Quadratic Optimal Control of Linear Complementarity Systems: First order necessary conditions and numerical analysis

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***Abstract*—This article is dedicated to the analysis of quadratic optimal control of linear complementarity systems (LCS), which are a class of strongly nonlinear and nonsmooth dynamical systems. Necessary first-order conditions are derived, that take the form of an LCS with inequality constraints, hence are numerically tractable. Sufficiency is proved in a specific sense. Numerical examples illustrate the theoretical developments and demonstrate the efficiency of the complementarity approach.**

***Index Terms*—Optimal control, Linear Complementarity System, Necessary conditions, Numerical analysis, MPEC**

I. Introduction

The objective of this article is to analyse the quadratic optimal control of Linear Complementarity Systems (LCS). More precisely, we wish to investigate properties and numerical resolution of the problem:

$$\min\ J(x,u,v) = \int_0^T (x(t)^\intercal Q x(t) + u(t)^\intercal U u(t) + v(t)^\intercal V v(t))dt, \tag{1}$$

$$\text{subject to } \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t), \\ w(t) = Cx(t) + Dv(t) + Eu(t), \\ 0 \le v(t) \perp w(t) \ge 0, \\ Mx(0) + Nx(T) = x_b, \end{cases} \tag{2}$$

where $T > 0$, $A, Q \in \mathbb{R}^{n\times n}$, $V, D \in \mathbb{R}^{m\times m}$, $U \in \mathbb{R}^{m_u\times m_u}$, $E \in \mathbb{R}^{m\times m_u}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times m_u}$, $C \in \mathbb{R}^{m\times n}$, $M, N \in \mathbb{R}^{2n\times n}$, $x : [0,T] \to \mathbb{R}^n$, $v : [0,T] \to \mathbb{R}^m$, $u : [0,T] \to \mathbb{R}^{m_u}$, $x_b \in \mathbb{R}^{2n}$. The notation $0 \le v \perp w \ge 0$ means that each component v_i and w_i of the vectors v and w comply with: $v_i, w_i \ge 0$, $v_i w_i = 0$. In order to avoid trivial cases, we assume that $(C,E) \neq (0,0)$ and x_b is in the image set $\mathrm{im}(M,N)$. Also, we choose U symmetric and positive-definite, and Q and V semi-positive definite. For the sake of simplicity, we omit here further equality or inequality constraints in (2), but these could be added in the analytical results presented in this article. LCS as (2) find applications in several important fields such as Nash equilibrium games, contact mechanics and electrical circuits [1], [2], [3]. The analytical study of such nonlinear and nonsmooth dynamical systems is well developed, highlighting properties of existence and uniqueness of solutions, stability and stabilization, passivity, periodic oscillations, observer design, output regulation, or non-zenoness of solutions, see [4], [5], [6], [7]. For further relations between LCS and other dynamical formalisms, see [8]. The so-called Mathematical Programs with Equilibrium Constraints (MPEC), which are the finite-dimensional counterpart of (1)(2) [9], are at the core of the numerical solvers for the discretized version of (1)(2). The problem of existence of solutions of the Optimal Control Problem (1)(2) is actually twofold. First, the existence of a trajectory for the LCS (2) is not straightforward, even if the system is expressed as an Initial Value Problem (IVP). Also the only available analysis about the controllability of (2) may be found in [10] (when D is a P-matrix), and in [11] (when $D = 0$) in a very particular case. Secondly and most importantly, the existence of solution for (1)(2) is still an open question. A famous result due to Filipov [12, Theorem 9.2i] states the existence of an optimal control under convexity of the so-called velocity set $\mathcal{V}(x)$. In our case, $\mathcal{V}(x) = \{(u,v) \in \mathbb{R}^{2m} | 0 \le v \perp Cx + Dv + Eu \ge 0\}$ is clearly not convex, due to the complementarity. *Therefore throughout this article, we admit that an optimal solution exists (in the sense of Definition 2 below)*, and the focus is on necessary conditions this optimal solution must comply with (relying strongly on the seminal work in [13]), together with their numerical computation which relies on MPEC algorithms.

Our contributions are the following: in Section II some stationary results for (1)(2) are proved, stated as complementarity problems, hence getting rid of the index sets defining the complementarity modes. These conditions are proved to be necessary and sufficient. Secondly, we develop two numerical methods for solving this problem in Section III: the first one is a direct method using MPEC solvers, the second one is a hybrid method that uses the stationary results obtained in Section II, which allows us to get fast and precise numerical solutions. An example is presented, which demonstrates the validity of the method and of the developed codes. Conclusions end the article in Section IV. The long version of this work with additional results and examples can be retrieved in [14].

Notation and definitions: $\mathbb{N}$ is the set of non-negative integers. For $n \in \mathbb{N}$, we denote by $\overline{n}$ the set $\{1, ..., n\}$. Given a set of indices $I \subset \overline{n}$ and $z \in \mathbb{R}^n$, we denote $z_I = \{z_i,\ i \in I\}$. For two set of indices $I \subset \overline{n}$, $J \subset \overline{m}$ and a matrix $M \in \mathbb{R}^{n\times m}$, M_{IJ} is the matrix formed by the rows indexed by I and the columns indexed by J. If $I = \overline{n}$, then we simply write $M_{\bullet J}$ (the same holds when $J = \overline{m}$). $\mathbb{R}^n_+$ is the positive orthant of $\mathbb{R}^n$. For $z \in \mathbb{R}^n$, we denote by $z^\intercal$ its transpose (the same notation holds for matrices). For any set $\Omega \subseteq \mathbb{R}^n$,

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$\mathrm{cl}(\Omega)$ is the closure of B, $\mathrm{co}\Omega$ is the convex hull of Ω, and $\mathrm{dist}_\Omega(z) = \inf_{y\in\Omega} \|y - z\|$ is the Euclidean distance from z to Ω, and for any $\delta > 0$, $B_\delta(z)$ denotes the open ball of radius δ and center z. For any matrix M, $\mathrm{im}(M)$ is the image set of M, $\ker(M)$ is the kernel of M. Given $x \in \mathrm{cl}\ \Omega$, the proximal normal cone to Ω at x is defined as: $\mathcal{N}_\Omega^P(x) = \{y \in \mathbb{R}^n : \exists\sigma > 0 \text{ s.t. } \langle y, z - x\rangle \le \sigma\|z - x\|^2\ \forall z \in \Omega\}$. When Ω is a convex set, $\mathcal{N}_\Omega^P(x)$ just reduces to the normal cone of convex analysis. Then we simply denote it as $\mathcal{N}_\Omega(x)$. For a set P included in a Hilbert space E, we denote by P^* its dual cone, defined by $P^* = \{y \in E : \forall x \in P,\ \langle y, x\rangle \ge 0\}$. For $D \in \mathbb{R}^{m\times m}$ and $q \in \mathbb{R}^m$, we call LCP(q, D) the problem of finding $\lambda \in \mathbb{R}^m$ such that $0 \le \lambda \perp D\lambda+q \ge 0$ and SOL(q, D) the (possibly empty) set of such λ.

II. First-order necessary conditions for the optimal control problem (1)(2)

A. Preliminaries

Let us first recall some basic definitions which will be used throughout the article.

