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# Output feedback stabilization of a clamped-free beam

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We consider a Euler–Bernoulli beam, clamped at one extremity and free at the other, to which are attached a piezoelectric actuator and a collocated sensor touching the clamped extremity. We provide an output feedback law and characterize the sensor/actuator lengths for which the strong stabilization holds. Finally, we prove that the energy decreases to zero in a polynomial way for almost all lengths, and in an exponential way for lengths admitting a certain coprime factorization.

## 1. Introduction

The Euler–Bernoulli equation provides a very commonly used model for the dynamic behaviour of a flexible beam, when the cross-sectional dimensions of the beam are small in comparison to its length. In the last decade, a lot of papers were devoted to the pointwise feedback stabilization of a beam; see e.g. Chen *et al*. (1987), Conrad (1990), Lagnese and Leugering (1991a, b), Morgül (1992), Rebarber (1995), Liu and Zheng (1999), Ammari and Tucsnak (2000) and de Queiroz *et al*. (2000) or to the modelling of piezoceramic actuation of beams (Crawley and Anderson 1990, Banks and Smith 1992) or to the controllability of such systems (Crépeau and Prieur (to appear), Tucsnak 1996). The piezoceramic actuation of beams or plates is investigated numerically and experimentally in Destuynder *et al*. (1992), Banks *et al*. (1993) and Destuynder (1999).

In this paper, we consider the control problem modelling the vibrations of a Euler–Bernoulli beam which is subject to the action of an attached piezoelectric actuator. If we assume that the beam is clamped at one end and free at the other, and that the actuator is touching the clamped end, then we obtain the system

$$
\begin{aligned}
w_{tt}(x,t) + w_{xxxx}(x,t) &= -h(t)\frac{\mathrm{d}\delta_\xi}{\mathrm{d}x}(x),\\
w(0,t) = w_x(0,t) = w_{xx}(\pi,t) &= w_{xxx}(\pi,t) = 0,\\
w(x,0) = w^0(x),\ \ w_t(x,0) &= w^1(x).
\end{aligned}
$$

In the above equations, $x \in (0,\pi)$ is the spatial coordinate, $t$ is time, $w$ stands for the transverse deflection of the beam, $w_t = \partial w/\partial t$, etc., 0 and $\xi$ are the coordinates of the extremities of the actuator ($0 < \xi \le \pi$), $\delta_y$ is the Dirac measure at the point $y$, and $h$ stands for the control input (see figure 1).

We are interested in the stabilization of the above system, the control $h$ being expressed as a function of the output $w_x(0,t) - w_x(\xi,t) = -w_x(\xi,t)$. This corresponds to the situation where the output comes from a piezoelectric sensor located on the same interval $(0,\xi)$ as the actuator. Formal computations on the variations of the energy

$$E(w,w_t) = \int_0^\pi (|w_{xx}|^2 + |w_t|^2)\,\mathrm{d}x$$

lead to take

$$h(t) := -Kw_{xt}(\xi,t)$$

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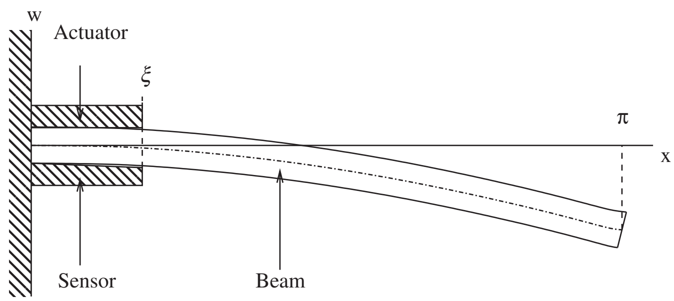


Figure 1. Clamped-free beam with collocated sensor/actuator.

where $K$ is some number whose range will be specified later. In short, we are interested in the stability properties of the closed-loop system

$$w_{tt}(x,t)+w_{xxxx}(x,t)=Kw_{xt}(\xi,t)\frac{\mathrm{d}\delta_\xi}{\mathrm{d}x}(x), \tag{1}$$

$$w(0,t)=w_x(0,t)=w_{xx}(\pi,t)=w_{xxx}(\pi,t)=0, \tag{2}$$

$$w(x,0)=w^0(x),\ w_t(x,0)=w^1(x). \tag{3}$$

The main goal of the paper is to characterize the lengths of the sensor/actuator leading to "good" stability properties for (1)–(3), i.e., with an exponential (or polynomial) energy decay rate.

The study of the controllability properties of a PDE is often considered as the first step towards the control theory of that PDE. The exact controllability of a beam hinged at both ends and subject to the action of a piezoelectric actuator is characterized in function of the location of the actuator in Tucsnak (1996). It is shown in that paper that the space of controllable states is related to Diophantine approximation properties for the extremities of the actuator. A similar study is performed in Crépeau and Prieur (to appear) in the more difficult framework of a clamped-free beam. The stabilization of a beam hinged at both ends and subject to a pointwise force has been deeply studied in Ammari and Tucsnak (2000). The method of proof consists of deducing the decay estimates from observability inequalities for the associated free problem via sharp trace regularity results.

When we compare our work with Ammari and Tucsnak (2000) we notice that: (i) the boundary conditions here, where only an asymptotic expansion of the eigenvalues for the free evolution of (1)–(3) is known, are more difficult to handle than the ones in Ammari and Tucsnak (2000) (where e.g. the eigenvectors for the free evolution reduce to sin functions); (ii) the method of proof used in Ammari and Tucsnak (2000) cannot be applied here due to an additional difficulty. The key observed quantity, namely the integral term $\int_0^T |w_{xt}(\xi,t)|^2\,\mathrm{d}t$, is of the same order as $\|(w^0,w^1)\|^2_{H^3\times H^1}$ for the free evolution ($K=0$), whereas it is of the same order as $\|(w^0,w^1)\|^2_{H^2\times H^0}$ for the controlled system ($K>0$). Thus, a spectral analysis resting on the free evolution as in Ammari and Tucsnak (2000) cannot be performed here.

To obtain our decay estimates, we follow the frequency-domain approach described in Liu and Zheng (1999) and based on a sharp estimation of a resolvent on the imaginary axis. More precisely, we combine the multiplier method to a careful computation of the value at $x=\pi$ of the solution to (1)–(3). Thanks to that calculation, we obtain an explicit polynomial decay estimate valid for almost every value of $\xi$, and an exponential decay estimate when $\xi$ admits a certain coprime factorization.

The paper is outlined as follows. The well-posedness of (1)–(3) in the energy space is investigated in § 2. The lengths $\xi$ of the actuator for which the strong stability holds are characterized in § 3. The last section contains the main results of the paper, namely Theorem 3 (resp., Theorem 4) which provides an exponential (resp., polynomial) energy decay rate.

## 2. Well-posedness of (1)–(3)

The well-posedness of (1)–(3) in the energy space is obtained in this section as a direct application of the classical semi-group theory. To specify the operator, we have to see for which function $w$ the r.h.s. of (1) belongs to $L^2(0,\pi)$. Let us introduce some notations. If $w$ is any function in $H^1(0,\xi)\cap H^1(\xi,\pi)$, we define $\{w_x\}\in L^2(0,\pi)$ by

$$\{w_x\}(x):=\begin{cases} w_x^{\mathcal{D}'(0,\xi)}(x) & \text{if } x\in(0,\xi),\\ w_x^{\mathcal{D}'(\xi,\pi)}(x) & \text{if } x\in(\xi,\pi),\end{cases}$$

where $w_x^{\mathcal{D}'(0,\xi)}$ (resp. $w_x^{\mathcal{D}'(\xi,\pi)}$) denotes the distributional derivative $\partial w/\partial x$ in $\mathcal{D}'(0,\xi)$ (resp. in $\mathcal{D}'(\xi,\pi)$). We also set $[w]_\xi := w(\xi^+) - w(\xi^-)$. It follows that

$$w_x = \{w_x\} + [w]_\xi \delta_\xi \quad \text{in } \mathcal{D}'(0,\pi).$$

Assume now that $w, v \in H^2(0,\pi)$, and define $u \in \mathcal{D}'(0,\pi)$ by $u := -w_{xxxx} + Kv_x(\xi)\mathrm{d}\delta_\xi/\mathrm{d}x$. If $u \in L^2(0,\pi)$, then the restriction of $w_{xxxx}$ to each of the intervals $(0,\xi)$ and $(\xi,\pi)$ has to be a square-integrable function; hence $w \in H^4(0,\xi) \cap H^4(\xi,\pi)$. Calculating $w_{xxx}$, $w_{xxxx}$ and $u$, we obtain

$$w_{xxx} = \{w_{xxx}\} + [w_{xx}]_\xi \delta_\xi,$$
$$w_{xxxx} = \{w_{xxxx}\} + [w_{xxx}]_\xi \delta_\xi + [w_{xx}]_\xi \frac{\mathrm{d}}{\mathrm{d}x}\delta_\xi,$$

and

$$u = -\{w_{xxxx}\} - [w_{xxx}]_\xi \delta_\xi - [w_{xx}]_\xi \frac{\mathrm{d}}{\mathrm{d}x}\delta_\xi + Kv_x(\xi)\frac{\mathrm{d}}{\mathrm{d}x}\delta_\xi. \tag{4}$$

Then $u \in L^2(0,\pi)$ provided that all the coefficients in front of the Dirac measures vanish, i.e.,

$$K\, v_x(\xi) = [w_{xx}]_\xi$$

and

$$[w_{xxx}]_\xi = 0.$$

We are now in a position to define the operator associated with (1)–(3).

Let $V = \{w \in H^2(0,\pi) \mid w(0) = w'(0) = 0\}$ (where $' = \mathrm{d}/\mathrm{d}x$) $(w_1, w_2)_V = \int_0^\pi w_1'' \overline{w_2''}\mathrm{d}x$, and $H = L^2(0,\pi)$, $(v_1, v_2)_H = \int_0^\pi v_1 \overline{v_2}\mathrm{d}x$. Then $\mathcal{H} = V \times H$, endowed with the usual product norm, is a (complex) Hilbert space. If we introduce $v := w_t$ and define the operator $\mathcal{A}$ with domain

$$\begin{aligned}\mathcal{D}(\mathcal{A}) = \big\{ & z = (w,v) \mid (w,v) \in H^2(0,\pi)^2, \\ & w \in H^4(0,\xi) \cap H^4(\xi,\pi), \\ & w(0) = w_x(0) = w_{xx}(\pi) = w_{xxx}(\pi) = 0,\ v(0) = v_x(0) = 0, \\ & Kv_x(\xi) = [w_{xx}]_\xi,\ [w_{xxx}]_\xi = 0 \big\}\end{aligned}$$

by

$$\mathcal{A}z = \left(v, -w_{xxxx} + Kv_x(\xi)\frac{\mathrm{d}}{\mathrm{d}x}\delta_\xi\right) = (v, -\{w_{xxxx}\}), \tag{5}$$

then we see that the closed-loop system (1)–(3) may be interpreted as the initial value problem for the abstract first-order evolution equation in $\mathcal{H}$

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathcal{A}z, & t > 0 \\ z(0) = (w^0, w^1). \end{cases}$$

The first result in this paper is the following one.

