Boundary Output Feedback Stabilization of Reaction-Diffusion PDEs with Delayed Boundary Measurement

Hugo Lhachemi[a] and Christophe Prieur[b]

[a]Université Paris-Saclay, CNRS, CentraleSupélec, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France; [b]Université Grenoble Alpes, CNRS, Grenoble-INP, GIPSA-lab, F-38000, Grenoble, France

ARTICLE HISTORY
Compiled June 25, 2021

ABSTRACT
This paper addresses the boundary output feedback stabilization of general 1-D reaction-diffusion PDEs with delayed boundary measurement. The output takes the form of a either Dirichlet or Neumann trace. The output delay can be arbitrarily large. The control strategy is composed of a finite-dimensional observer that is used to observe a delayed version of the first modes of the PDE and a predictor component which is employed to obtain the control input to be applied at current time. For any given value of the output delay, we assess the stability of the resulting closed-loop system provided the order of the observer is selected large enough. Taking advantage of this result, we discuss the extension of the control strategy to the case of simultaneous input and output delays.

KEYWORDS
Reaction-diffusion PDEs, output feedback, delayed measurement, boundary control.

1. Introduction

Time delays commonly arise in the design of control strategies due to either natural feedback processes or the active implementation of control laws. Moreover, time delays are well-known for their capability to introduce instabilities when not considered properly in the control design. For these reasons, the feedback control of finite-dimensional systems in the presence of delays has been extensively studied (Artstein, 1982; Richard, 2003). The extension of this problematic to Partial Differential Equations (PDEs) has been the topic of a number of papers in the recent years (Nicaise & Pignotti, 2008; Wang & Sun, 2018). In particular, the development of control strategies for the feedback stabilization of reaction-diffusion PDEs with an arbitrarily long delay in the either control input (Katz & Fridman, 2021b; Krstic, 2009; Lhachemi & Prieur, 2021a, 2021c; Lhachemi, Prieur, & Shorten, 2019, 2021; Qi & Krstic, 2020) or state (Hashimoto & Krstic, 2016; Kang & Fridman, 2017; Lhachemi & Shorten, 2020, 2021) has been intensively studied.

In this paper, we address the boundary output feedback stabilization of general 1-D reaction-diffusion PDEs with delayed boundary measurement. The control input and

Email: hugo.lhachemi@centralesupelec.fr

boundary conditions take the form of Dirichlet/Neumann/Robin boundary conditions. The output is selected as a either Dirichlet or Neumann boundary trace presenting an arbitrarily long delay. The control strategy couples a finite-dimensional observer (Balas, 1988; Curtain, 1982; Grüne & Meurer, 2021; Harkort & Deutscher, 2011) used to observe a finite number of modes of the PDE and a predictor component (Artstein, 1982; Karafyllis & Krstić, 2017). To design the finite-dimensional observer, we leverage the approach reported first in (Katz & Fridman, 2020) relying on spectral-reduction methods (Coron & Trélat, 2004, 2006; Russell, 1978), and more specifically on the scaling-based procedures described in (Lhachemi & Prieur, 2020, 2021b) that allow to handle Dirichlet/Neumann boundary measurement while performing, for very general 1-D reaction-diffusion PDEs, the control design directly with the actual control input u and not its time-derivative $v = \dot{u}$. We refer the reader, e.g., to (Curtain & Zwart, 2012, Sec. 3.3.) for a general introduction to the topic of boundary control systems.

It is worth noting that the robustness of the finite-dimensional control strategy reported in (Katz & Fridman, 2020) to small enough input and measurement delays was discussed in (Katz & Fridman, 2021a). However, the presence of an arbitrarily long output delay imposes more stringent constraints on the system and requires the development of a dedicated control strategy. This is achieved in this paper by leveraging a predictor design (Deng, Léchappé, Moulay, & Plestan, 2019; Karafyllis & Krstić, 2017). The possibility to couple a finite-dimensional observer with a predictor to handle arbitrary input delays was reported first in (Katz & Fridman, 2021b) in the very specific configuration of a Neumann boundary control, for a bounded output operator, and for system trajectories evaluated in L^2 norm. The case of general input delayed 1-D reaction-diffusion PDEs with Dirichlet/Neumann/Robin boundary control and Dirichlet/Neumann boundary measurement was solved in (Lhachemi & Prieur, 2021c) for PDE trajectories in H^1 norm. In this paper we address the dual problem of (Lhachemi & Prieur, 2021c), namely the output feedback stabilization of reaction-diffusion PDEs in the presence of an arbitrary output delay. The proposed control strategy is composed of a finite-dimensional observer that is used to observe a delayed version of the first modes of the PDE (this delayed observation matches with the measurement delay) and a predictor component which is employed to obtain the control input to be applied at current time. For a given value of the output delay, we derive a set of sufficient LMI conditions ensuring the exponential stability of the resulting closed-loop system for PDE trajectories evaluated in H^1 norm. For any given value of the output delay, these control design constraints are shown to be feasible provided the order of the observer is selected large enough. Combining the approach developed in this paper with the one reported in (Lhachemi & Prieur, 2021c) for the case of an input delay, we also discuss the extension of the method to the stabilization of reaction-diffusion PDEs in the presence of both input and output delays.

The paper is organized as follows. After introducing some definitions and properties, the control design problem addressed in this paper is presented in Section 2. The case of a delayed Dirichlet boundary measurement is reported in Section 3. The control design procedure is then extended to delayed Neumann boundary measurement in Section 4. A numerical illustration of these two settings is presented in Section 5. The extension of the obtained results to the case of input and output delays is discussed in Section 6. Finally, concluding remarks are formulated in Section 7.

2. Definitions and problem setting

2.1. Definitions and properties

2.1.1. Notation

Spaces $\mathbb{R}^n$ are equipped with the Euclidean norm denoted by $\|\cdot\|$. The associated induced norms of matrices are also denoted by $\|\cdot\|$. For any two vectors X and Y of arbitrary dimensions, $\mathrm{col}(X, Y)$ stands for the vector $[X^\top, Y^\top]^\top$. $L^2(0,1)$ stands for the space of square integrable functions on $(0,1)$ and is endowed with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$. The corresponding norm is denoted by $\|\cdot\|_{L^2}$. For an integer $m \geq 1$, $H^m(0,1)$ stands for the m-order Sobolev space and is endowed with its usual norm $\|\cdot\|_{H^m}$. For any symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P \succeq 0$ (resp. $P \succ 0$) indicates that P is positive semi-definite (resp. positive definite).

2.1.2. Properties of Sturm-Liouville operators

Let $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^1([0,1])$ and $q \in \mathcal{C}^0([0,1])$ with $p > 0$ and $q \geq 0$. Let the Sturm-Liouville operator $\mathcal{A} : D(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$ be defined by $\mathcal{A}f = -(pf')' + qf$ on the domain $D(\mathcal{A}) = \{f \in H^2(0,1) : c_{\theta_1} f(0) - s_{\theta_1} f'(0) = c_{\theta_2} f(1) + s_{\theta_2} f'(1) = 0\}$. Here we use the short notations $c_{\theta_i} = \cos\theta_i$ and $s_{\theta_i} = \sin\theta_i$. It is well-known that the eigenvalues λ_n, $n \geq 1$, of $\mathcal{A}$ are simple, non negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. Moreover the corresponding unit eigenvectors $\phi_n \in L^2(0,1)$ form a Hilbert basis. The domain of the operator $\mathcal{A}$ is equivalently characterized in terms of the above eigenstructures by $D(\mathcal{A}) = \{f \in L^2(0,1) : \sum_{n\geq 1} |\lambda_n|^2 |\langle f, \phi_n\rangle|^2 < +\infty\}$. Introducing $p_*, p^*, q^* \in \mathbb{R}$ so that $0 < p_* \leq p(x) \leq p^*$ and $0 \leq q(x) \leq q^*$ for all $x \in [0,1]$, we have $0 \leq \pi^2(n-1)^2 p_* \leq \lambda_n \leq \pi^2 n^2 p^* + q^*$ for all $n \geq 1$ (see, e.g., Orlov (2017)). Furthermore, with the additional assumption $p \in \mathcal{C}^2([0,1])$, we also have that $\phi_n(\xi) = O(1)$ and $\phi_n'(\xi) = O(\sqrt{\lambda_n})$ as $n \to +\infty$ for any given $\xi \in [0,1]$ (see, e.g., Orlov (2017)). Besides and under the assumption $q > 0$, an integration by parts and the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$ show the existence of constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_{H^1}^2 \leq \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle^2 = \langle \mathcal{A}f, f\rangle \leq C_2 \|f\|_{H^1}^2 \tag{1}$$

for all $f \in D(\mathcal{A})$. The latter inequalities and the Riesz-spectral nature of $\mathcal{A}$ imply that the series expansion $f = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n$ holds in $H^2(0,1)$ norm for any $f \in D(\mathcal{A})$. Invoking again the continuous embedding $H^1(0,1) \subset L^\infty(0,1)$, we deduce that $f(0) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n(0)$ and $f'(0) = \sum_{n\geq 1} \langle f, \phi_n\rangle \phi_n'(0)$.

