Feedback stabilization of a 1D linear reaction-diffusion equation with delay boundary control

Christophe Prieur and Emmanuel Trélat

***Abstract*—The goal of this work is to design a stabilizing feedback boundary control for a reaction-diffusion partial differential equation, where the boundary control is subject to a constant delay while the equation may be unstable without any control. For this system, which is equivalent to a parabolic equation coupled with a transport equation, a prediction-based control is explicitly computed by splitting the infinite-dimensional system into two parts: a finite-dimensional unstable part and a stable infinite-dimensional part. A finite-dimensional delayed controller is computed for the unstable part, and it is shown that this controller stabilizes the whole partial differential equation. The proof is based on an explicit expression of the classical Artstein transformation combined with an adequately designed Lyapunov function. A numerical simulation illustrates the constructive feedback design method.**

***Index Terms*—Reaction-diffusion equation, delay control, Lyapunov function, partial differential equation.**

I. Introduction and main result

A. Literature review and statement of the main result

There have been a number of works in the literature dealing with the stabilization of processes with input delays, mainly in finite dimension, but seemingly much less for processes driven by PDEs.

In [9] a stable PDE is controlled by means of a delayed bounded linear control operator (see also [21] for a semilinear case). In the present work, the control operator is unbounded (Dirichlet boundary control) and the open-loop system is unstable.

Unbounded control operators have been considered in [14], [15], [16] for heat and wave equations in which time-varying delays are allowed with a bound on the time-derivative of the delay function; see also [8] for a second-order evolution equation. In the present paper, a Lyapunov technique is developed in which, in addition to an exponential stability analysis, we also design a stabilizing controller from a finite-dimensional spectral truncation of (1) containing all unstable modes.

To the best of our knowledge, the first work dealing with input delayed unstable PDEs is [12] where a reaction-diffusion equation is considered and a backstepping approach is developed to stabilize it (see also [3] for a similar approach for a wave equation). In this paper we do not use backstepping and we exploit a decomposition of the state space into a stable part and a finite-dimensional unstable part.

Let us write more precisely the problem under study and state the main result of this work. Let $L>0$, let $c\in L^\infty(0,L)$ and let $D\geqslant 0$ be arbitrary. We consider the one-dimensional reaction-diffusion equation on $(0,L)$ with a delayed Dirichlet boundary control

$$\begin{aligned} &y_t = y_{xx} + c(x)y,\ t\geqslant 0,\ x\in(0,L),\\ &y(t,0)=0,\quad y(t,L)=u_D(t)=u(t-D),\ t\geqslant 0, \end{aligned} \tag{1}$$

where the state is $y(t,\cdot):[0,L]\to\mathbb{R}$ and the control is $u_D(t)=u(t-D)\in\mathbb{R}$, with $D>0$ a constant delay. Our objective is to design an exponentially stabilizing feedback control for (1).

By using a classical change of variables (see, e.g., [11]) this problem is equivalent to the problem of stabilizing the coupled system

$$\begin{aligned} &y_t = y_{xx} + c(x)y,\ t\geqslant 0,\ x\in(0,L),\\ &y(t,0)=0,\quad y(t,L)=z(t,0),\ t\geqslant 0,\\ &z_t = z_w,\ t\geqslant 0,\ t\geqslant 0,\ w\in(0,D),\\ &z(t,D)=u(t),\ t\geqslant 0, \end{aligned}$$

with $z(t,w)=u(t+w-D)$, where the first equation is (1) and the second equation is a transport equation causing the delay D in the control of (1). In other words our control objective can be seen as a boundary control problem of a coupled system obtained by writing in cascade a reaction-diffusion equation and a transport equation.

We assume that we are only interested in what happens for $t\geqslant 0$, and we consider an initial condition

$$y(0,\cdot)=y_0(\cdot),$$

and since the boundary control is retarded with the delay D, we assume that no control is applied (i.e., $u=0$) within the time interval $(0,D)$. For every $t>D$ on, a nontrivial control can then be applied.

In this paper, we establish the following result.

Theorem 1: *The delayed Dirichlet boundary control reaction-diffusion equation* (1) *is exponentially stabilizable, with a feedback control that is designed from a finite-dimensional autonomous linear control system with input delay. When closing the loop with this feedback, the PDE* (1) *is exponentially stable, that is, there exist $\mu>0$ and $C>0$ such that, for every $y_0(\cdot)\in H^1(0,L)$ satisfying $y_0(0)=0$,*

C. Prieur is with Univ. Grenoble Alpes, CNRS, Grenoble INP, Gipsa-lab, F-38000 Grenoble, France, christophe.prieur@gipsa-lab.fr

E. Trélat is with Sorbonne Université, Université Paris-Diderot SPC, CNRS, Inria, Laboratoire Jacques-Louis Lions, équipe CAGE, F-75005 Paris emmanuel.trelat@sorbonne-universite.fr

the solution of (1) *such that* $y(0,\cdot)=y_0(\cdot)$ *satisfies*

$$\|y(t,\cdot)\|_{H^1(0,L)} \leqslant Ce^{-\mu t}\|y_0(\cdot)\|_{H^1(0,L)} \qquad \forall t\geqslant 0.$$

Note that we do not make any smallness assumption on the delay D: for any $D\geqslant 0$ there exists a stabilizing feedback.

B. Presentation of the design method and organization of the paper

Our approach to build the delayed controller considered in Theorem 1 is a constructive design method. Our strategy, developed in Section II, starts with a spectral analysis of the operator underlying the control system (1) (compact perturbation of a Dirichlet-Laplacian), thanks to which we split the system into two parts. The first part of the system is finite dimensional and contains (at least) all unstable modes, whereas the second part is infinite dimensional and contains only stable modes. The stabilizing feedback is designed on the finite-dimensional part of the system: we use the Artstein model reduction and we design a Kalman gain matrix in a standard way with the pole-shifting theorem; then, following [4] we invert the Artstein transform and we obtain the desired feedback. This feedback control is such that its value $u(t-D)$ at time $t-D$ only depends on the values of $X_1(s)$ with $0<s<t-D$, where X_1 is identified with the unstable finite-dimensional part of the state.

By definition, this feedback stabilizes exponentially the finite-dimensional part of the system. Using an appropriate Lyapunov function, we then prove that it stabilizes as well the whole system. This is the core of the proof of Theorem 1.

The idea of designing a feedback on the unstable part of the system can be found in [18] and has been used for instance in [5], [6], [19] (for undelayed PDEs) where the efficiency of such a procedure has also been shown. Here, due to the presence of a delay, in practice one has to stabilize a finite-dimensional autonomous linear control system with input delay. In the existing literature, this classical issue has been investigated for instance in [2], [13] by a predictor approach. The recent paper [4] surveys on the numerical and practical aspects of this problem and shows that the designed controller can be computed numerically in particular thanks to a fixed point procedure. Here, we exploit this procedure to design a stabilizing controller for the unstable heat equation, by revisiting the delay input for the unstable finite-dimensional part of the state space, and by adapting it to the full boundary delay control.