**Theorem 1:** *If $K \geq 0$, then $\mathcal{A}$ generates a $C_0$-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ of contractions in $\mathcal{H}$.*

**Proof:** Obviously, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. According to a classical result (see e.g. Liu and Zheng (1999, [Theorem 1.2.4]), the theorem is proved if we show that $\mathcal{A}$ is dissipative and that $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Let us begin with the dissipativity property.

**Lemma 1:** *For any $z = (w,v) \in \mathcal{D}(\mathcal{A})$ we have that*

$$(\mathcal{A}z, z)_{\mathcal{H}} = 2i\,\mathrm{Im}\left(\int_0^\pi v_{xx}\overline{w_{xx}}\,\mathrm{d}x\right) - K|v_x(\xi)|^2. \tag{6}$$

*In particular,* $\mathrm{Re}(\mathcal{A}z, z)_{\mathcal{H}} = -K|v_x(\xi)|^2 \leq 0$, *i.e., $\mathcal{A}$ is dissipative.*

**Proof of Lemma 1:** Pick any pair of functions $(w,v) \in \mathcal{D}(\mathcal{A})$. Then

$$(\mathcal{A}(w,v),(w,v))_{\mathcal{H}} = \int_0^\pi v_{xx}\overline{w_{xx}}\,\mathrm{d}x - \int_0^\pi \{w_{xxxx}\}\overline{v}\,\mathrm{d}x.$$

After some integrations by parts on the intervals $(0,\xi)$ and $(\xi,\pi)$, we obtain that

$$\begin{aligned} -\int_0^\pi \{w_{xxxx}\}\overline{v} &= -\int_0^\xi \{w_{xxxx}\}\overline{v} - \int_\xi^\pi \{w_{xxxx}\}\overline{v} \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\,\mathrm{d}x + [w_{xx}\overline{v_x}]_{x=0}^\xi + [w_{xx}\overline{v_x}]_{x=\xi}^\pi \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\,\mathrm{d}x - [w_{xx}]_\xi \overline{v_x(\xi)}. \end{aligned}$$

Hence

$$\begin{aligned}(\mathcal{A}(w,v),(w,v))_{\mathcal{H}} &= \int_0^\pi (v_{xx}\overline{w_{xx}} - w_{xx}\overline{v_{xx}})\,\mathrm{d}x - [w_{xx}]_\xi\overline{v_x(\xi)} \\ &= 2i\,\mathrm{Im}\left(\int_0^\pi v_{xx}\overline{w_{xx}}\,\mathrm{d}x\right) - K|v_x(\xi)|^2.\end{aligned}$$

We now proceed to the study of $\mathcal{A}^{-1}$. The following result holds true. □

**Proposition 1:** $0 \in \rho(\mathcal{A})$.

**Proof:** We have to prove that the operator $\mathcal{A}: \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is one-to-one and onto, and that its inverse $\mathcal{A}^{-1}: \mathcal{H} \to \mathcal{H}$ is continuous. Let a pair $(f, g) \in \mathcal{H}$ be given, and let us investigate the equation $\mathcal{A}(w, v) = (f, g)$, where $(w, v)$ has to be found in $\mathcal{D}(\mathcal{A})$. We have to solve the system

$$\begin{cases} v = f \\ -\{w_{xxxx}\} = g \end{cases}$$

supplemented by adequate boundary conditions. To eliminate $g$, we introduce the (unique) solution $\tilde{w} \in H^4(0, \pi)$ of the following elliptic problem

$$\begin{cases} -\tilde{w}_{xxxx} = g \\ \tilde{w}(0) = \tilde{w}_x(0) = \tilde{w}_{xx}(\pi) = \tilde{w}_{xxx}(\pi) = 0. \end{cases}$$

Setting $w = \tilde{w} + \hat{w}$ and $C := Kv_x(\xi) = Kf_x(\xi)$, we have to solve

$$\{\hat{w}_{xxxx}\} = 0 \tag{7}$$

$$\hat{w}(0) = \hat{w}_x(0) = \hat{w}_{xx}(\pi) = \hat{w}_{xxx}(\pi) = 0 \tag{8}$$

$$[\hat{w}_{xx}]_\xi = C \tag{9}$$

$$[\hat{w}_{xxx}]_\xi = 0 \tag{10}$$

where $\hat{w}$ is to be found in $H^2(0, \pi) \cap H^4(0, \xi) \cap H^4(\xi, \pi)$. In particular,

$$[\hat{w}]_\xi = [\hat{w}_x]_\xi = 0. \tag{11}$$

We infer from (7) and (8) that there exist some constants $a_2$, $a_3$, $b_0$, and $b_1$ such that

$$\hat{w}(x) = \begin{cases} a_2 x^2 + a_3 x^3 & \text{if } 0 < x < \xi, \\ b_0 + b_1(x - \pi) & \text{if } \xi < x < \pi. \end{cases}$$

It follows from (10) that $a_3 = 0$. Equation (9) gives $-2a_2 = C$; hence $a_2 = -C/2$. Finally, it is easily seen that $b_0$ and $b_1$ are uniquely determined by the following system of linear equations (coming from (11))

$$\begin{cases} b_0 + b_1(\xi - \pi) = a_2\xi^2, \\ b_1 = 2a_2\xi. \end{cases}$$

This proves the existence and uniqueness of $\hat{w}$, and the existence and uniqueness of a pair $(w, v) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}(w, v) = (f, g)$. To see that $0 \in \rho(\mathcal{A})$, it remains to prove that the map $\mathcal{A}^{-1}: \mathcal{H} \to \mathcal{H}$ is continuous. Let $(w, v) = \mathcal{A}^{-1}(f, g)$. Then $\|v\|_H = \|f\|_H$, $\|\tilde{w}\|_{H^4(0,\pi)} \le \text{Const}\|g\|_H$, $\|\hat{w}\|_V \le \text{Const}|C| \le \text{Const}\|f\|_V$. Therefore,

$$\|(w, v)\|_{\mathcal{H}} \le \text{Const}\|(f, g)\|_{\mathcal{H}}.$$

This completes the proofs of Proposition 1 and of Theorem 1. □

**Proposition 2:** *The operator $\mathcal{A}^{-1}: \mathcal{H} \to \mathcal{H}$ is compact.*

**Proof:** Since

$$\|\mathcal{A}^{-1}(f, g)\|_{\mathcal{D}(\mathcal{A})} = \|\mathcal{A}^{-1}(f, g)\|_{\mathcal{H}} + \|(f, g)\|_{\mathcal{H}},$$

we see that $\mathcal{A}^{-1}$ is continuous from $\mathcal{D}$ into $\mathcal{D}(\mathcal{A})$, therefore it is sufficient to prove that the embedding $\mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is compact. Let $\left((w^n, v^n)\right)_{n\ge 0}$ be any bounded sequence in $\mathcal{D}(\mathcal{A})$. We have to prove that a subsequence $\left((w^{n'}, v^{n'})\right)$ converges (strongly) in $\mathcal{H}$. As $\|v^n\|_{H^2(0,\pi)} \le$ Const., there exist a subsequence $(v^{n'})$ and a function $v \in H = L^2(0, \pi)$ such that $v^{n'} \to v$ in $H$. On the other hand, $\|w^n\|_{H^2(0,\pi)} + \|\{w^n_{xxxx}\}\|_{L^2(0,\pi)} \le$ Const., so

$$\|w^n\|_{H^4(0,\xi)} + \|w^n\|_{H^4(\xi,\pi)} \le \text{Const}.$$

Extracting if needed a subsequence again denoted by $\left((w^{n'}, v^{n'})\right)$, we infer that there exists a function $w \in V$ such that $w^{n'} \rightharpoonup w$ in $H^2(0, \pi)$, and $w^{n'} \rightharpoonup w$ in $H^4(0, \xi)$ and in $H^4(\xi, \pi)$. It follows that $w^{n'} \to w$ in $H^2(0, \xi)$ and in $H^2(\xi, \pi)$. We conclude that $w^{n'} \to w$ in $V$. □

## 3. Strong stability

### 3.1 *Free evolution*

In this section we recall some useful facts about the *free* evolution of (1)–(3); i.e., when $K = 0$. Thus we consider the homogeneous Cauchy problem

$$\phi_{tt} + \phi_{xxxx} = 0, \tag{12}$$

$$\phi(0, t) = \phi_x(0, t) = \phi_{xx}(\pi, t) = \phi_{xxx}(\pi, t) = 0, \tag{13}$$

$$\phi(., 0) = \phi^0, \ \phi_t(., 0) = \phi^1. \tag{14}$$

Let $A: \mathcal{D}(A) \to L^2(0, \pi)$ be the operator with domain

$$\mathcal{D}(A) := \left\{\phi \in H^4(0, \pi) | \ \phi(0) = \phi_x(0) = \phi_{xx}(\pi) = \phi_{xxx}(\pi) = 0\right\}$$

and defined by $A\phi = \phi_{xxxx}$. Obviously, $A^{-1}$ is a compact symmetric operator on $L^2(0, \pi)$, hence there exists a countable orthonormal basis of $L^2(0, \pi)$ constituted of eigenvectors of $A^{-1}$ (and of $A$). The following result [(Crepeau and Prieur (to appear), Lemma 2.1] provides useful results about the eigenvectors of $A$.