In the sequel, we define for any integer $N \geq 1$ and any $f \in L^2(0,1)$ the quantity $\mathcal{R}_N f = \sum_{n\geq N+1} \langle f, \phi_n\rangle \phi_n$.

2.2. *Problem setting and spectral reduction*

2.2.1. *Problem setting*

Let the reaction-diffusion system with boundary control be described by

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}(x)z(t,x) \tag{2a}$$
$$c_{\theta_1} z(t,0) - s_{\theta_1} z_x(t,0) = 0 \tag{2b}$$
$$c_{\theta_2} z(t,1) + s_{\theta_2} z_x(t,1) = u(t) \tag{2c}$$
$$z(0,x) = z_0(x) \tag{2d}$$

for $t > 0$ and $x \in (0,1)$. Here $\theta_1, \theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q} \in \mathcal{C}^0([0,1])$. The state of the reaction-diffusion PDE at time t is $z(t,\cdot)$, the command is $u(t)$, and the initial condition is z_0. We define the initial command as $u_0 = c_{\theta_2} z_0(1) + s_{\theta_2} z_{0,x}(1)$. For some measurement delay $h > 0$, the system output is chosen as the either delayed Dirichlet or delayed Neumann trace. More precisely, in the case $\theta_1 \in (0, \pi/2]$, the delayed Dirichlet boundary measurement is defined by

$$y_D(t) = \begin{cases} z(t-h,0), & t \geq h \\ y_0(t-h), & 0 \leq t \leq h \end{cases} \tag{3}$$

Similarly but in the case $\theta_1 \in [0, \pi/2)$, the delayed Neumann boundary measurement is defined by

$$y_N(t) = \begin{cases} z_x(t-h,0), & t \geq h \\ y_0(t-h), & 0 \leq t \leq h \end{cases} \tag{4}$$

In both cases, $y_0 : [-h, 0] \to \mathbb{R}$ is the initial condition of the delayed boundary measurement and is assumed to be Lipschitz continuous.

Without loss of generality, we introduce $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ so that

$$\tilde{q}(x) = q(x) - q_c, \qquad q(x) > 0. \tag{5}$$

2.2.2. *Spectral reduction*

Based on the change of variable formula

$$w(t,x) = z(t,x) - \frac{x^2}{c_{\theta_2} + 2s_{\theta_2}} u(t) \tag{6}$$

the PDE (2) in original coordinates can be equivalently reformulated as the homogeneous PDE described by

$$v(t) = \dot{u}(t) \tag{7a}$$
$$w_t(t,x) = (p(x)w_x(t,x))_x - \tilde{q}(x)w(t,x) + a(x)u(t) + b(x)v(t) \tag{7b}$$
$$c_{\theta_1} w(t,0) - s_{\theta_1} w_x(t,0) = 0 \tag{7c}$$
$$c_{\theta_2} w(t,1) + s_{\theta_2} w_x(t,1) = 0 \tag{7d}$$
$$w(0,x) = w_0(x) \tag{7e}$$

Here we have $a(x) = \frac{1}{c_{\theta_2}+2s_{\theta_2}}\{2p(x)+2xp'(x)-x^2\tilde{q}(x)\}$, $b(x) = -\frac{x^2}{c_{\theta_2}+2s_{\theta_2}}$, and $w_0(x) = z_0(x) - \frac{x^2}{c_{\theta_2}+2s_{\theta_2}}u(0)$. Noting that $w(t,0) = z(t,0)$ and $w_x(t,0) = z_x(t,0)$, the boundary measurements are described for $t \geq h$ by

$$y_D(t) = w(t-h,0), \qquad y_N(t) = w_x(t-h,0). \tag{8}$$

Let us now define the coefficients of projection $z_n(t) = \langle z(t,\cdot), \phi_n \rangle$, $w_n(t) = \langle w(t,\cdot), \phi_n \rangle$, $a_n = \langle a, \phi_n \rangle$, and $b_n = \langle b, \phi_n \rangle$. Owing to (6), we infer that that

$$w_n(t) = z_n(t) + b_n u(t), \quad n \geq 1. \tag{9}$$

We now project the two PDEs representations (2) and (7) into the Hilbert basis $(\phi_n)_{n\geq 1}$. The former representation gives

$$\dot{z}_n(t) = (-\lambda_n + q_c) z_n(t) + \beta_n u(t) \tag{10}$$

where $\beta_n = a_n + (-\lambda_n + q_c) b_n = p(1)\{-c_{\theta_2}\phi_n'(1) + s_{\theta_2}\phi_n(1)\} = O(\sqrt{\lambda_n})$. The latter representation implies that

$$\dot{u}(t) = v(t) \tag{11a}$$
$$\dot{w}_n(t) = (-\lambda_n + q_c) w_n(t) + a_n u(t) + b_n v(t) \tag{11b}$$

Finally the delayed measurements (8) can be expressed for $t \geq h$ as the following series expansions:

$$y_D(t) = \sum_{n\geq 1} w_n(t-h)\phi_n(0), \qquad y_N(t) = \sum_{n\geq 1} w_n(t-h)\phi_n'(0). \tag{12}$$

3. Case of a delayed Dirichlet measurement

We address in this section the output feedback stabilization of the reaction-diffusion PDE described by (2) for $\theta_1 \in (0, \pi/2]$ with delayed Dirichlet measurement (3).

3.1. Control strategy

Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. Let $N \geq N_0 + 1$ be arbitrarily fixed and that will be specified later. Inspired by (Karafyllis & Krstić, 2017, Chap. 3) in the context of finite-dimensional systems, we first design an observer that is used to estimate from the delayed measurement $y_D(t)$ the N first modes $z_n(t-h)$ of the PDE at time $t-h$. The observer dynamics reads, for $t \geq 0$,

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n u(t-h) \tag{13a}$$
$$\begin{aligned}\dot{\hat{z}}_n(t) = {} & (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t-h) \\ & - l_n \left\{ \sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y_D(t) \right\}, \ 1 \leq n \leq N_0 \end{aligned} \tag{13b}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t-h), \ N_0 + 1 \leq n \leq N \tag{13c}$$

where $l_n \in \mathbb{R}$ are the observer gains and with $u(\tau) = u_0$ for $\tau \leq 0$. So $\hat{z}_n(t)$ is seen as the estimation of $z_n(t-h)$ for times $t \geq h$. Note that no control input is actually applied to the system (2) in negative time. The definition of u in negative time is only introduced here in order to make sure that the dynamics (13) is well-defined for all $t \geq 0$.

