High-gain observer design for some semilinear reaction-diffusion systems: a transformation-based approach

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Abstract— **The design problem of a high-gain observer is considered for some 2×2 and 3×3 semilinear reaction-diffusion systems, with possibly distinct diffusivities in the parabolic operator, and considering distributed measurement of part of the state. Due to limitations imposed by the parabolic operator, for the design of such an observer, an infinite-dimensional state transformation is first applied to map the system into a more suitable set of partial differential equations. The observer is then proposed including output correction terms and also spatial derivatives of the output of order depending on the number of distinct diffusivities. It ensures arbitrarily fast state estimation in some spatial norm. The result is illustrated with a simulated example of a Lotka-Volterra system.**

Keywords: high-gain observers, semilinear parabolic systems, Lotka-Volterra systems, C^k exponential stability.

I. Introduction

The problem of high-gain observer design for nonlinear finite-dimensional systems has been widely considered in the literature (see [13], [16], and references therein). Briefly, it relies on a tuning coefficient (gain), to be chosen large enough so as to ensure arbitrarily fast exponential convergence. In the recent papers [17], [18], this approach was extended to infinite-dimensional systems, more precisely, hyperbolic systems of balance laws, while utilizing distributed measurement of a part of the state.

For the case of linear distributed parameter systems, most techniques for the state estimation (the so–called late–lumping ones, see [27] for a survey on state estimation) rely on operator and semigroup-theoretic approaches, Lyapunov-based analysis and backstepping, see for instance [9], [10], [11], [24], [12], [15]. The case of state estimation for nonlinear infinte-dimensional systems, which is significantly more complicated, has been addressed in [25], [3], [7], [20], [2], [23], [6] amongst others, but mainly considering the full state vector on the boundaries as measurement, and without the high gain features.

The present paper aims at providing a solution to a high-gain observer design problem for some Lotka–Volterra–like semilinear parabolic cascade systems with 2 and 3 states, and considering measurement of the first state. This problem is technically nontrivial, since, as it was shown in [17], [18], an observability structure is not enough for the observer design, while system's differential operator imposes some limitations. More precisely, the distinct diffusivities of the parabolic operator do not allow the observer design to be directly feasible, as in the finite-dimensional approach. This work extends a recently introduced transformation-based methodology in [17] and [18] for 2×2 quasilinear and $n\times n$ linear hyperbolic systems to cases of 2×2 and 3×3 semilinear parabolic systems. Notice that the application of the present methodology to systems with more than 3 states is not solved, due to the nonlinearities, which impose extra difficulties. The class of systems that we study might describe biological predator-prey models and other population and social dynamics phenomena [5], [1]. Such systems have gained significant interest with respect to controllability and one can refer to [22] (see also [8]). For Lyapunov techniques on parabolic systems, one can refer to [19]. Moreover, observer design for finite-dimensional Lotka-Volterra systems has been addressed in [4].

The main contribution here is a solution to this high-gain observer design problem, in the presence of distinct diffusivities of the parabolic operator, and considering also semilinear dynamics. A fundamental idea of this paper is to first perform an appropriate infinite-dimensional and lower triangular transformation to map the considered systems into new sets of PDEs, where the parabolic operator is decomposed into a new one with only one diffusivity and a mapping, including spatial derivations of the measured state in its domain. As a consequence, this methodology results in requiring, additionally to the output correction terms, the injection of ouput's spatial derivatives in the high-gain observer dynamics, up to an order depending on the number of distinct diffusivities in the parabolic operator. The convergence of the observer is then proven for an appropriate spatial norm. Moreover, in the presence of the nonlinearities, we inject sufficiently smooth saturation functions in the observer nonlinearities, to tackle the lack of global Lipschitzness of the considered class of systems, while system's solutions remain uniformly bounded with a priori known bounds.

The paper is organized as follows. The sufficient conditions and a solution to the high-gain observer design problem are presented in Section II, where Theorem 1 constitutes the main result. The infinite-dimensional transformation that we use to obtain a target system for observer design, along with the proof of Theorem 1, are presented in Section III. In Section IV we apply our methodology to a 2×2 Lotka-Volterra system. Conclusions and perspectives are discussed in Section V.