1) MPEC constraints qualification: Simply speaking, MPECs are optimization programs of the form:

$$\begin{aligned} &\min\ f(z) \\ &\text{s.t. } 0 \le G(z) \perp H(z) \ge 0, \end{aligned} \tag{3}$$

for some scalar function f and vector functions G and H. Usual Constraints Qualifications (CQ) for this kind of programs, as for instance the Mangasarian Fromovitz Constraint Qualification (MFCQ), are violated at any point satisfying the complementarity conditions. Using the piecewise programming approach, other CQ specific for MPEC can be derived. We present here some definitions and properties. Further results can be found in [15], [16], [17], [9].

Definition 1. *Let $n, m \in \mathbb{N}$. The* complementarity cone *is defined as $\mathcal{C}^\ell = \{(v, w) \in \mathbb{R}^m \times \mathbb{R}^m : 0 \le v \perp w \ge 0\}$. Given a system of constraints $\Omega = \{z \in \mathcal{Q} : (G(z), H(z)) \in \mathcal{C}^\ell\}$ where $\mathcal{Q}$ is a closed subset in $\mathbb{R}^n$ and $G, H : \mathbb{R}^n \to \mathbb{R}^m$, we say that the* local error bound condition *holds at $\bar z \in \Omega$ if there exist $\tau > 0$ and $\delta > 0$ such that*

$$dist_\Omega(z) \le \tau dist_{\mathcal{C}^\ell}(G(z), H(z)),\ \forall z \in B_\delta(\bar z) \cap \mathcal{Q}. \tag{4}$$

Three different index sets are defined from these constraints, called the active sets and the degenerate set: $I^{+0}(\bar z) = \{i \in \overline{m} : G_i(\bar z) > 0 = H_i(\bar z)\}$, $I^{0+}(\bar z) = \{i \in \overline{m} : G_i(\bar z) = 0 < H_i(\bar z)\}$, $I^{00}(\bar z) = \{i \in \overline{m} : G_i(\bar z) = 0 = H_i(\bar z)\}$. The sets $I^{\bullet 0}(\bar z)$ and $I^{0\bullet}(\bar z)$ are defined as $I^{\bullet 0}(\bar z) = I^{+0}(\bar z) \cup I^{00}(\bar z)$, $I^{0\bullet}(\bar z) = I^{0+}(\bar z) \cup I^{00}(\bar z)$. The MPEC Linear Condition *holds if both functions $G(\cdot)$, $H(\cdot)$ are affine and $\mathcal{Q}$ is a union of finitely many polyhedral sets. When $\mathcal{Q} = \mathbb{R}^n$, the* MPEC Linear Independence Constraint Qualification (LICQ) *holds at $\bar z \in \Omega$ if the family of gradients $\{\nabla G_i(\bar z) : i \in I^{0\bullet}(\bar z)\} \cup \{\nabla H_i(\bar z) : i \in I^{\bullet 0}(\bar z)\}$ is linearly independent.*

2) Non-smooth optimal control:

Definition 2. *Let $n, m \in \mathbb{N}$. We refer to any absolutely continuous function as an arc. An* admissible pair *for* (2) *is a 3-tuple of functions (x, u, v) on $[0, T]$ for which u, v are controls and x is an arc, that satisfy all the constraints in* (2)*. Let us define the constraint set S, by $S = \left\{(x, u, v) \in \mathbb{R}^n \times \mathbb{R}^{m_u} \times \mathbb{R}^m\ :\ (v, Cx + Dv + Eu) \in \mathcal{C}^\ell\right\}$. Given a constant $R > 0$, we say that an admissible pair (x^*, u^*, v^*) is a* local minimizer of radius R *for* (1)(2) *if there exists $\varepsilon > 0$ such that for every pair (x, u, v) admissible for* (2)*, which also satisfies $\|x(t) - x^*(t)\| \le \varepsilon$, $\left\|\binom{u}{v} - \binom{u^*(t)}{v^*(t)}\right\| \le R$ a.e. $t \in [0, T]$ and $\int_0^T \|\dot x(t) - \dot x^*(t)\|dt \le \varepsilon$, we have $J(x^*, u^*, v^*) \le J(x, u, v)$. For every given $t \in [0, T]$, and constant scalars $\varepsilon > 0$ and $R > 0$, we define the neighborhood of the point $(x^*(t), u^*(t), v^*(t))$ as $S_*^{\varepsilon,R}(t) = \left\{(x, u, v) \in S :\ \|x - x^*(t)\| \le \varepsilon,\ \left\|\binom{u}{v} - \binom{u^*(t)}{v^*(t)}\right\| \le R\right\}$. The dependence on time of index sets is denoted as $I_t^{0+}(x, u, v) = \{i \in \overline{m} : v_i(t) > 0 = (Cx(t) + Dv(t) + Eu(t))_i\}$. The same definition follows for $I_t^{+0}(x, u, v)$, $I_t^{00}(x, u, v)$, $I_t^{\bullet 0}(x, u, v)$, $I_t^{0\bullet}(x, u, v)$. For a positive measurable function k_S defined for almost every $t \in [0, T]$,* the bounded slope condition *is defined as the following implication:*

$$\begin{aligned} &(x, u, v) \in S^{\varepsilon,R}(t), (\alpha, \beta, \gamma) \in \mathcal{N}_S^P(x, u, v) \\ &\qquad\qquad \Longrightarrow\ \|\alpha\| \le k_S(t) \left\|\begin{pmatrix}\beta\\ \gamma\end{pmatrix}\right\|. \end{aligned} \tag{5}$$

B. Necessary first-order conditions

The necessary first order conditions for a very general optimal control problem containing complementarity constraints have been derived in [13]. In this section, our goal is to show how the results in [13] particularize when we consider the problem (1)(2), as stated in Theorem 1 below. Before going on, let us derive conditions which guarantee that (5) holds.

Proposition 1. *Suppose $E \ne 0$ and $\mathrm{im}(C) \subseteq \mathrm{im}(E)$. Then the local error bound condition holds at every admissible point, and the bounded slope condition for* (2) *holds.*

In order to prove this, one needs first the following lemma:

Lemma 1. *Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m_u\times m}$ such that $B \ne 0$ and $\ker(B) \subseteq \ker(A)$. Then there exists $\alpha > 0$ such that: $\forall x \in \mathbb{R}^m$, $\|Ax\| \le \alpha\|Bx\|$.*

Proof. Let $x \in \mathbb{R}^m$. We decompose x in the following way: $x = x_B + x_A + x_\perp$, $x_B \in \ker B$, $x_A \in \ker A \backslash \ker B \cup \{0\}$, $x_\perp \in (\ker A)^\perp$ such that $Ax = Ax_\perp$, $Bx = B(x_A + x_\perp)$. Thus:

$$\begin{aligned} \|Ax\|^2 &\le \|A\|^2\|x_\perp\|^2 \\ &\le \|A\|^2\left[\|x_\perp\|^2 + \|x_A\|^2\right] \\ &\le \|A\|^2\|x_A + x_\perp\|^2 \text{ because } x_A \perp x_\perp \end{aligned}$$