Since the observer (13) estimates the first modes of the PDE at time $t-h$ while the feedback must be applied at current time t, we need to introduce a predictor component. Defining $\hat{Z}^{N_0} = \begin{bmatrix} \hat{z}_1 & \ldots & \hat{z}_{N_0} \end{bmatrix}^\top$ along with $A_0 = \operatorname{diag}(-\lambda_1 + q_c, \ldots, -\lambda_{N_0} + q_c)$ and $\mathfrak{B}_0 = \begin{bmatrix} \beta_1 & \ldots & \beta_{N_0} \end{bmatrix}^\top$, we introduce the following Artstein tranformation:

$$\hat{Z}_A^{N_0}(t) = e^{A_0 h} \hat{Z}^{N_0}(t) + \int_{t-h}^{t} e^{A_0(t-s)} \mathfrak{B}_0 u(s) \,\mathrm{d}s. \tag{14}$$

We can now define the control input as

$$u(t) = K \hat{Z}_A^{N_0}(t) \tag{15}$$

for all $t \geq 0$ where $K \in \mathbb{R}^{1 \times N_0}$ is the feedback gain.

Remark 1. The controller described by (13-15) takes a form similar to the one reported in (Lhachemi & Prieur, 2021c) in the case of an input delay. However, due to the output delay considered in this paper, the measurement $y_D(t)$ appearing in (13b) is a time delayed version of the Dirichlet trace as described by (3). Moreover, the delayed input $u(t-h)$ appearing in (13) is not reminiscent of an actual input delay, as the ones considered in (Lhachemi & Prieur, 2021c), but is due to the fact that $\hat{z}_n(t)$ does not estimate $z_n(t)$ but $z_n(t-h)$ for $t \geq h$, so that the measurement $y_D(t) = w(t-h, 0)$ can indeed be used to design a classical Luenberger observer. We refer to (Karafyllis & Krstić, 2017, Chap. 3) for general explanations of such a control design strategy in the context of output delayed finite-dimensional systems.

Remark 2. Equations (14-15) imply that the initial condition $\hat{Z}^{N_0}(0) \in \mathbb{R}^{N_0}$ of the N_0 first modes of the observer must be selected so that $u_0 = K \hat{Z}_A^{N_0}(0)$. For a given $u_0 \in \mathbb{R}$, the latter condition is equivalent to $K e^{A_0 h} \hat{Z}^{N_0}(0) = \left(1 - \int_{-h}^{0} K e^{-A_0 s} \mathfrak{B}_0 \,\mathrm{d}s\right) u_0$. This is possible as soon as $K \neq 0$.

Remark 3. The well-posedness of the closed-loop system composed of the plant (2), the delayed Dirichlet measurement (3), and the controller (13-15), is not trivial under this form due to the integral term $\varphi(t) = \int_{t-h}^{t} e^{A_0(t-s)} \mathfrak{B}_0 u(s) \,\mathrm{d}s$ appearing in (14). However, it is observed that such a function φ is the unique solution to the EDO

$$\dot{\varphi}(t) = A_0 \varphi(t) + \mathfrak{B}_0 K \hat{Z}_A^{N_0}(t) - e^{A_0 h} \mathfrak{B}_0 u(t-h) \tag{16}$$

associated with the initial condition $\varphi(0) = \int_{-h}^{0} e^{-A_0 s} \mathfrak{B}_0 u_0 \,\mathrm{d}s$. Hence considering the infinite-dimensional system described by the plant (2), the delayed Dirichlet measurement (3), the observer dynamics (13), the control input (15) with $\hat{Z}_A^{N_0}(t)$ define by

$$\hat{Z}_A^{N_0}(t) = e^{A_0 h} \hat{Z}^{N_0}(t) + \varphi(t),$$

along with the ODE (16), the well-posedness in terms of classical solutions for initial conditions $z_0 \in H^2(0,1)$ and $\hat{z}_n(0) \in \mathbb{R}$ so that $c_{\theta_1} z_0(0) - s_{\theta_1} z_0'(0) = 0$ and $c_{\theta_2} z_0(1) + s_{\theta_2} z_0'(1) = u_0 = K \hat{Z}_A^{N_0}(0)$, and any Lipschitz continuous $y_0 \in \mathcal{C}^0([-h,0])$ so that $y_0(0) = z_0(0)$, is now an immediate consequence of (Pazy, 2012, Thm. 6.3.1 and 6.3.3) and the use of a classical induction argument.

3.2. Truncated model for stability analysis

In order to complete the tuning of the controller gains and to perform the stability analysis, we need to introduce first a finite dimensional model capturing the N first modes of the PDE in z coordinates (2) and the controller dynamics (13-15) based on the delayed Dirichlet measurement (3). To do so we define the observation error of the n-th mode as $e_n(t) = z_n(t-h) - \hat{z}_n(t)$ for all $1 \leq n \leq N$ and all $t \geq h$. Defining $E^{N_0} = \begin{bmatrix} e_1 & \ldots & e_{N_0} \end{bmatrix}^\top$, the scaled error $\tilde{e}_n = \sqrt{\lambda_n} e_n$, and $\tilde{E}^{N-N_0} = \begin{bmatrix} \tilde{e}_{N_0+1} & \ldots & \tilde{e}_N \end{bmatrix}^\top$, we obtain from (13b) and (15) that

$$\dot{\hat{Z}}^{N_0}(t) = A_0 \hat{Z}^{N_0}(t) + \mathfrak{B}_0 u(t-h) + L C_0 E^{N_0}(t) + L \tilde{C}_1 \tilde{E}^{N-N_0}(t) + L\zeta(t-h) \quad (17)$$

for all $t \geq h$. Defining the residue of measurement as $\zeta = \sum_{n \geq N+1} w_n \phi_n(0)$, we have $\zeta(t-h) = \sum_{n \geq N+1} w_n(t-h)\phi_n(0)$ for all $t \geq h$. The different matrices are defined by $C_0 = \begin{bmatrix} \phi_1(0) & \ldots & \phi_{N_0}(0) \end{bmatrix}$, $\tilde{C}_1 = \begin{bmatrix} \frac{\phi_{N_0+1}(0)}{\sqrt{\lambda_{N_0+1}}} & \ldots & \frac{\phi_N(0)}{\sqrt{\lambda_N}} \end{bmatrix}$, and $L = \begin{bmatrix} l_1 & \ldots & l_{N_0} \end{bmatrix}^\top$. Invoking the Artstein transformation (14) and using (15) we infer that

$$\begin{aligned} \dot{\hat{Z}}_A^{N_0}(t) = (A_0 + \mathfrak{B}_0 K) \hat{Z}_A^{N_0}(t) + e^{A_0 h} L C_0 E^{N_0}(t) \\ + e^{A_0 h} L \tilde{C}_1 \tilde{E}^{N-N_0}(t) + e^{A_0 h} L \zeta(t-h) \end{aligned} \quad (18)$$

for all $t \geq h$. Besides, the combination of (10) evaluated at time $t-h$ and (17) gives

$$\dot{E}^{N_0}(t) = (A_0 - LC_0) E^{N_0}(t) - L\tilde{C}_1 \tilde{E}^{N-N_0}(t) - L\zeta(t-h) \quad (19)$$

for all $t \geq h$.

Based on (13c) and defining the scaled estimation $\tilde{z}_n = \hat{z}_n / \lambda_n$ and $\tilde{Z}^{N-N_0} = \begin{bmatrix} \tilde{z}_{N_0+1} & \ldots & \tilde{z}_N \end{bmatrix}^\top$, we deduce that

$$\dot{\tilde{Z}}^{N-N_0}(t) = A_1 \tilde{Z}^{N-N_0}(t) + \tilde{\mathfrak{B}}_1 u(t-h)$$

for $t \geq 0$ where $A_1 = \mathrm{diag}(-\lambda_{N_0+1} + q_c, \ldots, -\lambda_N + q_c)$ and $\tilde{\mathfrak{B}}_1 = \begin{bmatrix} \beta_{N_0+1}/\lambda_{N_0+1} & \ldots & \beta_N/\lambda_N \end{bmatrix}^\top$. Introducing the second Artstein tranformation:

$$\tilde{Z}_A^{N-N_0}(t) = e^{A_1 h} \tilde{Z}^{N-N_0}(t) + \int_{t-h}^{t} e^{A_1(t-s)} \tilde{\mathfrak{B}}_1 u(s) \, \mathrm{d}s \quad (20)$$

and owing to (15) we infer that

$$\dot{\tilde{Z}}_A^{N-N_0}(t) = A_1 \tilde{Z}_A^{N-N_0}(t) + \tilde{\mathfrak{B}}_1 K \hat{Z}_A^{N_0}(t). \quad (21)$$

Moreover, using (10) evaluated at time $t-h$ and (13c), the error dynamics reads

$$\dot{\tilde{E}}^{N-N_0}(t) = A_1 \tilde{E}^{N-N_0}(t) \tag{22}$$

for all $t \geq h$.

