Anti-Windup Design for a Reaction-Diffusion Equation

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Abstract— **This paper focuses on the anti-windup design for saturated one-dimensional linear reaction-diffusion equation. The considered open-loop system admits a finite number of unstable poles. We consider a scenario in which the system is controlled via a dynamic output feedback controller ensuring closed-loop exponential stability. Within this setting, a method is proposed to design a dynamic anti-windup compensator to maximize the region of attraction and minimize the effect of external perturbations. More precisely, the sufficient conditions for the local exponential stability of the closed-loop system are derived and expressed in terms of a set of matrix inequalities. Using generalized sector conditions and proper change of variables, the conditions are then recast as an optimization problem solving linear matrix inequalities. A numerical example is provided to showcase the proposed method and highlight its effectiveness on the system performance.**

I. Introduction

In various control engineering problems, the linearity of the modeled system is hindered by a nonlinear saturated control input. This constraint on the control input generates an inconsistency (also called a windup) described by an offset between the plant input and the unconstrained control signal. The main goal for introducing an anti-windup compensator is to compensate for this offset and therefore restore consistency within the closed-loop system. The early anti-windup technique was presented in [6] and then later extended to observer-based approaches in [1], [24]. There have been several advancements in different anti-windup schemes in the general context of finite-dimensional systems (see [5], [11], [23] for some literature on this topic).

A popular approach is LMI-based, in which the saturation is treated as a sector nonlinearity and Lyapunov methods are used to derive linear matrix inequalities (LMIs) conditions that can guarantee the desired regions of attraction for the system origin and performance levels. The method can be briefly described as measuring the difference between the control signal before and after saturation and injecting it into the compensator system which then recovers the discrepancy, in a manner that satisfies the LMI conditions, and emits a signal into the controller state equation. In some applications, the compensator emits a second signal into the saturated control input directly. In the general context of infinite-dimensional systems, the notion of anti-windup compensation is yet to be explored.

Infinite-dimensional systems emerge to be of utmost relevance when studying physical systems throughout all engineering domains such as quantum systems, fluid mechanical systems, wave propagation, diffusion phenomena. Furthermore, practical applications are subjected to delays, reaction-diffusion dynamics, thermodynamics, etc. Therefore, the control theory of infinite-dimensional systems remains a necessary area of research and its motivation has been well established (see more in [4], [14], [15]). The problem of output feedback control by means of an observer-based control, adaptive control or model predictive control has been previously presented in (see [3], [25]) and output-feedback control extensions to PDEs (see e.g., [2], [26]).

In this work, we focus on the output feedback stabilization of a specific type of partial differential equation called the reaction-diffusion equation. The global stability of a reaction-diffusion equation has been investigated in previous papers [7]–[9], [12], [19] and the extension to local stability in the case of saturated actuator was studied in [13]. We will use the regional, LMI-based anti-windup scheme presented in [21], [22] to achieve local exponential input-output stability for a class of distributed parameter systems. More precisely, a dynamic anti-windup compensator will be introduced to a stable closed-loop system consisting of a reaction-diffusion plant in feedback with a saturated dynamic controller. The control input acts on the domain of the partial differential equation. Using Lyapunov methods, dead-zone nonlinearities and associated sector conditions, we tackle two main issues. The first is estimating the region of attraction for the closed-loop system given in terms of linear matrix inequalities when the in-domain exogenous signal is considered to be null. The second is evaluating the performance level of each system by estimating the input-output stability (IOS) gain when the in-domain exogenous signal is different than zero and is energy-bounded. In the process of achieving these two goals, an optimization problem for the anti-windup design problem is presented which allows the optimization of the region of attraction and the stability gain. Finally, the efficiency of the proposed method is illustrated using numerical simulations which clearly demonstrate the benefits of the anti-windup compensator in a saturated control problem.

The remainder of this paper is organized as follows. Section II presents preliminary definitions and results and states the problem that we solve. Section IV presents the main results pertaining to stability analysis and anti-windup design. Section VI showcases the effectiveness of the proposed method through a numerical example. Finally, Section VII concludes the paper and offer some future insight.

Notation and basic notions: The symbol $\mathbb{S}_p^n$ denotes the set of real $n \times n$ symmetric positive definite matrices. We use the notation $\mathtt{He}(A) = A + A^\top$ for square

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matrices. For a symmetric matrix A, positive definiteness and positive semidefiniteness are denoted, respectively, by $A > 0$ and $A \geq 0$. Also, $\lambda_{\min}(A)$ (respectively $\lambda_{\max}(A)$) denotes its smallest (respectively largest) eigenvalue. The following shorthand notation is used $M^\top AM = (\bullet)^\top AM$. We use the equivalent notation for Euclidean vectors $(x, y) = \begin{bmatrix} x^\top & y^\top \end{bmatrix}^\top$. In partitioned symmetric matrices, the symbol $\star$ stands for symmetric blocks. For a vector $z \in \mathbb{R}^n$, $\|z\|$ denotes its Euclidean norm. For $U \subset \mathbb{R}$, $f\colon U \subset \mathbb{R} \to V$, we denote by $\|f\|_{\mathcal{L}^2} = (\int_U \|f(z)\|^2\, dz)^{\frac{1}{2}}$, the $\mathcal{L}^2$-norm of[1] f and the Fréchet derivative of f at z and is denoted by $Df(z)$. Given $f\colon U \subset \mathbb{R} \to V$, we say that $f \in \mathcal{L}^2$ if f is measurable and $\|f\|_{\mathcal{L}^2}$ is finite. The symbol $\mathcal{C}^k(U, V)$ denotes the set of functions $f\colon U \to V$ that are k-times continuously differentiable. Let p a positive integer, the symbol $\mathcal{H}^p(0,1)$, denotes the set of functions $f\colon [0,1] \to \mathbb{R}$ such that $f, \frac{d}{dz}f, \ldots, \frac{d^{p-1}}{dz^{p-1}}f$ are absolutely continuous on $(0,1)$ and $\frac{d^p}{dz^p}f \in \mathcal{L}^2$.