However, $x_A + x_\perp \in (\ker B)^\perp$. Therefore, the linear application $B : (\ker B)^\perp \to \mathrm{im}B$ (we write the same way the linear application and the associated matrix in the canonical basis) is an isomorphism, which admits an inverse $B^\dagger$, which is the Moore-Penrose inverse. It yields: $\exists! y \in \mathrm{im}B$; $B(x_A + x_\perp) = y \iff x_A + x_\perp = B^\dagger y$, and thus

$\|x_A + x_\perp\| \leq \|B^\dagger\|\|y\| = \|B^\dagger\|\|B(x_A + x_\perp)\|$. We can then prove that:

$$\begin{aligned}\|Ax\|^2 &\leq \|A\|^2\|x_A + x_\perp\|^2 \\ &\leq \|A\|^2\|B^\dagger\|^2\|B(x_A + x_\perp)\|^2 \\ &\leq \|A\|^2\|B^\dagger\|^2\|Bx\|^2.\end{aligned}$$

This yields the desired result. □

Proof of Proposition 1. Since the MPEC Linear Condition holds, the local error bound condition also holds at every admissible points (see [13, Proposition 2.3]). Applying [13, Proposition 3.7], a sufficient condition for the bounded slope condition to hold is:

$$\forall \lambda^H, \lambda^G \in \mathbb{R}^m,\ \|C^\intercal \lambda^H\| \leq k_S(t) \left\| \begin{matrix} \lambda^G + D^\intercal \lambda^H \\ E^\intercal \lambda^H \end{matrix} \right\|$$

Since $\mathrm{im}(C) \subseteq \mathrm{im}(E)$ (or equivalently, $\ker(E^\intercal) \subseteq \ker(C^\intercal)$), applying Lemma 1 with $C^\intercal$ and $E^\intercal$, one proves that: $\exists \alpha > 0,\ \forall \lambda^H \in \mathbb{R}^m,\ \|C^\intercal \lambda^H\|^2 \leq \alpha^2 \|E^\intercal \lambda^H\|^2$. It easily proves: $\exists \alpha > 0,\ \forall \lambda^G, \lambda^H \in \mathbb{R}^m,\ \|C^\intercal \lambda^H\|^2 \leq \alpha^2 \left\| \begin{matrix} \lambda^G + D^\intercal \lambda^H \\ E^\intercal \lambda^H \end{matrix} \right\|^2$ Therefore, the sufficient condition for the bounded slope condition hold. □

Let us now apply [13, Theorem 3.2], to the problem (1)(2).

Proposition 2. *Let (x^*, u^*, v^*) be a local minimizer of constant radius $R > 0$ for* (1)(2)*. Suppose* $\mathrm{im}(C) \subseteq \mathrm{im}(E)$*. Then there exist an arc $p : [0, T] \to \mathbb{R}^n$, a scalar $\lambda_0 \leq 0$ and measurable functions $\lambda^G : \mathbb{R} \to \mathbb{R}^m$, $\lambda^H : \mathbb{R} \to \mathbb{R}^m$ such that the following conditions hold:*

1) The non-triviality condition: $(\lambda_0, p(t)) \neq 0$, $\forall t \in [0, T]$.
2) The transversality condition: $\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} \in \mathrm{im} \begin{pmatrix} M^\intercal \\ N^\intercal \end{pmatrix}$.
3) The Euler adjoint equation: *for almost every* $t \in [0, T]$,

$$\begin{aligned} \dot{p}(t) &= -A^\intercal p - 2\lambda_0 Q x^* - C^\intercal \lambda^H \\ 0 &= F^\intercal p + 2\lambda_0 U u^* + E^\intercal \lambda^H \\ 0 &= B^\intercal p + 2\lambda_0 V v^* + \lambda^G + D^\intercal \lambda^H \\ 0 &= \lambda_i^G(t),\ \forall i \in I_t^{+0}(x^*, u^*, v^*) \\ 0 &= \lambda_i^H(t),\ \forall i \in I_t^{0+}(x^*, u^*, v^*). \end{aligned} \tag{6}$$

4) The Weierstrass condition for radius R: *for almost every* $t \in [t_0, t_1]$,

$$(x^*(t), u, v) \in S,\ \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^*(t) \\ v^*(t) \end{pmatrix} \right\| < R$$

$$\begin{aligned} &\implies \langle p(t), Ax^*(t) + Bv^*(t) + Fu^*(t)) \rangle + \\ &- \frac{1}{2}\left(x^*(t)^\intercal Q x^*(t) + u^*(t)^\intercal U u^*(t) + v^*(t)^\intercal V v^*(t)\right) \\ &\geq \langle p(t), Ax^*(t) + Bv + Fu) \rangle \\ &- \frac{1}{2}\left(x^*(t)^\intercal Q x^*(t) + u^\intercal U u + v^\intercal V v\right). \end{aligned} \tag{7}$$

Proof. Let us check that the problem complies with [13, Assumption 3.1]. Let $t \in [0, T]$ and $(x_1, u_1, v_1), (x_2, u_2, v_2) \in S_*^{\varepsilon, R}(t)$. First, let us check [13, Assumption 2.4(a)]:

$$\begin{aligned} &\|(Ax_1 + Bv_1 + Fu_1) - (Ax_2 + Bv_2 + Fu_2)\| \leq \\ &\quad \|A\|\|x_1 - x_2\| + \|B\|\|v_1 - v_2\| + \|F\|\|u_1 - u_2\|. \end{aligned}$$

Secondly, we must check the inequality concerning the cost in [13, Assumption 2.4(b)]. For that, remark first that:

$$\begin{aligned} |x_1^\intercal Q x_1 - x_2^\intercal Q x_2| &= |(x_1 + x_2)^\intercal Q (x_1 - x_2)| \\ &\leq \|x_1 - x^*(t) + x_2 - x^*(t) + 2x^*(t)\| \\ &\quad \|Q\|\|x_1 - x_2\| \\ &\leq (\|x_1 - x^*(t)\| + \|x_2 - x^*(t)\| + \\ &\quad 2\|x^*(t)\|)\|Q\|\|x_1 - x_2\| \\ &\leq 2\|Q\|\left(\|x^*(t)\| + \varepsilon\right)\|x_1 - x_2\|. \end{aligned}$$

Similarly, one proves that $|u_1^\intercal U u_1 - u_2^\intercal U u_2| \leq 2\|U\|(\|u^*(t)\| + R)\|u_1 - u_2\|$ and $|v_1^\intercal V v_1 - v_2^\intercal V v_2| \leq 2\|V\|(\|v^*(t)\| + R)\|v_1 - v_2\|$. Therefore:

$$\begin{aligned} &|(x_1^\intercal Q x_1 + u_1^\intercal U u_1 + v_1^\intercal V v_1) - \\ &\quad (x_2^\intercal Q x_2 + u_2^\intercal U u_2 + v_2^\intercal V v_2)| \\ &\leq k_x(t)\|x_1 - x_2\| + k_u(t)\|u_1 - u_2\| \\ &\quad + k_v(t)\|v_1 - v_2\|, \end{aligned}$$