**Proposition 3:** *The $L^2(0,\pi)$-normalized eigenfunctions of $A$ are the functions $(\psi_k)_{k\geq 1}$ defined by*

$$\psi_k(x) = \frac{1}{\sqrt{\pi}}\big\{\cos(\alpha_k x) - \cosh(\alpha_k x) + \mu_k\big(\sinh(\alpha_k x) - \sin(\alpha_k x)\big)\big\}, \tag{15}$$

*where $\alpha_k$ is the kth positive root of*

$$1 + \cos(\alpha_k \pi)\cosh(\alpha_k \pi) = 0, \tag{16}$$

*and*

$$\mu_k = \frac{\cos(\alpha_k\pi) + \cosh(\alpha_k\pi)}{\sin(\alpha_k\pi) + \sinh(\alpha_k\pi)},$$

*and the eigenvalue associated with $\psi_k$ is $\lambda_k = |\alpha_k|^4$. Moreover, we have as $k \to +\infty$*

$$\alpha_k = k - \frac{1}{2} + (-1)^{k+1}\frac{2e^{\pi/2}}{\pi}e^{-\pi k} + o(e^{-\pi k}), \tag{17}$$

$$\mu_k = 1 + 2(-1)^k e^{-\alpha_k\pi} + o(e^{-\alpha_k\pi}), \tag{18}$$

and

$$-\sinh(\alpha_k\rho) + \mu_k\cosh(\alpha_k\rho) = \begin{cases} e^{-\alpha_k\rho} + o(e^{-\alpha_k\rho}) & \text{if } 0 < \rho < \pi/2, \\ (-1)^k e^{-\alpha_k(\pi-\rho)} + o(e^{-\alpha_k(\pi-\rho)}) & \text{if } \pi/2 < \rho < \pi. \end{cases} \tag{19}$$

It is easily seen that if $\phi^0 = \sum_{k\geq 1}\phi_k^0\psi_k$, and $\phi^1 = \sum_{k\geq 1}\phi_k^1\psi_k$, then the solution $\phi = \phi(x,t)$ of (12)–(14) reads

$$\phi(x,t) = \sum_{k=1}^{+\infty}\left(\phi_k^0\cos(|\alpha_k|^2 t) + \frac{\phi_k^1}{|\alpha_k|^2}\sin(|\alpha_k|^2 t)\right)\psi_k(x). \tag{20}$$

### 3.2 *Statement of the strong stability result*

For any $k \geq 1$ let

$$\mathcal{S}_k := \{\xi \in (0,\pi] \mid \psi_k'(\xi) = 0\}, \tag{21}$$

and set

$$\mathcal{S} := \cup_{k\geq 1}\mathcal{S}_k.$$

**Definition 1:** We say that (1)–(3) is strongly stable in $\mathcal{H}$, if for any $(w^0, w^1) \in \mathcal{H}$ we have that

$$E(w(t), w_t(t)) = \|(w(t), w_t(t))\|_{\mathcal{H}}^2 \to 0$$

as $t \to +\infty$.

The following theorem provides a characterization of the lengths of the sensor/actuator for which the strong stability holds.

**Theorem 2:** *The system* (1)–(3) *is strongly stable in $\mathcal{H}$ if and only if $K > 0$ and $\xi \notin \mathcal{S}$.*

**Proof:** First, it follows from Lemma 1 that for any $(w^0, w^1) \in \mathcal{D}(A)$ and for any $t \geq 0$,

$$E(w(T), v(T)) - E(w^0, w^1) = -2K\int_0^T |v_x(\xi, t)|^2\,\mathrm{d}t. \tag{22}$$

Thus, the condition $K > 0$ is needed for the energy to decrease. On the other hand, if $\xi \in \mathcal{S}_k$ for some $k \geq 1$, then any state of the form $(\phi^0, \phi^1) = (\phi_k^0\psi_k, \phi_k^1\psi_k)$ gives rise to a solution of (1)–(3) whose energy does not tend to 0. Thus $K > 0$ together with $\xi \notin \mathcal{S}$ constitutes a necessary condition for (1)–(3) to be strongly stable. Let us now check that this condition is also sufficient. We need the following result, which extends Proposition 1. □

**Proposition 4** $i\mathbb{R} \subset \rho(\mathcal{A})$.

**Proof of Proposition 4:** Since $\mathcal{A}^{-1}$ is compact, the spectrum of $\mathcal{A}$ is constituted only of eigenvalues, so to prove the claim it is sufficient to check that for any $\beta \in \mathbb{R}^*$ the equation

$$(i\beta - \mathcal{A})(w, v) = (0, 0), \quad (w, v) \in \mathcal{D}(\mathcal{A}) \tag{23}$$

admits only the trivial solution $(w, v) = (0, 0)$. Using (23) and (6), we obtain that

$$0 = ((i\beta - \mathcal{A})(w, v), (w, v))_{\mathcal{H}} = i\left(\beta\|(w, v)\|_{\mathcal{H}}^2 - 2\,\mathrm{Im}\int_0^\pi v_{xx}\overline{w_{xx}}\,\mathrm{d}x\right) + K|v_x(\xi)|^2,$$

hence

$$[w_{xx}]_\xi = Kv_x(\xi) = 0$$

and $w \in H^4(0,\pi)$. On the other hand, (23) yields

$$\begin{cases} -\beta^2 w + w^{(4)} = 0 \\ w(0) = w'(0) = w''(\pi) = w'''(\pi) = 0, \end{cases}$$

so if $w \neq 0$, then $\beta^2 = \lambda_k$ and $w = C\psi_k$ for some $k \geq 1$, $C \in \mathbb{C}$. It follows that $0 = Kv'(\xi) = i\beta KC\psi_k'(\xi)$; hence $C = 0$ (since $K \neq 0$ and $\xi \notin \mathcal{S}$), contradicting the assumption $w \neq 0$. Therefore $w = v = 0$.

Applying a classical stability theorem due to Arendt–Batty (1988), we infer from Proposition 4 that the system (1)–(3) is strongly stable in $\mathcal{H}$. This completes the proof of Theorem 2. □

**Remark 1:**

(i) Proceeding as in Le Gall *et al.* (to appear), one may write a more elementary proof of Theorem 2 by using LaSalle's invariance principle (see Ammari and Tucsnak (2000) for strong stability results obtained this way).

(ii) If $\xi \in \mathcal{S}_k$ and $\xi \notin \mathcal{S}_{k'}$ for $k' \neq k$, then a strong stability result also holds true in the subspace $\mathcal{H}' := \mathrm{Span}\{(\psi_k, 0), (0, \psi_k)\}^{\perp}$ of codimension 2.

### 3.3 *Properties of the set $\mathcal{S}$ of critical lengths*

As the set $\mathcal{S}$ plays a crucial role in all the stability results, we collect some of its properties in the following proposition.

**Proposition 5:** *The set $\mathcal{S}$ is countable and dense in* $(0, \pi]$.

**Proof:** To prove that $\mathcal{S}$ is (at most) countable, it is sufficient to prove that each set $\mathcal{S}_k$ is finite. But $\mathcal{S}_k = (\psi_k')^{-1}(0) \cap (0, \pi]$, and the function $\psi_k'$, which is analytic, has only a finite number of zeros in the interval $(0, \pi]$ if it is not identically null. To check the last property, we need first to establish Claim 1.

**Claim 1** $(\mu_k \neq 1 \quad \forall k \geq 1)$**.** Argue by contradiction. If the claim is false, then there exists some $k \geq 1$ with

$$\mu_k = \frac{\cos(\alpha_k \pi) + \cosh(\alpha_k \pi)}{\sin(\alpha_k \pi) + \sinh(\alpha_k \pi)} = 1. \tag{24}$$

Let $x = \cos(\alpha_k \pi)$. Using (16) and (24), we arrive to the equation

$$x - x^{-1} = \pm\sqrt{1 - x^2} + \sqrt{x^{-2} - 1}$$

whose solutions are easily found to be $\pm 1$. Now, (16) has no solution if $\cos(\alpha_k \pi) = \pm 1$. The claim is proved.

Derivating in (15) we obtain

$$\begin{aligned}\psi_k'(x) &= \pi^{-1/2}\big( -\alpha_k \sin(\alpha_k x) - \alpha_k \sinh(\alpha_k x) \\ &\quad + \mu_k(\alpha_k \cosh(\alpha_k x) - \alpha_k \cos(\alpha_k x))\big) \\ &\sim \pi^{-1/2}\alpha_k(\mu_k \cosh(\alpha_k x) - \sinh(\alpha_k x)) \\ &\sim \frac{\pi^{-1/2}\alpha_k(\mu_k - 1)e^{\alpha_k x}}{2}\end{aligned}$$

as $x \to +\infty$, hence $\psi_k' \not\equiv 0$. The proof that $\mathcal{S}$ is at most countable is achieved.

Let us now check that $\mathcal{S}$ is dense in $(0, \pi]$ (hence not finite). Applying Proposition 3, we obtain that

$$\psi_k'(x) = -\sqrt{2}\pi^{-1/2}\alpha_k\Big(\sin\Big(\alpha_k x + \frac{\pi}{4}\Big) + O(e^{-\delta\alpha_k})\Big)$$

on each of the intervals $[0, \pi/2 - \varepsilon]$ and $[\pi/2 + \varepsilon, \pi]$ ($\varepsilon > 0$ being arbitrarily small), $\delta = \delta(\varepsilon) \in (0, \pi/2)$ denoting some appropriate constant. Applying the intermediate values theorem, we conclude that the function $\psi_k'$ vanishes between any pair of successive extrema of the function $\sin(\alpha_k x + \pi/4)$ on each of the above intervals. The density of $\mathcal{S}$ follows at once. □

## 4. Energy decay rates

In this section we provide explicit energy decay rates in function of the length $\xi$ of the sensor/actuator.

### 4.1 *Exponential decay rate*

An exponential decay rate is first derived when $\xi/\pi$ is rational with

$$\frac{\xi}{\pi} \neq \frac{4k' + 3}{4k + 2}, \quad \forall k, k' \in \mathbb{Z}. \tag{25}$$

**Theorem 3:** *Let $\xi \notin \mathcal{S}$ be such that $\xi/\pi \in \mathbb{Q}$ and* (25) *holds true. Then*

$$\sup_{|\beta| \geq 1} \|(i\beta - \mathcal{A})^{-1}\| < \infty. \tag{26}$$

*It follows that the semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is exponentially stable; i.e., there exist two constants $C > 0$ and $\delta > 0$ such that*

$$\|e^{t\mathcal{A}} z_0\|_{\mathcal{H}} \leq C e^{-\delta t} \|z_0\|_{\mathcal{H}}$$

*for any $z_0 = (w_0, v_0) \in \mathcal{H}$ and any $t \geq 0$.*

**Remark 2:** It is easy to see that $\xi := \pi$ does not belong to $\mathcal{S}$, and that (25) is satisfied for this choice of $\xi$. Unfortunately, checking that $\xi \notin \mathcal{S}$ for a $\xi$ fulfilling (25) seems to be a very hard task.