Overall, our stabilization procedure is carried out with a simple approach that is easy to implement. Some details and a numerical illustration are provided in Section III.

The paper is organized as follows. Section II is devoted to the proof of the main result and to the design of the delayed boundary controller. To do that we first decouple the reaction-diffusion equation into two coupled parts: one unstable finite dimensional part and one infinite-dimensional part, using a spectral decomposition. It allows us to explicitly compute a finite dimensional delay controller in Section II-B. When closing the loop with this delay input, we prove that the PDE (1) is exponentially stable by using an appropriate Lyapunov function. A numerical simulation is given in Section III, highlighting the applicability of this design method. Section IV contains the proof of an intermediate result. Finally Section V collects concluding remarks and gives possible open issues.

II. Construction of the feedback and proof of Theorem 1

A. Spectral reduction

First of all, in order to deal rather with a homogeneous Dirichlet problem (which is more convenient), we set

$$w(t,x)=y(t,x)-\frac{x}{L}u_D(t) \tag{2}$$

and we suppose that the control u_D is differentiable for all positive times (this will be true in the construction that we will carry out). This leads to

$$\begin{aligned}
&w_t=w_{xx}+cw+\frac{x}{L}cu_D-\frac{x}{L}u_D', \quad \forall t>0,\ \forall x\in(0,L),\\
&w(t,0)=w(t,L)=0, \quad \forall t>0,\\
&w(0,x)=y(0,x)-\frac{x}{L}u_D(0), \quad \forall x\in(0,L).
\end{aligned} \tag{3}$$

In what follows, we choose $u_D(0)=y_0(L)$, so that $w(0,L)=0$ and $w\in H_0^1(0,L)$. We define the operator

$$A=\partial_{xx}+c(\cdot)\mathrm{id} \tag{4}$$

on the domain $D(A)=H^2(0,L)\cap H_0^1(0,L)$. Then the above control system is

$$w_t(t,\cdot)=Aw(t,\cdot)+a(\cdot)u_D(t)+b(\cdot)u_D'(t) \tag{5}$$

with $a(x)=\frac{x}{L}c(x)$ and $b(x)=-\frac{x}{L}$ for every $x\in(0,L)$.

Noting that A is selfadjoint and of compact inverse, we consider a Hilbert basis $(e_j)_{j\geqslant 1}$ of $L^2(0,L)$ consisting of eigenfunctions of A, associated with the sequence of eigenvalues $(\lambda_j)_{j\geqslant 1}$. Note that

$$-\infty<\cdots<\lambda_j<\cdots<\lambda_1 \quad \text{and} \quad \lambda_j \underset{j\to+\infty}{\longrightarrow} -\infty$$

and that $e_j(\cdot)\in H_0^1(0,L)\cap C^2([0,L])$ for every $j\geqslant 1$. Every solution $w(t,\cdot)\in H^2(0,L)\cap H_0^1(0,L)$ of (5) can be expanded as a series in the eigenfunctions $e_j(\cdot)$, convergent in $H_0^1(0,L)$,

$$w(t,\cdot)=\sum_{j=1}^{\infty} w_j(t)e_j(\cdot),$$

and therefore (1) is equivalent to the infinite-dimensional control system

$$w_j'(t)=\lambda_j w_j(t)+a_j u_D(t)+b_j u_D'(t), \qquad \forall j\in\mathbb{N}\setminus\{0\}, \tag{6}$$

with

$$\begin{aligned}
a_j&=\langle a(\cdot),e_j(\cdot)\rangle_{L^2(0,L)}=\frac{1}{L}\int_0^L xc(x)e_j(x)\,dx,\\
b_j&=\langle b(\cdot),e_j(\cdot)\rangle_{L^2(0,L)}=-\frac{1}{L}\int_0^L xe_j(x)\,dx,
\end{aligned} \tag{7}$$

for every $j\in\mathbb{N}\setminus\{0\}$. We define

$$\alpha_D(t)=u_D'(t) \tag{8}$$

and we consider from now on $u_D(t)$ as a state and $\alpha_D(t)$ as a control (destinated to be a delayed feedback, with constant delay D), so that equations (6) and (8) form an infinite-dimensional control system controlled by α_D, written as

$$\begin{aligned} u_D'(t) &= \alpha_D(t) \\ w_1'(t) &= \lambda_1 w_1(t) + a_1 u_D(t) + b_1 \alpha_D(t) \\ &\vdots \\ w_j'(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j \alpha_D(t) \\ &\vdots \end{aligned} \tag{9}$$

and which is equivalent to (1).

Let $n \in \mathbb{N} \setminus \{0\}$ be the number of nonnegative eigenvalues and let $\eta > 0$ be such that

$$\forall k > n \quad \lambda_k < -\eta < 0. \tag{10}$$

Let π_1 be the orthogonal projection onto the subspace of $L^2(0, L)$ spanned by $e_1(\cdot), \dots, e_n(\cdot)$, and let

$$w^1(t) = \pi_1 w(t, \cdot) = \sum_{j=1}^{n} w_j(t) e_j(\cdot). \tag{11}$$

With the matrix notations

$$X_1 = \begin{pmatrix} u_D \\ w_1 \\ \vdots \\ w_n \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & \lambda_n \end{pmatrix}, \ B_1 = \begin{pmatrix} 1 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} \tag{12}$$

the n first equations of (9) form the finite-dimensional control system with input delay

$$X_1'(t) = A_1 X_1(t) + B_1 \alpha_D(t) = A_1 X_1(t) + B_1 \alpha(t - D). \tag{13}$$

Note that the state $X_1(t) \in \mathbb{R}^{n+1}$ involves the term $u_D(t)$ which contains the delay.

Our objective is to design a feedback control α exponentially stabilizing the infinite-dimensional system (9). As shortly explained in the previous section, the idea consists of first designing a feedback control exponentially stabilizing the finite-dimensional system (13), and then of proving that this feedback actually stabilizes the whole system (9). The idea underneath is that the finite-dimensional system (13) contains all unstable modes of the complete system (9), and thus has to be stabilized. It is however not obvious that this feedback stabilizing the unstable finite-dimensional part actually stabilizes as well the entire system (9). This fact will be proved thanks to an appropriate Lyapunov functional.

B. Stabilization of the unstable finite-dimensional part

Let us design a feedback control stabilizing the finite-dimensional linear autonomous control system with input delay (13) and let us also design a Lyapunov functional. First of all, following the so-called Artstein model reduction (see [1], [17]), we set, for every $t \in \mathbb{R}$,

$$\begin{aligned} Z_1(t) &= X_1(t) + \int_{t-D}^{t} e^{(t-s-D)A_1} B_1 \alpha(s)\, ds \\ &= X_1(t) + \int_0^D e^{-\tau A_1} B_1 \alpha(t - D + \tau)\, d\tau \end{aligned} \tag{14}$$

and we get that (13) is equivalent to

$$\dot{Z}_1(t) = A_1 Z_1(t) + e^{-DA_1} B_1 \alpha(t) \tag{15}$$

which is a usual linear autonomous control system without input delay in $\mathbb{R}^{n+1}$. The equivalence is due to the fact that the Artstein transformation (14) can be inverted (see further). Now, for this classical finite-dimensional system, we have the following result.