Let $p \in \mathcal{C}^1([0,1];\mathbb{R})$ and $q \in \mathcal{C}^0([0,1];\mathbb{R})$ with $p, q > 0$. Let the Sturm-Liouville operator $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset \mathcal{L}^2(0,1;\mathbb{R}) \to \mathcal{L}^2(0,1;\mathbb{R})$ be defined by

$$\mathbb{A}f \coloneqq -(pf')' + qf \tag{1}$$

on the domain $\mathcal{D}(\mathbb{A}) \subset \mathcal{L}^2(0,1;\mathbb{R})$ given by $\mathcal{D}(\mathbb{A}) \coloneqq \{f \in \mathcal{H}^2(0,1) : f'(0) = f(1) = 0\}$. The eigenvalues $\lambda_n, n \geq 1$ of $\mathbb{A}$ are simple, non-negative, and form an increasing sequence with $\lambda_n \to +\infty$ as $n \to +\infty$. Moreover the associated unit eigenvectors $\phi_n \in \mathcal{L}^2(0,1;\mathbb{R}^n)$ form an orthonormal basis and we also have $\mathcal{D}(\mathbb{A}) = \{f \in \mathcal{H}^2(0,1;\mathbb{R}) : \sum_{n\geq 1} |\lambda_n|^2 |\langle f, \phi_n\rangle|^2 < +\infty\}$ and $\mathbb{A}f = \sum_{n\geq 1} \lambda_n \langle f, \phi_n\rangle \phi_n$. Let $p_\star, p^\star, q^\star, q_\star \in \mathbb{R}$ be such that $0 < p_\star \leq p(x) \leq p^\star$ and $0 < q_\star \leq q(x) \leq q^\star$ for all $x \in [0,1]$, then it holds (see e.g., [16]), for all $n \geq 1$,

$$0 \leq \pi^2(n-1)^2 p_\star + q_\star \leq \lambda_n \leq \pi^2 n^2 p^\star + q^\star. \tag{2}$$

II. Problem Statement

We consider the stabilizability problem of a one-dimensional linear reaction-diffusion equation by means of a distributed control input u. The system model is given for all $t \geq 0$ and for $z \in (0,1)$:

$$\begin{aligned} w_t(t,z) =& (p(z)w_z(t,z))_z + (q_c - q(z))w(t,z) \\ &+ b(z)u(t) + m(z)d(t) \\ w_z(t,0) =& w(t,1) = 0 \\ y(t) =& w(t,0). \end{aligned} \tag{3}$$

The state-space of this system is $\mathcal{L}_2(0,1;\mathbb{R})$, and we assume that $q_c \in \mathbb{R}$ and $b, m \in \mathcal{L}^2(0,1;\mathbb{R})$. We suppose that the control input is subject to a symmetric magnitude limitation $\bar{u}_l$ such that $u \coloneqq \sigma(v) = \min(|v|, \bar{u}_l)\mathrm{sign}(v)$ where the input signal v is given by the output of the control system dynamics specified later. The dynamics are written in abstract form, and given an initial condition w_0, the Cauchy problem is written as:

$$\begin{aligned} \dot{w} &= -\mathbb{A}w + q_c w + bu + md \\ w(0) &= w_0. \end{aligned} \tag{4}$$

for the Sturm-Liouvile operator defined by (1).

A. Partition of the System into Stable and Unstable parts

The Sturm-Liouville operator $\mathbb{A}$ consists of positive single eigenvalues $(\lambda_n)_{n\geq 0}$ such that $\lambda_n \to \infty$, as $n \to \infty$. Now, introduce the coefficients of projection $w_n = \langle w(\cdot), \Phi_n\rangle$, $b_n = \langle b, \Phi_n\rangle$ and $m_n = \langle m, \Phi_n\rangle$ for $n \in \mathbb{N}^\star$. We have for all $w(t,\cdot) \in \mathcal{D}(\mathbb{A})$ and for all $t \geq 0$ and for $n \in \mathbb{N}^\star$:

$$\begin{aligned} &\dot{w}_n = (-\lambda_n + q_c)w_n + b_n u + m_n d, \\ &y = \sum_{i\geq 1} \Phi_i(0) w_i. \end{aligned} \tag{5}$$

Let $N_0 \geq 1$ and $\delta > 0$ be given such that $-\lambda_n + q_c < -\delta < 0$ for all $n \geq N_0 + 1$. We now introduce an arbitrary integer $N \geq N_0$ which will be further constrained later. We design an output feedback controller that will act on and modify the first N modes of the plant. First, we introduce the following vectors:

$$W^N \coloneqq \begin{bmatrix} w_1 & w_2 & \ldots w_N \end{bmatrix}^\top, B_1 \coloneqq \begin{bmatrix} b_1 & b_2 & \ldots b_N \end{bmatrix}^\top,$$
$$B_2 \coloneqq \begin{bmatrix} m_1 & m_2 & \ldots m_N \end{bmatrix}^\top, A_0 = \mathtt{Diag}(-\lambda_1 + q_c, \ldots, -\lambda_N + q_c)$$

and we focus on the following finite-dimensional truncation of (5):

$$\dot{W}^N = A_0 W^N + B_1 u + B_2 d. \tag{6}$$

When $d = 0$, if $q_c > \lambda_1$, then system (6) is unstable. The following assumptions are enforced henceforth.

