

Anti-windup strategy for reset control systems

Sophie Tarbouriech^{1,2,*}, Thomas Loquen³ and Christophe Prieur⁴

¹CNRS, LAAS, 7 Avenue du Colonel Roche F-31077, Toulouse Cedex 4, France

²Université de Toulouse, UPS, INSA, INP, ISAE, LAAS, F-31077, Toulouse Cedex 4, France

³ONERA, System Control and Flight Dynamics Department, 2 Avenue Edouard Belin,
F-31055 Toulouse Cedex 4, France

⁴Department of Automatic Control, Gipsa-lab, Domaine Universitaire, 961 rue de la Houille Blanche, BP 46,
38402 Grenoble Cedex, France

SUMMARY

This paper proposes an anti-windup strategy to deal with stability and performance requirements for a class of hybrid systems, such as those including a reset controller and subject to input saturation. The computation of the anti-windup compensator aiming at ensuring both \mathcal{L}_2 input-to-state stability and internal stability of the closed-loop system is carried out from the solution of matrix inequalities. Depending on the way chosen to describe reset rules, the conditions for designing the anti-windup compensator are expressed through nonlinear matrix inequalities or linear matrix inequalities. Some optimization criteria in both cases are considered for the synthesis purpose: maximization of the \mathcal{L}_2 -norm upper bound on the admissible disturbances for which the trajectories are assured to be bounded; minimization of the \mathcal{L}_2 -gain of the disturbance to the system-to-be-controlled output; and the maximization of an estimate of the domain of attraction of the origin. Copyright © 2011 John Wiley & Sons, Ltd.

Received 21 May 2010; Revised 18 March 2011; Accepted 22 March 2011

KEY WORDS: reset control systems; saturation; stability regions; anti-windup technique

1. INTRODUCTION

The anti-windup strategy is a well-known and efficient technique to cope with undesirable effects (on both performance and stability) induced by actuator saturation in control loops. The design of anti-windup compensators were first motivated by dealing with the degradation of the transient performance induced by saturation in feedback control systems containing integral actions [1]. More recently, the anti-windup compensators design through formal and systematic methods has emerged (see, for example, [2–6], for a large recent overview). In particular, these works present approaches based on LMI conditions (or quasi-linear matrix equality (LMI), i.e. non-convex condition but with simple nonlinearity). The advantage of the LMI-based methods lies on the fact that the anti-windup design can be carried out through convex optimization problems, allowing to address different optimal synthesis criteria (such as \mathcal{L}_2 -gain attenuation or enlargement of the basin of attraction) in an optimal way.

Moreover, in the quest for providing more flexible stabilizing tools, research efforts have focused on developing control algorithms, and using controllers that involve switching, or on-line adaptation. Adding some hybrid loops may be fruitful in comparison with linear controllers. See e.g. [7, 8] for some academic examples, and consider [9–12] for specific control applications where hybrid

*Correspondence to: Sophie Tarbouriech, CNRS, LAAS; 7 Avenue du Colonel Roche F-31077, Toulouse Cedex 4, France.

†E-mail: tarbour@laas.fr

controllers are used. Even if the analysis of the stability may become complicated, the performance improvement of hybrid strategies with respect to more classical controllers (i.e. regular linear or nonlinear controllers) is now proved (see, e.g. [13–18], for different use of hybrid controllers). However, such a class of feedback laws may introduce—possibly unrealistic—large transition input values. Indeed, many physical control systems are subject to magnitude-limitation in the input. A first work studying the stability issue of a nonlinear systems resulting from a FORE and a saturation in the input is proposed in [19].

In this paper, a single input single output (SISO) linear system controlled by a reset controller and subject to saturation is considered. The considered hybrid controller is that one that combines a continuous linear-based dynamic (represented by differential equations) with a reset dynamic (instantaneous jumps to zero of variables). Continuous trajectories are active in a subset of the state space, called the flow set. Resets of whole or part of system states correspond to an instantaneous jump and are only active in the subset of the state space, called jump set. The current paper proposes some extensions of previous works (in particular [7, 14]) to synthesize a dynamic anti-windup compensator. This paper proposes in particular an anti-windup strategy to deal with stability and performance requirements for a class of hybrid systems, such as those including a reset controller and subject to input saturation. At this aim, different types of reset rule and therefore different flow and jump sets are considered. In both cases, constructive conditions leading to quasi-LMIs or LMIs, based on the use of suitable Lyapunov functions, modified sector condition and the application of adequate congruence transformations, are proposed to compute anti-windup compensators. The computation of the anti-windup compensator aiming at ensuring both \mathcal{L}_2 input-to-state stability and internal stability of the closed-loop system is then carried out from the solution of convex optimization problems, including the maximization of the \mathcal{L}_2 -norm upper bound on the admissible disturbances for which the trajectories are bounded; the minimization of the \mathcal{L}_2 -gain of the disturbance to the system regulated output; and the maximization of an estimate of the domain of attraction of the origin.

This paper is organized as follows. The problem to be treated is formally stated in Section 2. In Section 3, theoretical results on the anti-windup synthesis are developed. Two reset conditions are considered. Section 4 addresses the computational and numerical issues regarding the theoretical conditions previously developed. Convex optimization schemes and an illustrative example are also presented. Finally, Section 5 contains some concluding remarks and presents forthcoming issues.

Notation

For two vectors x, y of \mathbb{R}^n , the notation $x \geq y$ means that $x_{(i)} - y_{(i)} \geq 0, i = 1, \dots, n$. For any vector $x \in \mathbb{R}^n$, $|x|$ is its absolute value (i.e. $|x|_{(i)} = |x_{(i)}|, i = 1, \dots, m$) and $\|x\|$ is its Euclidian norm. The identity matrix of order n is denoted I_n . The null $m \times n$ matrix is denoted $0_{m \times n}$. When no confusion is possible, identity and null matrices are simply denoted as I and 0 , respectively. For two symmetric matrices, A and B , $A > B$ (respectively $A \geq B$) means that $A - B$ is positive definite (respectively positive semi-definite). A' denotes the transpose of A . $\text{diag}(A, B)$ denotes the diagonal matrix obtained from matrices A and B . Furthermore, in the case of partitioned symmetric matrices, the symbol \star denotes symmetric blocks. The convex hull defined by the vertices $v_i \in \mathbb{R}^r$ are denoted as $\text{Co}\{v_i, i = 1, \dots, r\}$.

2. PROBLEM STATEMENT

Consider the following SISO plant:

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u + B_w w \\ y &= C_p x_p \\ z &= C_z x_p + D_z u \end{aligned} \quad (1)$$

where $x_p \in \mathbb{R}^{n_p}$ is the state vector, $y \in \mathbb{R}$ is the measured output of the plant, $u \in \mathbb{R}$ is the input of the plant, $w \in \mathbb{R}^{n_w}$ is the disturbance vector and $z \in \mathbb{R}^{n_z}$ is the regulated output for performance

purpose. In (1), A_p , B_p , C_p , C_z and D_z are constant matrices of appropriate dimensions. Pairs (A_p, B_p) and (C_p, A_p) are assumed to be controllable and observable, respectively. Furthermore, the disturbance vector w is assumed to be limited in energy, i.e. given a finite positive scalar δ :

$$\|w\|_2^2 = \int_0^\infty w(\tau)'w(\tau)d\tau \leq \frac{1}{\delta} \quad (2)$$

The input of the plant is supposed to be bounded in magnitude by $u_0 > 0$, i.e. $-u_0 \leq u \leq u_0$. In consequence, the control signal to be effectively injected in the system considering the controller output y_c is

$$u = \text{sat}(y_c) \quad (3)$$

The saturation map is the classical function defined as $\text{sat}(y_c) = \text{sign}(y_c) \min(|y_c|, u_0)$, where $u_0 > 0$ denotes the level of saturation.

Associated with system (1), we consider a reset controller that leads to mixed discrete/continuous dynamics. This kind of systems is in the class of hybrid systems. Hence, by using the hybrid framework introduced in [20], the reset controller is described by

$$\left. \begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ y_c &= C_c x_c + D_c y \\ \dot{\tau} &= 1 \end{aligned} \right\} \text{if } (x_p, x_c) \in \mathcal{F}_0 \text{ or } \tau \leq \rho \quad (4)$$

$$\left. \begin{aligned} x_c^+ &= 0 \\ \tau^+ &= 0 \end{aligned} \right\} \text{if } (x_p, x_c) \in \mathcal{J}_0 \text{ and } \tau \geq \rho$$

where $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, x_c is the state value before a jump and x_c^+ after a jump. In (4), A_c , B_c , C_c , D_c are constant matrices of appropriate dimensions, and constitute the so-called linear-based data of the hybrid controller. The variables y and y_c are the controller input and output, respectively. \mathcal{F}_0 and \mathcal{J}_0 correspond to the flow and jump sets. The variable τ corresponds to the so-called 'temporal regularization'. It allows to guarantee a lower bound of the time in between resets, called ρ . The objective to add this positive scalar ρ is to avoid an infinite number of switches, i.e. to avoid Zeno solutions [20].