Notation: For a given w in $\mathbb{R}^n$, $|w|$ denotes its usual Euclidean norm. For a given matrix A in $\mathbb{R}^{n\times n}$, A^{T} denotes its transpose, $|A| := \sup\{|Aw|, |w|=1\}$ is its induced

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norm and $\mathrm{Sym}(A) = \frac{A+A^{\mathrm{T}}}{2}$ stands for its symmetric part. By $\underline{\mathrm{eig}}(A)$ we denote the minimum eigenvalue of a matrix A. By I_n we denote the identity matrix of dimension n. For given $\xi : [0,+\infty) \times [l,L] \to \mathbb{R}^n$ and time $t \geq 0$ we use the notation $\xi(t)(x) := \xi(t,x)$, for all x in $[l,L]$. For a continuous (C^0) map $[l,L] \ni x \mapsto \xi(x) \in \mathbb{R}^n$ we adopt the notation $\|\xi\|_\infty := \max\{|\xi(x)|, x \in [l,L]\}$ (with $\|\xi\|_0 := \|\xi\|_\infty$). For a q - times continuously differentiable (C^q) map $[l,L] \ni x \mapsto \xi(x) \in \mathbb{R}^n$ we adopt the notation $\|\xi\|_q := \sum_{i=0}^{q} \|\partial_x^i \xi\|_\infty$. For a mapping f, we use the difference operator given by $\Delta_{\hat{\xi}}[f(\xi)] := f\left(\hat{\xi}\right) - f(\xi)$, parametrized by $\hat{\xi}$.

II. Problem Statement and Main Result

In this work, motivated by applications to information diffusion by multiple sources in social media [26], we consider a system of two or three one-dimensional semilinear parabolic equations, written in the following form:

$$u_t = Du_{xx} + A(u_1)u + f(u) \text{ on } (0,\infty) \times (l,L) \tag{1a}$$

$$u(0,x) = u^0(x), x \in [l,L] \tag{1b}$$

$$u_x(t,l) = 0, u_x(t,L) = 0, t \in (0,\infty). \tag{1c}$$

where $L > l \geq 0$ define the space domain, and $u = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^{\mathrm{T}}$ is the 2 or 3-dimensional vector state (namely $n = 2$ or $n = 3$),

$$D = \mathrm{diag}(d_1, \ldots, d_n); d_i > 0, i = 1, \ldots, n$$

$$A(u_1) := \begin{cases} \begin{pmatrix} 0 & a_{12}u_1 \\ 0 & 0 \end{pmatrix}, & \text{if } n = 2, \\ \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}, & \text{if } n = 3 \end{cases}$$

$$f(u) := \begin{cases} \begin{pmatrix} r_1 u_1 \left(1 - \frac{u_1}{K_n}\right) \\ r_2 u_2 \left(1 - \frac{u_2}{K_2}\right) + a_{21} u_2 u_1 \end{pmatrix}, & \text{if } n = 2, \\ \begin{pmatrix} r_1 u_1 \left(1 - \frac{u_1}{K_1}\right) \\ r_2 u_2 \left(1 - \frac{u_2}{K_2}\right) + a_{21} u_2 u_1 \\ r_3 u_3 \left(1 - \frac{u_3}{K_3}\right) + a_{31} u_3 u_1 + a_{32} u_3 u_2 \end{pmatrix}, & \text{if } n = 3 \end{cases}$$

for positive constants a_{12}, a_{23}, r_i, K_i. Note here that these two types of systems are henceforth parametrized by n, thus, whenever we say $n = 2$ or $n = 3$, we refer to the above 2×2 or 3×3 system, respectively. Assume, also, that initial conditions $u^0(\cdot)$, with $u^0(\cdot) \geq 0$, belong to the space $\mathcal{X}$, where

$$\mathcal{X} := C^2\left([l,L];\mathbb{R}^2\right), \text{when } n = 2 \text{ and}$$
$$\mathcal{X} := C^{q_0}([l,L];\mathbb{R}) \times C^2\left([l,L];\mathbb{R}^2\right), \text{when } n = 3;$$
$$q_0 := \max(2, 2q-2);$$
$$q := \min\{i : d_i = d_j, \forall j = i, i+1, \ldots, n\}.$$

with $\mathcal{X}$ being a Banach space, equipped with the natural norm, when $n = 2$, and

$$\|u\|_{\mathcal{X}} := \|u_1\|_{q_0} + \|(u_2,u_3)\|_2, \text{when } n = 3.$$

From the above definition of q, note that when diffusivities d_i are distinct, we get $q = n$ and we have $q = 1$, when all diffusivities are equal.