Assumption 1: *The pair (A_0, B_1) is controllable.*

Assumption 2: *Let $\xi > 0$. We suppose that the Lipschitz continuous exogenous disturbance d belongs to the following set of functions $\mathcal{S} \coloneqq \{d\colon \mathbb{R}_{\geq 0} \to \mathbb{R}\colon d^2(t) \leq \xi^{-1}, \forall t \geq 0\}$.*

Given $N \geq N_0$, consider the following state space representation for the continuous-time linear plant $\mathcal{P}$:

$$\begin{aligned} &\dot{W}^N = A_0 W^N + B_1\sigma(v) + B_2 d, \\ &\dot{w}_n = (-\lambda_n + q_c)w_n + b_n\sigma(v) + m_n d \quad n \geq N+1, \\ &y = \sum_{i\geq 1} \Phi_i(0) w_i. \end{aligned} \tag{7}$$

The closed-loop system is composed of the plant $\mathcal{P}$ with a linear time-invariant dynamic output feedback controller $\mathcal{K}_c$ given by:

$$\begin{aligned} &\dot{X}_c = A_c X_c + B_c y + v_x \\ &v = Y_c = C_c X_c \end{aligned} \tag{8}$$

where $X_c \in \mathbb{R}^N$ is the state of the controller and $A_c \in \mathbb{R}^{N\times N}, B_c \in \mathbb{R}^N$ and $C_c \in \mathbb{R}^{1\times N}$ and the term v_x, related to the anti-windup setting, is considered to be null at this point. We assume that the control parameters A_c, B_c, C_c, D_c are given such that, for no saturation limitation and without disturbances, the origin of the closed-loop system (7) is globally exponentially stable (the design of such matrices is done in e.g., [7]).

[1]In this paper, we only consider Lebesgue measurable functions.

The main objective of this work is to introduce and design an anti-windup compensator such that we meet the desired system performance levels in terms of stability gain and region of attraction. The general framework of the anti-windup compensator is presented in the next section.

III. General Set-Up for the Anti-Windup

We introduce an anti-windup compensator to the overall system such that the output of the anti-windup plant is plugged into the dynamics of the control state X_c. The extra input v_x is given in the following simplified direct anti-windup system $\mathcal{K}_a$ presented in [20, Chapter 7] called the direct linear anti-windup design:

$$\begin{aligned} \dot{X}_{aw} &= A_{aw}X_{aw} + B_{aw}(\sigma(Y_c) - Y_c) \\ v_x &= C_{aw}X_{aw} + D_{aw}(\sigma(Y_c) - Y_c), \end{aligned} \tag{9}$$

where $X_{aw} \in \mathbb{R}^{2N}$ is the anti-windup state such that the dimension of the anti-windup state is the sum of the dimensions of W^N and X_c and the output of the anti-windup plant Y_{aw} is injected into the dynamics of the controller state ($Y_{aw} = v_x$). The goal is to design suitable anti-windup parameters $A_{aw} \in \mathbb{R}^{2N\times 2N}, B_{aw} \in \mathbb{R}^{2N}, C_{aw} \in \mathbb{R}^{N\times 2N}$ and $D_{aw} \in \mathbb{R}^N$ so that the origin of system (3) in closed loop with (8), (9) achieves input-output stability with smaller IOS gain and larger region of attraction.

For all $\zeta_a \in \mathcal{H}^1(0,1) \times \mathbb{R}^N \times \mathbb{R}^{2N}$, we define the following norm: $\|\zeta_a\|_{\mathcal{H}^1_a} := \sqrt{\|w\|^2_{\mathcal{H}^1} + X_c^\top X_c + X_{aw}^\top X_{aw}}$. The deadzone nonlinearity is defined by $\phi(v) := \sigma(v) - v$. Denoting $C := (\Phi_1(0), \Phi_2(0), \dots \Phi_N(0))$ and $\tilde{y} := \sum_{i\geq N+1} \Phi_i(0)w_i$, the interconnection of (7), (8), and (9) can be formerly written as $(\mathcal{P}, \mathcal{K}_c, \mathcal{K}_a)$:

$$\begin{cases} \dot{X}_f = A_{11}X_f + B_{11}\phi(Y_c) + B_{12}d + B_{13}\tilde{y} \\ \dot{w}_n = (-\lambda_n + q_c)w_n + b_nC_cX_c + b_n\phi(Y_c) + m_nd \\ \qquad\qquad\qquad\qquad\qquad\qquad n \geq N+1 \\ Y_c = \mathbb{K}X_f \\ y = CW^N + \tilde{y}. \end{cases} \tag{10}$$

where $X_f := \begin{bmatrix} W^N & X_c & X_{aw} \end{bmatrix}^\top$ and

$$A_{11} := \left[\begin{array}{cc|c} A_0 & B_1C_c & 0 \\ B_cC & A_c & C_{aw} \\ \hline 0 & 0 & A_{aw} \end{array}\right] =: \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline 0 & A_{aw} \end{array}\right]$$

$$B_{11} := \begin{bmatrix} B_1 \\ D_{aw} \\ B_{aw} \end{bmatrix}, B_{12} := \begin{bmatrix} B_2 \\ 0 \\ 0 \end{bmatrix}, B_{13} := \begin{bmatrix} 0 \\ B_c \\ 0 \end{bmatrix}, \mathbb{K} := \begin{bmatrix} 0 \\ C_c \\ 0 \end{bmatrix}^\top. \tag{11}$$

A wellposedness result follows from [17, Theorem 1.6, Chapter 6, Page 189].

