factors of the general ellipsoid,* Phys. Rev., **67** (1945), 351–357.

[18] Augusto Visintin, *On Landau Lifschitz equation for ferromagnetism*, Japan Journal of Applied Mathematics, **1** (1985), 69–84.

Received October 2010; revised February 2011.

E-mail address: sdubey@iitm.ac.in
E-mail address: gilles.carbou@math.u-bordeaux1.fr
E-mail address: stephane.labbe@imag.fr
E-mail address: Christophe.Prieur@gipsa-lab.grenoble-inp.fr