Lemma 1: *For every $D \geqslant 0$, the pair $(A_1, e^{-DA_1} B_1)$ satisfies the Kalman condition, that is,*

$$\operatorname{rank}\left(e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \dots, A_1^n e^{-DA_1} B_1\right) = n + 1. \tag{16}$$

Proof. Since A_1 and e^{-DA_1} commute, and since e^{-DA_1} is invertible, we have

$$\begin{aligned} &\operatorname{rank}\left(e^{-DA_1} B_1, A_1 e^{-DA_1} B_1, \dots, A_1^n e^{-DA_1} B_1\right) \\ &= \operatorname{rank}\left(e^{-DA_1} B_1, e^{-DA_1} A_1 B_1, \dots, e^{-DA_1} A_1^n B_1\right) \\ &= \operatorname{rank}\left(B_1, A_1 B_1, \dots, A_1^n B_1\right), \end{aligned}$$

and hence it suffices to prove that the pair (A_1, B_1) satisfies the Kalman condition. A simple computation leads to

$$\begin{aligned} &\det\left(B_1, A_1 B_1, \dots, A_1^n B_1\right) \\ &\qquad = \prod_{j=1}^{n} (a_j + \lambda_j b_j)\, \mathrm{VdM}(\lambda_1, \dots, \lambda_n), \end{aligned} \tag{17}$$

where $\mathrm{VdM}(\lambda_1, \dots, \lambda_n)$ is a Van der Monde determinant, and thus is never equal to zero since the real numbers λ_j, $j = 1 \dots n$, are all distinct. On the other part, using the fact that every $e_j(\cdot)$ is an eigenfunction of A and belongs to $H_0^1(0, L)$, we have, for every integer j,

$$\begin{aligned} a_j + \lambda_j b_j &= \frac{1}{L} \int_0^L x \left(c(x) e_j(x) - \lambda_j e_j(x)\right) dx \\ &= -\frac{1}{L} \int_0^L x e_j''(x)\, dx = -e_j'(L) \end{aligned}$$

which is not equal to zero since $e_j(L) = 0$ and $e_j(\cdot)$ is a nontrivial solution of a linear second-order scalar differential equation. The lemma is proved. □

Since the linear control system (15) satisfies the Kalman condition, the well-known pole-shifting theorem implies the existence of a stabilizing gain matrix and of a Lyapunov functional (see, e.g., [10], [22], [23]). This yields the following corollary.

Corollary 1: *For every $D \geqslant 0$, there exists a $1 \times (n+1)$ matrix $K_1(D) = \big(k_0(D), k_1(D), \dots, k_n(D)\big)$ such that $A_1 + e^{-DA_1} B_1 K_1(D)$ admits -1 as an eigenvalue with order $n+1$. Moreover there exists a $(n+1) \times (n+1)$ symmetric positive definite matrix $P(D)$ such that*

$$\begin{aligned} &P(D)\left(A_1 + e^{-DA_1} B_1 K_1(D)\right) \\ &\quad + \left(A_1 + e^{-DA_1} B_1 K_1(D)\right)^\top P(D) = -I_{n+1}. \end{aligned} \tag{18}$$

In particular, the function

$$V_1(Z_1) = \frac{1}{2} Z_1^\top P(D) Z_1 \tag{19}$$

is a Lyapunov function for the closed-loop system

$$\dot{Z}_1(t) = (A_1 + e^{-DA_1} B_1 K_1(D)) Z_1(t).$$

Remark 1: It is even possible to choose $K_1(D)$ and $P(D)$ as smooth (i.e., of class C^∞) functions of D, but we do not need this property in this paper. ○

Remark 2: In the statement above, we chose -1 as an eigenvalue of $A_1 + e^{-DA_1}B_1K_1(D)$, but actually the pole-shifting theorem implies that, for every $(n+1)$-tuple $(\mu_0, \dots, \mu_n)$ of eigenvalues there exists a $1 \times (n+1)$ matrix $K_1(D)$ such that the eigenvalues $A_1 + e^{-DA_1}B_1K_1(D)$ are exactly $(\mu_0, \dots, \mu_n)$. The eigenvalue -1 was chosen here only for simplicity. What is important is to ensure that $A_1 + e^{-DA_1}B_1K_1(D)$ is a *Hurwitz* matrix (i.e., a matrix of which all eigenvalues have a negative real part).

In practice, other choices can be done, which can be more efficient according to such or such criterion. For instance, instead of using the pole-shifting theorem, one could design a stabilizing gain matrix K_1 by using a standard Riccati procedure. ○

Remark 3: From Corollary 1, we infer that, for every $D \geqslant 0$, there exists $C_1(D) > 0$ (depending smoothly on D) such that

$$\frac{d}{dt}V_1(Z_1(t)) = -\|Z_1(t)\|^2_{\mathbb{R}^{n+1}} \leqslant -C_1(D)\, V_1(Z_1(t)), \quad (20)$$

where $\| \|_{\mathbb{R}^{n+1}}$ is the usual Euclidean norm in $\mathbb{R}^{n+1}$. ○

Remark 4: By regularity of $K_1(D)$, $P(D)$ and $C_1(D)$ with respect to D (see Remark 1) and by the stability margin given by (20), we infer that the delay D does not need to be precisely known in practice. More precisely, if D is sufficiently close (in terms of the stability margin and in terms of the variation of all quantities depending on D) to a nominal value of the delay for which the feedback control has been computed, then the PDE (1) in closed-loop with this nominal feedback control is still exponentially stable for this value D of the delay. This result echoes what is well known when studying robustness properties of asymptotically stabilizing feedbacks for finite-dimensional systems (see, e.g., [10]) by means of strict Lyapunov functions. ○

From Corollary 1, the feedback $\alpha(t) = K_1(D)Z_1(t)$ stabilizes exponentially the control system (15). Since $\alpha(t-D)$ is used in the control system (13), and since in general we are only concerned with prescribing the future of a system, starting at time 0, we assume that the control system (13) is uncontrolled for $t < 0$, and from the starting time $t = 0$ on we let the feedback act on the system. In other words, we set

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ K_1(D)Z_1(t) & \text{if } t \geqslant D, \end{cases} \quad (21)$$

so that, with this control, the control system (13) with input delay is written as

$$X_1'(t) = A_1X_1(t) + \chi_{(D,+\infty)}(t)B_1K_1(D)Z_1(t-D)$$

with Z_1 given by (14). Here the notation χ_E stands for the characteristic function of E, that is the function defined by $\chi_E(t) = 1$ whenever $t \in E$ and $\chi_E(t) = 0$ otherwise. Using (14), the feedback α defined by (21) is such that, for every $t < D$,

$$\alpha(t) = 0 \quad (22a)$$

and, for every $t \geqslant D$,

$$\alpha(t) = K_1(D)X_1(t) + K_1(D)\int_{\max(t-D,D)}^t e^{(t-D-s)A_1}B_1\alpha(s)\,ds. \quad (22b)$$

In other words, the value of the feedback control α at time t depends on $X_1(t)$ and of the controls applied in the past (more precisely, of the values of α over the time interval $(\max(t-D, D), t)$).