Introducing the state vector

$$X = \operatorname{col}\left(\hat{Z}_A^{N_0}, E^{N_0}, \tilde{Z}_A^{N-N_0}, \tilde{E}^{N-N_0}\right), \tag{23}$$

we obtain from (18-19) and (21-22) that

$$\dot{X}(t) = FX(t) + \mathcal{L}\zeta(t-h) \tag{24}$$

for all $t \geq h$ where

$$F = \begin{bmatrix} A_0 + \mathfrak{B}_0 K & e^{A_0 h} L C_0 & 0 & e^{A_0 h} L \tilde{C}_1 \\ 0 & A_0 - L C_0 & 0 & -L\tilde{C}_1 \\ \tilde{\mathfrak{B}}_1 K & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} e^{A_0 h} L \\ -L \\ 0 \\ 0 \end{bmatrix}$$

With $\tilde{X}(t) = \operatorname{col}(X(t), \zeta(t-h))$ and based on (15) and (17), we also have

$$u(t) = \tilde{K}X(t),\ \forall t \geq h\ ; \qquad \begin{aligned} v(t) = \dot{u}(t) &= K\dot{\hat{Z}}_A^{N_0}(t),\ \forall t \geq 0 \\ &= E\tilde{X}(t),\ \forall t \geq h \end{aligned} \tag{25}$$

with $\tilde{K} = \begin{bmatrix} K & 0 & 0 & 0 \end{bmatrix}$ and $E = K\begin{bmatrix} A_0 + \mathfrak{B}_0 K & e^{A_0 h} L C_0 & 0 & e^{A_0 h} L\tilde{C}_1 & e^{A_0 h} L \end{bmatrix}$.

Remark 4. The application of the Hautus test shows that the pairs $(A_0, \mathfrak{B}_0)$ and (A_0, C_0) satisfy the Kalman condition. Hence one can always compute feedback and observer gains $K \in \mathbb{R}^{1\times N_0}$ and $L \in \mathbb{R}^{N_0}$ so that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with arbitrary pole assignment.

Remark 5. It is worth noting that the matrix F and the vector $\mathcal{L}$ of the truncated model (24) are identical to the ones obtained in (Lhachemi & Prieur, 2021c) in the case of an input delay (instead of an output delay). However, the reduced model derived in (Lhachemi & Prieur, 2021c) is delay-free and valid for all $t \geq 0$, essentially because the predictor components manage to completely compensate the input delay. This is not the case in the output delay setting studied in this paper due to the delayed residue of measurement $\zeta(t-h)$. Conversely, since no input delay appears in the original PDE dynamics (2), the dynamics of the coefficients of projection (11) are delay-free. This is in contrast with the input delay setting studied in (Lhachemi & Prieur, 2021c) where the dynamics of the modes present a time delay.

3.3. Main stability result

We are now in position to state the main result of this section.

Theorem 3.1. *Let $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q} \in \mathcal{C}^0([0,1])$. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (5) holds. Let $\delta > 0$ and $N_0 \geq 1$*

be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0 + 1$. Let $K \in \mathbb{R}^{1 \times N_0} \backslash \{0\}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. Let $h > 0$ be given. For a given $N \geq N_0 + 1$, assume that there exist $P \succ 0$, $\alpha > 1$, and $\beta, \gamma > 0$ such that

$$\Theta_1 \preceq 0, \quad \Theta_2 \leq 0 \tag{26}$$

where

$$\Theta_1 = \begin{bmatrix} F^\top P + PF + 2\delta P + \alpha\gamma \|\mathcal{R}_N a\|_{L^2}^2 \tilde{K}^\top \tilde{K} & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta e^{-2\delta h} \end{bmatrix} + \alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 E^\top E \tag{27a}$$

$$\Theta_2 = 2\gamma \left\{ -\left(1 - \frac{1}{\alpha}\right) \lambda_{N+1} + q_c + \delta \right\} + \beta M_\phi \tag{27b}$$

where $M_\phi = \sum_{n \geq N+1} \frac{|\phi_n(0)|^2}{\lambda_n} < +\infty$. Then there exists a constant $M > 0$ such that for any initial condition $z_0 \in H^2(0,1)$ and $\hat{z}_n(0) \in \mathbb{R}$ so that $c_{\theta_1} z_0(0) - s_{\theta_1} z_0'(0) = 0$ and $c_{\theta_2} z_0(1) + s_{\theta_2} z_0'(1) = u_0 = K\hat{Z}_A^{N_0}(0)$, and any Lipschitz continuous $y_0 \in \mathcal{C}^0([-h, 0])$ so that $y_0(0) = z_0(0)$, the trajectories of the closed-loop system composed of the plant (2), the delayed Dirichlet measurement (3), and the controller (13-15) satisfy

$$\|z(t,\cdot)\|_{H^1}^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 \leq M e^{-2\delta t} \left(\|z_0\|_{H^1}^2 + \sum_{n=1}^{N} \hat{z}_n(0)^2 + u_0^2 + \|y_0\|_\infty^2 \right) \tag{28}$$

for all $t \geq 0$. Moreover, for any given $h > 0$, the constraints (26) are always feasible for N selected large enough.

Proof. Let $V(t) = V_0(t) + V_1(t)$ be defined for $t \geq h$ by

$$V_0(t) = X(t)^\top P X(t) + \gamma \sum_{n \geq N+1} \lambda_n w_n(t)^2 \tag{29a}$$

$$V_1(t) = \beta \int_{(t-h)^+}^{t} e^{-2\delta(t-s)} \zeta(s)^2 \mathrm{d}s \tag{29b}$$

where $(t-h)^+ = \max(t-h, 0)$. The computation of the time derivative of V along the system trajectories (11) and (24) for $t > h$ gives

$$\begin{aligned} \dot{V} \leq\ & \tilde{X}^\top \begin{bmatrix} F^\top P + PF & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta e^{-2\delta h} \end{bmatrix} \tilde{X} - 2\delta V_1 + \beta \zeta^2 \\ & + 2\gamma \sum_{n \geq N+1} \lambda_n (-\lambda_n + q_c) w_n^2 + 2\gamma \sum_{n \geq N+1} \lambda_n \{a_n u + b_n v\} w_n. \end{aligned}$$

Using (25) and invoking Young inequality, we infer for any $\alpha > 0$ that

$$2 \sum_{n \geq N+1} \lambda_n a_n u w_n \leq \frac{1}{\alpha} \sum_{n \geq N+1} \lambda_n^2 w_n^2 + \alpha \|\mathcal{R}_N a\|_{L^2}^2 X^\top \tilde{K}^\top \tilde{K} X$$

and

$$2\sum_{n\geq N+1}\lambda_n b_n v w_n \leq \frac{1}{\alpha}\sum_{n\geq N+1}\lambda_n^2 w_n^2 + \alpha\|\mathcal{R}_N b\|_{L^2}^2 \tilde{X}^\top E^\top E\tilde{X}.$$

The above estimates imply

$$\dot{V}+2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \beta\zeta^2 + 2\gamma\sum_{n\geq N+1}\lambda_n\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_n + q_c + \delta\right\} w_n^2.$$