**Proof of Theorem 3:** The exponential stability of the semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is in fact equivalent to the resolvent estimate (26) by virtue of a well-known result due to Huang and Prüss, see Prüss (1984) and Huang (1985). Therefore, we shall focus on the proof of (26). As the usefulness of the condition (25) will appear quite far in the proof, we assume at the beginning of the proof that $\xi$ is any number in $(0, \pi]$. Let us argue by contradiction. If (26) is false, then there exist $\beta_n \in \mathbb{R}$, $(w_n, v_n) \in \mathcal{D}(\mathcal{A})$ for $n = 1, 2, \ldots$ such that

$$\|(w_n, v_n)\|_{\mathcal{H}} = 1, \quad |\beta_n| \to +\infty \tag{27}$$

and if we set $(f_n, g_n) := (i\beta_n - \mathcal{A})(w_n, v_n)$,

$$(f_n, g_n) \to (0, 0) \quad \text{in } \mathcal{H}, \tag{28}$$

i.e.

$$i\beta_n w_n - v_n = f_n \to 0 \quad \text{in } V, \tag{29}$$

$$i\beta_n v_n + \left\{w_n^{(4)}\right\} = g_n \to 0 \quad \text{in } H. \tag{30}$$

Replacing if necessary $\beta_n$, $w_n$ and $v_n$ by $-\beta_n$, $\overline{w}_n$ and $\overline{v}_n$, respectively, we may assume that $\beta_n > 0$ for all $n$. Our goal is to obtain a contradiction to (27). This will be done in four steps. In the first step, using the multiplier method we establish some estimates which show that we are done if $\beta_n w_n(\pi) \to 0$ as $n \to +\infty$. In the second step, we show that the couple $(w_n(\pi), w_n'(\pi))$ may be obtained as the solution of a $2 \times 2$ linear system by performing a direct integration of (29) and (30). In the two remaining steps we estimate the determinants involved in the resolution of that linear system. In what follows, the letters $C, C', C'', \ldots$ will denote positive constants which may vary from line to line.

**Step 1** (Basic estimates by the multiplier method)**:** We infer from (6) that

$$\begin{aligned} K|v_n'(\xi)|^2 &= \mathrm{Re}((i\beta_n - \mathcal{A})(w_n, v_n), (w_n, v_n))_{\mathcal{H}} \\ &= \mathrm{Re}((f_n, g_n), (w_n, v_n))_{\mathcal{H}}, \end{aligned} \tag{31}$$

hence, using (27) and (28),

$$|v_n'(\xi)|^2 \to 0 \quad \text{as } n \to +\infty. \tag{32}$$

On the other hand, (29) and (30) give as $n \to \infty$

$$i\beta_n \|w_n\|_V^2 - (v_n, w_n)_V \to 0, \tag{33}$$

$$i\beta_n \|v_n\|_H^2 + (\{w_n^{(4)}\}, v_n)_H \to 0. \tag{34}$$

Taking the difference of (33) and (34) we obtain

$$i\beta_n(\|w_n\|_V^2 - \|v_n\|_H^2) - \int_0^\pi v_n''\overline{w_n''}\,\mathrm{d}x - \int_0^\pi \{w_n^{(4)}\}\overline{v_n}\,\mathrm{d}x \to 0$$

but

$$\mathrm{Im}\left(\int_0^\pi v_n''\overline{w_n''}\,\mathrm{d}x + \int_0^\pi \{w_n^{(4)}\}\overline{v_n}\,\mathrm{d}x\right) = 0$$

by (6), hence

$$\beta_n(\|w_n\|_V^2 - \|v_n\|_H^2) \to 0. \tag{35}$$

Therefore, using (27) and (29) we conclude that

$$\lim_{n\to+\infty} \|w_n\|_V^2 = \lim_{n\to+\infty} \|v_n\|_H^2 = \lim_{n\to+\infty} \|\beta_n w_n\|_H^2 = \tfrac{1}{2}\cdot \tag{36}$$

Eliminating $v_n$ in (30) by using (29), we obtain

$$-\beta_n^2 w_n + \{w_n^{(4)}\} = g_n + i\beta_n f_n, \tag{37}$$

hence

$$\left(-\beta_n^2 w_n + \{w_n^{(4)}\}, q w_n'\right)_H = (g_n + i\beta_n f_n, q w_n')_H \tag{38}$$

for any real function $q \in C^3([0, \pi])$.

It follows from (36) and a classical interpolation inequality (see e.g. Lions and Magenes (1972)) that for any $s \in [0, 2)$

$$w_n \to 0 \quad \text{in } H^s(0, \pi). \tag{39}$$

In particular,

$$(g_n, q w_n')_H \to 0. \tag{40}$$

On the other hand,

$$\int_0^\pi f_n q \overline{w_n'}\,\mathrm{d}x = -\int_0^\pi (f_n q)'\overline{w_n}\,\mathrm{d}x + f_n(\pi)q(\pi)\overline{w_n(\pi)} \tag{41}$$

but, using (29) and (36), we obtain

$$\beta_n\left|\int_0^\pi (f_n q)'\overline{w_n}\,\mathrm{d}x\right| \le C\|f_n\|_{H^1(0,\pi)}\|\beta_n w_n\|_H \to 0, \tag{42}$$

and

$$\beta_n f_n(\pi)q(\pi)\overline{w_n(\pi)} = o(\beta_n w_n(\pi)). \tag{43}$$

It follows that

$$(i\beta_n f_n, q w_n')_H = o(\beta_n w_n(\pi)) + o(1). \tag{44}$$

Let us now turn to the left hand side of (38). We have

$$2\mathrm{Re}(-\beta_n^2 w_n, qw_n')_H = \beta_n^2 \int_0^\pi q'|w_n|^2\,\mathrm{d}x - \beta_n^2|w_n(\pi)|^2 q(\pi). \tag{45}$$

Integrating by parts in

$$\left(\{w_n^{(4)}\}, qw_n'\right)_H = \int_0^\xi w_n^{(4)} q\overline{w_n'}\,\mathrm{d}x + \int_\xi^\pi w_n^{(4)} q\overline{w_n'}\,\mathrm{d}x,$$

we arrive to

$$\begin{aligned}2\mathrm{Re}\left(\{w_n^{(4)}\}, qw_n'\right)_H &= 3\int_0^\pi |w_n''|^2 q'\,\mathrm{d}x + |w_n''(0)|^2 q(0)\\ &\quad - \int_0^\pi |w_n'|^2 q'''\,\mathrm{d}x\\ &\quad + q(\xi)\left[|w_n''|^2\right]_\xi + 2\mathrm{Re}\left(q'(\xi)\overline{w_n'(\xi)}\left[w_n''\right]_\xi\right)\\ &\quad + |w_n'(\pi)|^2 q''(\pi)\\ &= 3\int_0^\pi |w_n''|^2 q'\,\mathrm{d}x\\ &\quad + 2Kq(\xi)\mathrm{Re}\left(v_n'(\xi)\overline{w_n''(\xi^+)}\right) + o(1)\end{aligned} \tag{46}$$

by (39) and (32), provided that $q(0) = 0$. Gathering together (40), (44), (45) and (46) with $q(x) = x$, we obtain

$$\begin{aligned}&3\int_0^\pi |w_n''|^2\,\mathrm{d}x + \beta_n^2\int_0^\pi |w_n|^2\,\mathrm{d}x\\ &\quad = (\pi\beta_n\overline{w_n(\pi)} + o(1))\beta_n w_n(\pi)\\ &\qquad - 2K\xi\mathrm{Re}\left(v_n'(\xi)\overline{w_n''(\xi^+)}\right) + o(1).\end{aligned} \tag{47}$$

The above equality may be simplified thanks to the next result.

**Proposition 6:**

$$|w_n''(\xi^+)|^2 = O(|\beta_n w_n(\pi)|^2) + o(1). \tag{48}$$

**Proof:** Multiplying each term in (37) by $q\overline{w_n'}$ where $q \in C^3([\xi,\pi])$, and integrating over $(\xi,\pi)$, we arrive to

$$-\beta_n^2\int_\xi^\pi qw_n\overline{w_n'}\,\mathrm{d}x + \int_\xi^\pi qw_n^{(4)}\overline{w_n'}\,\mathrm{d}x = \int_\xi^\pi (g_n + i\beta_n f_n)q\overline{w_n'}\,\mathrm{d}x.$$

Clearly

$$\left|\int_\xi^\pi g_n q\overline{w_n'}\,\mathrm{d}x\right| \le C\|g_n\|_H\|w_n'\|_H \to 0$$

and

$$\int_\xi^\pi i\beta_n f_n q\overline{w_n'}\,\mathrm{d}x = -i\int_\xi^\pi (f_n q)'\beta_n\overline{w_n}\,\mathrm{d}x + [if_n q\beta_n\overline{w_n}]_\xi^\pi,$$

so using again (29) and (36),

$$\left|\int_\xi^\pi (g_n + i\beta_n f_n)q\overline{w_n'}\,\mathrm{d}x\right| \le \frac{\xi}{4}\left(|\beta_n w_n(\xi)|^2 + |\beta_n w_n(\pi)|^2\right) + o(1).$$

On the other hand, we obtain after some integrations by parts that

$$2\mathrm{Re}\int_\xi^\pi (-\beta_n^2)qw_n\overline{w_n'}\,\mathrm{d}x = \beta_n^2\left\{\int_\xi^\pi q'|w_n|^2\,\mathrm{d}x - \left[q|w_n|^2\right]_\xi^\pi\right\}$$

and that

$$\begin{aligned}2\mathrm{Re}\int_\xi^\pi w_n^{(4)} q\overline{w_n'}\,\mathrm{d}x &= 3\int_\xi^\pi q'|w_n''|^2\,\mathrm{d}x - \left[|w_n''|^2 q\right]_\xi^\pi\\ &\quad - \int_\xi^\pi q'''|w_n'|^2\,\mathrm{d}x + \left[q''|w_n'|^2\right]_\xi^\pi\\ &\quad - 2\mathrm{Re}\left[w_n'' q'\overline{w_n'}\right]_\xi^\pi + 2\mathrm{Re}\left[w_n''' q\overline{w_n'}\right]_\xi^\pi.\end{aligned}$$

Taking $q(x) = x$ and using the fact that $w_n''(\pi) = w_n'''(\pi) = 0$, we arrive to

$$\begin{aligned}&3\int_\xi^\pi |w_n''|^2\,\mathrm{d}x + \beta_n^2\int_\xi^\pi |w_n|^2\,\mathrm{d}x + \xi(|w_n''(\xi^+)|^2\\ &\quad + \tfrac{1}{2}|\beta_n w_n(\xi)|^2) + 2\mathrm{Re}\left((w_n''(\xi^+) - \xi w_n'''(\xi))\overline{w_n'(\xi)}\right)\\ &\quad = O(|\beta_n w_n(\pi)|^2) + o(1).\end{aligned} \tag{49}$$

The following claims are needed. □

**Claim 2:** $\beta_n w_n'(\xi) \to 0$ as $n \to \infty$.
Indeed, derivating once in (29) and evaluating at $x = \xi$, we infer that $i\beta_n w_n'(\xi) = v_n'(\xi) + f_n'(\xi) \to 0$, by (28) and (32).