It is supposed that the controller (4) was computed to stabilize system (1), without taking into account the saturation of the input of the plant: $u = y_c$. In order to cope with such a disregarding, an anti-windup loop is then considered to minimize the effect of the saturation on the closed-loop performance and stability. At this aim, the following anti-windup compensator is considered:

$$\begin{aligned} \dot{x}_a &= A_a x_a + B_a (\text{sat}(y_c) - y_c) \\ y_a &= C_a x_a + D_a (\text{sat}(y_c) - y_c) \end{aligned} \quad (5)$$

leading to the modified controller:

$$\left. \begin{aligned} \dot{x}_c &= A_c x_c + B_c y + y_a \\ y_c &= C_c x_c + D_c y \\ \dot{\tau} &= 1 \end{aligned} \right\} \text{if } (x_p, x_c, x_a) \in \mathcal{F} \text{ or } \tau \leq \rho \quad (6)$$

$$\left. \begin{aligned} x_c^+ &= 0 \\ \tau^+ &= 0 \end{aligned} \right\} \text{if } (x_p, x_c, x_a) \in \mathcal{J} \text{ and } \tau \geq \rho$$

where $x_a \in \mathbb{R}^{n_a}$, $(\text{sat}(y_c) - y_c) \in \mathbb{R}$, $y_a \in \mathbb{R}^{n_c}$ are the state, the input and the output vectors of the anti-windup compensator. We recall some basic ingredients on hybrid system theory, and the notion of solutions of (1) in closed loop with (5) and (6) (see [15, 20, 21] and references therein). Owing

to mixed discrete/continuous dynamics, a solution of (1), (5) and (6) will be defined on a mixed discrete/continuous time domain. More precisely:

Definition 1

A set \mathcal{D} is a *hybrid time domain* if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, \dots, J\})$ is a *compact hybrid time domain*, i.e. it can be written as

$$\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq \dots \leq t_J$. A solution to (1), (5) and (6) consists of a hybrid time domain \mathcal{D} and functions $(x_p, x_c, x_a, \tau): \mathcal{D} \rightarrow \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_a} \times \mathbb{R}$ such that $(x_p, x_c, x_a, \tau)(t, j)$ is absolutely continuous in t for a fixed j and $(t, j) \in \mathcal{D}$ satisfying

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \mathcal{D}$, we have

$$(x_p, x_c, x_a)(t, j) \in \mathcal{F} \quad \text{or} \quad \tau(t, j) \leq \rho$$

and

$$\begin{aligned} \dot{x}_p(t, j) &= A_p x_p(t, j) + B_p u(t, j) + B_w w(t) \\ \dot{x}_c(t, j) &= A_c x_c(t, j) + B_c y(t, j) + y_a(t, j) \\ \dot{x}_a(t, j) &= A_a x_a(t, j) + B_a (\text{sat}(y_c(t, j)) - y_c(t, j)) \\ \dot{\tau}(t, j) &= 1. \end{aligned}$$

(S2) For all $(t, j) \in \mathcal{D}$ such that $(t, j+1) \in \mathcal{D}$, we have

$$(x_p, x_c, x_a)(t, j) \in \mathcal{J} \quad \text{and} \quad \tau(t, j) \geq \rho$$

and

$$\begin{aligned} x_p(t, j+1) &= x_p(t, j) \\ x_c(t, j+1) &= 0 \\ x_a(t, j+1) &= x_a(t, j) \\ \tau(t, j+1) &= 0 \end{aligned}$$

This definition implies in particular that the state solution (x_p, x_c, x_a, τ) is parameterized by (t, j) where t is the ordinary time and j is an independent variable that corresponds to the number of jumps of the solution. When the state belongs to the intersection of the flow set and of the jump set, then the solution can either flow or jump. Throughout this paper, we will always use the notation \dot{x}_p for $\dot{x}_p(t, j)$ and x_p^+ for $x_p(t_{j+1}, j+1)$ (and similarly for the other components of the state).

This paper is devoted to the following problem.

Problem 1

Given flow and jump sets \mathcal{F} and \mathcal{J} .

1. When $w = 0$, determine the anti-windup compensator, i.e. matrices A_a, B_a, C_a and D_a , and a region \mathcal{E} , as large as possible, in which the asymptotic stability of the closed-loop system (1), (5) and (6) is guaranteed, i.e. \mathcal{E} is included in the basin of attraction of the origin of closed-loop system.
2. When $w \neq 0$, the \mathcal{L}_2 -gain from w to z , in the case of null initial condition, for all disturbances w satisfying (2) is such that: $\|z\|_2^2 \leq \gamma \|w\|_2^2$.

The implicit objective in the first item of Problem 1 is to maximize the size of the basin of attraction for the closed-loop system (1), (5) and (6) (with $w=0$) over the choice of matrices A_a , B_a , C_a and D_a . Such an objective can be accomplished indirectly by searching for an anti-windup compensator defined from A_a , B_a , C_a and D_a that leads to a region of stability \mathcal{E} for the closed-loop system as large as possible, in particular in the x_p or (x_p, x_c) directions. On the other hand, when the open-loop system is asymptotically stable, it can be possible to search for the controller matrices in order to ensure the global asymptotic stability of the origin of the closed-loop system.

Moreover, Problem 1 will be addressed by considering different types of reset rule and therefore different flow and jump sets. Hence, we will study what is the influence of the reset rule in the solution of Problem 1.

3. THEORETICAL ANTI-WINDUP DESIGN CONDITIONS

3.1. Preliminary results

Let us define the augmented state vector

$$x = \begin{bmatrix} x_p \\ x_c \\ x_a \end{bmatrix} \in \mathbb{R}^n \tag{7}$$

with $n = n_p + n_c + n_a$ and the following deadzone nonlinearity:

$$\phi(Kx) = \text{sat}(Kx) - Kx, \quad K = [D_c C_p \quad C_c | 0] = [K_1 | 0] \in \mathbb{R}^{1 \times n} \tag{8}$$

Hence, by combining (1), (5), (6), (7) and (8), the closed-loop system reads:

$$\left. \begin{aligned} \dot{x} &= A_F x + B_F \phi(Kx) + B_2 w \\ y_c &= Kx \\ y &= Cx \\ z &= C_2 x + D_z \phi(Kx) \\ \dot{\tau} &= 1 \end{aligned} \right\} \text{if } x \in \mathcal{F} \text{ or } \tau \leq \rho \tag{9}$$

$$\left. \begin{aligned} x^+ &= A_J x \\ \tau^+ &= 0 \end{aligned} \right\} \text{if } x \in \mathcal{J} \text{ and } \tau \geq \rho$$

with

$$A_F = \left[\begin{array}{cc|c} A_p + B_p D_c C_p & B_p C_c & 0 \\ B_c C_p & A_c & C_a \\ \hline 0 & 0 & A_a \end{array} \right] = \left[\begin{array}{c|c} A & R C_a \\ \hline 0 & A_a \end{array} \right], \quad B_F = \left[\begin{array}{c} B_p \\ D_a \\ \hline B_a \end{array} \right] = \left[\begin{array}{c} B + R D_a \\ \hline B_a \end{array} \right] \tag{10}$$

$$B_2 = \left[\begin{array}{c} B_w \\ 0 \\ \hline 0 \end{array} \right] = \left[\begin{array}{c} \mathbb{B}_w \\ \hline 0 \end{array} \right], \quad B = \left[\begin{array}{c} B_p \\ \hline 0 \end{array} \right], \quad R = \left[\begin{array}{c} 0 \\ \hline I_{n_c} \end{array} \right], \quad C = [C_p \quad 0 | 0] = [C_1 | 0]$$

$$C_2 = [C_z + D_z D_c C_p \quad D_z C_c | 0] = [C_z | 0]$$

Matrix A_J , the flow and jump sets \mathcal{F} and \mathcal{J} will be precisely defined from several ways in the sequel. System (9) clearly lies in the set of hybrid and nonlinear control systems. To ensure that the solutions are in the flow set after a jump (see [20]), we adopt the following assumption (see [14]).