Lemma 2: *When closing the loop with the feedback* (22)*, the control system* (13) *is exponentially stable.*

Proof. By construction $t \mapsto Z_1(t)$ converges exponentially to 0, and hence $t \mapsto \alpha(t)$ and thus $t \mapsto \int_{\max(t-D,D)}^t e^{(t-D-s)A_1)}B_1\alpha(s)\,ds$ converges exponentially to 0 as well. Then the equality (14) implies that $t \mapsto X_1(t)$ converges exponentially to 0. □

Inversion of the Artstein transform. We are going to invert the Artstein transform, with two motivations in mind:

- First of all, it is interesting to express the stabilizing control α (defined by (21)) directly as a feedback of X_1.
- Secondly, it is interesting to express the Lyapunov functional V_1 (defined by (19)) as a function of X_1.

For more details on how to invert the Artstein transform and how to use it in practice, we refer the readers to [4]. Here, we develop only what is required to perform our stabilization analysis.

We have to solve the fixed point implicit equality (22). For every locally integrable function f on $\mathbb{R}$, we define

$$(T_Df)(t) = K_1(D)\int_{\max(t-D,D)}^t e^{(t-D-s)A_1}B_1f(s)\,ds.$$

It follows from (22) that $\alpha(t) = K_1(D)X_1(t) + (T_D\alpha)(t)$ for every $t \geqslant D$. A purely formal computation yields the following Neumann series

$$\alpha(t) = \sum_{j=0}^{+\infty}(T_D^jK_1(D)X_1)(t).$$

The convergence of the series is not obvious and is established in the following lemma.

Lemma 3: *We have*

$$\alpha(t) = \begin{cases} 0 & \text{if } t < D, \\ \displaystyle\sum_{j=0}^{+\infty}(T_D^jK_1(D)X_1)(t) & \text{if } t \geqslant D, \end{cases} \quad (23)$$

and the series is convergent, whatever the value of the delay $D \geqslant 0$ may be.

Note that the value of the feedback α at time $t \geqslant D$,

$$\begin{aligned}\alpha(t) &= K_1(D)X_1(t) \\ &+ K_1(D)\int_{\max(t-D,D)}^t e^{(t-D-s)A_1}B_1K_1(D)X_1(s)\,ds \\ &+ K_1(D)\int_{\max(t-D,D)}^t e^{(t-D-s)A_1}B_1K_1(D) \\ &\int_{\max(s-D,D)}^s e^{(s-D-\tau)A_1}B_1K_1(D)X_1(\tau)\,d\tau\,ds \\ &+ \cdots\end{aligned}$$

depends on the past values of X_1 over the time interval (D,t). Since the feedback is retarded with the delay D, the term $\alpha(t-D)$ appearing at the right-hand side of (13) only depends on the values of $X_1(s)$ with $0<s<t-D$, as desired.

Proof. We define the functions φ_{D_j} iteratively by

$$\begin{aligned}&\varphi_{D1}(t,\tau)=1\\&\varphi_{Dj+1}(t,\tau)=\int_{\max(\tau,t-D)}^{\min(t,\tau+jD)}\varphi_{Dj}(s,\tau)\,ds\quad\forall j\in\mathbb{N}\setminus\{0\}\end{aligned}\tag{24}$$

for every $t\geqslant\tau$, and by $\varphi_{Dj}(t,\tau)=0$ if $t<\tau$ and $j\in\mathbb{N}\setminus\{0\}$.

Let us prove by induction that

$$\begin{aligned}&\left|(T_D^jK_1(D)X_1)(t)\right|\leqslant\|B_1\|^j_{\mathbb{R}^{n+1}}\|K_1(D)\|^{j+1}_{\mathbb{R}^{n+1}}\\&\times\int_{\max(t-jD,D)}^t\varphi_{Dj}(t,\tau)e^{(t-jD-\tau)\|A_1\|}\|X_1(\tau)\|_{\mathbb{R}^{n+1}}\,d\tau\end{aligned}\tag{25}$$

for every $j\in\mathbb{N}\setminus\{0\}$. This is clearly true for $j=1$, since

$$\begin{aligned}&|(T_DK_1(D)X_1)(t)|\\&=\left|K_1(D)\int_{\max(t-D,D)}^te^{(t-D-s)A_1}B_1K_1(D)X_1(s)\,ds\right|\\&\leqslant\|B_1\|_{\mathbb{R}^{n+1}}\|K_1(D)\|^2_{\mathbb{R}^{n+1}}\\&\qquad\times\int_{\max(t-D,D)}^te^{(t-D-s)\|A_1\|}\|X_1(s)\|_{\mathbb{R}^{n+1}}\,ds.\end{aligned}$$

Assume that this is true for an integer $j\in\mathbb{N}\setminus\{0\}$, and let us derive the estimate for $j+1$. Since

$$\begin{aligned}&(T_D^{j+1}K_1(D)X_1)(t)\\&=K_1(D)\int_{\max(t-D,D)}^te^{(t-D-s)A_1}B_1(T_D^jK_1(D)X_1)(s)\,ds\end{aligned}$$

we get

$$\begin{aligned}&\left|(T_D^{j+1}K_1(D)X_1)(t)\right|\leqslant\|B_1\|^{j+1}_{\mathbb{R}^{n+1}}\|K_1(D)\|^{j+2}_{\mathbb{R}^{n+1}}\\&\times\int_{\max(t-D,D)}^te^{(t-D-s)\|A_1\|}\\&\times\int_{\max(s-jD,D)}^s\varphi_{Dj}(s,\tau)e^{(s-jD-\tau)\|A_1\|}\|X_1(\tau)\|_{\mathbb{R}^{n+1}}\,d\tau\,ds\end{aligned}$$

and, by the Fubini theorem, noting that (τ,s) is such that

$$\left\{\begin{array}{l}\max(s-jD,D)\leqslant\tau\leqslant s\\\max(t-D,D)\leqslant s\leqslant t\end{array}\right.$$

if and only if

$$\left\{\begin{array}{l}\max(t-(j+1)D,D)\leqslant\tau\leqslant t\\\max(\tau,t-D)\leqslant s\leqslant\min(t,\tau+jD),\end{array}\right.$$

we get the estimate

$$\begin{aligned}&\left|(T_D^{j+1}K_1(D)X_1)(t)\right|\leqslant\|B_1\|^{j+1}_{\mathbb{R}^{n+1}}\|K_1(D)\|^{j+2}_{\mathbb{R}^{n+1}}\\&\quad\times\int_{\max(t-(j+1)D,D)}^t\left(\int_{\max(\tau,t-D)}^{\min(t,\tau+jD)}\varphi_{Dj}(s,\tau)\,ds\right)\\&\quad\times\ e^{(t-(j+1)D-\tau)\|A_1\|}\|X_1(\tau)\|_{\mathbb{R}^{n+1}}\,d\tau\end{aligned}$$

and the desired estimate for $j+1$ follows by definition of φ_{Dj+1}.