**Claim 3:** $\|w_n\|_{H^4(\xi,\pi)} \le C\beta_n$.
Indeed, it follows from (30) and (36) that

$$\|\beta_n^{-1} w_n^{(4)}\|_{L^2(\xi,\pi)} \le \beta_n^{-1}\|g_n\|_{L^2(\xi,\pi)} + \|v_n\|_{L^2(\xi,\pi)} \le C.$$

Since $\|w_n\|_V$ is bounded, the claim follows.

We infer from Claim 3 that $|w_n''(\xi^+)| + |w_n'''(\xi)| \le C\beta_n$ which, combined to Claim 2, yields

$$\mathrm{Re}\left((w_n''(\xi^+) - \xi w_n'''(\xi))\overline{w_n'(\xi)}\right) \to 0 \text{ as } n \to \infty.$$

Therefore, (49) yields

$$3\int_{\xi}^{\pi}|w_n''|^2\,\mathrm{d}x+\beta_n^2\int_{\xi}^{\pi}|w_n|^2\,\mathrm{d}x+\xi\left(|w_n''(\xi^+)|^2+\tfrac{1}{2}|\beta_n w_n(\xi)|^2\right)$$
$$=O(|\beta_n w_n(\xi)|^2)+o(1),$$

hence (48) follows. The proof of Proposition 6 is complete. □

Using (32), (47) and (48), we arrive to

$$3\int_{0}^{\pi}|w_n''|^2\,\mathrm{d}x+\beta_n^2\int_{0}^{\pi}|w_n|^2\,\mathrm{d}x=O(|\beta_n w_n(\pi)|^2)+o(1). \quad (50)$$

Therefore, we obtain a contradiction with (36) if $\beta_n w_n(\pi)\to 0$. In the next steps, we show that $\beta_n w_n(\pi)\to 0$.

**Step 2** Computation of $w_n(\pi)$**:** To compute $w_n(\pi)$ we integrate (37) on $(0,\pi)$. More precisely, setting $F_n := g_n + i\beta_n f_n$, we solve the system

$$\begin{cases} -\beta_n^2 w_n + \{w_n^{(4)}\} = F_n & \text{on } (0,\pi),\\ w_n(0)=w_n'(0)=w_n''(\pi)=w_n'''(\pi)=0, & \\ [w_n]_\xi = [w_n']_\xi = [w_n''']_\xi = 0, & \\ [w_n'']_\xi = K v_n'(\xi) = K(i\beta_n w_n'(\xi) - f_n'(\xi)). & \end{cases}$$

To simplify the notations we drop the subscript $n$ in what follows. Let $\lambda$ and $\mu$ be given numbers. We first solve the following (backwards) Cauchy Problem on $(\xi,\pi)$

$$(S_1)\quad \begin{cases} w^{(4)}-\beta^2 w = F \quad \text{in } (\xi,\pi),\\ w(\pi)=\lambda,\ w'(\pi)=\mu,\ w''(\pi)=w'''(\pi)=0, \end{cases}$$

and next the following (backwards) Cauchy Problem on $(0,\xi)$

$$(S_2)\quad \begin{cases} w^{(4)}-\beta^2 w = F \ \text{ in } (0,\xi),\\ w(\xi^-)=w(\xi^+),\ w'(\xi^-)=w'(\xi^+),\ w'''(\xi^-)=w'''(\xi^+),\\ w''(\xi^-)=w''(\xi^+)-K(i\beta w'(\xi^+)-f'(\xi)). \end{cases}$$

Then $\lambda$ and $\mu$ have to be chosen so that $w(0)=w'(0)=0$.

The ODE $w^{(4)}-\beta^2 w = F$ may be written as

$$\begin{pmatrix} w\\ w'\\ w''\\ w''' \end{pmatrix}' = M\begin{pmatrix} w\\ w'\\ w''\\ w''' \end{pmatrix} + \begin{pmatrix} 0\\0\\0\\F \end{pmatrix}, \quad \text{with}$$

$$M=\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ \beta^2&0&0&0 \end{pmatrix}.$$

Easy computations show that

$$e^{xM}=\begin{pmatrix} a & \beta^{-2}a''' & \beta^{-2}a'' & \beta^{-2}a'\\ a' & a & \beta^{-2}a''' & \beta^{-2}a''\\ a'' & a' & a & \beta^{-2}a'''\\ a''' & a'' & a' & a \end{pmatrix}, \quad (51)$$

where $a(x) := (\cos(bx)+\cosh(bx))/2$ and $b=\sqrt{\beta}$. An integration of $(S_1)$ yields

$$\begin{pmatrix} w\\ w'\\ w''\\ w''' \end{pmatrix}(x) = e^{(x-\pi)M}\begin{pmatrix} \lambda\\ \mu\\ 0\\ 0 \end{pmatrix} + \int_{\pi}^{x} e^{(x-y)M}\begin{pmatrix} 0\\0\\0\\F(y) \end{pmatrix}\mathrm{d}y, \quad x\in(\xi,\pi),$$

hence

$$\begin{pmatrix} w\\ w'\\ w''\\ w''' \end{pmatrix}(\xi^+) = e^{(\xi-\pi)M}\begin{pmatrix} \lambda\\ \mu\\ 0\\ 0 \end{pmatrix} + \int_{\pi}^{\xi} e^{(\xi-y)M}\begin{pmatrix} 0\\0\\0\\F(y) \end{pmatrix}\mathrm{d}y. \quad (52)$$

On the other hand, an integration of $(S_2)$ leads to

$$\begin{pmatrix} w\\ w'\\ w''\\ w''' \end{pmatrix}(x) = e^{(x-\xi)M}\begin{pmatrix} w\\ w'\\ w''\\ w''' \end{pmatrix}(\xi^-) + \int_{\xi}^{x} e^{(x-y)M}\begin{pmatrix} 0\\0\\0\\F(y) \end{pmatrix}\mathrm{d}y. \quad (53)$$

Gathering (52) and (53), we obtain

$$\begin{pmatrix} w \\ w' \\ w'' \\ w''' \end{pmatrix}(0) = e^{-\xi M}\left(\begin{pmatrix} w \\ w' \\ w'' \\ w''' \end{pmatrix}(\xi^+) + \begin{pmatrix} 0 \\ 0 \\ -[w'']_\xi \\ 0 \end{pmatrix}\right) + \int_\xi^0 e^{-yM}\begin{pmatrix} 0 \\ 0 \\ 0 \\ F(y) \end{pmatrix}\mathrm{d}y$$

$$= e^{-\pi M}\begin{pmatrix} \lambda \\ \mu \\ 0 \\ 0 \end{pmatrix} + \int_\pi^0 e^{-yM}\begin{pmatrix} 0 \\ 0 \\ 0 \\ F(y) \end{pmatrix}\mathrm{d}y + e^{-\xi M}\begin{pmatrix} 0 \\ 0 \\ -K\big(i\beta w'(\xi^+) - f'(\xi)\big) \\ 0 \end{pmatrix}. \qquad (54)$$

By (51) and (52),

$$w'(\xi^+) = a'(\xi-\pi)\lambda + a(\xi-\pi)\mu + \int_\pi^\xi \beta^{-2}a''(\xi-y)F(y)\,\mathrm{d}y,$$

hence, keeping only the two first equations in (54), we arrive to

$$\begin{pmatrix} w(0) \\ w'(0) \end{pmatrix} = \begin{pmatrix} a(-\pi) & \beta^{-2}a'''(-\pi) \\ a'(-\pi) & a(-\pi) \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} + \int_\pi^0 \begin{pmatrix} \beta^{-2}a'(-y) \\ \beta^{-2}a''(-y) \end{pmatrix}F(y)\,\mathrm{d}y + \begin{pmatrix} \beta^{-2}a''(-\xi) \\ \beta^{-2}a'''(-\xi) \end{pmatrix}$$
$$\times\Big(Kf'(\xi) - iK\beta\big(a'(\xi-\pi)\lambda + a(\xi-\pi)\mu + \int_\pi^\xi \beta^{-2}a''(\xi-y)F(y)\,\mathrm{d}y\big)\Big).$$

Therefore, $w(0) = w'(0) = 0$ if and only if

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where

$$m_{11} = a(-\pi) - iK\beta^{-1}a''(-\xi)a'(\xi-\pi)$$
$$m_{12} = \beta^{-2}a'''(-\pi) - iK\beta^{-1}a''(-\xi)a(\xi-\pi)$$
$$m_{21} = a'(-\pi) - iK\beta^{-1}a'''(-\xi)a'(\xi-\pi)$$
$$m_{22} = a(-\pi) - iK\beta^{-1}a'''(-\xi)a(\xi-\pi)$$
$$c_1 = -\int_\pi^0 \beta^{-2}a'(-y)F(y)\,\mathrm{d}y + K\beta^{-2}a''(-\xi) \times\left(-f'(\xi) + i\beta\int_\pi^\xi \beta^{-2}a''(\xi-y)F(y)\,\mathrm{d}y\right)$$
$$c_2 = -\int_\pi^0 \beta^{-2}a''(-y)F(y)\,\mathrm{d}y + K\beta^{-2}a'''(-\xi) \times\left(-f'(\xi) + i\beta\int_\pi^\xi \beta^{-2}a''(\xi-y)F(y)\,\mathrm{d}y\right).$$

Letting

$$N = c_1m_{22} - c_2m_{12}$$
$$D = m_{11}m_{22} - m_{12}m_{21}$$

we obtain by Cramer rule $\lambda = N/D$. In the next step we show that $|D| \geq Ce^{b\pi}$ if the number $\xi/\pi$ is rational and fulfills (25).