where $k_x(t) = 2\|Q\|\left(\|x^*(t)\| + \varepsilon\right)$, $k_u(t) = 2\|U\|(\|u^*(t)\| + R)$ and $k_v(t) = 2\|V\|(\|v^*(t)\| + R)$. k_x, k_u, k_v are measurable functions of time, and $\|A\|$, $\|B\|$ and $\|F\|$ are constant. Since $\mathrm{im}(C) \subseteq \mathrm{im}(E)$, and using Proposition 1, the bounded slope condition holds, with a positive constant k_S. The terms $k_S[\|B\| + \|F\| + k_u]$, k_x and $\|A\|$ are all integrable on $[0, T]$, and obviously there exists a positive number η such that $R \geq \eta k_S$ on $[0, T]$ (just take $\eta = R/k_S$). Since all involved functions are smooth, all conditions of measurability and differentiability are met. Furthermore, since the MPEC Linear Condition holds (see Definition 1), the error bound condition for the system (2) holds at $(x^*(t), u^*(t), v^*(t))$ for all $t \in [0, T]$. Calculations of the non-triviality and Weierstrass conditions are straightforward. Since all functions are differentiable, the Clarke subdifferential used in [13] contains only the gradient, i.e.,
$\nabla_{x,u,v}\left(\langle p(t), Ax + Bv + Fu \rangle - \lambda_0(x^\intercal Q x + u^\intercal U u)\right)$, and $U(\cdot)$ is in our case the whole space $\mathbb{R}^{2m}$, so the normal cone reduces to $\{0\}$. Simple calculations yield the Euler equation (6). Concerning the transversality condition, notice first that we did not impose any boundary cost. Denote $P_b = \left\{ \left(\begin{smallmatrix} x_0 \\ x_T \end{smallmatrix}\right) : Mx_0 + Nx_T = x_b \right\}$, then for $\left(\begin{smallmatrix} x_0 \\ x_T \end{smallmatrix}\right) \in P_b$, since P_b is an affine vector space,

$$\begin{aligned} \mathcal{N}_{P_b}\begin{pmatrix} x_0 \\ x_T \end{pmatrix} &= -\left(P_b - \begin{pmatrix} x_0 \\ x_T \end{pmatrix}\right)^* \\ &= -\ker(M\ N)^* = \mathrm{im}\begin{pmatrix} M^\intercal \\ N^\intercal \end{pmatrix}. \end{aligned}$$

□

Remark. *The tuple consisting of a trajectory and the associated multipliers solution of* (6) *is called an extremal. The case $\lambda_0 = 0$ is often called the abnormal case, and the corresponding extremal an abnormal extremal. In this case, no information can be derived from these necessary conditions. In other cases, one can choose this value most conveniently, since the adjoint state p is defined up to a multiplicative positive*

constant. In the rest of this article, λ_0 will always be chosen as $-\frac{1}{2}$.

The Weierstrass condition (7) can be re-expressed as searching a local maximizer of the following MPEC:

$$\begin{aligned}\max_{u,v}\ & \langle p(t), Ax^*(t)+Bv+Fu\rangle + \lambda_0(x^*(t)^\intercal Qx^*(t) \\ & \qquad + u^\intercal Uu + v^\intercal Vv) \\ \text{s.t. } & 0 \le v \perp Cx^*(t)+Dv+Eu \ge 0.\end{aligned} \tag{8}$$

For each $t\in[0,T]$, this is an MPEC, as presented in Section II-A1. These programs admit first-order conditions that are specific, called weak and strong stationarity: this motivates the next definition.

Definition 3. *Let (x^*,u^*,v^*) be an admissible pair for* (2)*. Then:*

- *The FJ-type W(eak)-stationarity holds at (x^*,u^*,v^*) if there exist an arc p, a scalar $\lambda_0\le 0$ and measurable functions λ^G, λ^H such that Proposition 2 (1)-(4) hold.*
- *The FJ-type S(trong)-stationarity holds at (x^*,u^*,v^*) if (x^*,u^*,v^*) is FJ-type W-stationary with arc p and there exist measurable functions η^G, η^H such that, for almost every $t\in[0,T]$,*

$$\begin{aligned}0 &= F^\intercal p + 2\lambda_0 Uu^* + E^\intercal \eta^H \\ 0 &= B^\intercal p + 2\lambda_0 Vv^* + \eta^G + D^\intercal \eta^H \\ 0 &= \eta^G_i(t),\ \forall i\in I_t^{+0}(x^*,u^*,v^*) \\ 0 &= \eta^H_i(t),\ \forall i\in I_t^{0+}(x^*,u^*,v^*),\end{aligned}$$

$$\eta^G_i(t)\ge 0,\ \eta^H_i(t)\ge 0,\ \forall i\in I_t^{00}(x^*,u^*,v^*).$$

- *We simply call W-stationarity or S-stationarity the FJ-type W- or S-stationarity with $\lambda_0=-\frac{1}{2}$.*

The multipliers η^G, η^H can be different in measure from the corresponding λ^G, λ^H in Proposition 2. The next theorem, whose proof follows directly from [13, Theorem 3.6], addresses this problem:

Proposition 3. *[13] Let (x^*,u^*,v^*) be a local minimizer of radius R for* (1)(2)*. Suppose that for almost every $t\in[0,T]$, the MPEC LICQ holds at $(u^*(t),v^*(t))$ for problem* (8)*, i.e., the family of gradients*

$$\left\{\begin{pmatrix}0\\ e_i\end{pmatrix} : i\in I_t^{0\bullet}(x^*,u^*,v^*)\right\}\bigcup\left\{\begin{pmatrix}(E_{i\bullet})^\intercal\\ (D_{i\bullet})^\intercal\end{pmatrix} : i\in I_t^{\bullet 0}(x^*,u^*,v^*)\right\} \tag{9}$$

is linearly independent, where e_i is a vector such that its j-th component is equal to δ_i^j, the Kronecker delta. Then the S-stationarity holds at (x^*,u^*,v^*). Moreover, in this case, the multipliers η^G, η^H can be taken equal to λ^G, λ^H, respectively, almost everywhere.

We can now state the following result:

Corollary 1. *Suppose $m=m_u$ and E is invertible. Then the local minimum (x^*,u^*,v^*) is S-stationary, and the multipliers η^G, η^H can be chosen equal to λ^G, λ^H almost everywhere.*

Proof. Suppose first that $\operatorname{rank}\begin{pmatrix}0_m & E^\intercal\\ I_m & D^\intercal\end{pmatrix}=2m$. This rank condition ensures the fact that the family $\left\{\begin{pmatrix}0\\ e_i\end{pmatrix} : 1\le i\le m\right\}\bigcup\left\{\begin{pmatrix}(E_{i\bullet})^\intercal\\ (D_{i\bullet})^\intercal\end{pmatrix} : 1\le i\le m\right\}$ is linearly independent. The family in (9) being a subfamily of this one, it is necessarily linearly independent. So the MPEC LICQ in Definition 1 holds, and (x^*,u^*,v^*) is S-stationary. Let us show now that this rank condition is equivalent to E being invertible. First notice that $2m=\operatorname{rank}\begin{pmatrix}0_m & E^\intercal\\ I_m & D^\intercal\end{pmatrix}=\operatorname{rank}\begin{pmatrix}0_m & I_m\\ E & D\end{pmatrix}$. Since $\binom{0}{E}$ and $\binom{I_m}{D}$ are linearly independent, we have:

$$\begin{aligned}2m=\operatorname{rank}\begin{pmatrix}0_m & I_m\\ E & D\end{pmatrix} &= \operatorname{rank}\begin{pmatrix}0_m\\ E\end{pmatrix}+\operatorname{rank}\begin{pmatrix}I_m\\ D\end{pmatrix} \\ &= \operatorname{rank}(E)+m.\end{aligned}$$