Now, we claim that

$$0\leqslant\varphi_{Dj}(t,\tau)\leqslant\frac{(t-\tau)^{j-1}}{(j-1)!}\tag{26}$$

for every $j\in\mathbb{N}\setminus\{0\}$. Indeed, nonnegativity is obvious and the right-hand side estimate easily follows from the fact that $\varphi_{Dj+1}(t,\tau)\leqslant\int_\tau^t\varphi_{Dj}(s,\tau)\,ds$ and from a simple iteration argument.

Finally, from (25) and (26), we infer that

$$\begin{aligned}&\left|(T_D^jK_1(D)X_1)(t)\right|\\&\leqslant\|B_1\|^j_{\mathbb{R}^{n+1}}\|K_1(D)\|^{j+1}_{\mathbb{R}^{n+1}}\\&\quad\times\int_{\max(t-jD,D)}^t\frac{(t-\tau)^{j-1}}{(j-1)!}e^{(t-jD-\tau)\|A_1\|}\|X_1(\tau)\|_{\mathbb{R}^{n+1}}\,d\tau\\&\leqslant\|B_1\|^j_{\mathbb{R}^{n+1}}\|K_1(D)\|^{j+1}_{\mathbb{R}^{n+1}}\frac{(t-D)^j}{(j-1)!}\max_{D\leqslant s\leqslant t}\|X_1(s)\|_{\mathbb{R}^{n+1}}\end{aligned}$$

whence the convergence of the series in (23). □

Remark 5: It is also interesting to express Z_1 in function of X_1, that is, to invert the equality

$$Z_1(t)=X_1(t)+\int_{(t-D,t)\cap(D,+\infty)}e^{(t-s-D)A_1}B_1K_1(D)Z_1(s)\,ds\tag{27}$$

coming from (14) and (21). Although it is technical and not directly useful to derive the exponential stability of Z_1, it will however allow us to express the Lyapunov functional V_1 defined by (19). Note that

$$(t-D,t)\cap(D,+\infty)=\left\{\begin{array}{ll}\emptyset&\text{if }t<D,\\(D,t)&\text{if }D<t<2D,\\(t-D,t)&\text{if }2D<t.\end{array}\right.\tag{28}$$

In particular if $t<D$ then $Z_1(t)=X_1(t)$. We have the following result.

Lemma 4: *For every $t\in\mathbb{R}$, there holds*

$$X_1(t)=Z_1(t)-\int_{(t-D,t)\cap(D,+\infty)}f(t-s)X_1(s)\,ds\tag{29}$$

where f is defined as the unique solution of the fixed point equation

$$f(r)=f_0(r)+(\tilde{T}_Df)(r)$$

with $f_0(r)=e^{(r-D)A_1}B_1K_1(D)$ and

$$(\tilde{T}_Df)(r)=\int_0^re^{(r-\tau-D)A_1}B_1K_1(D)f(\tau)\,d\tau.$$

Moreover, we have

$$
\begin{aligned}
f(r) &= \sum_{j=0}^{+\infty} (\tilde{T}_D^j f_0)(r) \\
&= e^{(r-D)A_1} B_1 K_1(D) \\
&\quad + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) e^{(\tau-D)A_1} B_1 K_1(D)\, d\tau \\
&\quad + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) \\
&\quad \times \int_0^\tau e^{(\tau-s-D)A_1} B_1 K_1(D) e^{(s-D)A_1} B_1 K_1(D)\, ds\, d\tau \\
&\quad + \cdots
\end{aligned}
$$

and the series is convergent, whatever the value of the delay $D \geqslant 0$ *may be.*

The proof of this lemma is done in Section IV.

With Lemma 4 and using (27) in Remark 5, the feedback control α defined in (21) can be as well written as

$$
\begin{aligned}
\alpha(t) &= \chi_{(D,+\infty)}(t) K_1(D) Z_1(t) \\
&= \chi_{(D,+\infty)}(t) K_1(D) X_1(t) \\
&\quad + K_1(D) \int_{(t-D,t)\cap(D,+\infty)} f(t-s) X_1(s)\, ds
\end{aligned}
$$

and we recover of course the expression (23) derived in Lemma 3. ○

Plugging this feedback into the control system (13) yields, for $t > D$, the closed-loop system

$$
\begin{aligned}
X_1'(t) &= A_1 X_1(t) + B_1 \alpha(t-D) \\
&= A_1 X_1(t) + B_1 K_1(D) X_1(t-D) \\
&\quad + B_1 K_1(D) \int_{(t-2D,t-D)\cap(D,+\infty)} f(t-D-s) X_1(s)\, ds
\end{aligned}
\tag{30}
$$

which is, as said above, exponentially stable. Moreover, the Lyapunov function V_1, which is exponentially decreasing according to Remark 3, can be written as

$$
\begin{aligned}
V_1(t) = \frac{1}{2} &\left(X_1(t) + \int_{I_t(D)} f(t-s) X_1(s)\, ds \right)^\top P(D) \\
&\times \left(X_1(t) + \int_{I_t(D)} f(t-s) X_1(s)\, ds \right)
\end{aligned}
$$

with $I_t(D) = (t-D,t) \cap (D,+\infty)$. We stress once again that the above feedback and Lyapunov functional stabilize the system whatever the value of the delay may be.

C. Exponential stability of the entire system in closed-loop

In order to prove that the feedback α designed above stabilizes the whole system (9), we have to take into account the rest of the system, consisting of modes that are naturally stable. What has to be checked is whether the delay control part might destabilize this infinite-dimensional part or not.

Let $(u_D(\cdot), w(\cdot))$ denote a solution of (5) in which we choose the control α in the feedback form designed previously. Here, $w(t)$ designates the solution $w(t,\cdot) \in H^2(0,L) \cap H_0^1(0,L)$ satisfying

$$
\begin{aligned}
&u_D' = \alpha, \quad w' = Aw + a u_D + b\alpha, \\
&u_D(0) = y_0(L), \quad w(0,\cdot) = y_0(x) - \frac{x}{L} y_0(L).
\end{aligned}
\tag{31}
$$

Let $M(D)$ be a positive real number such that

$$
\begin{aligned}
M(D) \;>\; &\|b\|^2_{L^2(0,L)} \|K_1(D)\|^2_{\mathbb{R}^{n+1}} \\
&+ \max\left(2\|a\|^2_{L^2(0,L)}, \frac{\max(\lambda_1,\ldots,\lambda_n)}{\lambda_{\min}(P(D))} \right) \\
&\times \max\left(1, D e^{2D\|A_1\|} \|B_1\|^2_{\mathbb{R}^{n+1}} \|K_1(D)\|^2_{\mathbb{R}^{n+1}} \right)
\end{aligned}
\tag{32}
$$

where

$$
\|K_1(D)\|^2_{\mathbb{R}^{n+1}} = \sum_{j=0}^n k_j(D)^2, \quad \|B_1\|^2_{\mathbb{R}^{n+1}} = 1 + \sum_{j=1}^n b_j^2,
$$

with $\|A_1\|$ the usual matrix norm induced from the Euclidean norm of $\mathbb{R}^{n+1}$ and $\lambda_{\min}(P(D)) > 0$ the smallest eigenvalue of the symmetric positive definite matrix $P(D)$. The precise value of $M(D)$ is not important however. What is important in what follows is that $M(D) > 0$ be large enough.