We consider a distributed measurement of the first state, written as follows:

$$y(t,x) = Cu(t,x), \tag{2}$$

where $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$, if $n = 2$, or $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, if $n = 3$.

Remark 1: *We observe that the above–considered system for $n = 2$ is a Lotka-Volterra system. When $a_{21} > 0$, it is called a cooperative system, whereas for $a_{21} < 0$, it is called competitive [5]. We considered a simpler $A(\cdot)$ for the 3×3 system, not depending anymore on the states, and, thus, not satisfying the exact form of classical 3×3 Lotka–Volterra systems. We conjecture that a natural extension to 3×3 Lotka-Volterra systems is not feasible. Moreover, the application of the methodology we adopt here to systems with more than 3 states remains open problem, due to the nonlinearities in the parabolic dynamics. An extension of the present methodology to $n \times n$ nonlinear systems, by following methods inspired by [18] for linear (hyperbolic) systems, for instance, is not possible.*

Let us assume that initial conditions u^0 satisfy compatibility conditions for the space $\mathcal{X}$. By invoking classical results presented in [21], it can be proven that system satisfies the sufficient conditions provided therein, which along with the assumed regularity, guarantee unique existence of classical nonnegative solutions. The required extra regularity of these solutions is easy to prove by using classical arguments. More precisely, we have the following local existence result.

Fact 1: *There exists $T > 0$ and a unique solution u of* (1) *on $[0,T)$, with $u \in C^1\left([0,T) \times [l,L];\mathbb{R}^2\right)$, if $n = 2$ and in $C^{\max(1,q_0-2)}([0,T) \times [l,L];\mathbb{R}) \times C^1\left([0,+\infty) \times [l,L];\mathbb{R}^2\right)$, if $n = 3$, satisfying $u(t,\cdot) \in \mathcal{X}$. Furthermore, the solution is nonnegative.*

We next make the following assumption on global existence and uniform boundedness of the solutions (the reader can refer to [21] and references therein for exact conditions to allow such property). This is a prerequisite for observer design. Furthermore, we assume that, for the considered 2×2 Volterra system, there exists a lower positive bound for the first state. The latter guarantees system's observability, presenting an analogy to the finite–dimensional systems, see [14].

Assumption 1: *System* (1) *admits a unique and uniformly bounded solution u in $C^1\left([0,+\infty) \times [l,L];\mathbb{R}^2\right)$, for all $t \in [0,+\infty)$ if $n = 2$ and in $C^{\max(1,q_0-2)}([0,+\infty) \times [l,L]) \times C^1\left([0,+\infty) \times [l,L];\mathbb{R}^2\right)$, for all $t \in [0,+\infty)$, if $n = 3$, with $u(t,\cdot) \in \mathcal{X}$. Moreover, whenever $n = 2$, for solution u_1, we have*

$$\inf_{(t,x) \in [0,+\infty) \times [l,L]} u_1(t,x) > 0.$$

Remark 2: *By invoking [21], the existence of global classical solutions to the considered classes of systems can be guaranteed by assuming, for instance, that $a_{21} = -a_{12}$ for the 2×2 Lotka–Volterra system and that*

$\begin{pmatrix} \frac{r_i}{K_i} & \frac{a_{i+1,i}}{2} \\ \frac{a_{i+1,i}}{2} & \frac{r_{i+1}}{K_{i+1}} \end{pmatrix}$; $i = 1, 2$ *is positive semi–definite for the* 3×3 *system.*

We next provide some intuition on the methodology we need to follow for the observer design problem to be solvable.

Let us first considered a matrix P, which is used as a Lyapunov matrix in the observer error asymptotic stability analysis, that is subsequent to the observer design. This matrix is assumed to be symmetric and positive definite, satisfying a Lyapunov equation of the following form for $(t, x) \in [0, +\infty) \times [l, L]$:

$$\mathrm{Sym}\left(PA(y(t,x))\right) - C^{\mathrm{T}}C \leq -\frac{\eta}{2} I_n, \tag{3}$$