Proposition 1: *[17] Let d Lipschitz continuous in $\mathcal{L}^1(\mathtt{dom}d;\mathbb{R})$ and $w_0 \in \mathcal{D}(\mathbb{A})$. Then, the closed-loop system consisting of* (4), (8) *and* (9) *has a unique strong solution pair*[2] $((w, X_c, X_a), d) \in \mathcal{C}^1(\mathtt{dom}w; \mathcal{L}^2(0,1;\mathbb{R})) \times \mathcal{C}^1(\mathtt{dom}X_c, \mathbb{R}^N) \times \mathcal{C}^1(\mathtt{dom}X_c, \mathbb{R}^{2N}) \times \mathcal{C}^{1,1}(\mathtt{dom}d;\mathbb{R})$ *where* $\mathtt{dom}d = \mathtt{dom}w = \mathtt{dom}X_c = \mathtt{dom}X_a$ *is an interval of* $\mathbb{R}_{\geq 0}$ *including zero.* ⋄

There are two main issues to tackle in system (10). The first issue is in the case when $d = 0$, where the goal is to optimize the region of attraction. The second issue is in the case when the energy bounded exogenous signal $d \neq 0$, where the goal is to minimize the effect of the disturbance signal on the input-output stability property. We are now able to formally state the problem we solve in this paper.

Problem 1: *Given $p \in \mathcal{C}^2([0,1];\mathbb{R})$, $q \in \mathcal{C}^0([0,1];\mathbb{R})$ with $p, q > 0$ and $q_c \in \mathbb{R}$. Given the control parameters $\begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}$. Design the anti-windup parameters $A_{aw}, B_{aw}, C_{aw}, D_{aw}$ such that the following properties hold for system* (3) *in closed loop with* (8) *and* (9)*:*

- *the origin of the closed-loop system is zero-input locally exponentially stable with region of attraction $\mathcal{R}_a$,*
- *for some (solution independent) $\psi_a, \upsilon_a, \rho_a > 0$, for each strong solution pair $(\zeta_a, d) \in \mathcal{R}_a \times \mathcal{S}$ to the closed-loop system, the bound:*

$$|y(t)| \leq \psi_a e^{-\upsilon_a t} \|\zeta_a(0)\|_{\mathcal{H}^1_a} + \rho_a \sqrt{\int_0^t d(\theta)^2 d\theta} \tag{12}$$

holds for all $t \in \mathbb{R}_{\geq 0}$.

Inequality (12) corresponds to an input-output stability (IOS) bound for the closed-loop system. The main contribution of this paper is to design an anti-windup system $\mathcal{K}_a$ in order to further minimize the effect of the gain ρ_a for $d \neq 0$ and further maximize the region of attraction $\mathcal{R}_a$ for $d = 0$. In the next section, we provide an explicit estimate of the IOS gain ρ_a and the region of attraction $\mathcal{R}_a$.

IV. Input-Output Lyapunov Stability Analysis

A. Sufficient Conditions

The following section presents sufficient conditions for the solution to Problem 1. The result relies on an exponential dissipation inequality. This is done by proving the following proposition:

Proposition 2: *Assume there exist a Fréchet differentiable functional $\mathcal{V}\colon \mathcal{H}^1(0,1;\mathbb{R}) \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R}_{\geq 0}$ and $c_1, c_2, c_3, c_4, \chi \in \mathbb{R}_{>0}$ such that*

$$\chi^2 < \xi c_3 c_4, \tag{13}$$

and such that for each $d \in \mathcal{S}$ and ζ_a satisfying $\mathcal{V}(\zeta_a) \leq c_4$, the following hold:

$$c_1 \|\zeta_a\|^2_{\mathcal{H}^1_a} \leq \mathcal{V}(\zeta_a) \leq c_2 \|\zeta_a\|^2_{\mathcal{H}^1_a}, \tag{14}$$

$$D\mathcal{V}(\zeta_a)\dot{\zeta}_a \leq -c_3\mathcal{V}(\zeta_a) + \chi^2 d^2. \tag{15}$$

Then, the origin of the closed-loop system (10) is zero-input exponentially stable with region of attraction containing $\{\zeta,\ \mathcal{V}(\zeta) \leq c_4\}$. In particular, (12) *holds with:*

$$\psi_a = \sqrt{\frac{2c_2}{c_1}}, \qquad \upsilon_a = \frac{c_3}{2}, \qquad \rho_a = \sqrt{2}\frac{\chi}{\sqrt{c_1}}. \tag{16}$$

Proof: Define $\mathcal{R}_a = \{\zeta_a,\ \mathcal{V}(\zeta_a) \leq c_4\}$. First we consider a strong solution pair $(\zeta_a(t), d(t))$; i.e, $\zeta_a \in \mathcal{R}_a$

[2] A pair $((w, X_c, X_a), d)$ is a strong solution pair to (10) if (w, X_c, X_a) is differentiable almost everywhere and it satisfies its dynamics for a.e. t.