Since, by definition, $\zeta = \sum_{n\geq N+1} w_n\phi_n(0)$ we obtain from Cauchy-Schwarz inequality that $\beta\zeta^2 \leq \beta M_\phi \sum_{n\geq N+1}\lambda_n w_n^2$. Hence we have that

$$\dot{V}+2\delta V \leq \tilde{X}^\top \Theta_1 \tilde{X} + \sum_{n\geq N+1}\lambda_n\Gamma_n w_n^2 \tag{30}$$

where $\Gamma_n = 2\gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_n + q_c + \delta\right\} + \beta M_\phi$. Since $\alpha > 1$, we observe that $\Gamma_n \leq \Gamma_{N+1} = \Theta_2$ for all $n \geq N+1$. Owing to (26) we deduce that $\dot{V}+2\delta V \leq 0$ for all $t > h$. Hence $V(t) \leq e^{-2\delta(t-h)}V(h)$ for all $t \geq h$. Using standard arguments similar to the ones reported in the proof of (Pazy, 2012, Thm. 6.3.3) to estimate the trajectories of the closed-loop system on the time interval $[0,h]$, we infer the existence of a constant $c > 0$, independent of the initial conditions, such that

$$\sum_{n\geq 1}\lambda_n w_n(t)^2 + \sum_{n=1}^{N}\hat{z}_n(t)^2 \leq c\left(\sum_{n\geq 1}\lambda_n w_n(0)^2 + \sum_{n=1}^{N}\hat{z}_n(0)^2 + u_0^2 + \|y_0\|_\infty^2\right)$$

for all $t \in [0,h]$. The claimed stability estimate (28) is now obtained from the definition of V, the estimates (1), and by invoking the Artstein transformations (14) and (20).

It remains to show that the constraints (26) are feasible provided the dimension $N \geq N_0+1$ of the observer is selected sufficiently large. First, applying the Lemma reported in Appendix to the matrix $F+\delta I$, we infer for any $N \geq N_0+1$ the existence of a matrix $P \succ 0$ so that $F^\top P + PF + 2\delta P = -I$ and $\|P\| = O(1)$ as $N \to +\infty$. Let $\alpha > 1$ be arbitrarily fixed. For any given $N \geq N_0+1$ we set $\beta = \sqrt{N} > 0$ and $\gamma = 1/N > 0$. In this case, we observe that $\Theta_2 \to -\infty$ as $N \to +\infty$ showing that $\Theta_2 \leq 0$ for N large enough. Moreover, since $\|\tilde{K}\| = \|K\|$ and $\|\mathcal{L}\|$ are independent of N while $\|P\| = O(1)$ and $\|E\| = O(1)$ as $N \to +\infty$, the application of the Schur complement shows that $\Theta_1 \preceq 0$ for sufficiently large $N \geq N_0+1$. This completes the proof. $\square$

Remark 6. For a given $N \geq N_0+1$ and fixing arbitrarily the value of $\alpha > 1$, the constraints (26) now take the form of LMIs. Moreover, following the proof of Theorem 3.1, this latter LMI formulation remains feasible provided N is selected large enough.

4. Case of a delayed Neumann measurement

We address in this section the output feedback stabilization of the reaction-diffusion PDE described by (2) for $\theta_1 \in [0, \pi/2)$ with delayed Neumann measurement (4).

4.1. Control strategy

Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. Let $N \geq N_0 + 1$ be arbitrarily fixed and that will be specified later. The observer dynamics is described for $t \geq 0$ by

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n u(t-h) \tag{31a}$$

$$\begin{aligned} \dot{\hat{z}}_n(t) = {} & (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t-h) \\ & - l_n \left\{ \sum_{k=1}^{N} \hat{w}_k(t)\phi_k'(0) - y_N(t) \right\},\ 1 \leq n \leq N_0 \end{aligned} \tag{31b}$$

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t-h),\ N_0 + 1 \leq n \leq N \tag{31c}$$

where $l_n \in \mathbb{R}$ are the observer gains. The command input is then defined based on the Artstein transformation (14) and the feedback (15). The well-posedness of the resulting closed-loop system follows the same arguments that the ones reported in Remark 3.

4.2. Truncated model for stability analysis

Proceeding as in Subsection 3.2 while replacing the definition of ζ, $\tilde{e}_n$, C_0 and $\tilde{C}_1$ by the following: $\zeta = \sum_{n \geq N+1} w_n \phi_n'(0)$, $\tilde{e}_n = \lambda_n e_n$, $C_0 = \begin{bmatrix} \phi_1'(0) & \ldots & \phi_{N_0}'(0) \end{bmatrix}$, and $\tilde{C}_1 = \begin{bmatrix} \frac{\phi_{N_0+1}'(0)}{\lambda_{N_0+1}} & \ldots & \frac{\phi_N'(0)}{\lambda_N} \end{bmatrix}$, we infer that the representation (24) holds for all $t \geq h$.

4.3. Main stability result

We now state the main result of this section.

Theorem 4.1. *Let $\theta_1 \in [0, \pi/2)$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q} \in \mathcal{C}^0([0,1])$. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (5) holds. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0 + 1$. Let $K \in \mathbb{R}^{1 \times N_0} \backslash \{0\}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. Let $h > 0$ be given. For a given $N \geq N_0 + 1$, assume that there exist $\epsilon \in (0, 1/2]$, $P \succ 0$, $\alpha > 1$, and $\beta, \gamma > 0$ such that*

$$\Theta_1 \preceq 0, \quad \Theta_2 \leq 0, \quad \Theta_3 \geq 0 \tag{32}$$

where Θ_1 is defined by (27a) while

$$\Theta_2 = 2\gamma \left\{ -\left(1-\frac{1}{\alpha}\right)\lambda_{N+1} + q_c + \delta \right\} + \beta M_\phi(\epsilon)\lambda_{N+1}^{1/2+\epsilon} \tag{33a}$$

$$\Theta_3 = 2\gamma\left(1-\frac{1}{\alpha}\right) - \frac{\beta M_\phi(\epsilon)}{\lambda_{N+1}^{1/2-\epsilon}} \tag{33b}$$

where $M_\phi(\epsilon) = \sum_{n\geq N+1} \frac{|\phi_n'(0)|^2}{\lambda_n^{3/2+\epsilon}} < +\infty$. Then there exists a constant $M>0$ such that for any initial condition $z_0 \in H^2(0,1)$ and $\hat{z}_n(0)\in\mathbb{R}$ so that $c_{\theta_1} z_0(0) - s_{\theta_1} z_0'(0) = 0$ and $c_{\theta_2} z_0(1) + s_{\theta_2} z_0'(1) = u_0 = K\hat{Z}_A^{N_0}(0)$, and any Lipschitz continuous $y_0 \in \mathcal{C}^0([-h,0])$ so that $y_0(0) = z_{0,x}(0)$, the trajectories of the closed-loop system composed of the plant (2), the delayed Neumann measurement (4), and the controller composed of (31) and (14-15) satisfy

$$\|z(t,\cdot)\|_{H^1}^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 \leq M e^{-2\delta t}\left(\|\mathcal{A}w_0\|_{L^2}^2 + \sum_{n=1}^{N}\hat{z}_n(0)^2 + u_0^2 + \|y_0\|_\infty^2 \right) \tag{34}$$

for all $t\geq 0$. Moreover, for any given $h>0$, the constraints (32) are always feasible for N selected large enough.