Assumption 1

For system (9), the reset matrix A_J is such that $x \in \mathcal{J} \implies x^+ \in \mathcal{F}$.

3.2. Flow and jump sets case 1

We consider the case where the flow and jump sets are defined from the classical reset rule depending on the sign of the product between y (output of the plant) and y_c (output of the controller) (see, for example, [18, 19]). Hence, the sets \mathcal{F} and \mathcal{J} are defined by:

$$\begin{aligned} \mathcal{F} &= \{x \in \mathbb{R}^n; x' M x \geq 0\} \\ \mathcal{J} &= \{x \in \mathbb{R}^n; x' M x \leq 0\} \end{aligned} \tag{11}$$

In (11), matrix M is a reset matrix of appropriate dimensions satisfying $M = Q' \mathcal{M} Q$ with:

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \mathbb{R}^{2 \times 2} \\ Q &= \begin{bmatrix} C_p & 0 & 0 \\ D_c C_p & C_c & 0 \end{bmatrix} = \begin{bmatrix} C \\ K \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ K_1 & 0 \end{bmatrix} = [Q_1 \ 0] \in \mathbb{R}^{2 \times n} \end{aligned} \tag{12}$$

In the current case, two possible structures for the desired matrix $A_J \in \mathbb{R}^{n \times n}$ satisfying Assumption 1 can be considered:

$$A_J = \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{J0} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A_J = \begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_a} \end{bmatrix} = \begin{bmatrix} A_{J0} & 0 \\ 0 & I_{n_a} \end{bmatrix} \tag{13}$$

This means that both the state of the controller and of the anti-windup compensator are reset (first structure) or only the state of the controller is reset (second structure).

Let us first suppose that the anti-windup compensator (5) is given (that is matrices A_a, B_a, C_a, D_a are known). The following general result can be stated in the regional (local) context.

Proposition 2

If there exist two symmetric positive-definite matrices $W \in \mathbb{R}^{n \times n}$, $M_1 \in \mathbb{R}^{2 \times 2}$, a matrix $Z \in \mathbb{R}^{1 \times n}$ and positive scalars $S, \delta, \gamma, \tau_F, \tau_R$ satisfying

$$\begin{bmatrix} WA'_F + A_F W & B_F S - Z' & B_2 & WC'_2 & \tau_F W Q' \\ \star & -2S & 0 & SD'_z & 0 \\ \star & \star & -I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -\tau_F M_1^{-1} \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} W & WK' - Z' \\ \star & \delta u_0^2 \end{bmatrix} \geq 0 \tag{15}$$

$$\begin{bmatrix} -W & WA'_J & \tau_R W Q' \\ \star & -W & 0 \\ \star & \star & -\tau_R M_1^{-1} \end{bmatrix} \leq 0 \tag{16}$$

$$-M_1 \leq \mathcal{M} \leq M_1 \tag{17}$$

then:

1. When $w=0$, the nonlinear closed-loop system (9), with A_J as in (13), remains stable for any initial condition belonging to $\mathcal{E}(P, \delta)$, with

$$\mathcal{E}(P, \delta) = \{x \in \mathbb{R}^n; x'Px \leq \delta^{-1}\}, \quad P = W^{-1} \tag{18}$$

2. When $w \neq 0$, for $x(0)=0$,

- the closed-loop trajectories remain bounded in $\mathcal{E}(P, \delta)$,
- the map from z to w is finite gain \mathcal{L}_2 stable with

$$\int_0^T z(t)'z(t) dt \leq \gamma \int_0^T w(t)'w(t) dt \quad \forall T \geq 0 \tag{19}$$

Proof

Consider the quadratic Lyapunov function $V(x) = x'Px$, with $P = P' = W^{-1} > 0$ and the following change of variables: $GW = Z$. The satisfaction of relation (15) guarantees that the ellipsoid $\mathcal{E}(P, \delta)$ is included in the polyhedral set $S(u_0) = \{x \in \mathbb{R}^n; -u_0 \leq (K - G)x \leq u_0\}$. By using Lemma 1 in [22], we can verify that $-\phi(Kx)S^{-1}(\phi(Kx) + Gx) \geq 0$, with S a positive scalar, provided that $x \in S(u_0)$. Thus, for any $x \in \mathcal{E}(P, \delta)$ one gets: $\dot{V}(x) + (1/\gamma)z'z - w'w \leq \dot{V}(x) + (1/\gamma)z'z - w'w - 2\phi(Kx)S^{-1}(\phi(Kx) + Gx)$.

By developing the right-hand term of this inequality one obtains:

$$\begin{aligned} & \begin{bmatrix} W^{-1}x \\ S^{-1}\phi(Kx) \\ w \end{bmatrix}' \left\{ \begin{bmatrix} WA'_F + A_F W & B_F S - Z' & B_2 \\ \star & -2S & 0 \\ \star & \star & -I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} WC'_2 \\ SD'_z \\ 0 \end{bmatrix} \begin{bmatrix} C_2 W & D_z S & 0 \end{bmatrix} \right\} \\ & \times \begin{bmatrix} W^{-1}x \\ S^{-1}\phi(Kx) \\ w \end{bmatrix} \end{aligned}$$

By using the Schur complement it follows:

$$\mathcal{L} = \begin{bmatrix} WA'_F + A_F W & B_F S - Z' & B_2 & WC'_2 \\ \star & -2S & 0 & SD'_z \\ \star & \star & -I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} \tag{20}$$

Let us check that $\mathcal{L} < 0$ when the system (9) is flowing. Let \tilde{t} be a time of flow. Two cases may occur: $x(\tilde{t}) \in \mathcal{F}$ or $\tau \leq \rho$.

Case 1. If $x(\tilde{t}) \in \mathcal{F}$ then $\dot{V}(x) + (1/\gamma)z'z - w'w < 0$ as soon as $\mathcal{L} < 0$ for all $x \in \mathcal{F}$. Hence, by using the S-procedure, this is equivalent to satisfy:

$$\mathcal{L}_0 = \begin{bmatrix} WA'_F + A_F W + \tau_F W Q' \mathcal{M} Q W & B_F S - Z' & B_2 & WC'_2 \\ \star & -2S & 0 & SD'_z \\ \star & \star & -I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0$$

Hence, from relation (17), matrix \mathcal{L}_0 can be over-bounded as follows:

$$\mathcal{L}_0 \leq \begin{bmatrix} WA'_F + A_F W + \tau_F W Q' M_1 Q W & B_F S - Z' & B_2 & WC'_2 \\ \star & -2S & 0 & SD'_z \\ \star & \star & -I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix}$$

By using the Schur complement, the right-hand term of this inequality reads:

$$\mathcal{L}_1 = \begin{bmatrix} WA'_F + A_F W & B_F S - Z' & B_2 & WC'_2 & \tau_F W Q' \\ \star & -2S & 0 & SD'_z & 0 \\ \star & \star & -I & 0 & 0 \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -\tau_F M_1^{-1} \end{bmatrix}$$

If one ensures $\mathcal{L}_1 < 0$, which holds whether (14) is verified, then it follows that $\dot{V}(x) + (1/\gamma)z'z - w'w < 0$.

Case 2. Suppose that one gets $\tau \leq \rho$. Consider $\underline{t} \leq \tilde{t}$ the smallest time where we are flowing in $[\underline{t}, \tilde{t}]$. We have $x(\underline{t}) \in \mathcal{F}$ since either $\underline{t} = 0$, or we have a jump at \underline{t} (and due to Assumption 1 for all $x \in \mathcal{J}$, we have $A_J x \in \mathcal{F}$). Then, there exists a positive scalar η such that, for all $t \in [\underline{t}, \rho]$, we have $x(t) \in \mathcal{F}_\eta$ with $\mathcal{F}_\eta = \{x \in \mathbb{R}^n; x' M x + \eta x' x \geq 0\}$. Thus, we remain in $\mathcal{E}(P, \delta)$ and we have $\dot{V}(x(t)) + (1/\gamma)z'z - w'w < 0$ for all $t \in [\underline{t}, \rho]$ as soon as $\mathcal{L} < 0$ (with \mathcal{L} defined in (20)) for all $x \in \mathcal{F}_\eta$, which is equivalent to verify:

$$\begin{bmatrix} WA'_F + A_F W + \tau_F W(Q' M Q + \eta I_n)W & B_F S - Z' & B_2 & WC'_2 \\ \star & -2S & 0 & SD'_z \\ \star & \star & -I & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} < 0 \tag{21}$$

Moreover, note that $\eta \rightarrow 0$ as $\rho \rightarrow 0$. Therefore, by considering a small enough value for ρ , we can consider small enough η to be neglected and it follows that the satisfaction of relation (14) guarantees that inequality (21) holds.