and $d \in \mathcal{S} \cap \mathcal{C}^{1,1}(\mathtt{dom} d; \mathbb{R})$ for all $t \in \mathtt{dom}\zeta_a$ where $\mathtt{dom}\zeta_a$ is an interval of $\mathbb{R}_{\geq 0}$ including zero. Now, consider the function $\mathcal{W}\colon \mathtt{dom}\,\zeta_a \to \mathbb{R}$ defined by $\mathcal{W}(t) = (\mathcal{V} \circ \zeta_a)(t)$ Then, since $\mathcal{V}\colon \mathcal{H}^1(0,1;\mathbb{R}) \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is Fréchet differentiable everywhere and $\zeta_a : \mathtt{dom}\zeta_a \to \mathcal{H}^1(0,1;\mathbb{R}) \times \mathbb{R}^N$ is differentiable almost everywhere, it follows that $\dot{\mathcal{W}}(t) = D\mathcal{V}(\zeta_a)\dot{\zeta}_a(t)$. Thus we have for almost all $t \in \mathtt{dom}\zeta_a$

$$\dot{\mathcal{W}}(t) = D\mathcal{V}(\zeta_a)\begin{bmatrix} -\mathbb{A}w(t) + q_c w(t) + bu(t) + md(t) \\ A_c X_c(t) + B_c y(t) + v_x \\ A_{aw} X_{aw}(t) + B_{aw}(t)\phi(Y_c(t)) \end{bmatrix}$$

Using (15), one gets, for all $t \in \mathtt{dom}\zeta_a$, $\dot{\mathcal{W}}(t) \leq -c_3\mathcal{W}(t) + \chi^2 d(t)^2$. With (13), using a similar argument as [10, Lemma 9.2, page 347], one has that ζ_a cannot leave the set $\mathcal{R}_a$. Therefore, since $\mathcal{W}$ is continuous on $\mathtt{dom}\,\zeta_a$, from comparison lemma [10, Page 102], we have:

$$\mathcal{W}(t) \leq e^{-c_3 t}\mathcal{W}(0) + \chi^2 \int_0^t e^{-c_3(t-\theta)} d(\theta)^2 d\theta, \quad \forall t \in \mathtt{dom}\zeta_a.$$

The latter, thanks to (14), ensures that the origin of the closed-loop system is locally exponentially stable with respect to the $\mathcal{H}_a^1$-norm and with a region of attraction $\mathcal{R}_a$ when $d = 0$. The rest of the proof follows the proof done in [19, Proposition 2]. ■

The result given next provides the sufficient conditions for local exponential stability under the form of quadratic matrix inequalities.

Theorem 1: *Suppose there exist $P \in \mathbb{S}_p^{4N}, T \in \mathbb{R}_{>0}, G \in \mathbb{R}^{1\times 4N}$, $A_{aw} \in \mathbb{R}^{2N\times 2N}, B_{aw} \in \mathbb{R}^{2N}, C_{aw} \in \mathbb{R}^{1\times 2N}$ and $\alpha, \beta, , \gamma, \tau_1, \tau_2 \in \mathbb{R}_{>0}$ such that:*

$$\Theta_a := \begin{bmatrix} A_1 & PB_{11} - G^\top T & PB_{12} & PB_{13} \\ \star & \alpha' \|b\|_{\mathcal{L}^2}^2 - 2T & 0 & 0 \\ \star & \star & \alpha' \|m\|_{\mathcal{L}^2}^2 - \tau_2 & 0 \\ \star & \star & \star & -\beta \end{bmatrix} \leq 0 \tag{17}$$

$$\begin{bmatrix} P & \mathbb{K}^\top - G^\top \\ \star & \bar{u}_l^2 \end{bmatrix} \geq 0 \tag{18}$$

$$\tau_2 \xi^{-1} - \tau_1 < 0 \tag{19}$$

$$\Gamma_n := \lambda_n\left(-\lambda_n + q_c + \tau_1 + \frac{3}{\alpha} + \frac{\beta}{2\gamma}M_\phi\right) \leq 0, \quad \forall n \geq N+1 \tag{20}$$

where $A_1 := \mathtt{He}(PA_{11}) + \tau_1 P + A_{22}$, $A_{22} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha'\|b\|^2 C_c^\top C_c & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $M_\Phi := \sum_{i\geq N+1} \frac{\Phi_i(0)^2}{\lambda_i}$. Then, the parameters $A_{aw}, B_{aw}, C_{aw}, D_{aw}$ solve Problem 1. In particular, (12) holds with:

$$\begin{aligned} \rho_a &= \frac{\sqrt{2\tau_2}}{\sqrt{\min\{\lambda_{\min}(P), \gamma p_\star, \gamma q_\star\}}}, \qquad v_a = \frac{\tau_1}{2} \\ \psi_a &= \sqrt{\frac{2\max\{\lambda_{\max}(P), \gamma p^\star, \gamma q^\star\}}{\min\{\lambda_{\min}(P), \gamma p_\star, \gamma q_\star\}}}. \end{aligned} \tag{21}$$

Proof: The proof of the result hinges upon Proposition 2 with the following selection of the Lyapunov functional:

$$\begin{aligned} &\mathcal{V}\colon \mathcal{H}^1(0,1;\mathbb{R}) \times \mathbb{R}^N \times \mathbb{R}^{2N} \to \mathbb{R} \\ &\begin{bmatrix} w \\ X_c \\ X_{aw} \end{bmatrix} \mapsto X_f^\top P X_f + \gamma \sum_{n\geq N+1} \lambda_n \langle w, \Phi_n\rangle^2. \end{aligned} \tag{22}$$

Condition (14) holds for $c_1 := \min\{\lambda_{\min}(P), \gamma p_\star, \gamma q_\star\}$ and $c_2 := \max\{\lambda_{\max}(P), \gamma p^\star, \gamma q^\star\}$ which are strictly positive. Now we show that under the assumptions of the result, (15) holds. In particular, let:

$$\dot{\mathcal{V}}(w, X_c, X_{aw}, d) := D\mathcal{V}(w, X_c, X_{aw})\begin{bmatrix} \dot{w} \\ \dot{X}_c \\ \dot{X}_{aw} \end{bmatrix} \tag{23}$$