for some constant $\eta > 0$. Such an inequality is always feasible for $A(y)$ and C satisfying particular structures as the ones we already assumed. More explicitly, for systems with two states ($n = 2$), feasibility of such an inequality requires additionally $y = u_1$ to satisfy Assumption 1 (having upper and lower bounds). To prove the existence of such a P, one can refer to [14] for the case of $A(\cdot)$ satisfying analogous conditions as in the present case. For the case $n = 3$ such an inequality is feasible, because of the observability of the pair A, C. Furthermore, let us note that such a P is never diagonal. We indicate now how this matrix P is linked to the observer design. For observable systems (1), (2) that we consider here, observer design cannot be directly applied, although both systems (for $n = 2$ and $n = 3$) satisfy a triangular observability structure. As it was previouly mentioned in [18], this is a result of the fact that for infinite–dimensional systems of the form $\dot{\xi} = \mathcal{A}\xi + A(\xi_1)\xi + f(\xi), y = \xi_1$, with $\mathcal{A}$ being system's differential operator, additionally to an observability structure satisfied by $A(\xi_1)$ and $f(\xi)$, we require that PM_i is symmetrix, with P being a symmetric matrix satisfying (3), and M_i being the components of system's differential operator, namely, $\mathcal{A} := \sum_{i=1}^{k} M_i \partial_x^i$ (in system (1) we have a parabolic operator satisfying $k = 2$ and $M_1 := 0, M_2 := D$). This P may not be diagonal, as we mentioned before, and cannot commute with matrix of diffusivities D. This symmetricity property is essential to hold, in order to proceed to the Lyapunov analysis, more particularly in the integration by parts that is applied. For this reason, we propose a transformation into a new system where the differential operator is decomposed into a sum between a new differential operator, satisfying this required symmetricity property, another differential operator (possibly nonlinear) including only the first measured state in its domain and a bilinear mapping between a function of the unmeasured state and a differential operator, including only the first state in its domain. Furthermore, we require that this transformation preserves the observability structure of A and f. That kind of transformation is obviously infinite–dimensional and, moreover, lower triangular, in order not to lose observability structure. More precisely, we show the existence of a linear bounded injective transformation $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, with bounded inverse, which maps initial system into a target system v, as follows:

$$v = \mathcal{T}u; \qquad \text{with } v_1 = u_1. \tag{4}$$

Such an infinite–dimensional nonlinear transformation always exists for the considered systems with $n = 2$ or 3, as shown in the sequel, although existence of a transformation (possibly nonlinear) for more general $n \times n$ Lotka–Volterra systems with distinct diffusivities remains open (see also Remark 1).

The target system (T) of PDEs, which is suitable for observer design, satisfies the following equations on $[0, +\infty) \times [l, L]$:

$$\text{(T)} \begin{cases} v_t(t,x) = d_n v_{xx}(t,x) + A(v_1(t,x))v(t,x) \\ +f(v(t,x)) + \mathcal{M}_1[v_1(t)](x) + \mathcal{M}_2[v_1(t)](x)v(t,x) \\ v_x(l) = \mathcal{K}v_1(l), v_x(L) = \mathcal{K}v_1(L), \\ y_v(t,x) = y(t,x) = Cu(t,x), \end{cases}$$

with initial condition $v(0,x) := v^0(x) = \mathcal{T}u^0(x)$, where $\mathcal{M}_1 : C^{q_0}([l,L];\mathbb{R}) \to C^0([l,L];\mathbb{R}^n), \mathcal{M}_2 : C^{q_0}([l,L];\mathbb{R}) \to C^{\max(0,2q-4)}([l,L];\mathbb{R}^{n\times n}), \mathcal{K} : C^{q_0}([l,L];\mathbb{R}) \to \mathbb{R}^n$ are nonlinear differential operators acting on v_1, to be determined in the sequel, depending on the choice of $\mathcal{T}$, and y_v is target system's output, which remains equal to original system's output y. The existence of such a transformation $\mathcal{T}$ is shown in the Section III below.