Proof. Let $V(t) = V_0(t) + V_1(t)$ for $t \geq h$ be defined by (29). The first part of the proof follows the same lines that the one of Theorem 3.1. However, since ζ is now defined by $\zeta = \sum_{n\geq N+1} w_n \phi_n'(0)$, its estimate is replaced by the following: $\beta\zeta^2 \leq \beta M_\phi(\epsilon)\sum_{n\geq N+1}\lambda_n^{3/2+\epsilon} w_n^2$. This implies that (30) holds for all $t>h$ with $\Gamma_n = 2\gamma\left\{-\left(1-\frac{1}{\alpha}\right)\lambda_n + q_c + \delta\right\} + \beta M_\phi(\epsilon)\lambda_n^{1/2+\epsilon}$. Recalling that $\epsilon\in(0,1/2]$, we obtain for any $n\geq N+1$ that $\lambda_n^{1/2+\epsilon} = \lambda_n/\lambda_n^{1/2-\epsilon} \leq \lambda_n/\lambda_{N+1}^{1/2-\epsilon}$. Since $\Theta_3\geq 0$, this implies that $\Gamma_n \leq -\Theta_3\lambda_n + 2\gamma\{q_c+\delta\} \leq \Gamma_{N+1} = \Theta_2 \leq 0$ for all $n\geq N+1$. The proof of the stability estimate (34) is now obtained using similar arguments that the ones reported in the proof of Theorem 3.1. We use here in particular for some $\alpha_0\in(3/4,1)$ the fact that, based on standard arguments similar to the ones reported in the proof of (Pazy, 2012, Thm. 6.3.3), we have the existence of a constant $c>0$, independent of the initial conditions, such that

$$\sum_{n\geq 1}\lambda_n^{2\alpha_0} w_n(t)^2 + \sum_{n=1}^{N}\hat{z}_n(t)^2 \leq c\left(\sum_{n\geq 1}\lambda_n^{2\alpha_0} w_n(0)^2 + \sum_{n=1}^{N}\hat{z}_n(0)^2 + u_0^2 + \|y_0\|_\infty^2\right)$$

for all $t\in[0,h]$. Finally, the feasibility of the constraints (32) for N large enough is obtained similarly by setting $\epsilon = 1/8$, $\beta = N^{1/8}$, and $\gamma = 1/N^{3/16}$. □

5. Numerical example

For numerical illustration of the main results of this paper, we set the parameters $p=1$, $\tilde{q}=-5$, $\theta_1=\pi/5$, $\theta_2=0$ (Dirichlet boundary control), and the input delay $h=2\,\text{s}$. The resulting reaction-diffusion PDE given by (2) is open-loop unstable.

We set the feedback gain $K=-1.6037$. The observer gain is set as $L=4.0832$ in the case of the delayed Dirichlet measurement (3) while $L = 2.9666$ in the case of the delayed Neumann measurement (4).

With fix the prescribed decay rate $\delta = 0.5$. In the case of the Dirichlet measurement (3), the constraints of Theorems 3.1 are found feasible for an observer of dimension $N = 3$, ensuring the exponential stability of the closed-loop system in H^1 norm. Dealing with the case of the Neumann boundary measurement (4), the constraints of Theorem 4.1 are found feasible for an observer of dimension $N = 15$, ensuring the exponential decay of the closed-loop system in H^1 norm.

We complete this numerical illustration by depicting the closed-loop system behavior in the case of the delayed Dirichlet measurement (3). We set the initial conditions $z_0(x) = 5x^2(x-3/4)$ and $y_0(\tau) = 3\cos(10\pi(\tau+h))\sin(3\pi\tau)$ for $\tau \leq 0$, while $\hat{z}_n(0)$ are fixed so that $u_0 = K\hat{Z}_A^{N_0}(0)$. In this setting, the time domain evolution of the closed-loop system is depicted in Fig. 1. As predicted by Theorem 3.1, we observe the exponential decay of the both state of the PDE and observation error in spite of the $h = 2\,\mathrm{s}$ output delay.

6. Extension to input and output delays

We briefly discuss in this section how the output feedback boundary stabilization of general 1-D reaction-diffusion PDEs in the presence of both input and output delays can be achieved by merging the techniques developed in this paper for an output delay and the ones reported in (Lhachemi & Prieur, 2021c) for an input delay. We focus the presentation on the Dirichlet measurement but the same procedure can be used to address the case of the Neumann measurement. Consider the reaction-diffusion system with boundary control described by

$$z_t(t,x) = (p(x)z_x(t,x))_x - \tilde{q}(x)z(t,x) \tag{35a}$$
$$c_{\theta_1}z(t,0) - s_{\theta_1}z_x(t,0) = 0 \tag{35b}$$
$$c_{\theta_2}z(t,1) + s_{\theta_2}z_x(t,1) = u(t-h_i) \tag{35c}$$
$$z(0,x) = z_0(x) \tag{35d}$$

for $t > 0$ and $x \in (0,1)$. The different parameters are defined as in (2) while $h_i > 0$ is an input delay. We assume that $u(\tau) = 0$ for all $\tau \leq 0$. In the case case $\theta_1 \in (0, \pi/2]$, the delayed Dirichlet boundary measurement is defined by

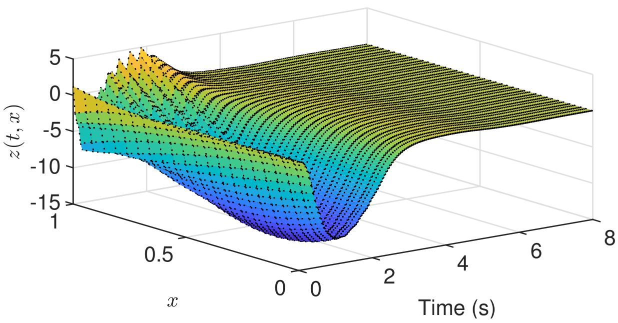
$$y_D(t) = \begin{cases} z(t-h_o,0), & t \geq h_o \\ y_0(t-h_o), & 0 \leq t \leq h_o \end{cases} \tag{36}$$

with output delay $h_o > 0$. We introduce $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ so that (5) holds. Defining the change of variable

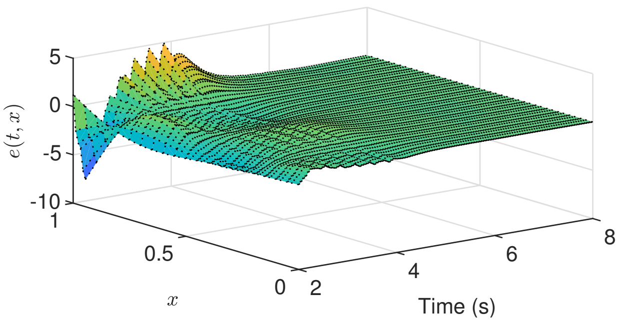
$$w(t,x) = z(t,x) - \frac{x^2}{c_{\theta_2} + 2s_{\theta_2}} u(t-h_i). \tag{37}$$

we infer that

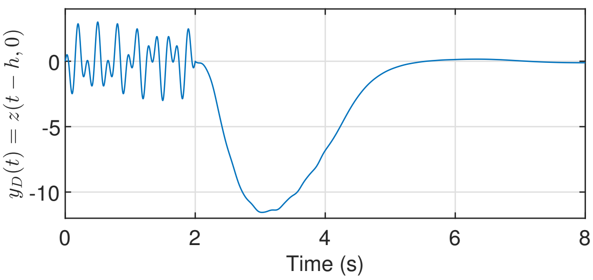
$$w_n(t) = z_n(t) + b_n u(t-h_i), \quad n \geq 1. \tag{38}$$



(a) State of the reaction-diffusion system $z(t,x)$



(b) Error of observation $e(t,x) = z(t-h,x) - \hat{z}(t,x)$ for $t \geq h$



(c) Delayed measurement $y_D(t) = z(t-h,0)$ with delay $h = 2\,\mathrm{s}$

Figure 1. Time evolution of the closed-loop system for Dirichlet measurement $y_D(t) = z(t-h,0)$ with delay $h = 2\,\mathrm{s}$

The projections of the PDE in z coordinates gives

$$\dot{z}_n(t) = (-\lambda_n + q_c)z_n(t) + \beta_n u(t - h_i) \tag{39}$$

while, in w coordinates,

$$\dot{u}(t) = v(t) \tag{40a}$$
$$\dot{w}_n(t) = (-\lambda_n + q_c)w_n(t) + a_n u(t - h_i) + b_n v(t - h_i). \tag{40b}$$