To this end, let $V_1(X_f) := {X_f}^\top P X_f$. Then, one gets:

$$DV_1(X_f)\dot{X}_f = \begin{bmatrix} X_f \\ d \\ \tilde{y} \\ \phi \end{bmatrix}^\top \begin{bmatrix} \mathtt{He}(PA_{11}) & PB_{12} & PB_{22} & PB_{23} \\ `\star & 0 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \begin{bmatrix} X_f \\ d \\ \tilde{y} \\ \phi \end{bmatrix}.$$

Let $G := \begin{bmatrix} G_1 & G_2 \end{bmatrix} \in \mathbb{R}^{1\times 2N} \times \mathbb{R}^{1\times 2N}$. Note that, due to condition (18), for any X_f satisfying $X_f^\top P X_f \leq 1$, it holds $X_f \in \{X_f \in \mathbb{R}^{4N}; |\mathbb{K}X_f - GX_f| \leq \bar{u}_l\}$. Therefore, by using [20, Lemma 1.6, Page 43] with $v_1 = \mathbb{K}X_f$ and $v_2 = GX_f$, it holds

$$\phi(\mathbb{K}X_f)^\top T(\phi(\mathbb{K}X_f) + GX_f) \leq 0 \tag{24}$$

for all X_f satisfying $X_f^\top P X_f \leq 1$. In particular $(w, X_c, X_{aw}) \in \mathcal{R}_a$, then we have that $X_f^\top P X_f \leq 1$ and so (24) holds, and we get the following inequality

$$\begin{aligned} &DV_1(X_f)\dot{X}_f - \tau_2 d^\top d \leq \\ &DV_1(X_f)\dot{X}_f - \tau_2 d^\top d - 2\phi T(\phi(\mathbb{K}_a X_f) + GX_f). \end{aligned} \tag{25}$$

Now, let $V_2(w) := \gamma \sum_{n\geq N+1} \lambda_n \langle w, \Phi_n\rangle$ with $\gamma > 0$. Then

$$DV_2(w)\dot{w} = 2\gamma \sum_{n\geq N+1} \lambda_n((-\lambda_n + q_c)w_n^2 + b_n\sigma(Y_c)w_n + m_n d w_n).$$

Thus, bearing in mind that $\mathcal{V} = V_1 + V_2$, using (25) for all $(w, X_c, X_{aw}) \in \mathcal{R}_a$, one gets:

$$\begin{aligned} &\dot{\mathcal{V}}(w, X_c, X_{aw}, d) + \tau_1\mathcal{V}(w, X_c, X_{aw}) - \tau_2 d^2 \leq \\ &\pi^\top \begin{bmatrix} \mathtt{He}(PA_{11}) + \tau_1 P & PB_{11} - G^\top T & PB_{12} & PB_{13} \\ \star & -2T & 0 & 0 \\ \star & \star & -\tau_2 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \underbrace{\begin{bmatrix} X_f \\ \phi \\ d \\ \tilde{y} \end{bmatrix}}_{\pi} \\ &+ 2\gamma \sum_{n\geq N+1} \lambda_n[(-\lambda_n + q_c + \tau_1)w_n^2 + b_n\mathbb{K}X_f w_n + b_n\phi w_n \\ &+ m_n d w_n] \end{aligned} \tag{26}$$

The rest of the proof follows the proof done in [19, Theorem 4]. Thus, for all $(w, X_c, X_{aw}) \in \mathcal{R}_a$ and $d \in \mathbb{R}$,

$$\begin{aligned} &\dot{\mathcal{V}}(w, X_c, X_{aw}, d) + \tau_1\mathcal{V}(w, X_c, X_{aw}) - \tau_2 d^2 \leq \\ &\pi^\top\Theta_a\pi + 2\gamma \sum_{n\geq N+1} \lambda_n\left(-\lambda_n + q_c + \tau_1 + \frac{3}{\alpha} + \frac{\beta}{2\gamma}M_\Phi\right)w_n^2 \end{aligned}$$

where Θ_a is defined in (17). Therefore, due to the satisfaction of (17), (18), and (20), one gets, for all $\forall\zeta_a \in \mathcal{R}_a, d \in \mathbb{R}$:

$$\dot{\mathcal{V}}(w, X_c, X_{aw}, d) + \tau_1\mathcal{V}(w, X_c, X_{aw}) - \tau_2 d^2 \leq 0$$

which corresponds to (15) with $\chi = \tau_2$, $c_3 = \tau_1$, and $c_4 = 1$. To conclude, observe that with the above selection of the parameters in Proposition 2, (19) matches (13). Hence, the result is established. ■

V. Compensator Design and Optimization Issues

Theorem 1 provides sufficient conditions for the solution to Problem 1. However, it turns out that the matrix Θ_a in (17) is nonlinear in the decision variables $P, A_{aw}, B_{aw}, C_{aw}, G$, and T. Thus, we transform the quadratic conditions found in Theorem 1 into linear matrix inequalities that can be exploited numerically in order to calculate the anti-windup parameters.

A. LMI-based Compensator Design

Using Theorem 1 and getting inspiration from [18], we can use a change of variables and derive a sufficient condition, so that Problem 1 is numerically tractable, as stated in the following proposition.