We are now in a position to propose a high–gain observer for target system (T) satisfying the following equations on $[0, +\infty) \times [l, L]$:

$$\begin{aligned} \hat{v}_t(t,x) &= d_n \hat{v}_{xx}(t,x) + A(y(t,x))\hat{v}(t,x) \\ &+ \mathcal{M}_2[y(t)](x)\hat{v}(t,x) + f\left(\mathrm{sat}_\delta(\hat{v}(t,x))\right) \\ &+ \Theta P^{-1}C^{\mathrm{T}}\left(y(t,x) - C\hat{v}(t,x)\right) + \mathcal{M}_1[y(t)](x), \end{aligned} \tag{6a}$$

$$\hat{v}_x(l) = \mathcal{K}y(l), \hat{v}_x(L) = \mathcal{K}y(L) \tag{6b}$$

with initial condition $\hat{v}^0(x) := \hat{v}(0,x)$ (for a function $\hat{v}^0$ in $\mathcal{X}$, satisfying compatibility conditions for this space), where

$$\Theta := \mathrm{diag}\left\{\theta, \theta^2, \ldots, \theta^n\right\}, \tag{7}$$

with $\theta > 1$ being the candidate high–gain constant of the observer, to be selected sufficiently large and precisely determined in the sequel, and P being symmetric and positive definite, satisfying (3) for some $\eta > 0$. In addition, we injected in the nonlinear dynamics a vector–valued function $\mathbb{R}^n \ni v \mapsto \mathrm{sat}_\delta(v) = \left(\mathrm{sat}_\delta^1(v_1), \cdots, \mathrm{sat}_\delta^n(v_n)\right)$, parametrized by δ, where

$$\delta := \sup_{t\in[0,+\infty)} \|v(t,\cdot)\|_{\mathcal{X}},$$

which is a real number by Assumption 1, in conjunction with (4). We assume that sat_δ is in C^2 and satisfies the following properties (there is no need, however, to inject this function in the appearing linear terms, since they are already globally Lipschitz, although this is not explicitly written here):

1) For every $\delta > 0$ and $w, \hat{w}$ in $\mathbb{R}$, such that $|w| \leq \delta$, there exists $\omega_\delta > 0$, such that the following inequality is satisfied:

$$|\mathrm{sat}_\delta^i(\hat{w}) - w| \leq \omega_\delta |\hat{w} - w|, i = 1, \ldots, n. \tag{8a}$$

2) There exists M, such that for every v in $\mathbb{R}$,

$$|\frac{\partial^j}{\partial v^j}\mathrm{sat}_\delta^i(v)| \leq M, j = 0, 1, 2, i = 1, \ldots, n. \tag{8b}$$

We are now in a position to present our main result on the convergence of the proposed high–gain observer.

Theorem 1: *Consider system* (1) *with output* (2) *and suppose that Assumption 1 holds. Then, there exists a linear bounded injective operator $\mathcal{T}$ with bounded inverse, transforming system* (1) *into system (T). Let also P be positive definite and symmetric, satisfying* (3) *for some $\eta > 0$. Then, for θ large enough, $\mathcal{T}^{-1}\hat{v}$ admits a unique solution on the one hand, providing an estimate for the state u of system* (1) *on the other hand. More precisely, for every $\kappa > 0$, there exists $\theta_0 \geq 1$, such that for every $\theta \geq \theta_0$, the following holds for all $t \geq 0$:*

$$\|\mathcal{T}^{-1}\hat{v}(t,\cdot) - u(t,\cdot)\|_\infty \leq l e^{-\kappa t}\|\mathcal{T}^{-1}\hat{v}^0(\cdot) - u^0(\cdot)\|_\infty, \tag{9}$$

for some $l > 0$, polynomial in θ.

III. Observer Convergence Proof

In this section we prove Theorem 1.

We show, first, the existence of $\mathcal{T}$ of the form (4) mapping (1), (2) into target system (T) of the previous section. Let us choose

$$\mathcal{T} := \begin{cases} I_2, & n = 2, \\ \begin{pmatrix} 1 & 0 & 0 \\ b\partial_x^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & n = 3 \end{cases};$$

$$\text{where } b := \frac{d_3 - d_2}{a_{12}}$$

Note that $\mathcal{T}$ is obviously invertible.