We set

$$
\begin{aligned}
V_D(t) &= M(D) \left(V_1(t) + \int_{(t-D,t)\cap(D,+\infty)} V_1(s)\, ds \right) \\
&\quad - \frac{1}{2} \langle w(t), Aw(t) \rangle_{L^2(0,L)} \\
&= \frac{M(D)}{2} \Big(Z_1(t)^\top P(D) Z_1(t) \\
&\quad + \int_{(t-D,t)\cap(D,+\infty)} Z_1(s)^\top P(D) Z_1(s)\, ds \Big) \\
&\quad - \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j w_j(t)^2.
\end{aligned}
\tag{33}
$$

We are going to prove that $V_D(t)$ is positive and decreases exponentially to 0. This Lyapunov functional consists of three terms. The two first terms stand for the unstable finite-dimensional part of the system. As we will see, the integral term is instrumental in order to tackle the delayed terms. The third term stands for the infinite-dimensional part of the system. In this infinite sum actually all modes are involved, in particular those that are unstable. Then the two first terms of (33), weighted with $M(D) > 0$, can be seen as corrective terms and this weight $M(D) > 0$ is chosen large enough so that $V_D(t)$ be indeed positive. More precisely,

$$
-\sum_{j=1}^{+\infty} \lambda_j w_j^2 = -\sum_{j=1}^{n} \lambda_j w_j^2 - \sum_{j=n+1}^{\infty} \lambda_j w_j^2 \tag{34}
$$

where $\lambda_j \geqslant 0$ for every $j \in \{1,\ldots,n\}$ and $\lambda_j < -\eta < 0$ for every $j > n$ (see (10)). Therefore the second term of (34) is positive and the first term, which is nonpositive, is actually compensated by the first term of $V_D(t)$ since $M(D)$ is large enough, as proved in the following more precise lemma.

Lemma 5: *There exists* $C_2(D) > 0$ *such that*

$$
V_D(t) \geqslant C_2(D) \left(u_D(t)^2 + \|w(t)\|^2_{H_0^1(0,L)} \right) \quad \forall t \geqslant 0. \tag{35}
$$

Proof. First of all, by definition of $\lambda_{\min}(P(D))$, we have

$$\lambda_{\max}(P(D))\left(\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}+\int_{t-D}^t\|Z_1(s)\|^2_{\mathbb{R}^{n+1}}\,ds\right)$$
$$\geqslant Z_1(t)^\top P(D)Z_1(t)+\int_{t-D}^t Z_1(s)^\top P(D)Z_1(s)\,ds$$
$$\geqslant \lambda_{\min}(P(D))\left(\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}+\int_{t-D}^t\|Z_1(s)\|^2_{\mathbb{R}^{n+1}}\,ds\right), \tag{36}$$

for every $t\geqslant 0$. Besides, recall that, from (27), one has

$$X_1(t)=Z_1(t) - \int_{(t-D,t)\cap(D,+\infty)} e^{(t-s-D)A_1}B_1K_1(D)Z_1(s)\,ds$$

and therefore, using the Cauchy-Schwarz inequality and the inequality $(a+b)^2\leqslant 2a^2+2b^2$, it follows that

$$\|X_1(t)\|^2_{\mathbb{R}^{n+1}}\leqslant C_3(D)\left(\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}+\int_{t-D}^t\|Z_1(s)\|^2_{\mathbb{R}^{n+1}}\,ds\right) \tag{37}$$

with

$$C_3(D)=\max\left(2,2De^{2D\|A_1\|}\|B_1\|^2_{\mathbb{R}^{n+1}}\|K_1(D)\|^2_{\mathbb{R}^{n+1}}\right). \tag{38}$$

We then infer from (36) and (37) that

$$\frac{M(D)}{2}Z_1(t)^\top P(D)Z_1(t) + \frac{M(D)}{2}\int_{t-D}^t Z_1(s)^\top P(D)Z_1(s)\,ds \geqslant M(D)\frac{\lambda_{\min}(P(D))}{2C_3(D)}\|X_1(t)\|^2_{\mathbb{R}^{n+1}} \tag{39}$$

for every $t\geqslant 0$.

Using (34) and the definition of X_1 in (12), we have

$$-\frac{1}{2}\sum_{j=1}^{+\infty}\lambda_j w_j(t)^2\geqslant -\frac{1}{2}\sum_{j=n+1}^{\infty}\lambda_j w_j(t)^2 -\frac{1}{2}\max_{1\leqslant j\leqslant n}(\lambda_j)\|X_1(t)\|^2_{\mathbb{R}^{n+1}} \tag{40}$$

and therefore, using (39), we get

$$2V_D(t)\geqslant\left(M(D)\frac{\lambda_{\min}(P(D))}{C_3(D)}-\max_{1\leqslant j\leqslant n}(\lambda_j)\right)\|X_1(t)\|^2_{\mathbb{R}^{n+1}} -\sum_{j=n+1}^{\infty}\lambda_j w_j(t)^2,$$

for every $t\geqslant 0$. By definition of $M(D)$ (see (32)) and by definition of $C_3(D)$ (see (38)), one has $M(D)\frac{\lambda_{\min}(P(D))}{2C_3(D)}-\frac{1}{2}\max_{1\leqslant j\leqslant n}(\lambda_j)>0$ and hence there exists $C_4(D)>0$ such that

$$V_D(t)\geqslant C_4(D)\left(\|X_1(t)\|^2_{\mathbb{R}^{n+1}}-\frac{1}{2}\sum_{j=n+1}^{\infty}\lambda_j w_j(t)^2\right). \tag{41}$$

Using the series expansion $w(t,\cdot)=\sum_{i=1}^{+\infty}w_i(t)e_i(\cdot)$, we have

$$\|w(t)\|^2_{H^1_0(0,L)}=\sum_{(i,j)\in(\mathbb{N}\setminus\{0\})^2}w_i(t)w_j(t)\int_0^L e_i'(x)e_j'(x)\,dx.$$