Proposition 3: *Assume there exist* $X, Y \in \mathbb{S}_p^{2N}, K \in \mathbb{R}^{2N\times 2N}, M \in \mathbb{R}^{N\times 2N}$, $Q_1 \in \mathbb{R}^{N\times 1}$, $Q_2 \in \mathbb{R}^{2N\times 1}$, Z_1, $Z_2 \in \mathbb{R}^{1\times 2N}$, $S, \beta, \alpha', \alpha'' \in \mathbb{R}_{>0}$ *and* $\tau_1, \tau_2 \in \mathbb{R}_{>0}$ *satisfying* (19) *and:*

$$\Xi := \begin{bmatrix} A_2 & \mathfrak{B}_1 - \mathfrak{Z} & \mathfrak{B}_2 & \mathfrak{B}_3 & \|b\|_{\mathcal{L}^2}\,\Pi^\top C_c^\top \\ \star & \alpha'' - 2S & 0 & 0 & 0 \\ \star & \star & \alpha'\,\|m\|^2_{\mathcal{L}^2} - \tau_2 & 0 & 0 \\ \star & \star & \star & -\beta & 0 \\ \star & \star & \star & \star & -\frac{1}{\alpha'} \end{bmatrix} \leq 0 \tag{27}$$

$$\begin{bmatrix} Y & I & YC_2^\top - Z_1^\top \\ \star & X & C_2^\top - Z_2^\top \\ \star & \star & \bar{u}_l^2 \end{bmatrix} > 0 \tag{28}$$

where $A_2 := \mathtt{He}(\mathfrak{A}) + \tau_1$ *and the following matrices are defined as:*

$$\mathfrak{A} := \begin{bmatrix} \mathcal{A}Y + \bar{B}M & \mathcal{A} \\ K & X\mathcal{A} \end{bmatrix}, \mathfrak{X} := \begin{bmatrix} Y & I \\ I & X \end{bmatrix}, \mathfrak{Z} := \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix};$$

$$\mathfrak{B}_1 := \begin{bmatrix} B_1'S + \bar{B}Q_1 \\ Q_2 \end{bmatrix}, \mathfrak{B}_2 := \begin{bmatrix} B_2' \\ XB_2' \end{bmatrix}, \mathfrak{B}_3 := \begin{bmatrix} B_3' \\ XB_3' \end{bmatrix}$$

and $C_2 := \begin{bmatrix} 0 & C_c \end{bmatrix}$, $\Pi := \begin{bmatrix} Y\bar{B} & \bar{B} \end{bmatrix}^\top$. *Then,* $I - YX$ *is nonsingular. Let* $U, V \in \mathbb{R}^{2N\times 2N}$ *be nonsingular matrices such that* $YX + VU^\top = I$ *and* $\gamma, \alpha > 0$ *such that* $\alpha' = \alpha\gamma$.

Assume moreover that (20) *holds. Then, the anti-windup parameters defined by*

$$\begin{bmatrix} A_{aw} & B_{aw} \\ C_{aw} & D_{aw} \end{bmatrix} = \begin{bmatrix} U & X\bar{B} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} K & L \\ M & E \end{bmatrix} \begin{bmatrix} V^{-\top} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} X\mathcal{A}Y & 0 \\ 0 & 0 \end{bmatrix} \tag{29}$$

where $\bar{B} := \begin{bmatrix} 0 & I \end{bmatrix}^\top$, $E := Q_1 S^{-1}$, $L := Q_2 S^{-1} - XB_1'$, *solve Problem 1 with* (21).

Proof: Let $T = S^{-1}$. First note that, due to (28), $I - YX$ is nonsingular. Let $\mathcal{Y} := \begin{bmatrix} Y & I \\ V^\top & 0 \end{bmatrix}$ and $P := \begin{bmatrix} X & U \\ U^\top & \widehat{X} \end{bmatrix}$, where $\widehat{X} := U^\top (X - Y^{-1})^{-1} U$. Since V is nonsingular, it follows that $\mathcal{Y}$ is so. Simple manipulations yield

$$\Theta_a' := \Lambda^\top \Theta_a \underbrace{\mathtt{Diag}(\mathcal{Y}, S, I, I)}_{\Lambda}$$

$$= \begin{bmatrix} A_3 & \mathcal{Y}^\top PB_{11}S - \mathcal{Y}^\top G^\top & \mathcal{Y}^\top PB_{12} & \mathcal{Y}^\top PB_{13} \\ \star & \alpha'\,\|b\|^2_{\mathcal{L}^2} S^2 - 2S & 0 & 0 \\ \star & \star & \alpha'\,\|m\|^2_{\mathcal{L}^2} - \tau_2 & 0 \\ \star & \star & \star & -\beta \end{bmatrix}$$

with $G := \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}^\top \mathcal{Y}^{-1} \in \mathbb{R}^{1\times 4N}$ and $A_3 := \mathcal{Y}^\top(\mathtt{He}(PA_{11}) + A_{22} + \tau_1 P)\mathcal{Y}$. We may check that

$$\begin{aligned} &\mathcal{Y}^\top(\mathtt{He}(PA_{11}))\mathcal{Y} = \mathtt{He}(\mathfrak{A}); \qquad \mathcal{Y}^\top P\mathcal{Y} = \mathfrak{X}; \\ &\mathcal{Y}^\top PB_{11}S = \begin{bmatrix} B_1' + \bar{B}D_{aw} \\ X(B_1' + \bar{B}D_{aw}) + UB_{aw} \end{bmatrix} S = \mathfrak{B}_1; \\ &\mathcal{Y}^\top PB_{12} = \mathfrak{B}_2; \qquad \mathcal{Y}^\top PB_{13} = \mathfrak{B}_3; \qquad \mathcal{Y}^\top G^\top = \mathfrak{Z}; \\ &\mathcal{Y}^\top A_{22}\mathcal{Y} = \Pi^\top \alpha'\,\|b\|^2_{\mathcal{L}^2}\, C_c^\top C_c \Pi. \end{aligned} \tag{30}$$

Thus, using Schur complement lemma, the following equivalence holds $\Theta_a' < 0 \iff \Xi < 0$. Therefore, under (27), (17) holds. Moreover, notice that from (28), it holds $\mathfrak{X} > 0$. Hence, thanks to the second relationship in (30) and $\mathcal{Y}$ being nonsingular, it follows that $P > 0$.