Then, applying this transformation to the initial system, we obtain system (T) with

$$\mathcal{M}_1[v_1] := \begin{cases} \begin{pmatrix} (d_1 - d_3)\partial_x^2 v_1 \\ 0 \end{pmatrix}, & n = 2, \\ \begin{pmatrix} (d_1 - d_3 - a_{12}b)\partial_x^2 v_1 \\ \begin{pmatrix} \left[b(d_1 - d_3)\partial_x^4 + b(r_1 - r_2)\partial_x^2\right] v_1 - \\ -b\left(a_{21} + 2\frac{r_1}{K_1}\right) u_1 \partial_x^2 v_1 - \\ -2b\frac{r_1}{K_1}(\partial_x v_1)^2 - b^2\frac{r_2}{K_2}(\partial_x^2 v_1)^2 \end{pmatrix} \\ 0 \end{pmatrix}, & n = 3, \end{cases}$$

$$\mathcal{M}_2[v_1] := \begin{cases} 0, & n = 2, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2b\frac{r_2}{K_2}\partial_x^2 v_1 & 0 \\ 0 & 0 & -ba_{32}\partial_x^2 v_1 \end{pmatrix}, & n = 3, \end{cases}$$

$$\mathcal{K} = 0, \text{if } n = 2,$$

$$\mathcal{K} = \begin{pmatrix} 0 \\ b\partial_x^2 \\ 0 \end{pmatrix}, \text{if } n = 3.$$

We note here some properties, that are invoked in the observer well–posedeness and its convergence proof. Due to continuity of the nonlinear operators $\mathcal{M}_1, \mathcal{M}_2$, we get

$$\sup_{\|y\|_{q_0} \leq \delta} |\mathcal{M}_1[y]| < +\infty,$$
$$s_\delta := \sup_{\|y\|_{q_0} \leq \delta} |\mathcal{M}_2[y]| < +\infty \tag{10}$$

We obtain similar properties as the above for the spatial derivatives of $\mathcal{M}_2$, up to order $\max(0, 2q - 4)$. Furthermore, for any $\delta > 0$, there exists constant $L_\delta > 0$, such that for every $v, \hat{v}$ in $C^0([l, L]; \mathbb{R}^n)$, with $\|v\|_\infty \leq \delta$, the following inequality is satisfied for all x in $[l, L]$:

$$|\Delta_{\mathrm{sat}_\delta(\hat{v}(x))} f(v(x))| \leq L_\delta |\hat{v}(x) - v(x)|. \tag{11}$$

Invoking again [21], in conjunction with the sufficient regularity we have assumed, property (10), and the global Lipschitzness of the nonlinearities $f \circ \mathrm{sat}_\delta$, we can prove well–posedeness of observer system (6) with initial conditions $\hat{v}^0 \in \mathcal{X}$ in the sense that there exists a T^*, such that system admits a unique solution $\hat{v}$ on $[0, T^*)$, with $\hat{v} \in C^1\left([0, T^*) \times [l, L]; \mathbb{R}^2\right)$, if $n = 2$ and in $C^{\max(1, q_0 - 2)}\left([0, T^*) \times [l, L]; \mathbb{R}\right) \times C^1\left([0, T^*) \times [l, L]; \mathbb{R}^2\right)$, if $n = 3$, with $\hat{v}(t, \cdot) \in \mathcal{X}$.

Let us now proceed to the stability proof.

We choose a scaled observer error as

$$\varepsilon := \Theta^{-1}(\hat{v} - v),$$

for which we derive the following parabolic equation on $[0, +\infty) \times [l, L]$:

$$\varepsilon_t(t,x) = d_n \varepsilon_{xx}(t,x) + \theta\left(A(y(t,x)) - P^{-1}C^{\mathrm{T}}C\right)\varepsilon(t,x)$$
$$+\mathcal{M}_2[y(t)](x)\varepsilon(t,x) + \Theta^{-1}\Delta_{\mathrm{sat}_\delta(\hat{v}(t,x))} f(v(t,x)), \tag{12a}$$
$$\varepsilon_x(l) = \varepsilon_x(L) = 0. \tag{12b}$$

To prove the error's exponential stability with respect to its origin, we adopt a Lyapunov–based approach. Let us define a Lyapunov functional $\mathcal{W}_{0,p} : \mathcal{X} \to \mathbb{R}$ by

$$\mathcal{W}_{0,p}[\varepsilon] := \left(\int_l^L \left(\varepsilon^{\mathrm{T}}(x)P\varepsilon(x)\right)^p \mathrm{d}x\right)^{1/p}.$$

with p in $\mathbb{N}$. Denoting $W_{0,p}(t) := \mathcal{W}_{0,p}[\varepsilon(t)], t \in [0, T^*]$, we calculate the time–derivative $\dot{W}_{0,p}$ along the solutions to the error equations (12) as follows:

$$\dot{W}_{0,p} = \frac{1}{p}W_{0,p}^{1-p}\int_l^L p\left(\varepsilon^{\mathrm{T}}(x)P\varepsilon(x)\right)^{p-1}\left(\varepsilon_t(x)^{\mathrm{T}}P\varepsilon(x)\right.$$
$$\left.+\varepsilon(x)^{\mathrm{T}}P\varepsilon_t(x)\right)\mathrm{d}x.$$