Moreover, we want also to prove that $V(x^+) - V(x) \leq 0$ for $x \in \mathcal{J}$. This condition holds if the following relation is satisfied:

$$x'(A'_J P A_J - P)x - \tau_R x' M x = x'(A'_J P A_J - P)x - \tau_R x' Q' M Q x \leq 0 \tag{22}$$

which reads, by using the Schur complement upon pre- and post-multiplying (22) by W :

$$\begin{bmatrix} -W - \tau_R W Q' M Q W & WA'_J \\ \star & -W \end{bmatrix} \leq 0 \tag{23}$$

From relation (17) one can over-bound the term $-\tau_R W Q' M Q W$ by $\tau_R W Q' M_1 Q W$. Thus, by using the Schur complement, condition (23) can be rewritten as:

$$\begin{bmatrix} -W & WA'_J & \tau_R W Q' \\ \star & -W & 0 \\ \star & \star & -\tau_R M_1^{-1} \end{bmatrix} \leq 0 \tag{24}$$

Hence, the satisfaction of relations (14), (15), (16) and (17) ensures that along the trajectories of the hybrid system (9) the Lyapunov function satisfies

$$\begin{aligned} \dot{V}(x(t)) + \frac{1}{\gamma} z'z - w'w < 0 \quad \text{if } x \in \mathcal{F} \\ \Delta V(x) \leq 0 \quad \text{if } x \in \mathcal{J} \end{aligned}$$

for all w satisfying (2). Note that since the flow set \mathcal{F} and the jump set \mathcal{J} are closed (see (11)), and with the regularity on the data of (9) we get, with [20, Theorem S3], that each maximal solution is either complete (i.e. with a unbounded time domain) or blows up in finite time. But with the previous estimation on the time-evolution of V , the solutions cannot blow up in finite time and thus the solutions of (9) are defined for all time. Moreover, it follows that:

- for $x(0)=0$, one gets $V(0)=0$ and $V(T)=x(T)'Px(T) \leq \int_0^T w(t)'w(t)dt \leq \delta^{-1}$, $\forall T \geq 0$. In other words, the trajectories of the closed-loop system (9) do not leave the set $\mathcal{E}(P, \delta)$.
- When $T \rightarrow \infty$, one gets $\int_0^\infty z(t)'z(t)dt \leq \gamma \int_0^\infty w(t)'w(t)dt$, therefore, the \mathcal{L}_2 -gain of the system is less than γ .
- If $w=0$ then one obtains $\dot{V}(x) < -(1/\gamma)x'Px < 0$. The nonlinear closed-loop system (9) with A_J defined from (13) remains stable for any initial condition belonging to $\mathcal{E}(P, \delta)$.

This completes the proof of Proposition 2. □

Let us focus on the synthesis issue as stated in Problem 1. It is clear that it is not possible to determine simultaneously the analysis variable W and the anti-windup matrices A_a, B_a, C_a and D_a from the conditions of Proposition 2. By considering the case where $n_a = n_p + n_c$, these conditions can be simplified. In particular, different frameworks can be adapted in order to obtain quasi-convex or convex conditions in the case $n_a = n_p + n_c$: one can, for example, cite the framework used in [2, 23], based on the projection lemma, or that one based on some change of variables [24, 25]. For the ease of the presentation, we choose the above-mentioned second framework.

The following theorem exhibits solution in the sense to design the anti-windup compensator satisfying Proposition 2.

Theorem 3

If there exist three symmetric positive-definite matrices $X \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $Y \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $M_1 \in \mathbb{R}^{2 \times 2}$, matrices $Z_1 \in \mathbb{R}^{1 \times (n_p+n_c)}$, $\mathcal{Z} \in \mathbb{R}^{1 \times (n_p+n_c)}$, $\Omega_1 \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $\Omega_2 \in \mathbb{R}^{(n_p+n_c) \times 1}$, $\Omega_3 \in \mathbb{R}^{n_c \times (n_p+n_c)}$, $\Omega_4 \in \mathbb{R}^{n_c \times 1}$, positive scalars $S, \tau_F, \tau_R, \delta, \gamma$ such that relation (17) and the following conditions hold:

$$\begin{bmatrix} XA' + AX & XA' + AY + R\Omega_3 + \Omega_1 & BS + R\Omega_4 + \Omega_2 - \mathcal{Z}' & \mathbb{B}_w & XC'_z & \tau_F X Q'_1 \\ \star & AY + YA' + R\Omega_3 + \Omega_3 R' & BS + R\Omega_4 - Z'_1 & \mathbb{B}_w & YC'_z & \tau_F Y Q'_1 \\ \star & \star & -2S & 0 & SD'_z & 0 \\ \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -\tau_F M_1^{-1} \end{bmatrix} < 0 \quad (25)$$

$$\begin{bmatrix} X & X & XK'_1 - \mathcal{Z}' \\ \star & Y & YK'_1 - Z'_1 \\ \star & \star & \delta u_0^2 \end{bmatrix} \geq 0 \quad (26)$$

$$\begin{bmatrix} -X & -X & XA'_{J0} & XA'_{J0} & \tau_R X Q'_1 \\ \star & -Y & YA'_{J0} & YA'_{J0} & \tau_R Y Q'_1 \\ \star & \star & -X & -X & 0 \\ \star & \star & \star & -Y & 0 \\ \star & \star & \star & \star & -\tau_R M_1^{-1} \end{bmatrix} \leq 0 \quad (27)$$

then the anti-windup controller defined by

$$\begin{aligned} A_a &= (\mathcal{U}')^{-1} X^{-1} \Omega_1 N^{-1} \\ B_a &= (\mathcal{U}')^{-1} X^{-1} \Omega_2 S^{-1} \\ C_a &= \Omega_3 N^{-1} \\ D_a &= \Omega_4 S^{-1} \end{aligned} \quad (28)$$

where matrices \mathcal{U} , N verify $N'\mathcal{U} = I - YX^{-1}$, is solution to Problem 1, that is,

1. When $w=0$, the nonlinear closed-loop system (9), with A_J defined as the first structure in (13), remains stable for any initial condition belonging to the set $\mathcal{E}(P, \delta)$, defined as in (18) with

$$P = W^{-1} = \begin{bmatrix} X^{-1} & \mathcal{U}' \\ \star & F \end{bmatrix}, \quad W = \begin{bmatrix} Y & N' \\ \star & H \end{bmatrix} \quad (29)$$

2. When $w \neq 0$, for $x(0)=0$,

- the closed-loop trajectories remain bounded in $\mathcal{E}(P, \delta)$,
- the map from z to w is finite gain \mathcal{L}_2 stable and satisfies (19).

Proof

Let us define:

$$W = \begin{bmatrix} Y & N' \\ \star & H \end{bmatrix} \quad \text{and} \quad P = W^{-1} = \begin{bmatrix} X^{-1} & \mathcal{U}' \\ \star & F \end{bmatrix}$$

where X , Y , N , \mathcal{U} , F and H are decision variables of appropriate dimensions. Then, it follows: $YX^{-1} + N'\mathcal{U} = I$, $Y\mathcal{U}' + N'F = 0$, $NX^{-1} + H\mathcal{U} = 0$ and $N\mathcal{U}' + HF = I$. Similarly to [25], we use the matrix

$$\Gamma = \begin{bmatrix} I & I \\ \mathcal{U}X & 0 \end{bmatrix}.$$

From this, one obtains:

$$\begin{aligned} \Gamma'W &= \begin{bmatrix} X & 0 \\ Y & N' \end{bmatrix}, \quad \Gamma'W\Gamma = \begin{bmatrix} X & X \\ X & Y \end{bmatrix} \\ \Gamma'WA'_F\Gamma &= \begin{bmatrix} XA' & XA' \\ YA' + N'C'_aR' + N'A'_a\mathcal{U}X & YA' + N'C'_aR' \end{bmatrix} \\ \Gamma'(B_FS - Z') &= \begin{bmatrix} BS + RD_aS + X\mathcal{U}'B_aS - (Z'_1 + X\mathcal{U}'Z'_2) \\ BS + RD_aS - Z'_1 \end{bmatrix} \\ \Gamma'B_2 &= \begin{bmatrix} \mathbb{B}_w \\ \mathbb{B}_w \end{bmatrix}, \quad \Gamma'WC'_2 = \begin{bmatrix} XC'_z \\ YC'_z \end{bmatrix}, \quad \Gamma'WQ' = \begin{bmatrix} XQ'_1 \\ YQ'_1 \end{bmatrix} \end{aligned}$$

where the variable Z can be partitioned as $Z = [Z_1 \ Z_2] \in \mathbb{R}^{1 \times n}$, with $Z_1 \in \mathbb{R}^{1 \times (n_p + n_c)}$ and $Z_2 \in \mathbb{R}^{1 \times (n_p + n_c)}$ (since $n_a = n_p + n_c$).