In addition, again by relying on the nonsingularity of $\mathcal{Y}$, multiplying the matrix in the left-hand side of (18) by $\mathcal{Y}^\top$ and $\mathcal{Y}$ on the right-hand side, we obtain that (18) is implied by (28). Therefore all the assumptions in Theorem 1 hold, thereby concluding the proof. ■

Remark 1: Notice here also that the matrix in (27) is nonlinear in the terms X, Y, τ_1, α'. The nonlinear terms $\tau_1 X, \tau_1 Y, \frac{1}{\alpha'}$, become linear if τ_1, α' are fixed by performing a line search on $\tau_1, \alpha' \in \mathbb{R}_{\geq 0}$. By choosing $\gamma = \beta$, we can deduce the value of α from α'. ∘

B. Optimal compensator design

The minimization problem of the effect of the external perturbations on system (3) in closed loop with (8) and (9) boils down to designing the anti-windup with minimal ρ_a in (12). Due to (21), such minimization can be achieved by solving the following optimization problem:

$$\begin{aligned} &\inf \quad \tau_2 + r - \beta \\ &\text{subject to: } (19), (20), (27), (28), X, Y \in \mathbb{S}_p^{2N}, \\ &\begin{bmatrix} -rI & V^\top & 0 \\ \star & -Y & I \\ \star & \star & -X \end{bmatrix} \leq 0. \end{aligned} \tag{31}$$

Minimizing the value of r is equivalent to maximizing $\lambda_{\min}(P)$ with P as in Proposition 3. Indeed, it turns out that

$$P^{-1} = \begin{bmatrix} Y & V \\ \star & V^\top (Y - X^{-1})^{-1} V \end{bmatrix}$$

and the last constraint in (31) is equivalent to $P^{-1} - rI \leq 0$ which means $\frac{1}{\lambda_{\min}(P)} \leq r$. Other optimization problems could be solved, as the maximization of the size of the region of attraction for the local exponential stability of the origin of (3) in closed loop with (8) and (9).

Remark 2: The basic algorithm to design the anti-windup compensator is as follows. For the given system, we numerically solve the linear conditions (19), (27), (28) for $N \geq 0$. Then, we confirm that (20) holds for the chosen N. If not, we increment N and repeat the procedure. The anti-windup parameters are then deduced using (29). ∘

VI. Numerical Simulation

In this section, we use the YALMIP package in Matlab to solve the LMIs and derive a feasible solution to Problem 1.

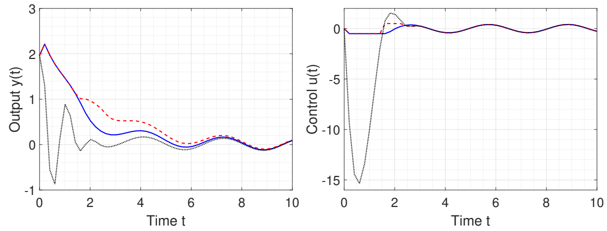

Fig. 1. Left: time evolution of the output y with anti-windup (solid-blue line), without anti-windup (dashed-red line) and without saturation (dotted-black line). Right: time evolution of the saturated control signal $u(t)$.

Consider (3) with $b(z) = m(z) = p(z) = q(z) \equiv 1$. We illustrate the result of Section III using a modal approximation that captures the 50 dominant modes of the reaction-diffusion plant with an in-domain disturbance given by, for all $t \geq 0$, $d(t) = 0.5\sin(2t)$, so that Assumptions 1 and 2 hold with $\xi = 1$. The saturation limit is $\bar{u}_l = 1$. Choose $q_c = 3$ such that the open-loop plant is unstable with $N_0 = 1$ and select the dimension of the finite-dimensional controller $N = 2$. We consider a given output feedback controller rendering the closed-loop, system without saturation, exponentially stable. For this selection of the controller, solving (31) gives:

$$\begin{aligned} A_{aw} &= \begin{bmatrix} 0.53 & 0.01 & -0.18 & 5.17 \\ -0.02 & -22 & 5.05 & -1.87 \\ -0.25 & 1.36 & -149 & 1.65 \\ -7.42 & 9.23 & -24 & -1.4 \end{bmatrix}, B_{aw} = \begin{bmatrix} -81 \\ -14 \\ 34 \\ -195 \end{bmatrix} \\ C_{aw} &= \begin{bmatrix} -18 & -34 & -12 & -28 \\ -4 & -23 & 15 & -29 \end{bmatrix}, D_{aw} = \begin{bmatrix} 197 \\ 124 \end{bmatrix} \end{aligned} \tag{32}$$

Figure 1 shows that the norm of the output of the closed-loop system with anti-windup converges to zero faster and smoother than that without anti-windup, imitating the linear behavior. Also, the effect of the disturbance on the steady-state reflects the bound in (12). In addition, Figure 1 shows that the control input in the closed-loop system with anti-windup tends to saturate for a longer time. This makes the effect of the proposed anti-windup compensator clear.

VII. Conclusion

In this paper, we designed an anti-windup compensator for a reaction diffusion equation with in-domain saturated control inputs. The proposed compensator allowed to compensate for the control saturation. Sufficient conditions for regional exponential stability and input-output stability were devised. A numerical affordable approach to the design of the compensator was propoed.

The use of more general Lyapunov functionals and alternative anti-windup schemes are currently part of our research.

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