**Step 3** Estimation of $D$**:** Substituting the expressions of $m_{11}, m_{12}, m_{21}$ and $m_{22}$ into $D$ we obtain

$$\begin{aligned} D &= (a(-\pi) - iK\beta^{-1}a''(-\xi)a'(\xi-\pi))(a(-\pi) - iK\beta^{-1}a'''(-\xi)a(\xi-\pi)) - (\beta^{-2}a'''(-\pi) \\ &\quad - iK\beta^{-1}a''(-\xi)a(\xi-\pi))(a'(-\pi) - iK\beta^{-1}a'''(-\xi)a'(\xi-\pi)) \\ &= a(-\pi)^2 - \beta^{-2}a'''(-\pi)a'(-\pi) \\ &\quad - iK\beta^{-1}\{a''(-\xi)a'(\xi-\pi)a(-\pi) + a(-\pi)a'''(-\xi)a(\xi-\pi) \\ &\quad - a''(-\xi)a(\xi-\pi)a'(-\pi) - \beta^{-2}a'''(-\pi)a'''(-\xi)a'(\xi-\pi)\} \\ &=: D_1 + iD_2. \end{aligned}$$

Substituting the values of $a$ and of its derivatives in $D_1$ we obtain

$$\begin{aligned} D_1 &= \left(\frac{1}{2}(\cos b\pi + \cosh b\pi)\right)^2 - \beta^{-2}\frac{b^3}{2}(-\sin b\pi - \sinh b\pi) \times\frac{b}{2}(\sin b\pi - \sinh b\pi) \\ &= \frac{1}{4}\big(\cos^2 b\pi + \cosh^2 b\pi + 2\cos b\pi\cosh b\pi + \sin^2 b\pi - \sinh^2 b\pi\big) \\ &= \frac{1}{2}(1 + \cos b\pi\cosh b\pi). \end{aligned} \qquad (55)$$

Let us now turn to the estimation of $D_2$.

$$D_2 = -K\beta^{-1}\big\{a'(\xi-\pi)\big(a''(-\xi)a(-\pi) - \beta^{-2}a'''(-\pi)a'''(-\xi)\big) + a(\xi-\pi)(a(-\pi)a'''(-\xi) - a''(-\xi)a'(-\pi))\big\}.$$

To estimate $D_2$ we introduce the functions $G, H\colon [0,\pi] \to \mathbb{R}$ defined by

$$G(y) = a''(-\xi)a(-y) - \beta^{-2}a'''(-y)a'''(-\xi)$$
$$H(y) = a(-y)a'''(-\xi) - a''(-\xi)a'(-y).$$

Then the following result holds.

**Lemma 2:**

$$G(y) = \frac{b^2}{4}\left(\cosh b(y-\xi) + \frac{e^{b\xi}}{2}(\cos by - \sin by) - \frac{e^{by}}{2}(\cos b\xi + \sin b\xi) + O(1)\right), \tag{56}$$

$$H(y) = \frac{b^3}{4}\left(\sinh b(y-\xi) - \frac{e^{b\xi}}{2}(\cos by + \sin by) - \frac{e^{by}}{2}(\cos b\xi + \sin b\xi) + O(b)\right), \tag{57}$$

*where $O(1)$ denotes a term which is bounded in $y$, $\xi$, and $b$.*

**Proof of Lemma 2:** Expanding $a$ in the expression of $G$, we obtain

$$\begin{aligned} G(y) &= \frac{b^2}{4}\{(-\cos b\xi + \cosh b\xi)(\cos by + \cosh by) \\ &\quad -(\sin by + \sinh by)(\sin b\xi + \sinh b\xi)\} \\ &= \frac{b^2}{4}\{\cosh by \cosh b\xi - \sinh by \sinh b\xi \\ &\quad + \frac{1}{2}(e^{b\xi} + e^{-b\xi})\cos by - \frac{1}{2}(e^{by} + e^{-by})\cos b\xi \\ &\quad - \frac{1}{2}(e^{by} - e^{-by})\sin b\xi - \frac{1}{2}(e^{b\xi} - e^{-b\xi})\sin by \\ &\quad - \cos b\xi \cos by - \sin by \sin b\xi\} \\ &= \frac{b^2}{4}\left\{\cosh b(y-\xi) + \frac{e^{b\xi}}{2}(\cos by - \sin by) - \frac{e^{by}}{2}(\cos b\xi + \sin b\xi) + O(1)\right\}. \end{aligned}$$

Thus (56) is proved. (57) may be obtained by noticing that $H(y) = G'(y)$. □

Applying Lemma 2 with $y = \pi$, we obtain

$$\begin{aligned} D_2 &= -K\beta^{-1}(a'(\xi-\pi)G(\pi) + a(\xi-\pi)H(\pi)) \\ &= -K\beta^{-1}\left\{\frac{b}{2}\left(-\sin b(\xi-\pi) + \frac{e^{b(\xi-\pi)} - e^{b(\pi-\xi)}}{2}\right)\right. \\ &\quad \times \frac{b^2}{4}\left(\frac{e^{b(\pi-\xi)}}{2} + \frac{e^{b\xi}}{2}(\cos b\pi - \sin b\pi) - \frac{e^{b\pi}}{2}(\cos b\xi + \sin b\xi) + O(1)\right) \\ &\quad + \frac{1}{2}\left(\cos b(\xi-\pi) + \frac{e^{b(\xi-\pi)} + e^{b(\pi-\xi)}}{2}\right) \\ &\quad \left.\times \frac{b^3}{4}\left(\frac{e^{b(\pi-\xi)}}{2} - \frac{e^{b\xi}}{2}(\cos b\pi + \sin b\pi) - \frac{e^{b\pi}}{2}(\cos b\xi + \sin b\xi) + O(1)\right)\right\}. \end{aligned}$$

Expanding $D_2$, we notice that the coefficients in front of the exponential terms $e^{b(2\pi-\xi)}$ and $e^{2b(\pi-\xi)}$ vanish, so that the leading exponential term is $e^{b\pi}$, and the remaining exponential terms read $e^{b(\pi-\xi)}$, $e^{b(2\xi-\pi)}$, $e^{b\xi}$, $e^0$, and $e^{b(\xi-\pi)}$. We conclude that if $\xi < \pi$, then

$$D_2 = \frac{Kb}{16}(\cos b\pi + (\cos b\xi + \sin b\xi) \times (\cos b(\xi-\pi) - \sin b(\xi-\pi))e^{b\pi} + O(e^{b(\pi-\delta)}) \tag{58}$$

for some $\delta > 0$, and if $\xi = \pi$, then

$$D_2 = \frac{Kb}{4}\big((\cos b\pi + \sin b\pi)e^{b\pi} + O(b)\big). \tag{59}$$

It is clear that $|D_1 + iD_2| \geq Ce^{b\pi}$ for $b$ large enough if $\xi = \pi$, for $(\cos b\pi)^2 + (\cos b\pi + \sin b\pi)^2 \geq C > 0$ for all $b \in \mathbb{R}$.

Let us assume from now on that $\xi < \pi$, and that the number $\xi/\pi$ is rational and admits a coprime factorization $\xi/\pi = p/q$, with $p, q \in \mathbb{N}^*$, $0 < p < q$, and let us introduce the functions

$$c(b) := \cos b\pi + (\cos b\xi + \sin b\xi) \times (\cos b(\xi-\pi) - \sin b(\xi-\pi)), \tag{60}$$

$$h(b) := (\cos b\pi)^2 + (c(b))^2. \tag{61}$$

**Lemma 3:** $\inf_{b\in\mathbb{R}} |h(b)| > 0$ *if and only if $p/q$ is not of the form $(4k'+3)/(4k+2)$, $k, k' \in \mathbb{Z}$.*

**Proof of Lemma 3:** Since $h$ is $2q$-periodic, we only have to characterize the rational numbers $p/q$ for which $h(b) > 0$ for all $b \in \mathbb{R}$. As $h$ is the sum of two squared terms, we may restrict ourselves to the $b$'s for which $\cos b\pi = 0$, namely $b = (1/2) + k$, $k \in \mathbb{Z}$. Then $h(b) = 4\cos^2((1/2+k)\xi - \pi/4)\cos^2((1/2+k)\times(\xi-\pi)+\pi/4)$, and a straightforward calculation shows that $h(b) = 0$ if and only if $p/q$ is of the form $(4k'+3)/(4k+2)$, $k, k' \in \mathbb{Z}$. □

Applying Lemma 3, we infer that as $b \to +\infty$

$$\begin{aligned}|D|^2 &= D_1^2 + D_2^2\\ &\geq Ch(b)e^{2b\pi} + O\big(e^{b(2\pi-\delta)}\big)\\ &\geq C'e^{2b\pi}\end{aligned}$$

provided that the condition (25) is fulfilled.

**Step 4** Estimation of $N$**:** We have to bound the quantity $N = c_1 m_{22} - c_2 m_{12}$. Letting

$$Z_1 := -\int_\pi^0 \beta^{-2} a'(-y)F(y)\,\mathrm{d}y$$

$$Z_2 := -\int_\pi^0 \beta^{-2} a''(-y)F(y)\,\mathrm{d}y$$

and $$Z_3 := i\beta^{-1}\int_\pi^\xi a''(\xi-y)F(y)\,\mathrm{d}y - f'(\xi),$$

we obtain that

$$c_1 = Z_1 + K\beta^{-2}a''(-\xi)Z_3, \quad c_2 = Z_2 + K\beta^{-2}a'''(-\xi)Z_3,$$

hence

$$\begin{aligned}N &= (Z_1 + K\beta^{-2}a''(-\xi)Z_3)(a(-\pi) - iK\beta^{-1}a'''(-\xi)a(\xi-\pi))\\
&\quad -(Z_2 + K\beta^{-2}a'''(-\xi)Z_3)(\beta^{-2}a'''(-\pi)\\
&\quad - iK\beta^{-1}a''(-\xi)a(\xi-\pi))\\
&= K\beta^{-2}\big(a''(-\xi)a(-\pi) - \beta^{-2}a'''(-\xi)a'''(-\pi)\big)Z_3\\
&\quad + Z_1\big(a(-\pi) - iK\beta^{-1}a'''(-\xi)a(\xi-\pi)\big)\\
&\quad - Z_2\big(\beta^{-2}a'''(-\pi) - iK\beta^{-1}a''(-\xi)a(\xi-\pi)\big)\\
&= -K\beta^{-2}G(\pi)f'(\xi) + iK\beta^{-3}G(\pi)\int_\pi^\xi a''(\xi-y)F(y)\,\mathrm{d}y\\
&\quad -\beta^{-2}\int_\pi^0 \big\{a'(-y)(a(-\pi) - iK\beta^{-1}a'''(-\xi)a(\xi-\pi))\\
&\quad - a''(-y)(\beta^{-2}a'''(-\pi) - iK\beta^{-1}a''(-\xi)a(\xi-\pi))\big\}F(y)\,\mathrm{d}y\\
&= -K\beta^{-2}G(\pi)f'(\xi)\\
&\quad -\beta^{-2}\int_\pi^0 (a'(-y)a(-\pi) - a''(-y)\beta^{-2}a'''(-\pi))F(y)\,\mathrm{d}y\\
&\quad + iK\beta^{-3}\int_\xi^0 (a'(-y)a'''(-\xi) - a''(-y)\\
&\quad \times a''(-\xi))a(\xi-\pi)F(y)\,\mathrm{d}y\\
&\quad + iK\beta^{-3}\int_\pi^\xi \big\{G(\pi)a''(\xi-y) + \big(a'(-y)a'''(-\xi) - a''(-y)\\
&\quad \times a''\big(-\xi)\big)a(\xi-\pi)\big\}F(y)\,\mathrm{d}y\big\}\\
&=: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