After substituting error equations (12) and applying an integration by parts, $\dot{W}_{0,p}$ can be written as follows:

$$\dot{W}_{0,p} = W_{0,p}^{1-p}\left(\frac{1}{p}T_{1,p} + T_{2,p} + T_{3,p}\right), \tag{13}$$

where

$$T_{1,p} := \left(\left[(\varepsilon^{\mathrm{T}} P \varepsilon)_x\right]_l^L\right)^p,$$

$$T_{2,p} := \int_l^L (\varepsilon^{\mathrm{T}} P\varepsilon)^{p-1} \left[2\varepsilon^{\mathrm{T}} P \Theta^{-1} \right.$$
$$\times \Delta_{\mathrm{sat}_\delta(\hat{v})}\left[f(v)\right] + \varepsilon^{\mathrm{T}} \left(P\mathcal{M}_2[y] + \mathcal{M}_2[y]P\right)\varepsilon$$
$$\left. -2\varepsilon_x^{\mathrm{T}} P \varepsilon_x\right] \mathrm{d}x,$$

$$T_{3,p} := 2\theta \int_l^L (\varepsilon^{\mathrm{T}} P\varepsilon)^{p-1}\varepsilon^{\mathrm{T}}\left[\mathrm{Sym}\left(PA(y)\right) - C^{\mathrm{T}} C\right]\varepsilon \mathrm{d}x.$$

By use of boundary conditions (12b), the Poincaré inequality, and also (3), (10), and (11), we get

$$T_{1,p} = 0,$$
$$T_{2,p} \le \sigma W_{0,p}^p,$$
$$T_{3,p} \le -\theta \frac{\eta}{|P|} W_{0,p}^p,$$

with

$$\sigma := 2\frac{|P|}{\underline{\mathrm{eig}}(P)}\left(\sqrt{n}L_\delta + s_\delta\right) - 2\frac{\underline{\mathrm{eig}}(P)}{|P|}\frac{\pi^2}{(L-l)^2}.$$

Now, selecting

$$\theta > \theta_0 := \max\left(1, \frac{|P|}{\eta}\sigma\right),$$

by (13), there exists $\kappa_\theta > 0$, such that

$$\dot{W}_{0,p}(t) \le -2\kappa_\theta W_{0,p}(t), \forall t \ge 0. \tag{15}$$

By comparison lemma, we get the following estimate:

$$W_{0,p}(t) \le e^{-2\kappa_\theta t} W_{0,p}(0), t \in [0, T^*].$$

By this estimate, we deduce that solutions to the observer equations (12) exist globally in time and, therefore, stability result holds for all $t \ge 0$. We invoke, next, the following property

$$\lim_{p\to\infty} W_{0,p} = \|\varepsilon^{\mathrm{T}}(\cdot)P\varepsilon(\cdot)\|_\infty,$$

holding for continuous ε, which along with the boundedness of $\mathcal{T}$ and its inverse, leads to stability inequality (9).

Remark 3: *One would expect a stability result as the one of Theorem 1, but for the norm of the space $\mathcal{X}$, in accordance with the system's regularity. Here, we prove the stability in the* sup*–norm instead, which requires less tedious calculations. Intuition on how to prove stability in the norm of space $\mathcal{X}$ can be acquired by works for 2×2 quasilinear hyperbolic systems [17] (C^1–norm) and $n \times n$ linear hyperbolic systems [18] (C^k–norm), by use of appropriate Lyapunov p–functionals.*

The proof of Theorem 1 is complete.