Hence, relation (25) is obtained by pre- and post-multiplying relation (14) by $\text{diag}(\Gamma', I, I, I, I)$ and by $\text{diag}(\Gamma, I, I, I, I)$, respectively and by using the following change of variables:

$$\Omega_1 = X\mathcal{U}'A_aN, \quad \Omega_2 = X\mathcal{U}'B_aS, \quad \Omega_3 = C_aN, \quad \Omega_4 = D_aS, \quad \mathcal{L} = Z_1 + Z_2\mathcal{U}X \quad (30)$$

By the same way, relation (26) is obtained by pre- and post-multiplying relation (15) by $\text{diag}(\Gamma', I)$ and by $\text{diag}(\Gamma, I)$ and by using the change of variables (30).

Finally, relation (27) is obtained by pre- and post-multiplying relation (16) by $\text{diag}(\Gamma', I, I)$ and by $\text{diag}(\Gamma, I, I)$ and by considering the first structure of matrix A_J given in (13). Then, it follows:

$$\Gamma' W A'_J \Gamma = \begin{bmatrix} X A'_{J0} & X A'_{J0} \\ Y A'_{J0} & Y A'_{J0} \end{bmatrix}$$

Similar arguments regarding flow and jump sets are used as in the proof of Proposition 2. \square

Remark 4

If the second structure for matrix A_J given in (13) is used, relation (27) in Theorem 3 is slightly modified: the block (2,3) is changed as $Y A'_{J0} + X - Y$.

Remark 5

When the plant is asymptotically stable (i.e. matrix A_p is Hurwitz), it is possible to seek the assurance of the global stability of the closed-loop system. The closed-loop system is finite gain \mathcal{L}_2 stable for any admissible w satisfying (2) and any initial condition $x_0 \in \mathbb{R}^n$, i.e. the origin of the system is globally asymptotically stable and one obtains:

$$\int_0^T z(t)' z(t) dt \leq \gamma \int_0^T w(t)' w(t) dt + \gamma x_0' P x_0 \quad \forall T \geq 0 \quad \forall x_0 \in \mathbb{R}^n \tag{31}$$

Hence, the global asymptotic stability can be addressed by considering $Z = K W$ in Proposition 2. In this case, one chooses $\mathcal{L} = Z_1 = K_1 W$ and therefore one replaces condition (26) by $Y - X > 0$ in Theorem 3.

3.3. Flow and jump sets case 2

We consider the case where the flow and jump sets are defined from the region of stability $\mathcal{E}(P, \delta) = \{x \in \mathbb{R}^n; x' P x \leq \delta^{-1}\}$ with $P = P' > 0, P \in \mathbb{R}^{n \times n}$. By using similar tools as in Theorem 3, the following theorem provides convex conditions to exhibit both the suitable anti-windup matrices A_a, B_a, C_a, D_a and the reset matrix A_J .

Let us first define the structure of the desired matrix A_J :

$$A_J = \begin{bmatrix} A_{J0} & 0 \\ 0 & 0 \end{bmatrix} + E_r \Theta F_r \tag{32}$$

with

$$A_{J0} = \begin{bmatrix} I_{n_p} & 0 \\ 0 & 0 \end{bmatrix}, \quad E_r = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad F_r = \begin{bmatrix} C_p & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ R' & 0 \\ 0 & I \end{bmatrix} \tag{33}$$

$$\Theta = \begin{bmatrix} A_{r1} & B_{r1} & C_{r1} \\ A_{r2} & B_{r2} & C_{r2} \end{bmatrix} = [A_r \ B_r \ C_r]$$

In the proposed structure (33), matrices $A_r \in \mathbb{R}^{(n_p+2n_c) \times 1}, B_r \in \mathbb{R}^{(n_p+2n_c) \times n_c}, C_r \in \mathbb{R}^{(n_p+2n_c) \times (n_p+n_c)}$ are decision variables, which correspond to the jump equations:

$$\begin{aligned} x_p^+ &= x_p \\ x_c^+ &= A_{r1} C_p x_p + B_{r1} x_c + C_{r1} x_a \\ x_a^+ &= A_{r2} C_p x_p + B_{r1} x_c + C_{r1} x_a \end{aligned} \tag{34}$$

Furthermore, the flow and jump sets are defined as

$$\begin{aligned} \mathcal{F} &= \mathcal{E}(P, \delta) \\ \mathcal{J} &= \text{clos}(\mathbb{R}^n \setminus \mathcal{E}(P, \delta)) \end{aligned} \tag{35}$$

where $\text{clos}(\cdot)$ stands for the closure.

Theorem 6

If there exist four symmetric positive-definite matrices $X \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $Y \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $P_{11} \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $P_{13} \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, matrices $Z_1 \in \mathbb{R}^{1 \times (n_p+n_c)}$, $\mathcal{L} \in \mathbb{R}^{1 \times (n_p+n_c)}$, $P_{12} \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $\Omega_1 \in \mathbb{R}^{(n_p+n_c) \times (n_p+n_c)}$, $\Omega_2 \in \mathbb{R}^{(n_p+n_c) \times 1}$, $\Omega_3 \in \mathbb{R}^{n_c \times (n_p+n_c)}$, $\Omega_4 \in \mathbb{R}^{n_c \times 1}$, $\Omega_5 \in \mathbb{R}^{(2n_c+n_p) \times 1}$, $\Omega_6 \in \mathbb{R}^{(2n_c+n_p) \times n_c}$, $\Omega_7 \in \mathbb{R}^{(2n_c+n_p) \times (n_c+n_p)}$, positive scalars S, δ, γ satisfying

$$\begin{bmatrix} XA' + AX & XA' + AY + R\Omega_3 + \Omega_1 & BS + R\Omega_4 + \Omega_2 - \mathcal{L}' & \mathbb{B}_w & XC'_z \\ \star & AY + YA' + R\Omega_3 + \Omega_3'R' & BS + R\Omega_4 - Z'_1 & \mathbb{B}_w & YC'_z \\ \star & \star & -2S & 0 & SD'_z \\ \star & \star & \star & -I & 0 \\ \star & \star & \star & \star & -\gamma I \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} X & X & XK'_1 - \mathcal{L}' \\ \star & Y & YK'_1 - Z'_1 \\ \star & \star & \delta u_0^2 \end{bmatrix} \geq 0 \tag{37}$$

$$\begin{bmatrix} P_{11} & P_{12} & A'_{J0} + C'_1\Omega'_5 + R\Omega'_6 & A'_{J0} \\ \star & P_{13} & \Omega'_7 & 0 \\ \star & \star & X & X \\ \star & \star & \star & Y \end{bmatrix} \geq 0 \tag{38}$$

then the anti-windup controller defined by

$$\begin{aligned} A_a &= (\mathcal{U}')^{-1} X^{-1} \Omega_1 N^{-1} \\ B_a &= (\mathcal{U}')^{-1} X^{-1} \Omega_2 S^{-1} \\ C_a &= \Omega_3 N^{-1} \\ D_a &= \Omega_4 S^{-1} \end{aligned} \tag{39}$$

is solution to Problem 1 with A_J defined from (32) with

$$\begin{aligned} A_r &= (\mathcal{U}')^{-1} X^{-1} \Omega_5 \\ B_r &= (\mathcal{U}')^{-1} X^{-1} \Omega_6 \\ C_r &= (\mathcal{U}')^{-1} X^{-1} \Omega_7 \end{aligned} \tag{40}$$

where matrices \mathcal{U}, N verify $N'\mathcal{U} = I - YX^{-1}$. In other words:

1. When $w=0$, the nonlinear closed-loop system (9) remains stable for any initial condition belonging to the set $\mathcal{E}(P, \delta) \cup \mathcal{E}(P_1, \delta)$, where $\mathcal{E}(P, \delta)$ is defined as in (18) and $\mathcal{E}(P_1, \delta) = \{x \in \mathbb{R}^n; x' P_1 x \leq \delta^{-1}\}$ with

$$P = W^{-1} = \begin{bmatrix} X^{-1} & \mathcal{U}' \\ \star & F \end{bmatrix}, \quad W = \begin{bmatrix} Y & N' \\ \star & H \end{bmatrix}, \quad P_1 = \begin{bmatrix} P_{11} & P_{12} \\ \star & P_{13} \end{bmatrix} \tag{41}$$

2. When $w \neq 0$, for $x(0)=0$,

- the closed-loop trajectories remain bounded in $\mathcal{E}(P, \delta)$,
- the map from z to w is finite gain \mathcal{L}_2 stable and satisfies (19).