Thus, $\operatorname{rank}(E)=m$, so E is invertible. □

From now on, it will be assumed that $m=m_u$. Let us now state a result that allows us to reformulate the S-stationarity conditions through a complementarity system, in order to remove the active sets. One can simply see it that way: for almost all $t\in[0,T]$, the conditions on the multipliers λ^H and λ^G are:

$$\begin{aligned}&\lambda^G_i(t)=0,\ \forall i\in I_t^{+0}(x,u,v)\\ &\lambda^H_i(t)=0,\ \forall i\in I_t^{0+}(x,u,v)\\ &\lambda^G_i(t)\ge 0,\ \lambda^H_i(t)\ge 0,\ \forall i\in I_t^{00}(x,u,v).\end{aligned} \tag{10}$$

The presence of the active and degenerate sets is bothersome, since they depend on the optimal solution, not in a useful way. Nonetheless, the conditions in (10) look almost like a linear complementarity problem. What is missing is the sign of λ^G_i for $i\in I_t^{0+}(x,u,v)$ (and the same with λ^H_i on $I_t^{+0}(x,u,v)$). On these index sets, the multipliers could be negative. But we could for instance create new variables, say α and β, that will both be non-negative and comply with these conditions. This is the purpose of the next proposition.

Proposition 4. *Suppose (x,u,v) is an S-stationary trajectory. Then there exist measurable functions $\beta:[0,T]\to\mathbb{R}^m$, $\zeta:[0,T]\to\mathbb{R}$ such that:*

$$u(t)=U^{-1}\left(F^\intercal p(t)+E^\intercal\beta(t)-\zeta(t)E^\intercal v(t)\right)$$

and

$$\begin{aligned}\begin{pmatrix}\dot x\\ \dot p\end{pmatrix} &= \begin{pmatrix}A & FU^{-1}F^\intercal\\ Q & -A^\intercal\end{pmatrix}\begin{pmatrix}x\\ p\end{pmatrix} \\ &\quad + \begin{pmatrix}B-\zeta FU^{-1}E^\intercal\\ \zeta C^\intercal\end{pmatrix}v+\begin{pmatrix}FU^{-1}E^\intercal\\ -C^\intercal\end{pmatrix}\beta\end{aligned}$$

$$\begin{aligned}0\le\begin{pmatrix}v\\ \beta\end{pmatrix}\perp\begin{pmatrix}D-\zeta EU^{-1}E^\intercal & EU^{-1}E^\intercal\\ D-\zeta EU^{-1}E^\intercal & EU^{-1}E^\intercal\end{pmatrix}\begin{pmatrix}v\\ \beta\end{pmatrix}+ \\ \begin{pmatrix}C & EU^{-1}F^\intercal\\ C & EU^{-1}F^\intercal\end{pmatrix}\begin{pmatrix}x\\ p\end{pmatrix}\ge 0\end{aligned}$$

$$\begin{aligned}0\le v\perp\zeta(D+D^\intercal+V-\zeta EU^{-1}E^\intercal)v+(\zeta EU^{-1}E^\intercal-D^\intercal)\beta+ \\ (\zeta EU^{-1}F^\intercal-B^\intercal)p+\zeta Cx\ge 0.\end{aligned} \tag{11}$$

To prove this proposition, we first need the following lemma.

Lemma 2. *Let (x, u, v) be an S-stationary trajectory, and λ^G, λ^H be the associated multipliers. Then there exists a measurable function $\zeta : [0, T] \to \mathbb{R}$ such that $\begin{pmatrix} \lambda^G(t) + \zeta(t)w(t) \\ \lambda^H(t) + \zeta(t)v(t) \end{pmatrix} \geq 0$, where w is defined in* (2).

Proof. First, remark that, for all $t \in [0, T]$, a candidate $\zeta(t)$ has been defined in [18, Theorem 3.1]. Denote $F : [0, T] \times \mathbb{R} \to \mathbb{R}^{2m}$, $F(t, \zeta) = \begin{pmatrix} \lambda^G(t) + \zeta w(t) \\ \lambda^H(t) + \zeta v(t) \end{pmatrix}$. F is a Carathéodory mapping, since: λ^G, λ^H, v and w are measurable, so $F(\cdot, \zeta)$ is measurable for each fixed $\zeta \in \mathbb{R}$, and $F(t, \cdot)$ is an affine function, and as such it is continuous, for each fixed t. By the Implicit Measurable Function Theorem [19, Theorem 14.16], there exists a measurable function $\zeta : [0, T] \to \mathbb{R}$ such that $F(t, \zeta(t)) \in \mathbb{R}^{2m}_+$, which is the intended result. □

Proof of Proposition 4. As proved in Lemma 2, there exists a measurable function $\zeta : [0, T] \to \mathbb{R}$ such that:

$$\begin{pmatrix} \lambda^G(t) + \zeta(t)w(t) \\ \lambda^H(t) + \zeta(t)v(t) \end{pmatrix} \geq 0.$$

Define $\alpha, \beta : [0, T] \to \mathbb{R}^m$ as $\alpha = \lambda^G + \zeta w$, $\beta = \lambda^H + \zeta v$. The variables α and β are, by construction, measurable and non-negative. From the fact that (x, u, v) is an S-stationary trajectory, we also have that, for almost every $t \in [0, T]$, $\lambda^G_i(t)v_i(t) = 0$ and $\lambda^H_i(t)w_i(t) = 0$ for all $i \in \overline{m}$. Therefore, let us deduce that:

$$\begin{cases} \lambda^G = \alpha - \zeta w \\ \lambda^H = \beta - \zeta v \\ 0 \leq \alpha \perp v \geq 0 \\ 0 \leq \beta \perp w \geq 0. \end{cases} \tag{12}$$

In (6), we can isolate u, since we supposed U symmetric positive definite. Inserting the redefinition of λ^H yields:

$$\begin{aligned} u(t) &= U^{-1}(F^\intercal p(t) + E^\intercal \lambda^H(t)) \\ &= U^{-1}\left(F^\intercal p(t) + E^\intercal \beta(t) - \zeta(t)E^\intercal v(t)\right). \end{aligned} \tag{13}$$

Recall that $w = Cx + Dv + Eu$. Inserting this and (13) in (12), we obtain:

$$\begin{aligned} &\lambda^G = \alpha - \zeta(Cx + Dv + Eu) \\ &= \alpha - \zeta(Cx + (D - \zeta EU^{-1}E^\intercal)v + EU^{-1}F^\intercal p + EU^{-1}E^\intercal \beta)). \end{aligned}$$

Inserting (12) and (13) into (2) and (6) allows us to rewrite the differential equations defining x and p as :

$$\begin{cases} \dot{x} &= Ax + Bv + Fu \\ &= Ax + FU^{-1}F^\intercal p + (B - \zeta FU^{-1}E^\intercal)v + FU^{-1}E^\intercal \beta \\ \dot{p} &= -A^\intercal p + Qx - C^\intercal \lambda^H \\ &= -A^\intercal p + Qx + \zeta C^\intercal v - C^\intercal \beta. \end{cases}$$