By definition, one has $e_n''+ce_n=\lambda_n e_n$ and $e_n(0)=e_n(L)=0$, for every $n\in\mathbb{N}\setminus\{0\}$. Integrating by parts and using the orthonormality property, we get

$$\int_0^L e_i'(x)e_j'(x)\,dx=\int_0^L c(x)e_i(x)e_j(x)\,dx-\lambda_j\delta_{ij}$$

with $\delta_{ij}=1$ whenever $i=j$ and $\delta_{ij}=0$ otherwise, and thus, for all $t\geqslant 0$,

$$\|w(t)\|^2_{H^1_0(0,L)}=\int_0^L c(x)w(t,x)^2\,dx-\sum_{j=1}^{\infty}\lambda_j w_j(t)^2. \tag{42}$$

Since $c\in L^\infty(0,L)$, it follows that

$$\|w(t)\|^2_{H^1_0(0,L)}$$
$$\leqslant \|c\|_{L^\infty(0,L)}\ \|w(t)\|^2_{L^2(0,L)}-\sum_{j=1}^{n}\lambda_j w_j(t)^2-\sum_{j=n+1}^{\infty}\lambda_j w_j(t)^2$$
$$\leqslant \|c\|_{L^\infty(0,L)}\sum_{j=1}^{\infty}w_j(t)^2-\sum_{j=n+1}^{\infty}\lambda_j w_j(t)^2$$
$$\leqslant \|c\|_{L^\infty(0,L)}\|X_1(t)\|^2_{\mathbb{R}^{n+1}}-\sum_{j=n+1}^{\infty}(\lambda_j-\|c\|_{L^\infty(0,L)})w_j(t)^2$$

and since $\lambda_j\to-\infty$ as j tends to $+\infty$, there exists $C_5>0$ such that

$$\|w(t)\|^2_{H^1_0(0,L)}\leqslant C_5\left(\|X_1(t)\|^2_{\mathbb{R}^{n+1}}-\frac{1}{2}\sum_{j=n+1}^{\infty}\lambda_j w_j(t)^2\right).$$

Then (35) follows from (41). □

Using (28), note that if $t<D$ then the integral term of (33) is equal to 0 and $Z_1(t)=X_1(t)$, and hence

$$V_D(t)=\frac{M(D)}{2}X_1(t)^\top P(D)X_1(t)-\frac{1}{2}\sum_{j=1}^{+\infty}\lambda_j w_j(t)^2$$

for every $t<D$. This remark leads to the following lemma.

Lemma 6: *There exists $C_6(D)>0$ such that*

$$V_D(t)\leqslant C_6(D)(u_D(t)^2+\|w(t)\|^2_{H^1_0(0,L)})\quad \forall t<D. \tag{43}$$

Proof. Using (42), one has

$$-\sum_{j=1}^{+\infty}\lambda_j w_j(t)^2\leqslant\|w(t)\|^2_{H^1_0(0,L)}+\|c\|_{L^\infty(0,L)}\ \|w(t)\|^2_{L^2(0,L)}$$
$$\leqslant C_8(D)\|w(t)\|^2_{H^1_0(0,L)}$$

and the lemma follows from the inequality $\|w(t)\|^2_{L^2(0,L)}\leqslant L\|w(t)\|^2_{H^1_0(0,L)}$ derived from the Poincaré inequality. □

Lemma 7: *The functional V_D decreases exponentially to* 0.

Proof. Let us compute $V_D'(t)$ for $t>2D$ and state a differential inequality satisfied by V_D. First of all, it follows from (18) (in Corollary 1) that

$$\frac{d}{dt}\frac{M(D)}{2}Z_1(t)^\top P(D)Z_1(t)=-M(D)\|Z_1(t)\|^2_{\mathbb{R}^{n+1}}$$

and thus

$$\frac{d}{dt}\frac{M(D)}{2}\int_{t-D}^{t} Z_1(s)^\top P(D) Z_1(s)\,ds = -M(D)\int_{t-D}^{t}\|Z_1(s)\|^2_{\mathbf{R}^{n+1}}\,ds.$$

Then, using (31), (33) and the fact that A is selfadjoint, we get

$$\begin{aligned} V_D'(t) &= -M(D)\|Z_1(t)\|^2_{\mathbf{R}^{n+1}} \\ &\quad - M(D)\int_{t-D}^{t}\|Z_1(s)\|^2_{\mathbf{R}^{n+1}}\,ds \\ &\quad - \|Aw(t)\|^2_{L^2(0,L)} - \langle Aw(t), a\rangle_{L^2(0,L)} u_D(t) \\ &\quad - \langle Aw(t), b\rangle_{L^2(0,L)} K_1(D) Z_1(t), \end{aligned} \tag{44}$$

for every $t > 2D$. From the Young inequality, we derive the estimates

$$\begin{aligned} &\left|\langle Aw(t), a\rangle_{L^2(0,L)} u_D(t)\right| \\ &\quad\leqslant \frac{1}{4}\|Aw(t)\|^2_{L^2(0,L)} + \|a\|^2_{L^2(0,L)}\|X_1(t)\|^2_{\mathbf{R}^{n+1}} \end{aligned} \tag{45}$$

and

$$\begin{aligned} &\left|\langle Aw(t), b\rangle_{L^2(0,L)} K_1(D) Z_1(t)\right| \\ &\leqslant \frac{1}{4}\|Aw(t)\|^2_{L^2(0,L)} + \|b\|^2_{L^2(0,L)}\|K_1(D)\|^2_{\mathbf{R}^{n+1}}\|Z_1(t)\|^2_{\mathbf{R}^{n+1}}. \end{aligned} \tag{46}$$

With the estimates (45), (46) and (37), we infer from (37) and from (44) that

$$\begin{aligned} V_D'(t) \leqslant &-\Big(M(D) - \|b\|^2_{L^2(0,L)}\|K_1(D)\|^2_{\mathbf{R}^{n+1}} \\ &-\|a\|^2_{L^2(0,L)} C_3(D)\Big)\|Z_1(t)\|^2_{\mathbf{R}^{n+1}} \\ &-\Big(M(D) - \|a\|^2_{L^2(0,L)} C_3(D)\Big)\textstyle\int_{t-D}^{t}\|Z_1(s)\|^2_{\mathbf{R}^{n+1}}\,ds \\ &-\tfrac{1}{2}\|Aw(t)\|^2_{L^2(0,L)}. \end{aligned}$$

From (32) and (38), the real number $M(D)$ has been chosen large enough so that

$$M(D) - \|b\|^2_{L^2(0,L)}\|K_1(D)\|^2_{\mathbf{R}^{n+1}} - \|a\|^2_{L^2(0,L)} C_3(D) > 0$$

and

$$M(D) - \|a\|^2_{L^2(0,L)} C_3(D) > 0.$$

Therefore, there exists $C_7(D) > 0$ such that

$$\begin{aligned} V_D'(t) \leqslant -C_7(D)\left(\|Z_1(t)\|^2_{\mathbf{R}^{n+1}} + \int_{t-D}^{t}\|Z_1(s)\|^2_{\mathbf{R}^{n+1}}\,ds\right) \\ - \frac{1}{2}\|Aw(t)\|^2_{L^2(0,L)}. \end{aligned} \tag{47}$$