By virtue of Lemma 2,

$$\begin{aligned}|G(\pi)| &= \frac{b^2}{4}\left|\cosh b(\pi-\xi) + \frac{e^{b\xi}}{2}(\cos b\pi - \sin b\pi)\right.\\
&\quad \left. - \frac{e^{b\pi}}{2}(\cos b\xi + \sin b\xi) + O(1)\right|\\
&\leq Cb^2 e^{b\pi},\end{aligned}$$

hence

$$|I_1| \leq Cb^{-2}e^{b\pi}\|f\|_V.$$

Let $K(y) := a'(-y)a(-\pi) - a''(-y)\beta^{-2}a'''(-\pi)$ for $y \in [0,\pi]$. Then

$$I_2 = \beta^{-2}\int_0^\pi K(y)g(y)\,\mathrm{d}y + i\beta^{-1}\int_0^\pi K(y)f(y)\,\mathrm{d}y =: I_2^1 + I_2^2.$$

Since

$$\begin{aligned}K(y) &= \frac{b}{4}\big\{(\sin by - \sinh by)(\cos b\pi + \cosh b\pi)\\
&\quad -(-\cos by + \cosh by)(-\sin b\pi - \sinh b\pi)\big\}\\
&= \frac{b}{4}\left\{\cosh by \sinh b\pi - \sinh by\cosh b\pi\right.\\
&\quad - \frac{e^{by} - e^{-by}}{2}\cos b\pi + \frac{e^{b\pi} + e^{-b\pi}}{2}\sin by\\
&\quad - \frac{e^{b\pi} - e^{-b\pi}}{2}\cos by + \frac{e^{by} + e^{-by}}{2}\sin b\pi\\
&\quad \left. + \sin by\cos b\pi - \cos by \sin b\pi\right\}\\
&= \frac{b}{4}\left\{\sinh b(\pi - y) + \frac{e^{b\pi}}{2}(\sin by - \cos by)\right.\\
&\quad \left. + \frac{e^{by}}{2}(-\cos b\pi + \sin b\pi) + O(1)\right\}\end{aligned}$$

we see that

$$\begin{aligned}|I_2^1| &\leq \beta^{-2}\|K(y)\|_H\|g\|_H\\
&\leq Cb^{-3}\big(\|\sinh b(\pi-y)\|_H + e^{b\pi}\|\sin by - \cos by\|_H\\
&\quad + \|e^{by}\|_H + O(1)\big)\|g\|_H\\
&\leq Cb^{-3}e^{b\pi}\|g\|_H.\end{aligned}$$

On the other hand, integrating by parts in $I_2^2$ yields

$$I_2^2 = i\beta^{-1}\left(-\int_0^\pi \tilde{K}(y)f'(y)\,\mathrm{d}y + \big[\tilde{K}(y)f(y)\big]_0^\pi\right)$$

with

$$\begin{aligned}\tilde{K}(y) &= \int_0^y K(z)\,dz \\ &= \frac{1}{4}\left\{-\cosh b(\pi-y) - \frac{e^{b\pi}}{2}(\cos by + \sin by)\right. \\ &\quad \left. + \frac{e^{by}}{2}(-\cos b\pi + \sin b\pi) + O(1)\right\}.\end{aligned}$$

Thus

$$|I_2^2| \le C\beta^{-1}e^{b\pi}(\|f'\|_H + |f(\pi)|) \le C\beta^{-1}e^{b\pi}\|f\|_V.$$

We now turn to the next term $I_3$. We notice that

$$\begin{aligned}I_3 &= iK\beta^{-3}a(\xi-\pi)\int_\xi^0 (-H'(y))(g(y)+i\beta f(y))\,\mathrm{d}y \\ &= iK\beta^{-3}a(\xi-\pi)\int_0^\xi H'(y)g(y)\,\mathrm{d}y - K\beta^{-2}a(\xi-\pi) \\ &\quad \times \int_0^\xi H'(y)f(y)\,\mathrm{d}y =: I_3^1 + I_3^2.\end{aligned}$$

Since

$$\begin{aligned}H'(y) &= \frac{b^4}{4}\left\{\cosh b(y-\xi) + \frac{e^{b\xi}}{2}(\sin by - \cos by)\right. \\ &\quad \left. - \frac{e^{by}}{2}(\cos b\xi + \sin b\xi) + O(b^2)\right\}\end{aligned}$$

we have that

$$\begin{aligned}|I_3^1| &\le Cb^{-2}e^{b(\pi-\xi)}\big(\|\cosh b(y-\xi)\|_{L^2(0,\xi)} \\ &\quad + e^{b\xi}\|\sin by - \cos by\|_{L^2(0,\xi)} \\ &\quad + \|e^{by}\|_{L^2(0,\xi)} + O(b^2)\big)\|g\|_H \\ &\le Cb^{-2}e^{b\pi}\|g\|_H.\end{aligned}$$

Integrating twice by parts in $I_3^2$ yields

$$\begin{aligned}I_3^2 &= -K\beta^{-2}a(\xi-\pi) \\ &\quad \times \left(\int_0^\xi G(y)f''(y)\,\mathrm{d}y + [H(y)f(y) - G(y)f'(y)]_0^\xi\right)\end{aligned}$$

(since $G' = H$). Using (56) and the fact that $f(0) = f'(0) = 0$, we obtain that

$$\begin{aligned}|I_3^2 + K\beta^{-2}a(\xi-\pi)H(\xi)f(\xi)| &\le C\beta^{-2}e^{b(\pi-\xi)} \\ &\quad \times \big(b^2e^{b\xi}\|f''\|_H + b^2e^{b\xi}|f'(\xi)|\big) \\ &\le C\beta^{-1}e^{b\pi}\|f\|_V.\end{aligned}$$

Using (57) to estimate $H(\xi)$, we conclude that

$$I_3^2 = \frac{K}{16}b^{-1}e^{b\pi}(\cos b\xi + \sin b\xi)f(\xi) + O\big(\beta^{-1}e^{b\pi}\|f\|_V\big).$$

It remains to bound

$$\begin{aligned}I_4 &= iK\beta^{-3}\int_\pi^\xi L(y)\big(g(y) + i\beta f(y)\big)\,\mathrm{d}y \\ &= iK\beta^{-3}\int_\pi^\xi L(y)g(y)\,\mathrm{d}y - K\beta^{-2}\int_\pi^\xi L(y)f(y)\,\mathrm{d}y \\ &=: I_4^1 + I_4^2\end{aligned}$$

where we have set

$$L(y) := G(\pi)a''(\xi-y) - H'(y)a(\xi-\pi).$$

We first estimate the function $L$.

$$\begin{aligned}L(y) &= \frac{b^4}{8}\left\{\Big(\cosh b(\pi-\xi) + \frac{e^{b\xi}}{2}(\cos b\pi - \sin b\pi)\right. \\ &\quad - \frac{e^{b\pi}}{2}(\cos b\xi + \sin b\xi) + O(1)\Big) \\ &\quad \times (-\cos b(\xi-y) + \cosh b(\xi-y)) \\ &\quad - \Big(\cosh b(y-\xi) + \frac{e^{b\xi}}{2}(\sin by - \cos by) \\ &\quad - \frac{e^{by}}{2}(\cos b\xi + \sin b\xi) + O(b^2)\Big) \\ &\quad \left. \times (\cos b(\xi-\pi) + \cosh b(\xi-\pi))\right\} \\ &= \frac{b^4}{8}\left\{e^{b\pi}\left(\frac{1}{2}(\cos b\xi + \sin b\xi)\cos b(\xi-y)\right.\right. \\ &\quad \left. + \frac{1}{4}(\cos by - \sin by)\right) \\ &\quad + e^{by}\left(\frac{1}{2}(\cos b\xi + \sin b\xi)\cos b(\xi-\pi)\right. \\ &\quad \left. + \frac{1}{4}(\cos b\pi - \sin b\pi)\right) \\ &\quad \left. + e^{b(\pi+\xi-y)}\left(-\frac{1}{4}\right)(\cos b\xi + \sin b\xi) + O\big(e^{b(\pi-\delta)}\big)\right\}\end{aligned}$$

for every $y \in [\xi, \pi]$ and some $\delta > 0$. Thus

$$\begin{aligned}|I_4^1| &\le Cb^{-2}\|g\|_H\Big(e^{b\pi}(\|\cos b(\xi-y)\|_{L^2(\xi,\pi)} \\ &\quad + \|\cos by - \sin by\|_{L^2(\xi,\pi)}\Big) \\ &\quad + \|e^{by}\|_{L^2(\xi,\pi)} + \|e^{b(\pi+\xi-y)}\|_{L^2(\xi,\pi)} + O(e^{b(\pi-\delta)})\Big) \\ &\le Cb^{-2}e^{b\pi}\|g\|_H.\end{aligned}$$