Proof

This proof mimicks in part that ones of Proposition 2 and Theorem 3. Similar definitions and changes of variables as in the proof of Theorem 3 are used to derive the conditions in the continuous part (flow case).

Let us focus on the discrete part (jump case) of the proof. We want to verify that $x_0^+ \in \mathcal{E}(P, \delta)$ if $x_0 \in \mathcal{E}(P_1, \delta)$, that is,

$$x_0' A_J' P A_J x_0 \leq \delta^{-1} \quad \text{if } x_0' P_1 x_0 \leq \delta^{-1}$$

By using the S-procedure, it follows that this condition will be satisfied if one gets: $\delta(P_1 - A_J' P A_J) \geq 0$, or equivalently, $P_1 - A_J' P A_J \geq 0$. From the definition of A_J given by (32)–(33) and by applying the Schur complement one gets the following relation:

$$\begin{bmatrix} P_1 & (A_J + E_r \Theta F_r)' \\ \star & W \end{bmatrix} \geq 0 \quad (42)$$

Hence, relation (38) is obtained by pre- and post-multiplying relation (42) by $\text{diag}(I, \Gamma')$ and by $\text{diag}(I, \Gamma)$, respectively, and by using the following change of variables:

$$\Omega_5 = X \mathcal{U}' A_r, \quad \Omega_6 = X \mathcal{U}' B_r, \quad \Omega_7 = X \mathcal{U}' C_r \quad (43)$$

with

$$\Gamma = \begin{bmatrix} I & I \\ \mathcal{U} X & 0 \end{bmatrix}.$$

Remark 7

The reset controller defined from (34) uses the measured output of the plant ($y = C_p x_p$), and the states x_c and x_a of the controller and of the anti-windup compensator. Moreover, the flow and jump sets defining the reset rules in (35) depend on the complete state $x = [x_p' \ x_c' \ x_a']'$ along time. This requires that all system states are measurable. States x_c and x_a can be easily obtained by taking care during controller implementation. However, that is not the case for x_p since by definition of system (1) only $C_p x_p$ is actually measured. Hence, a next step could be, for example, to use an observer and to redefine adequately the flow and jump sets from the observed state \hat{x}_p .

Remark 8

It is important to note that Theorem 6 could easily apply to MIMO systems contrary to Theorem 3. This is mainly due to the way chosen for describing the flow and jump sets.

4. COMPUTATIONAL AND NUMERICAL ISSUES

Theorem 3 presents non-convex conditions: one of the nonlinearities (or non-convexity) is due to the presence of both M_1 and M_1^{-1} in the conditions, $M_1 \in \mathbb{R}^{2 \times 2}$ being a decision variable. Nevertheless, a way to overcome such a problem consists in fixing M_1 as a diagonal positive-definite matrix. Then, a search for the optimal solution over a bi-dimensional grid (composed of the two elements of M_1) can be considered. A sub-case can be to choose $M_1 = m_1 I$ implying to use an iterative line search. An alternative solution consists in using a relaxation scheme, which translates the problem into a sequence of iterative LMI problems fixing M_1 or the other variables at each step. In this case, the convergence of the procedure is always ensured, but not necessarily to the global optimal value. Furthermore, the convergence value will depend on the initialization of M_1

in the iterative procedure. Then, a more adulterated approach based on a linearization algorithm, as detailed in [26], could allow to tackle the presence of both M_1 and M_1^{-1} in the conditions.

The two other nonlinearities are due to the products $\tau_F X$, $\tau_F Y$, $\tau_R X$, $\tau_R Y$. Nevertheless, positive parameters τ_F and τ_R related to the use of the S-procedure may be *a priori* fixed. Although their numerical value may have an impact on the obtained solution, it remains limited and the *a priori* choice does not prevent from obtaining an admissible solution.

At this stage, it is interesting to point out that Theorem 6 provides LMI conditions.

Based on Theorems 3 and 6, several LMI-based optimization problems can be proposed to compute the anti-windup controller optimizing one of the following criteria: maximization of the \mathcal{L}_2 bound on the admissible disturbances (disturbance tolerance maximization that corresponds to minimize δ); the minimization of the induced \mathcal{L}_2 -gain between the disturbance w and the regulated output z (disturbance rejection maximization that corresponds to minimize γ for a given δ) or the maximization of the region where the asymptotic stability of the closed-loop system is ensured (maximization of the region of attraction).

Remark 9

Conditions of Theorem 3 (and Theorem 6) imply the satisfaction $WA'_F + A_F W < 0$. Owing to the structure of matrix A_F , this guarantees the asymptotic stability of the matrix A_a of the anti-windup compensator. In the spirit of [27], it would then be possible to modify conditions of theorems in order to force the poles of the anti-windup controller (i.e. the poles of matrix A_a) to be located in a certain region.

4.1. Maximization of the \mathcal{L}_2 bound

In the case of Theorem 3, the following convex optimization problem can be considered: Fix M_1 for which (17) holds; Choose small enough $\tau_F > 0$ and $\tau_R > 0$; Solve $\min \delta$, subject to relations (25), (26), (27), with respect to the decision variables X , Y , δ , S , Z_1 , \mathcal{L} , Ω_1 , Ω_2 , Ω_3 , Ω_4 , γ . In the case of Theorem 6, we solve $\min \delta$, subject to relations (36), (37), (38), with respect to the decision variables X , Y , P_{11} , P_{12} , P_{13} , δ , S , Z_1 , \mathcal{L} , Ω_1 , Ω_2 , Ω_3 , Ω_4 , Ω_5 , Ω_6 , Ω_7 , γ .

4.2. Minimization of the induced \mathcal{L}_2 -gain

Given δ , we want to solve the following optimization: $\min \gamma$ subject to relations of Theorem 3 or 6.

4.3. Maximization of the region of attraction

At this aim, we consider the disturbance-free case (i.e. $w=0$). Hence, a modified version of Theorem 3 or 6 is considered: in relation (25) or (36), the two lines and columns related to the disturbance and the output z are removed. Different linear optimization criteria $J(\cdot)$, associated with the size of $\mathcal{E}(P, \delta)$, can be considered, like the volume, the size of the minor axis or the maximization of a scaling factor allowing to include a given shape in the ellipsoid [28, 29]. In particular, it is interesting to address this problem in the plant space. For the sake of simplicity, in this case, we set $\delta=1$. Choose a set of interesting directions $v_i \in \mathbb{R}^{n_p}$, $i=1, \dots, q$. Define $\tilde{v}_i = N_a v_i \in \mathbb{R}^{n_p+n_c}$, with $N_a = \begin{bmatrix} I_{n_p} \\ 0 \end{bmatrix}$, one considers the additional constraint $\mu - [\tilde{v}_i' \ 0] P \begin{bmatrix} \tilde{v}_i \\ 0 \end{bmatrix} \geq 0$, $i=1, \dots, q$, or equivalently from the definition of P :

$$\begin{bmatrix} \mu & v_i' N_a' \\ \star & X \end{bmatrix} \geq 0, \quad i=1, \dots, q \quad (44)$$

Thus, we want to solve the following optimization problem: $\min \mu$ subject to (44) and modified relations of Theorems 3 or 6.

4.4. Reduced-order anti-windup compensator

Let us now focus on a special case where the matrices A_a and C_a of the anti-windup compensator are *a priori* fixed. This allows not only to control the order of the anti-windup compensator but

also to reduce the computational efforts. For example, in relation (14) of Proposition 2 the main nonlinearity involving W , A_a and C_a can be removed as soon as the matrices A_a and C_a of the anti-windup controller are fixed. The main difficulty however resides in choosing the matrices A_a and C_a adequately. According to the approach developed in [23], for instance, (see also [30, 31]), this choice may be carried out by considering the poles of the anti-windup controller. These poles can be chosen by selecting a part of poles obtained in the full order design case ($n_a = n_p + n_c$). Typically, the slow and fast dynamics are eliminated. Alternatively, an iterative procedure starting from the static case can be used. The list of poles is then progressively enriched until the gap between the full and reduced order cases becomes small enough. Note then that the order of the controller is now given by $n_a = n_1 + 2n_2$, where n_1 and n_2 correspond, respectively, to real and complex poles and have to be chosen sufficiently small, so that $n_a < n_p + n_c$. The algorithm proposed in [31] is based on a different decomposition of the anti-windup controller using dyadic forms and the associated procedure requires the user to specify output pole directions which may not be trivial.