The only equation left is the third equation in (6). Replacing λ^G and λ^H with the expressions (12) yields:

$$\begin{aligned} &B^\intercal p - Vv + \alpha - \zeta(Cx + (D - \zeta EU^{-1}E^\intercal)v + EU^{-1}F^\intercal p \\ &\quad + EU^{-1}E^\intercal \beta) + D^\intercal(\beta - \zeta v) = 0 \\ &\implies \alpha = (\zeta EU^{-1}F^\intercal - B^\intercal)p + \zeta Cx \\ &+ (\zeta EU^{-1}E^\intercal - D^\intercal)\beta + \zeta(D + D^\intercal + V - \zeta EU^{-1}E^\intercal)v. \end{aligned}$$

Replacing α and u in the complementarity conditions appearing in (2) and in (12) yields the complementarity conditions in (11). □

The three complementarity conditions defining β and v in (11) are not written as a classical Variational Inequality (VI), since they involve $2m$ unknowns but $3m$ complementarity problems. The next proposition addresses this problem.

Proposition 5. *Let r be any given positive scalar. Denote (P) the complementarity conditions appearing in* (11)*, and denote (P_r) the problem:*

$$\begin{cases} 0 \leq \beta + rv \quad \perp (D - \zeta EU^{-1}E^\intercal)v + EU^{-1}E^\intercal \beta \\ \qquad\qquad + EU^{-1}F^\intercal p + Cx \geq 0 \\ 0 \leq v \qquad \perp \zeta(D + D^\intercal + V - \zeta EU^{-1}E^\intercal)v \\ \qquad\qquad + (\zeta EU^{-1}E^\intercal - D^\intercal)\beta \\ \qquad\qquad + (\zeta EU^{-1}F^\intercal - B^\intercal)p + \zeta Cx \geq 0 \\ \beta \geq 0. \end{cases} \tag{P_r}$$

Then (v, β) is a solution of (P) if and only if it is a solution of (P_r).

Proof. We rewrite more simply the two problems as follows:

$$\begin{cases} 0 \leq v \perp \tilde{D}v + \tilde{U}\beta + q_1 \geq 0 & (P1) \\ 0 \leq \beta \perp \tilde{D}v + \tilde{U}\beta + q_1 \geq 0 & (P2) \\ 0 \leq v \perp \tilde{D}_2 v + \tilde{U}_2 \beta + q_2 \geq 0 & (P3) \end{cases}$$

$$\begin{cases} 0 \leq \beta + rv \perp \tilde{D}v + \tilde{U}\beta + q_1 \geq 0 & (P_r1) \\ \beta \geq 0 & (P_r2) \\ 0 \leq v \perp \tilde{D}_2 v + \tilde{U}_2 \beta + q_2 \geq 0 & (P_r3) \end{cases}$$

where $\tilde{D} = (D - \zeta EU^{-1}E^\intercal)$, $\tilde{U} = EU^{-1}E^\intercal$, $\tilde{D}_2 = \zeta(D + D^\intercal + V - \zeta EU^{-1}E^\intercal)$, $\tilde{U}_2 = (\zeta EU^{-1}E^\intercal - D^\intercal)$, $q_1 = EU^{-1}F^\intercal p + Cx$, $q_2 = (\zeta EU^{-1}F^\intercal - B^\intercal)p + \zeta Cx$. Let (v, β) be a solution of (P). Denote: $I^{0+} = \{i : v_i = \beta_i = 0,\ (\tilde{D}v + \tilde{U}\beta + q_1)_i > 0\}$, $I^{\bullet 0} = \{i : (\tilde{D}v + \tilde{U}\beta + q_1)_i = 0\}$. These two sets form a partition of $\{1, ..., m\}$. Since CP $(P3)$ and (P_r3) are the same problem, (v, β) is also solution of (P_r3). Using $(P2)$, we find that β complies with (P_r2). We are just left with (P_r1). By assumption it follows that $\forall i \in I^{0+}, \beta_i + rv_i = 0,\ (\tilde{D}v + \tilde{U}\beta + q_1)_i > 0$ and $\forall i \in I^{\bullet 0}, \beta_i + rv_i \geq 0,\ (\tilde{D}v + \tilde{U}\beta + q_1)_i = 0$. So (v, β) is also a solution of (P_r1). This proves that (v, β) is a solution of (P_r).

Conversely, let (v, β) be a solution of (P_r). Since it is a solution of (P_r1), denote $I^{0+}_r = \{i : \beta_i + rv_i = 0,\ (\tilde{D}v + \tilde{U}\beta + q_1)_i > 0\}$ and $I^{\bullet 0}_r = \{i : (\tilde{D}v + \tilde{U}\beta + q_1)_i = 0\}$. These two sets form a partition of $\{1, ..., m\}$. Since CP (P_r3) and $(P3)$ are the same problem, (v, β) is also solution of $(P3)$. For all $i \in I^{0+}_r$, $\beta_i + rv_i = 0$ and $(\tilde{D}v + \tilde{U}\beta + q_1)_i > 0$. Thanks to (P_r3) and (P_r2), we know that $\beta_i \geq 0$, $v_i \geq 0$. Since $r > 0$, we have a sum of positive terms that must equal 0, so $\beta_i = v_i = 0$. For all $i \in I^{\bullet 0}_r$, $(\tilde{D}v + \tilde{U}\beta + q_1)_i = 0$ and using (P_r3) and (P_r2), $\beta_i \geq 0$, $v_i \geq 0$. So (v, β) is also a solution of $(P1)$ and $(P2)$. It proves that (v, β) is a solution of (P). □

Let us define $\tilde{\beta} = \beta + rv$ and replace β in (P_r). Also, define $\mathcal{D} = \begin{pmatrix} EU^{-1}E^\intercal & D-(\zeta+r)EU^{-1}E^\intercal \\ \zeta EU^{-1}E^\intercal - D^\intercal & \zeta(D+V)+(\zeta+r)\left(D^\intercal - \zeta EU^{-1}E^\intercal\right)\end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} C & EU^{-1}F^\intercal \\ \zeta C & \zeta EU^{-1}F^\intercal - B^\intercal\end{pmatrix}$. To sum up, by Propositions 2, 4 and 5, the following theorem holds:

Theorem 1. *Let (x^*, u^*, v^*) be a local minimizer of constant radius $R > 0$ for* (1)(2)*. Suppose $m = m_u$, E is invertible and (x^*, u^*, v^*) is not the projection of an abnormal extremal. Define $\mathcal{D}$ and $\mathcal{C}$ as above. Then there exist an arc $p : [0,T] \to \mathbb{R}^n$, and measurable functions $\tilde{\beta} : [0,T] \to \mathbb{R}^m$, $\zeta : [0,T] \to \mathbb{R}$ such that, for an arbitrary scalar $r > 0$:*

$$u^*(t) = U^{-1}\left(F^\intercal p(t) + E^\intercal \tilde{\beta}(t) - (\zeta(t)+r)E^\intercal v^*(t)\right)$$

and the following conditions hold:

1) The transversality condition: $\begin{pmatrix} p(0) \\ -p(T)\end{pmatrix} \in \operatorname{im}\begin{pmatrix} M^\intercal \\ N^\intercal\end{pmatrix}$.
2) The Euler adjoint equation: *for almost every $t \in [0,T]$,*

$$\begin{pmatrix}\dot{x}^* \\ \dot{p}\end{pmatrix} = \begin{pmatrix} A & FU^{-1}F^\intercal \\ Q & -A^\intercal\end{pmatrix}\begin{pmatrix} x^* \\ p\end{pmatrix} + \begin{pmatrix} FU^{-1}E^\intercal & B-(\zeta+r)FU^{-1}E^\intercal \\ -C^\intercal & (\zeta+r)C^\intercal\end{pmatrix}\begin{pmatrix}\tilde{\beta} \\ v^*\end{pmatrix} \quad (14)$$

$$\begin{cases} 0 \le \begin{pmatrix}\tilde{\beta} \\ v^*\end{pmatrix} \perp \quad \mathcal{D}\begin{pmatrix}\tilde{\beta} \\ v^*\end{pmatrix} + \mathcal{C}\begin{pmatrix} x^* \\ p\end{pmatrix} \ge 0 \\ \tilde{\beta} \ge rv^*. \end{cases}$$

3) The Weierstrass condition for radius R: *for almost every $t \in [t_0, t_1]$,*

$$(x^*(t), u, v) \in S, \ \left\| \begin{pmatrix} u \\ v\end{pmatrix} - \begin{pmatrix} u^*(t) \\ v^*(t)\end{pmatrix}\right\| < R$$

$$\begin{aligned} &\implies \langle p(t), Ax^*(t) + Bv^*(t) + Fu^*(t))\rangle \\ &- \frac{1}{2}\left(x^*(t)^\intercal Qx^*(t) + u^*(t)^\intercal Uu^*(t) + v^*(t)^\intercal Vv^*(t)\right) \\ &\ge \langle p(t), Ax^*(t) + Bv + Fu)\rangle \\ &- \frac{1}{2}\left(x^*(t)^\intercal Qx^*(t) + u^\intercal Uu + v^\intercal Vv\right). \end{aligned}$$

This result will be used in an *indirect method* in order to compute numerically an approximate solution with high accuracy. However, this indirect method needs a first guess in order to converge, which is provided by the *direct method*. These methods are presented briefly Section III.

C. Sufficiency of the W-stationarity

Surprisingly, the weakest form of stationarity for the problem (1)(2) turns to be also sufficient, in some sense. For this, we need to define trajectories with the same *history*. The development shown here is directly inspired by [20, Proposition 3.1] and by [21].

Definition 4. *Let (x, u, v) and (x^*, u^*, v^*) be two admissible trajectories for* (2) *(associated with $w = Cx + Dv + Eu$ and w^*, defined the same way). We say that they have the same history on $[0,T]$ if the following condition holds for almost every $t \in [0,T]$, $[v_i(t) = 0 \iff v_i^*(t) = 0]$ and $[w_i(t) = 0 \iff w_i^*(t) = 0]$*

From the point of view of the switching systems, two trajectories have the same history on $[0,T]$ if they visit the same modes at the same time along $[0,T]$. In the following sufficient condition for optimality, the comparison of the different trajectories is done with respect to this history condition.

Theorem 2. *Suppose that (x^*, u^*, v^*) is an admissible W-stationary trajectory (with $\lambda_0 = -\frac{1}{2}$). Then, (x^*, u^*, v^*) solves* (1)(2) *among all admissible trajectories for* (2) *having the same history.*

Proof. Since (x^*, u^*, v^*) is a W-stationary trajectory, there exist an arc p and measurable functions λ^G and λ^H satisfying (6). Notice that (6) implies, for almost all $t \in [0,T]$ and all $i \in \overline{m}$,

$$\lambda_i^G(t)v_i^*(t) = 0 \text{ and } \lambda_i^H(t)w_i^*(t) = 0. \quad (15)$$

Let (x, u, v) be a second admissible trajectory for (2) with the same history as (x^*, u^*, v^*). Since they both have the same history, it also satisfies, for almost all $t \in [0,T]$ and all $i \in \overline{m}$:

$$\lambda_i^G(t)v_i(t) = 0 \text{ and } \lambda_i^H(t)w_i(t) = 0. \quad (16)$$

Denote $L(x,u,v) = \frac{1}{2}\left(x^\intercal Qx + u^\intercal Uu + v^\intercal Vv\right)$. The goal is to prove that $\Delta = \int_0^T \left(L(x(t),u(t),v(t)) - L(x^*(t),u^*(t),v^*(t))\right)dt \ge 0$. For this, let us first reexpress the expression of Δ.

$$\begin{aligned} \Delta = &\int_0^T \left(L(x(t),u(t),v(t)) - L(x^*(t),u^*(t),v(t))\right)dt \\ &+ \int_0^T p(t)^\intercal(\dot{x}(t) - Ax(t) - Bv(t) - Fu(t) \\ &\quad - (\dot{x}^*(t) - Ax^*(t) - Bv^*(t) - Fu^*(t)))dt \\ = &\int_0^T \left(L(x(t),u(t),v(t)) - L(x^*(t),u^*(t),v^*(t))\right)dt \\ &+ \int_0^T \frac{d}{dt}\left[p(t)^\intercal(x(t) - x^*(t))\right]dt \\ &- \int_0^T \dot{p}(t)^\intercal(x(t) - x^*(t))dt \\ &- \int_0^T p(t)^\intercal(A(x(t) - x^*(t)) \\ &\quad + B(v(t) - v^*(t)) + F(u(t) - u^*(t)))dt \end{aligned}$$

The last equality is obtained by integration by parts of $\int_0^T p^\intercal(\dot{x} - \dot{x}^*)$. Therefore:

$$\Delta = \int_0^T \Bigg(L(x(t),u(t),v(t)) - L(x^*(t),u^*(t),v^*(t)) - \begin{pmatrix} \dot p(t) + A^\intercal p(t) + C^\intercal \lambda^H(t) \\ F^\intercal p(t) + E^\intercal \lambda^H(t) \\ B^\intercal p(t) + \lambda^G(t) + D^\intercal \lambda^H(t) \end{pmatrix}^\intercal \begin{pmatrix} x(t)-x^*(t) \\ u(t)-u^*(t) \\ v(t)-v^*(t) \end{pmatrix} \Bigg) dt + \int_0^T \lambda^H(t)^\intercal (C(x(t)-x^*(t)) + D(v(t)-v^*(t)) + E(u(t)-u^*(t)))dt + \int_0^T \lambda^G(t)^\intercal (v(t)-v^*(t))dt + \int_0^T \frac{d}{dt}\left[p(t)^\intercal (x(t)-x^*(t))\right] dt$$

Using Proposition 2, $\begin{pmatrix} \dot p(t) + A^\intercal p(t) + C^\intercal \lambda^H(t) \\ F^\intercal p(t) + E^\intercal \lambda^H(t) \\ B^\intercal p(t) + \lambda^G(t) + D^\intercal \lambda^H(t) \end{pmatrix} = \nabla L(x^*(t),u^*(t),v^*(t))$. Since L is a convex function, it yields for almost all t in $[0,T]$:

$$L(x(t),u(t),v(t)) - L(x^*(t),u^*(t),v^*(t)) - (\nabla L(x^*(t),u^*(t),v^*(t)))^\intercal \begin{pmatrix} x(t)-x^*(t) \\ u(t)-u^*(t) \\ v(t)-v^*(t) \end{pmatrix} \geq 0$$