The delayed measurement is expressed for $t \geq h_o$ by

$$y_D(t) = \sum_{n \geq 1} w_n(t - h_o)\phi_n(0). \tag{41}$$

Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. Let $N \geq N_0 + 1$ be arbitrarily given. The observer dynamics, used to estimate the N first modes $z_n(t - h_o)$ of the PDE at time $t - h_o$, is described for $t \geq 0$ by

$$\hat{w}_n(t) = \hat{z}_n(t) + b_n u(t - h_{io}) \tag{42a}$$
$$\begin{aligned}\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t - h_{io}) \\ - l_n \left\{ \sum_{k=1}^{N} \hat{w}_k(t)\phi_k(0) - y_D(t) \right\},\ 1 \leq n \leq N_0\end{aligned} \tag{42b}$$
$$\dot{\hat{z}}_n(t) = (-\lambda_n + q_c)\hat{z}_n(t) + \beta_n u(t - h_{io}),\ N_0 + 1 \leq n \leq N \tag{42c}$$

where $l_n \in \mathbb{R}$ are the observer gains and $h_{io} = h_i + h_o > 0$. Introducing the predictor component defined by

$$\hat{Z}_A^{N_0}(t) = e^{A_0 h_{io}}\hat{Z}^{N_0}(t) + \int_{t-h_{io}}^{t} e^{A_0(t-s)}\mathfrak{B}_0 u(s)\,\mathrm{d}s, \tag{43}$$

we define the control input as

$$u(t) = K\hat{Z}_A^{N_0}(t) \tag{44}$$

for all $t \geq 0$ where $K \in \mathbb{R}^{1\times N_0}$ is the feedback gain. Then proceeding as in Subsection 3.2 but with $\tilde{Z}_A^{N-N_0}$ defined by

$$\tilde{Z}_A^{N-N_0}(t) = e^{A_1 h_{io}}\tilde{Z}^{N-N_0}(t) + \int_{t-h_{io}}^{t} e^{A_1(t-s)}\tilde{\mathfrak{B}}_1 u(s)\,\mathrm{d}s, \tag{45}$$

we infer that

$$\dot{X}(t) = FX(t) + \mathcal{L}\zeta(t - h_o) \tag{46}$$

for all $t \geq h_0$ where X is defined by (23) and $\zeta = \sum_{n\geq N+1} w_n\phi_n(0)$ while the matrix

F and the vector $\mathcal{L}$ are defined by

$$F = \begin{bmatrix} A_0 + \mathfrak{B}_0 K & e^{A_0 h_{io}} L C_0 & 0 & e^{A_0 h_{io}} L \tilde{C}_1 \\ 0 & A_0 - L C_0 & 0 & -L \tilde{C}_1 \\ \tilde{\mathfrak{B}}_1 K & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \qquad \mathcal{L} = \begin{bmatrix} e^{A_0 h_{io}} L \\ -L \\ 0 \\ 0 \end{bmatrix}.$$

Defining $\tilde{X}(t) = \mathrm{col}\,(X(t), \zeta(t-h_o))$, we have $u(t) = \tilde{K}X(t)$ and $v(t) = \dot{u}(t) = K\dot{\hat{Z}}_A^{N_0}(t)$ for all $t \geq 0$ where $\tilde{K} = \begin{bmatrix} K & 0 & 0 & 0 \end{bmatrix}$. Moreover, we also have $\dot{\hat{Z}}_A^{N_0}(t) = E\tilde{X}(t)$ for all $t \geq h_o$ with $E = \begin{bmatrix} A_0 + \mathfrak{B}_0 K & e^{A_0 h_{io}} L C_0 & 0 & e^{A_0 h_{io}} L\tilde{C}_1 & e^{A_0 h_{io}} L \end{bmatrix}$.

Combining now the approaches developed in this paper and in (Lhachemi & Prieur, 2021c) to handle the output delay $h_o > 0$ appearing in (46) and the input delay $h_i > 0$ occurring in (40), respectively, we arrive at the following theorem.

Theorem 6.1. *Let $\theta_1 \in (0, \pi/2]$, $\theta_2 \in [0, \pi/2]$, $p \in \mathcal{C}^2([0,1])$ with $p > 0$, and $\tilde{q} \in \mathcal{C}^0([0,1])$. Let $q \in \mathcal{C}^0([0,1])$ and $q_c \in \mathbb{R}$ be such that (5) holds. Let $\delta > 0$ and $N_0 \geq 1$ be such that $-\lambda_n + q_c < -\delta$ for all $n \geq N_0 + 1$. Let $K \in \mathbb{R}^{1 \times N_0}$ and $L \in \mathbb{R}^{N_0}$ be such that $A_0 + \mathfrak{B}_0 K$ and $A_0 - LC_0$ are Hurwitz with eigenvalues that have a real part strictly less than $-\delta < 0$. Let $h_i, h_o > 0$ be given. For a given $N \geq N_0 + 1$, assume that there exist $P \succ 0$, $Q_1, Q_2 \succeq 0$, $\alpha > 1$, and $\beta, \gamma > 0$ such that*

$$\Theta_1 \preceq 0, \quad \Theta_2 \leq 0, \quad R_1 \preceq 0, \quad R_2 \preceq 0 \tag{47}$$

where

$$\Theta_1 = \begin{bmatrix} F^\top P + PF + 2\delta P + \tilde{Q}_1 & P\mathcal{L} \\ \mathcal{L}^\top P & -\beta e^{-2\delta h_o} \end{bmatrix} + E^\top Q_2 E \tag{48a}$$

$$\Theta_2 = 2\gamma \left\{ -\left(1 - \frac{1}{\alpha}\right) \lambda_{N+1} + q_c + \delta \right\} + \beta M_\phi \tag{48b}$$

$$R_1 = -e^{-2\delta h_i} Q_1 + \alpha\gamma \|\mathcal{R}_N a\|_{L^2}^2 K^\top K \tag{48c}$$

$$R_2 = -e^{-2\delta h_i} Q_2 + \alpha\gamma \|\mathcal{R}_N b\|_{L^2}^2 K^\top K \tag{48d}$$

where $\tilde{Q}_1 = \mathrm{diag}(Q_1, 0, 0, 0)$ and $M_\phi = \sum_{n \geq N+1} \frac{|\phi_n(0)|^2}{\lambda_n} < +\infty$. Then there exists a constant $M > 0$ such that for any initial condition $z_0 \in H^2(0,1)$ and $\hat{z}_n(0) \in \mathbb{R}$ so that $c_{\theta_1} z_0(0) - s_{\theta_1} z_0'(0) = 0$ and $c_{\theta_2} z_0(1) + s_{\theta_2} z_0'(1) = 0$, and any Lipschitz continuous $y_0 \in \mathcal{C}^0([-h_o, 0])$ so that $y_0(0) = z_0(0)$, the trajectories of the closed-loop system composed of the plant (35), the delayed Dirichlet measurement (36), and the controller (42-44) with zero control in negative time and zero initial condition for the observer ($u(\tau) = 0$ for $\tau < 0$ and $\hat{z}_n(0) = 0$) satisfy

$$\|z(t,\cdot)\|_{H^1}^2 + \sum_{n=1}^{N} \hat{z}_n(t)^2 \leq M e^{-2\delta t} \left(\|z_0\|_{H^1}^2 + \|y_0\|_\infty^2 \right) \tag{49}$$

for all $t \geq 0$. Moreover, for any given $h_i, h_o > 0$, the constraints (47) are always feasible for N selected large enough.

7. Conclusion

This paper solved the problem of output feedback stabilization of 1-D reaction-diffusion PDEs in the presence of an arbitrary output delay. The proposed setting embraces general Dirichlet/Neumann/Robin boundary condition/control along with Dirichlet/Neumann boundary measurement. While the output feedback stabilization of general 1-D reaction-diffusion PDEs was reported in (Lhachemi & Prieur, 2021c) for an arbitrary input delay and in (Lhachemi & Shorten, 2021) for an arbitrary state delay in the reaction term, the results presented in this paper complete the full picture by addressing the case of an arbitrary output delay. Furthermore, we showed how the combination of the techniques developed in this paper with the ones reported in (Lhachemi & Prieur, 2021c) allows to address the case of simultaneous input and output delays.