Another interesting way consists in imposing a particular architecture to the anti-windup compensator, quite different from the anti-windup scheme (5) and copying in part the plant as follows [2, 32, 33]:

$$\begin{aligned} \dot{x}_a &= (A_p + B_p F_a)x_a + B_p(\text{sat}(y_c + y_{a1}) - (y_c + y_{a1})) \\ y_{a1} &= F_a x_a \\ y_{a2} &= C_p x_a \end{aligned} \quad (45)$$

leading to the modified controller:

$$\left. \begin{aligned} \dot{x}_c(t, j) &= A_c x_c(t, j) + B_c(y(t) + y_{a2}) \\ y_c(t, j) &= C_c x_c(t, j) + D_c(y(t) + y_{a2}) \\ \dot{\tau}(t, j) &= 1 \end{aligned} \right\} \text{if } (x_p, x_c, x_a) \in \mathcal{F} \quad \text{or} \quad \tau \leq \rho \quad (46)$$

$$\left. \begin{aligned} x_c(t_{j+1}, j+1) &= 0 \\ \tau(t_{j+1}, j+1) &= 0 \end{aligned} \right\} \text{if } (x_p, x_c, x_a) \in \mathcal{J} \quad \text{and} \quad \tau \geq \rho$$

where $x_a \in \mathbb{R}^{n_p}$, $\text{sat}(y_c + y_{a1}) - (y_c + y_{a1})$, $y_{a2} \in \mathbb{R}^1$, $y_{a1} \in \mathbb{R}^1$ are the state, the input and the outputs of the anti-windup compensator. Using the augmented state vector x , as defined in (7), a similar closed-loop system as described in (9) can be obtained. The anti-windup design then consists in synthesizing the gain F_a such that matrix $A_p + B_p F_a$ is Hurwitz. From this, Proposition 2 can be used. Similar to Section 3.3, we can adapt the conditions in the case where the flow and jump sets are defined from the region of stability.

Remark 10

The static anti-windup case can be considered: by choosing $n_a = 0$, $A_a = 0$, $B_a = 0$, $C_a = 0$ and by computing the gain D_a .

4.5. Numerical examples

We consider the system proposed in [34] characterized by the linear open-loop unstable system:

$$\begin{aligned} \dot{x}_p &= 0.1x_p + u \\ y &= z = x \end{aligned} \quad (47)$$

where u satisfies relation (3) with $u_0 = 1$ and the stabilizing PI controller

$$\begin{aligned} \dot{x}_c &= -0.2y \\ y_c &= x_c - 2y \end{aligned} \quad (48)$$

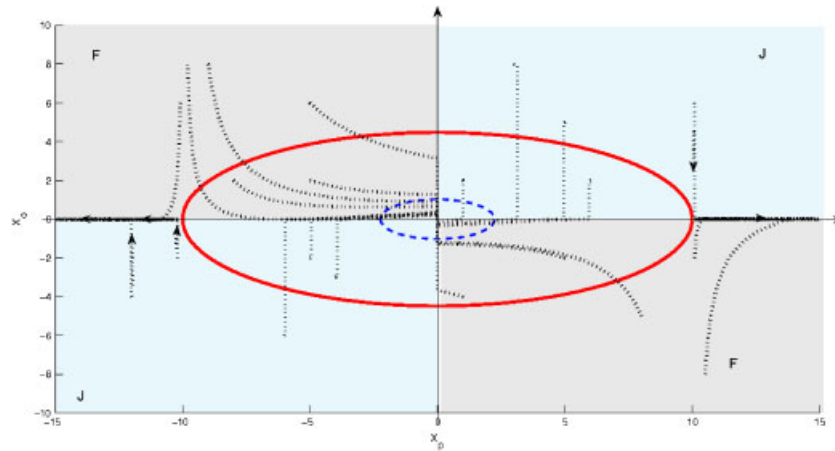


Figure 1. Region of attraction of the reset system with anti-windup (50) (solid) and without (dotted).

4.5.1. Flow and jump set case 1. We first aim at applying Theorem 3 to solve the optimization problem proposed in Section 4.3. System (47)–(48) is subject to a reset control law satisfying (11)–(12). We define the set of interesting directions, denoted Ξ_0 , as a square region in \mathbb{R}^2 :

$$\Xi_0 = \text{Co} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} \quad (49)$$

To apply Theorem 3, we suppose scalars δ , τ_F and τ_R *a priori* fixed (as $\delta=1$, $\tau_F=1.9 \cdot 10^{-6}$ and $\tau_R=1.25 \cdot 10^{-8}$) and $M_1=m_1 I$. The estimation of the stability domain is depicted in Figure 1, where the dotted ellipsoid represents the estimation of the region of attraction without anti-windup. By applying the numerical procedure proposed in Section 4.3, one obtains the following matrices:

$$A_a = \begin{bmatrix} -0.0843 & 0.0130 \\ -0.1549 & -0.1907 \end{bmatrix}, \quad B_a = \begin{bmatrix} -0.4e^{-3} \\ 0.17e^{-3} \end{bmatrix}, \quad C_a = [2.4 \quad 2.3], \quad D_a = 0.11, \quad (50)$$

The associated domain of stability is in solid line in Figure 1. Note that we obtain a better estimation of the basin of attraction thanks to the anti-windup layer. Trajectories starting from several initial states are also plotted in Figure 1. They illustrate that the set $\mathcal{E}(P, \delta)$ is a good approximation of the basin of attraction of the origin. Note that some points of instability are outside of $\mathcal{E}(P, \delta)$ but very close to its boundary.

Considering the performance analysis, i.e. solving the optimization problem proposed in Section 4.2 subject to relation of Theorem 3, we obtain, for the same stability region, the values of γ as follows: $\gamma=2.7$ in the linear case, $\gamma=0.8$ with reset and $\gamma=1.03$ with reset and anti-windup (see also temporal simulations in Figure 2). We emphasize that the anti-windup compensator is designed to improve the performance. If the obtained \mathcal{L}_2 gain is actually better than the linear case one, the system behavior is better with reset and without anti-windup. Indeed, in this case, the anti-windup compensator acts to smooth the reset controller output y_c to avoid saturation. As a consequence, states of the closed-loop system remain in the flow set, and we do not benefit of jump action to improve output convergence. For this example, anti-windup is efficient to enlarge the stability region, but not really to minimize the \mathcal{L}_2 -gain.

4.5.2. Flow and jump set case 2. We are now applying Theorem 6. The objective is to synthesize a new reset control rule and new flow and jump sets in order to obtain a stability domain as large as possible. We consider system (47)–(48) with the reset control law (34). Flow and jump sets satisfy (35). We solve the optimization problem proposed in Section 4.3 subject to relations of Theorem 6 to obtain sets $\mathcal{E}(P, 1)$ and $\mathcal{E}(P_1, 1)$ as large as possible. The anti-windup controller (39) leads to

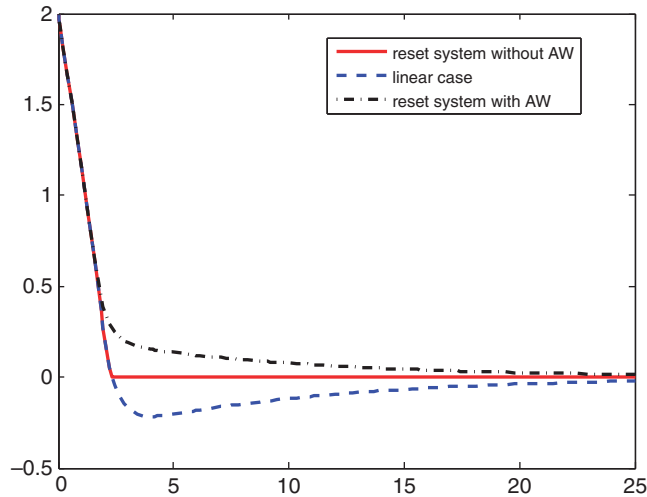


Figure 2. Output $y(t)$ of system (47)–(48) in the linear case (dashed line, $\gamma=2.7$), with reset (solid line $\gamma=0.8$) and with reset and anti-windup (dotted line, $\gamma=1.03$).