Therefore, this proves: $\Delta \geq \int_0^T \frac{d}{dt}\left[p(t)^\intercal (x(t)-x^*(t))\right] dt$, the last inequality being obtained with (15) and (16). Furthermore: $\int_0^T \frac{d}{dt}\left[p(t)^\intercal (x(t)-x^*(t))\right] dt = -\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix}^\intercal \begin{pmatrix} x(0)-x^*(0) \\ x(T)-x^*(T) \end{pmatrix}$ But the boundary conditions in (2) yield $\begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} x(0)-x^*(0) \\ x(T)-x^*(T) \end{pmatrix} = 0$, such that $\begin{pmatrix} x(0)-x^*(0) \\ x(T)-x^*(T) \end{pmatrix} \in \ker\begin{pmatrix} M & N \end{pmatrix} = \left(\operatorname{im}\begin{pmatrix} M^\intercal \\ N^\intercal \end{pmatrix}\right)^\perp$. And since $\begin{pmatrix} p(0) \\ -p(T) \end{pmatrix} \in \operatorname{im}\begin{pmatrix} M^\intercal \\ N^\intercal \end{pmatrix}$, it proves $\int_0^T \frac{d}{dt}\left[p(t)^\intercal (x(t)-x^*(t))\right] dt = 0$. Finally, we conclude that $\Delta \geq 0$. □

III. Numerical methods

A. Direct method and hybrid approach

The direct method consists in discretizing directly the problem (1)(2) in order to solve a finite-dimensional optimization problem. The complementarity is discretized the following way: $0 \leq v_k \perp Cx_k + Dv_k + Eu_k \geq 0, \ \forall k$. It differs from the implicit methods found in [22], [1]. For this optimal control problem, the complementarity should not be seen as a way to express the variable v_k, but as a mixed constraint. Therefore, its discretization must hold at all discrete times t_k, and the trajectory, solution of this discretized LCS, will be computed not step by step but for all k at once. The problem is then to solve an MPEC numerically. To this end, we use the algorithm found in [23]. The idea behind this algorithm is to relax the complementarity, creating a sequence of optimization problems converging to a stationary point. The reason to use this algorithm is that, under some hypothesis, it converges to S-stationarity points. In the report [14], consistency of this scheme with an S-stationary trajectory solution of (1)(2) is proved.

The indirect method consists in solving the first-order necessary conditions derived in Section II-B in order to solve the optimal control problem. As pointed out in [24], it has the advantage that the numerical solutions are usually very precise, but the method suffers from a huge sensitivity on the initial guess. Indeed, if the initial guess is not close enough to a solution, then the method may fail to converge. A natural approach is then to use both the direct and the indirect methods in order to obtain a very precise solution, taking advantage of both methods: this is called the hybrid approach. In our framework, we have to face two problems. First, the active index sets appearing in the Euler equations (6), used to impose conditions on the multipliers, are not useful as they are. This problem has been tackled by re-expressing these equations in Theorem 1. The presence of the unknown variable ζ can be overcome by fixing a constant value to it, big enough, guessed from the direct method (see [14] for details). Secondly, we often have to solve a Boundary Value Problem (BVP). The report [14] proposes a method for solving BVP expressed in the form of an LCS.

B. Numerical results

The two methods (direct and hybrid) were tested on the following example, where the obtained optimal controller in discontinuous:

$$\min \int_0^1 \left(\|x(t)\|_2^2 + u(t)^2\right) dt, \quad \text{s.t.} \begin{cases} \dot x(t) = \begin{pmatrix} 1 & -3 \\ -8 & 10 \end{pmatrix} x(t) + \begin{pmatrix} -3 \\ -1 \end{pmatrix} v(t) + \begin{pmatrix} 4 \\ 8 \end{pmatrix} u(t), \\ 0 \leq v(t) \perp w(t) = \begin{pmatrix} 1 & -3 \end{pmatrix} x(t) + 5v(t) + 3u(t) \geq 0, \\ x(0) = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}, \ x(T) \text{ free}, \end{cases} \tag{17}$$

The numerical results are shown in Figure 1. The complementarity constraint is satisfied, and the associated multipliers suggest that the trajectory indeed is an S-stationary trajectory. It is clear that u admits a discontinuous switch around $t_1 = 0.112$. It is noteworthy to take a look at the different modes activated along the solution. In this case where the complementarity constraint is of dimension 1, we have three possible cases : $v = 0 < w$ (happening on $[0, t_1]$), $v > 0 = w$ (happening on $[t_1, 0.87]$ approximately), $v = 0 = w$ (happening on $[0.87, 1]$). It shows that, compared with some other methods for optimal control of switching systems (see for instance [25], [26]), this method does not require to guess a priori the number of switches nor the times of commutation in order to approximate the solution. The tracking of the switches is taken care of by the MPEC solver. This is a major advantage of the complementarity approach over event-driven, hybrid-like approaches.

Other examples, including convergence order graphics and showing computational times, can be found in [14]. All the codes were designed using the library CasADi [27].

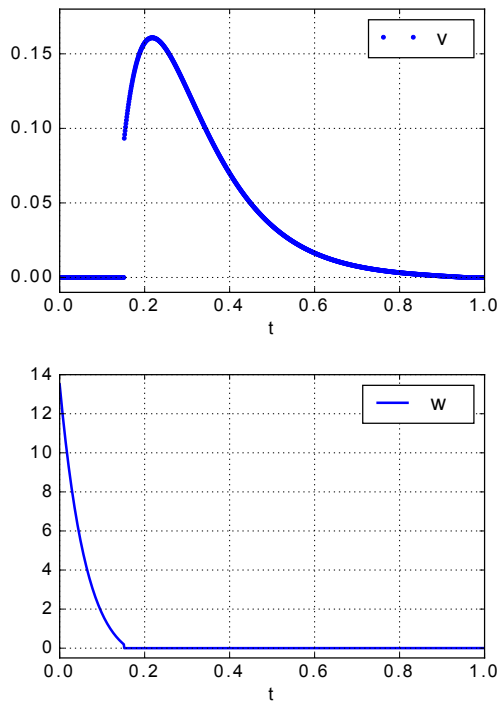

Figure 1: Numerical results for Example 17 using hybrid approach, $h = 5.10^{-4}$.

IV. Conclusion

This paper focuses on the quadratic optimal control of LCS. Necessary (and sufficient in some sense) first-order conditions are presented, and two numerical algorithms providing fast and accurate numerical approximation are developed and proposed. Several examples prove the efficiency of the approach. Future investigations should concern several aspects. One could add some equality and inequality constraints in the Problem (1)(2). The analysis would not change much, but the numerical results could become more engaged. The optimal solution found here is an open-loop solution. Trajectory tracking algorithms for LCS should be studied to stabilize the optimal solution. Closed-loop optimal control via the Hamilton-Jacobi equations might also be analysed.

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