Next we introduce the function $\tilde{L}(y) := G(\pi)a(\xi - y) - G(y)a(\xi - \pi)$ which fulfills $\tilde{L}'' = L$. Easy calculations show that

$$\begin{aligned}\tilde{L}(y) = \frac{b^2}{8}\Big\{ & -e^{b\pi}\Big(\frac{1}{2}(\cos b\xi + \sin b\xi)\cos b(\xi - y) \\ & + \frac{1}{4}(\cos by - \sin by)\Big) \\ & + e^{by}\Big(\frac{1}{2}(\cos b\xi + \sin b\xi)\cos b(\xi - \pi) \\ & + \frac{1}{4}(\cos b\pi - \sin b\pi)\Big) \\ & + e^{b(\pi+\xi-y)}\Big(-\frac{1}{4}\Big)(\cos b\xi + \sin b\xi) + O(e^{b(\pi-\delta)})\Big\}\end{aligned}$$

and that

$$\begin{aligned}\tilde{L}'(y) = \frac{b^3}{8}\Big\{ & e^{b\pi}\Big(\frac{1}{2}(\cos b\xi + \sin b\xi)\sin b(y - \xi) \\ & + \frac{1}{4}(\cos by + \sin by)\Big) \\ & + e^{by}\Big(\frac{1}{2}(\cos b\xi + \sin b\xi)\cos b(\xi - \pi) \\ & + \frac{1}{4}(\cos b\pi - \sin b\pi)\Big) \\ & - e^{b(\pi+\xi-y)}\Big(-\frac{1}{4}\Big)(\cos b\xi + \sin b\xi) + O\big(e^{b(\pi-\delta)}\big)\Big\},\end{aligned}$$

hence for all $y \in [\xi, \pi]$

$$|\tilde{L}(y)| \leq Cb^2e^{b\pi} \quad \text{and} \quad |\tilde{L}'(y)| \leq Cb^3e^{b\pi}.$$

Integrating twice by parts in $I_4^2$, we obtain

$$I_4^2 = K\beta^{-2}\Big(\int_\xi^\pi \tilde{L}(y)f''(y)\,dy + \big[\tilde{L}'(y)f(y) - \tilde{L}(y)f'(y)\big]_\xi^\pi\Big),$$

hence

$$\begin{aligned}\Big|I_4^2 - K\beta^{-2}\big[\tilde{L}'(y)f(y)\big]_\xi^\pi\Big| &\leq C\beta^{-2}\big(b^2e^{b\pi}\|f''\|_H \\ &\quad + b^2e^{b\pi}(|f'(\pi)| + |f'(\xi)|)\big) \\ &\leq C\beta^{-1}e^{b\pi}\|f\|_V.\end{aligned}$$

On the other hand

$$\begin{aligned}K\beta^{-2}\big[\tilde{L}'(y)f(y)\big]_\xi^\pi = \frac{K}{16}b^{-1}e^{b\pi}\{ & -(\cos b\xi + \sin b\xi)f(\xi) \\ & + c(b)f(\pi) + O(e^{-b\delta})\|f\|_V\},\end{aligned}$$

where the function $c(b)$ is defined in (60) and $\delta > 0$ is small enough. It follows that

$$I_3^2 + I_4^2 = \frac{K}{16}b^{-1}e^{b\pi}c(b)f(\pi) + O(\beta^{-1}e^{b\pi}\|f\|_V).$$

We conclude that

$$N = \frac{K}{16}b^{-1}e^{b\pi}c(b)f(\pi) + O(\beta^{-1}e^{b\pi}(\|f\|_V + \|g\|_H)) \quad (62)$$

and that (writing again the subscript $n$)

$$\begin{aligned}|\beta_n w_n(\pi)| &= \Big|\beta_n\frac{N}{D}\Big| \\ &\leq \Big|\frac{(K/16)b_ne^{b_n\pi}c(b_n)f_n(\pi)}{D_1 + iD_2}\Big| + C(\|f_n\|_V + \|g_n\|_H) \\ &\leq C(\|f_n\|_V + \|g_n\|_H) \\ &\to 0 \quad \text{as } n \to \infty\end{aligned}$$

by (28) and (58). Using (50), we obtain the promised contradiction to (36). The proof of Theorem 3 is achieved. □

### 4.2 *Polynomial decay rate*

The next result asserts that a polynomial decay rate still holds true for almost every value of $\xi$.

**Theorem 4:** *For almost every $\xi \in (0, \pi)$ we have for every $l > 1$*

$$\sup_{|\beta|\geq 1}|\beta|^{-l}\|(i\beta - \mathcal{A})^{-1}\| < \infty. \quad (63)$$

*This implies by Liu and Rao* (2005) *that for every positive integer k there exists a constant $C_k > 0$ such that*

$$\|e^{t\mathcal{A}}z_0\|_{\mathcal{H}} \leq C_k\Big(\frac{\ln t}{t}\Big)^{k/l}(\ln t)\|z_0\|_{\mathcal{D}(\mathcal{A}^k)} \qquad \forall z_0 \in \mathcal{D}(\mathcal{A}^k).$$

**Proof of Theorem 4:** Notice first that it is sufficient to prove (63) for all numbers $l \in (1, +\infty) \cap \mathbb{Q}$ and for a.e. $\xi \in (0, \pi)$. Pick any $l \in (1, +\infty) \cap \mathbb{Q}$, any $\xi \in (0, \pi)$ (to be specified later) and argue by contradiction as in the proof of Theorem 3. If (63) is false, then there exist $\beta_n \in (0, +\infty)$, $(w_n, v_n) \in \mathcal{D}(\mathcal{A})$ for $n = 1, 2, \ldots$ such that

$$\|(w_n, v_n)\|_{\mathcal{H}} = 1, \qquad \beta_n \to +\infty, \quad (64)$$

and such that, setting $(f_n, g_n) := (i\beta_n - \mathcal{A})(w_n, v_n)$,

$$\beta_n^l(f_n, g_n) \to (0, 0) \text{ in } \mathcal{H}. \quad (65)$$

Then (36) and (50) hold true, and to obtain a contradiction to (36) it is sufficient to prove that $\beta_n w_n(\pi) \to 0$. The same calculations as in the proof of Theorem 3 lead to $w(\pi) = N/D$ (the subscript $n$ being again omitted), where $N$ fulfills (62), and $D = D_1 + iD_2$ with $D_1$ and $D_2$ given by (55) and (58), respectively. We aim to prove that $|D| \geq Ce^{b\pi}/b^{2+2\varepsilon}$ for any $\varepsilon > 0$, a.e. $\xi \in (0, \pi)$, and every $b > 0$ large enough. To this end, we notice first that

$$(\cos b\xi + \sin b\xi)(\cos b(\xi - \pi) - \sin b(\xi - \pi))$$
$$= 2\sin\pi\left(b\frac{\xi}{\pi} + \frac{1}{4}\right)\sin\pi\left(b\left(1 - \frac{\xi}{\pi}\right) + \frac{1}{4}\right).$$

Let $|||x||| = \inf_{k\in\mathbb{Z}} |x - k|$. We need the following result, which is a variant of a classical result in Diophantine approximations (compare [Lang (1995, Theorem 4]). Its proof may be found in Crépeau and Prieur (to appear).

**Lemma 5:** *Pick any* $\varepsilon > 0$. *Then for almost every* $\alpha \in (0, 1)$, *there is only a finite number of solutions* $q \in \mathbb{N}^*$ *to the inequality*

$$\left|\left|\left|\left(q + \frac{1}{2}\right)\alpha + \frac{1}{4}\right|\right|\right| < \frac{1}{q^{1+\varepsilon}}.$$

In particular, for a.e. $\alpha \in (0, 1)$ one may find a positive constant $C$ such that for all $q \in \mathbb{N}^*$

$$\left|\left|\left|\left(q + \frac{1}{2}\right)\alpha + \frac{1}{4}\right|\right|\right| \geq \frac{C}{q^{1+\varepsilon}}.$$

Applying that property to $\xi/\pi$ and to $1 - \xi/\pi$, we conclude that for a.e. $\xi \in (0, \pi)$, we have for some constant $C > 0$ and for all numbers $b = q + \frac{1}{2}$, $q \in \mathbb{N}^*$,

$$\left|\sin\pi\left(b\frac{\xi}{\pi} + \frac{1}{4}\right)\right| \geq \frac{C}{b^{1+\varepsilon}} \text{ and } \left|\sin\pi\left(b\left(1 - \frac{\xi}{\pi}\right) + \frac{1}{4}\right)\right| \geq \frac{C}{b^{1+\varepsilon}},$$

hence $|c(b)| \geq Cb^{-2-2\varepsilon}$ and

$$|D_2(b)| \geq C\frac{e^{b\pi}}{b^{1+2\varepsilon}}$$

if $b$ is large enough. Since the function $c(b)$ is uniformly Lipschitz continuous on $\mathbb{R}$, the same inequalities hold (with different constants) for $b$ (large enough) in

$$\bigcup_{q\in\mathbb{N}^*}\left(q + \frac{1}{2} - \frac{C'}{q^{2+2\varepsilon}}, q + \frac{1}{2} + \frac{C'}{q^{2+2\varepsilon}}\right),$$

if the constant $C' > 0$ is small enough. On the other hand, one may associate with that constant $C' > 0$ a constant $C'' > 0$ such that

$$|D_1(b)| \geq C''\frac{e^{b\pi}}{b^{2+2\varepsilon}}$$

for $b$ large enough fulfilling $|b - (q + \frac{1}{2})| \geq C'/q^{2+2\varepsilon}$ for each $q \in \mathbb{N}^*$. It follows that for a.e. $\xi \in (0, \pi)$, we have

$$|D(b)| \geq C\frac{e^{b\pi}}{b^{2+2\varepsilon}} \tag{66}$$

for $b$ large enough. Gathering together (62) and (66), we arrive to

$$|\beta_n w_n(\pi)| \leq Cb_n^{2+2\varepsilon}\left(\|g_n\|_H + \|f_n\|_V\right) \to 0$$

as $n \to +\infty$ by (65), if we pick $\varepsilon := l - 1$. Thus we have obtained the desired contradiction to (36). The proof of Theorem 4 is complete. □

## 5. Conclusion

The paper was devoted to the output stabilization of a clamped-free beam with collocated piezoelectric sensor/actuator. Under the assumption that the actuator is touching the clamped extremity of the beam, it has been proved that the strong stability holds provided that the length $\xi$ of the actuator does not belong to a certain dense countable set of $(0, \pi)$. Under this assumption it has been shown that the energy decreases exponentially if the ratio $\xi/\pi$ belongs to a certain set of rational numbers. The question whether this last assumption may be dropped will be the purpose of further research in a near future.

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