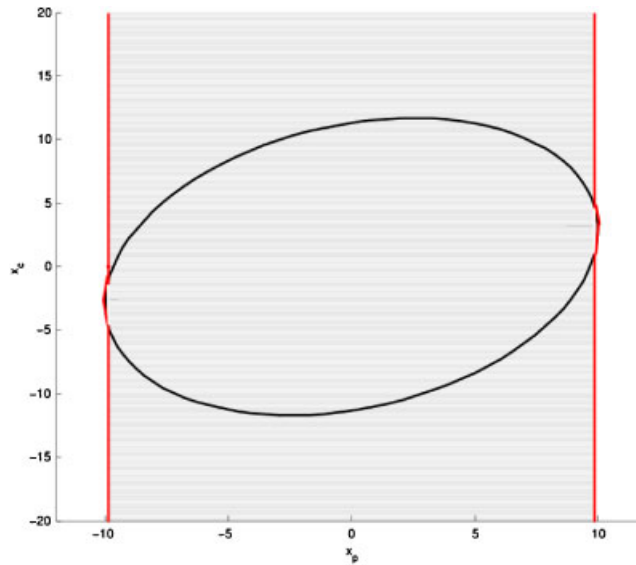


Figure 3. Stability domain of system (47)–(48) with the reset control law (33).

$\mathcal{E}(P, 1)$, the black continuous line ellipsoid in Figure 3. Note that this stability domain is close to the one obtained with a static anti-windup in [34]. By computing the reset law (40), we obtain a stability domain $\mathcal{E}(P_1, 1)$ quasi tangent to $\mathcal{E}(P, 1)$, which allows to extend the region of stability in the directions x_c . The region of stability for the closed-loop system is the union of these two sets, included in the red continuous line (the grey zone of Figure 3).

5. CONCLUSION

This paper dealt with the anti-windup design technique for a SISO linear system controlled by a reset controller and subject to saturation. In this context, different types of reset rule and therefore different flow and jump sets were considered. Constructive conditions leading to quasi-LMIs or LMIs, based on the use of suitable Lyapunov functions, modified sector condition and the

application of adequate congruence transformations, were proposed to exhibit the matrices defining the anti-windup compensator. The computation of the anti-windup compensator aiming at ensuring both finite gain \mathcal{L}_2 stability and internal stability of the closed-loop system was then carried out from the solution of some convex optimization problems.

A natural extension of this work could be to consider MIMO reset systems, leading to revisit the definition of reset rules, and to deal with more complex Lyapunov functions, as, for example, piecewise Lyapunov functions.

ACKNOWLEDGEMENTS

This work is supported in part by the ANR project ArHyCo, ARPEGE, contract number ANR-2008 SEGI 004 01-30011459.

REFERENCES

1. Glattfelder AH, Schaufelberger W. *Control Systems with Input and Output Constraints*. Springer: Amsterdam, 2003.
2. Galeani S, Tarbouriech S, Turner MC, Zaccarian L. A tutorial on modern anti-windup design. *European Journal of Control* 2009; **15**(3–4):418–440.
3. Grimm G, Hatfield J, Postlethwaite I, Teel AR, Turner MC, Zaccarian L. Antiwindup for stable linear systems with input saturation: an LMI-based synthesis. *IEEE Transactions on Automatic Control* 2003; **48**(9):1509–1525.
4. Hu T, Teel AR, Zaccarian L. Anti-windup synthesis for linear control systems with input saturation: achieving regional, nonlinear performance. *Automatica* 2008; **44**(2):515–519.
5. Tarbouriech S, Garcia G, Glattfelder AH (eds). *Advanced Strategies in Control Systems with Input and Output Constraints*, Lecture Notes in Control and Information Sciences, vol. 346. Springer: Berlin, 2007.
6. Tarbouriech S, Turner MC. Anti-windup design: an overview of some recent advances and open problems. *IET Control Theory and Application* 2009; **3**(1):1–19.
7. Beker O, Hollot CV, Chait Y. Plant with an integrator: an example of reset control overcoming limitations of linear feedback. *IEEE Transactions on Automatic Control* 2001; **46**:1797–1799.
8. Feuer A, Goodwin GC, Salgado M. Potential benefits of hybrid control for linear time invariant plants. *American Control Conference*, New Mexico, U.S.A., vol. 5, June 1997; 2790–2794.
9. Bobrow JE, Jabbari F, Thai K. An active truss element and control law for vibration suppression. *Smart Materials and Structures* 1995; **4**(4):264–269.
10. Bupp RT, Bernstein DS, Chellaboina VS, Haddad WM. Resetting virtual absorbers for vibration control. *American Control Conference*, New Mexico, U.S.A., vol. 5, June 1997; 2647–2651.
11. Haddad WM, Chellaboina VS, Kablar NA. Active control of combustion instabilities via hybrid resetting controllers. *American Control Conference*, June 2000; 2378–2382.
12. Zheng Y, Chait Y, Hollot CV, Steinbuch M, Norg M. Experimental demonstration of reset control design. *Control Engineering Practice* 2000; **8**(2):113–120.
13. Clegg JC. A nonlinear integrator for servomechanisms. *Transactions, A.I.E.E* 1958; **77**(Part II):41–42.
14. Nešić D, Zaccarian L, Teel AR. Stability properties of reset systems. *Automatica* 2008; **44**(8):2019–2026.
15. Prieur C, Goebel R, Teel AR. Hybrid feedback control and robust stabilization of nonlinear systems. *IEEE Transactions on Automatic Control* 2007; **52**(11):2103–2117.
16. Prieur C, Teel AR. Uniting local and global output feedback controllers. *IEEE Transactions on Automatic Control* 2011; DOI: 10.1109/TAC.2010.2091436.
17. Goebel R, Prieur C, Teel AR. Smooth patchy control Lyapunov functions. *Automatica* 2009; **45**(3):675–683.
18. Zaccarian L, Nešić D, Teel AR. First order reset elements and the Clegg integrator revisited. *American Control Conference*, Portland, OR, U.S.A., June 2005; 563–568.
19. Loquen T, Tarbouriech S, Prieur C. Stability analysis for reset systems with input saturation. *Conference on Decision and Control*, New Orleans, LA, U.S.A., 2007; 3272–3277.
20. Goebel R, Sanfelice RG, Teel AR. Hybrid dynamical systems. *IEEE Control Systems Magazine* 2009; **29**(2):28–93.
21. Prieur C, Tarbouriech S, Zaccarian L. Guaranteed stability for nonlinear systems by means of a hybrid loop. *IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Bologna, Italy, September 2010; 72–77.
22. Tarbouriech S, Prieur C, Gomes da Silva Jr JM. Stability analysis and stabilization of systems presenting nested saturations. *IEEE Transactions on Automatic Control* 2006; **51**(8):1364–1371.
23. Biannic J-M, Tarbouriech S. Optimization and implementation of dynamic anti-windup compensators with multiple saturations in flight control systems. *Control Engineering Practice* 2009; **17**:703–713.
24. Bender FA, Gomes da Silva Jr JM, Tarbouriech S. A convex framework for the design of dynamic anti-windup for state-delayed systems. *American Control Conference*, Baltimore, U.S.A., June 2010; 6763–6768.
25. Scherer C, Gahinet P, Chilali M. Multi-objective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control* 1997; **42**(7):896–911.

26. El Ghaoui L, Oustry F, Ait Rami M. A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Transactions on Automatic Control* 1997; **42**(8):1171–1176.
27. Roos C, Biannic J-M. A convex characterization of dynamically-constrained anti-windup controllers. *Automatica* 2008; **44**(9):2449–2452.
28. Hu T, Lin Z. *Control Systems with Actuator Saturation: Analysis and Design*. Birkhäuser: Boston, MA, 2001.
29. Gomes da Silva Jr JM, Tarbouriech S. Local stabilization of discrete-time linear systems with saturating controls: an LMI-based approach. *IEEE Transactions on Automatic Control* 2001; **46**(1):119–124.
30. Turner MC, Postlethwaite I. A new perspective on static and low order anti-windup synthesis. *International Journal of Control* 2004; **77**(1):27–44.
31. Kerr ML, Turner MC, Postlethwaite I. Practical approaches to low-order anti-windup compensator design: a flight control comparison. *World IFAC Congress*, Seoul, Korea, 2008.
32. Teel AR, Kapoor N. The \mathcal{L}_2 anti-windup problem: its definition and solution. *European Control Conference*, Brussels, Belgium, July 1997.
33. Zaccarian L, Teel AR. A common framework for anti-windup, bumpless transfer and reliable designs. *Automatica* 2002; **38**(10):1735–1744.
34. Gomes da Silva Jr JM, Tarbouriech S. Anti-windup design with guaranteed regions of stability: an LMI-based approach. *IEEE Transactions on Automatic Control* 2005; **50**(1):106–111.