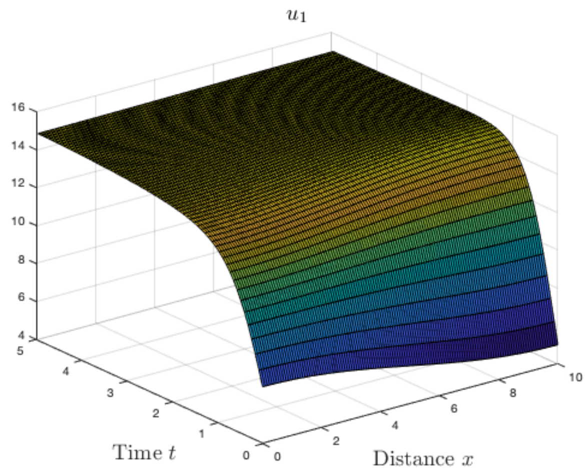

Fig. 1. Time and space evolution of system's output

IV. Simulation

In this section we apply the proposed high–gain observer design to a 2×2 Lotka–Volterra system (1) as an illustration, with $l = 0, L = 10$, diffusivities $d_1 = 2, d_2 = 1$ and $a_{12} = 0.2, a_{21} = -0.2, K_1 = 15, K_2 = 0.1, r_1 = 0.5, r_2 = 0.01$. We choose initial conditions $u_1^0(x) = \cos(\pi x/10) + 6, u_1^0(x) = -3\cos(\pi x/10) + 9$, such that Assumption 1 is satisfied with a global bound for the solution, which can be known a priori, to be $\delta = 20$. The corresponding output is represented in Fig. 1.

The proposed high–gain observer has the form (6). Exploiting the a priori known bounds of system's output, we choose $\eta = 0.5, P = \begin{pmatrix} 13.4 & 0.667 \\ 0.667 & 0.14 \end{pmatrix}$, in order that (3) holds. We next apply Theorem 1, with observer given by (6) and $\theta = 4$. As expected, the convergence of observer state to the unknown state u is guaranteed.

In Figures 2, 3 we see the observation errors for each of the states u_1, u_2, after choosing arbitrary observer's initial conditions, satisfying also observer's boundary conditions.

V. Conclusion

In this contribution, a high-gain observer for a class of 2×2 and 3×3 observable semilinear parabolic systems of Lotka-Volterra type, with possibly distinct diffusivities has been presented, considering distributed measurement of part of the state. This result constitutes an extension of the high-gain observer design for finite-dimensional systems to a class of nonlinear parabolic systems and, also, an extension of previous works towards this direction for hyperbolic systems. To overcome a technical obstacle imposed by the parabolic operator, the parabolic system is first mapped into a target system of PDEs and an observer for this system is designed, utilizing output correction terms and injection of output spatial derivatives. The extension of this methodology to more complex infinite-dimensional systems will be subject for our future work.

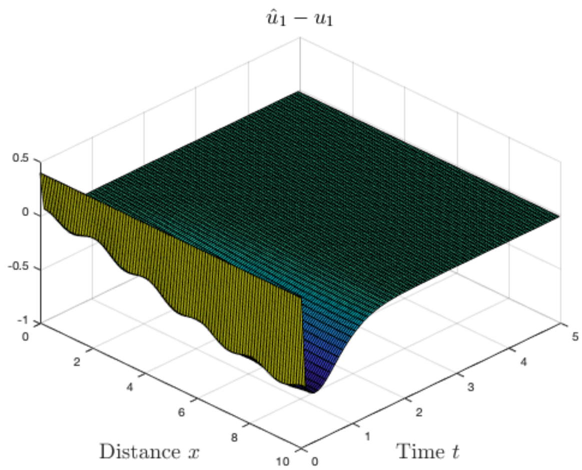

Fig. 2. Time and space evolution of the first estimation error

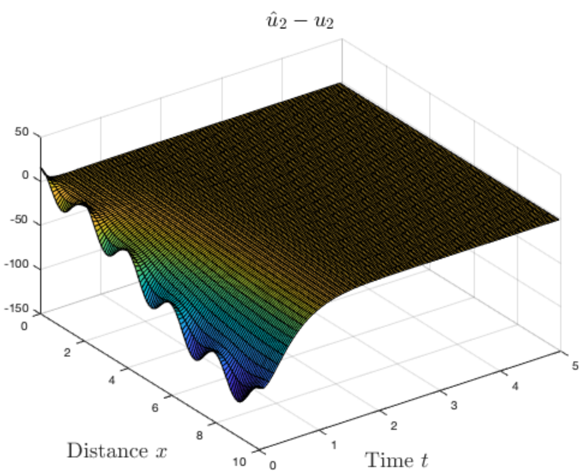

Fig. 3. Time and space evolution of the second estimation error

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