It is worth noting that the main results of this paper can be extended in a straightforward manner to any $\theta_1, \theta_2 \in [0, \pi)$. This can be achieved by 1) selecting q in (5) sufficiently large positive so that (1) holds true; 2) adapt the change of variable formula (6) to avoid a possible division by 0 by using $w(t,x) = z(t,x) - \frac{x^\alpha}{c_{\theta_2} + \alpha s_{\theta_2}} u(t)$ where $\alpha > 1$ is selected such that $c_{\theta_2} + \alpha s_{\theta_2} \neq 0$.

Funding

The work of C. Prieur has been partially supported by MIAI@Grenoble Alpes (ANR-19-P3IA-0003)

8. References

References

Artstein, Z. (1982). Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, *27*(4), 869–879.

Balas, M. J. (1988). Finite-dimensional controllers for linear distributed parameter systems: exponential stability using residual mode filters. *Journal of Mathematical Analysis and Applications*, *133*(2), 283–296.

Coron, J.-M., & Trélat, E. (2004). Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM Journal on Control and Optimization*, *43*(2), 549–569.

Coron, J.-M., & Trélat, E. (2006). Global steady-state stabilization and controllability of 1D semilinear wave equations. *Communications in Contemporary Mathematics*, *8*(04), 535–567.

Curtain, R. (1982). Finite-dimensional compensator design for parabolic distributed systems with point sensors and boundary input. *IEEE Transactions on Automatic Control*, *27*(1), 98–104.

Curtain, R., & Zwart, H. (2012). *An introduction to infinite-dimensional linear systems theory* (Vol. 21). Springer Science & Business Media.

Deng, Y., Léchappé, V., Moulay, E., & Plestan, F. (2019). Prediction-based control with delay estimation of lti systems with input-output delays. In *2019 american control conference (acc)* (pp. 3702–3707).

Grüne, L., & Meurer, T. (2021). Finite-dimensional output stabilization of linear diffusion-reaction systems–a small-gain approach. *arXiv preprint arXiv:2104.06102*.

Harkort, C., & Deutscher, J. (2011). Finite-dimensional observer-based control of linear distributed parameter systems using cascaded output observers. *International journal of control*, *84*(1), 107–122.

Hashimoto, T., & Krstic, M. (2016). Stabilization of reaction diffusion equations with state delay using boundary control input. *IEEE Transactions on Automatic Control*, *61*(12), 4041–4047.

Kang, W., & Fridman, E. (2017). Boundary control of delayed ODE-heat cascade under actuator saturation. *Automatica*, *83*, 252–261.

Karafyllis, I., & Krstić, M. (2017). *Predictor feedback for delay systems: Implementations and approximations*. Springer.

Katz, R., & Fridman, E. (2020). Constructive method for finite-dimensional observer-based control of 1-D parabolic PDEs. *Automatica*, *122*, 109285.

Katz, R., & Fridman, E. (2021a). Delayed finite-dimensional observer-based control of 1-D parabolic PDEs. *Automatica*, *123*, 109364.

Katz, R., & Fridman, E. (2021b). Sub-predictors and classical predictors for finite-dimensional observer-based control of parabolic PDEs. *arXiv preprint arXiv:2104.13294*.

Krstic, M. (2009). Control of an unstable reaction-diffusion PDE with long input delay. *Systems & Control Letters*, *58*(10-11), 773–782.

Lhachemi, H., & Prieur, C. (2020). Finite-dimensional observer-based boundary stabilization of reaction-diffusion equations with a either Dirichlet or Neumann boundary measurement. *submitted to Automatica*. Retrieved from http://www.gipsa-lab.grenoble-inp.fr/%7Echristophe.prieur/Papers/submitted.pdf

Lhachemi, H., & Prieur, C. (2021a). Feedback stabilization of a class of diagonal infinite-dimensional systems with delay boundary control. *IEEE Transactions on Automatic Control*, *66*(1), 105–120.

Lhachemi, H., & Prieur, C. (2021b). Nonlinear boundary output feedback stabilization of reaction diffusion PDEs. *arXiv preprint arXiv:2105.08418*.

Lhachemi, H., & Prieur, C. (2021c). Predictor-based output feedback stabilization of an input delayed parabolic PDE with boundary measurement. *arXiv preprint arXiv:2105.08431*.

Lhachemi, H., Prieur, C., & Shorten, R. (2019). An LMI condition for the robustness of constant-delay linear predictor feedback with respect to uncertain time-varying input delays. *Automatica*, *109*, 108551.

Lhachemi, H., Prieur, C., & Shorten, R. (2021). Robustness of constant-delay predictor feedback for in-domain stabilization of reaction–diffusion PDEs with time-and spatially-varying input delays. *Automatica*, *123*, 109347.

Lhachemi, H., & Shorten, R. (2020). Boundary feedback stabilization of a reaction–diffusion equation with robin boundary conditions and state-delay. *Automatica*, *116*, 108931.

Lhachemi, H., & Shorten, R. (2021). Boundary output feedback stabilization of state delayed reaction-diffusion PDEs. *arXiv preprint arXiv:2105.15056*.

Nicaise, S., & Pignotti, C. (2008). Stabilization of the wave equation with boundary or internal distributed delay. *Differential and Integral Equations*, *21*(9-10), 935–958.

Orlov, Y. (2017). On general properties of eigenvalues and eigenfunctions of a Sturm–Liouville operator: comments on "ISS with respect to boundary disturbances for 1-D parabolic PDEs". *IEEE Transactions on Automatic Control*, *62*(11), 5970–5973.

Pazy, A. (2012). *Semigroups of linear operators and applications to partial differential equations* (Vol. 44). Springer Science & Business Media.

Qi, J., & Krstic, M. (2020). Compensation of spatially-varying input delay in distributed control of reaction-diffusion PDEs. *IEEE Transactions on Automatic Control*.

Richard, J.-P. (2003). Time-delay systems: an overview of some recent advances and open problems. *Automatica*, *39*(10), 1667–1694.

Russell, D. L. (1978). Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Review*, *20*(4), 639–739.

Wang, J.-W., & Sun, C.-Y. (2018). Delay-dependent exponential stabilization for linear distributed parameter systems with time-varying delay. *Journal of Dynamic Systems, Measurement, and Control*, *140*(5), 051003.

Appendix A. Technical lemma

The following Lemma is an immediate generalization of the result presented in (Katz & Fridman, 2020).

Lemma A.1. *Let $n, m, N \geq 1$, $M_{11} \in \mathbb{R}^{n\times n}$ and $M_{22} \in \mathbb{R}^{m\times m}$ Hurwitz, $M_{12} \in \mathbb{R}^{n\times m}$, $M_{14}^N \in \mathbb{R}^{n\times N}$, $M_{24}^N \in \mathbb{R}^{m\times N}$, $M_{31}^N \in \mathbb{R}^{N\times n}$, $M_{33}^N, M_{44}^N \in \mathbb{R}^{N\times N}$, and*

$$F^N = \begin{bmatrix} M_{11} & M_{12} & 0 & M_{14}^N \\ 0 & M_{22} & 0 & M_{24}^N \\ M_{31}^N & 0 & M_{33}^N & 0 \\ 0 & 0 & 0 & M_{44}^N \end{bmatrix}.$$

We assume that there exist constants $C_0, \kappa_0 > 0$ such that $\|e^{M_{33}^N t}\| \leq C_0 e^{-\kappa_0 t}$ and $\|e^{M_{44}^N t}\| \leq C_0 e^{-\kappa_0 t}$ for all $t \geq 0$ and all $N \geq 1$. Moreover, we assume that there exists a constant $C_1 > 0$ such that $\|M_{14}^N\| \leq C_1$, $\|M_{24}^N\| \leq C_1$, and $\|M_{31}^N\| \leq C_1$ for all $N \geq 1$. Then there exists a constant $C_2 > 0$ such that, for any $N \geq 1$, there exists a symmetric matrix $P^N \in \mathbb{R}^{n+m+2N}$ with $P^N \succ 0$ such that $P^N F^N + (F^N)^\top P^N = -I$ and $\|P^N\| \leq C_2$.