Let us provide an estimate of $\|Aw(t)\|^2_{L^2(0,L)}$. Since $-\lambda_j \leqslant \lambda_j^2$ for any j large enough, it follows that there exists $C_8 > 0$ such that

$$\begin{aligned} &-\langle w(t), Aw(t)\rangle_{L^2(0,L)} \\ &= -\sum_{j=1}^{n}\lambda_j w_j(t)^2 - \sum_{j=n+1}^{+\infty}\lambda_j w_j(t)^2 \leqslant -\sum_{j=n+1}^{+\infty}\lambda_j w_j(t)^2 \\ &\qquad \leqslant \frac{1}{C_8}\sum_{j=1}^{+\infty}\lambda_j^2 w_j(t)^2 \leqslant \frac{1}{C_8}\|Aw\|^2_{L^2(0,L)}. \end{aligned}$$

Hence it follows from (47) that

$$\begin{aligned} V_D'(t) \leqslant -C_7(D)\left(\|Z_1(t)\|^2_{\mathbf{R}^{n+1}} + \int_{t-D}^{t}\|Z_1(s)\|^2_{\mathbf{R}^{n+1}}\,ds\right) \\ + \frac{C_8}{2}\langle w(t), Aw(t)\rangle_{L^2(0,L)}. \end{aligned}$$

Finally, using (36), there exists $C_9(D) > 0$ such that

$$V_D'(t) \leqslant -C_9(D) V_D(t) \qquad \forall t > 2D$$

and hence $V_D(t)$ decreases exponentially to 0. □

From Lemma 7, $V_D(t)$ decreases exponentially to 0. It follows from Lemmas 5 and 6 that there exists $C_{10}(D) > 0$ and $\mu > 0$ such that

$$\begin{aligned} &u_D(t)^2 + \|w(t)\|^2_{H_0^1(0,L)} \\ &\qquad \leqslant C_{10}(D) e^{-\mu t}\left(u_D(0)^2 + \|w(0)\|^2_{H_0^1(0,L)}\right) \end{aligned}$$

for every $t \geqslant 0$. Using (2), Theorem 1 follows.

III. Numerical simulation

In this section, we illustrate Theorem 1 with an example and a numerical simulation. We take $c(x) = 0.5$, for all $x \in (0, L)$, $L = 2\pi$ and $D = 1$. It is easily checked that, with a null boundary control, there is only one eigenvalue that is positive and thus there is only one mode of (1) that is unstable. Using a simple pole-shifting controller on the finite-dimensional linear control system resulting of the unstable part of (12) (with $(-0.5, -1)$ as poles for the closed-loop system), we compute (with `Matlab`) a stabilizing delay input for the infinite-dimensional system (1). The overall numerical procedure to compute the controller is based on the discretization of the explicit form of the Artstein transformation for the finite-dimensional unstable part of (1) (as done in [4] for finite dimensional control system with input delay). Then we discretize the reaction-diffusion equation (1) using the first 6 modes, when closing the loop with this delay controller. We take as initial condition $y^0(x) = x(L - x)$.

The time evolution of the obtained solution is given on Figure 1 and the delayed boundary controller u_D is given on Figure 2. It can be checked on Figure 1 that, as expected, the solution converges to equilibrium.

IV. Proof of Lemma 4

Let us search the kernel Φ_D such that

$$X_1(t) = Z_1(t) - \int_{-\infty}^{t}\Phi_D(t,s) X_1(s)\,ds,$$

postulating that $\Phi_D(t,s) = 0$ whenever $s > t$. Using (27) we must have

$$\begin{aligned} &X_1(t) + \int_{-\infty}^{t}\Phi_D(t,s) X_1(s)\,ds \\ &\quad = X_1(t) + \int_{(t-D,t)\cap(D,+\infty)} e^{(t-s-D)A_1} B_1 K_1(D) \\ &\qquad\qquad \times\left(X_1(s) + \int_{-\infty}^{s}\Phi_D(s,\tau) X_1(\tau)\right) d\tau. \end{aligned}$$

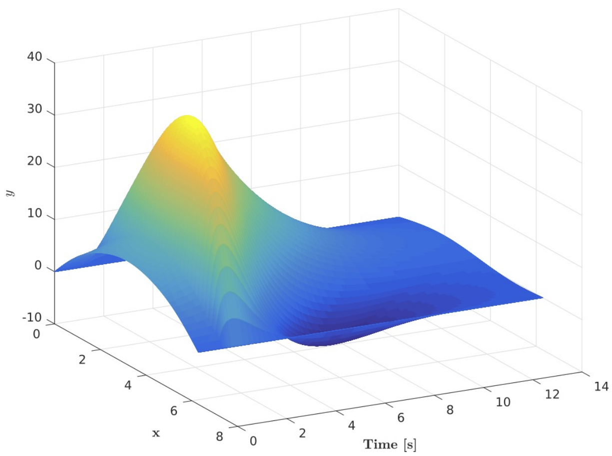

Fig. 1. Time-evolution of the solution of (1) with delay boundary controller

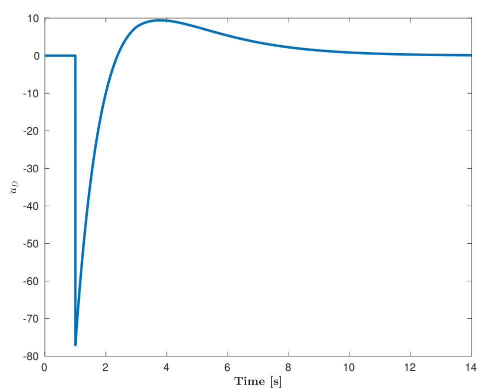

Fig. 2. Delay control u_D for (1)

We have already noted that if $t < D$ then $Z_1(t) = X_1(t)$, and hence in that case $\Phi_D(t,s) = 0$. Hence in what follows we assume that $t > D$. Using the Fubini theorem, we get

$$\int_{-\infty}^{t} \Phi_D(t,s) X_1(s)\, ds = \int_{\max(t-D,D)}^{t} e^{(t-s-D)A_1} B_1 K_1(D) X_1(s)\, ds + \int_{-\infty}^{t}\int_{\max(t-D,D,s)}^{t} e^{(t-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau,s)\, d\tau\, X_1(s)\, ds.$$

Since we would like this equality to hold true for every X_1, there must hold

$$\Phi_D(t,s) = e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(\max(t-D,D),t)}(s) + \int_{\max(t-D,D,s)}^{t} e^{(t-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau,s)\, d\tau \quad (48)$$

for every $t > D$. Let us now solve the implicit equation (48).

First of all, if $D < t < 2D$ then $\max(t-D,D) = D$ and (48) yields

$$\Phi_D(t,s) = e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(D,t)}(s) + \int_{\max(s,D)}^{t} e^{(t-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau,s)\, d\tau.$$

There are two subcases.

If $s < D$ or if $s > t$ then clearly $\Phi_D(t,s) = 0$ is a solution.

If $D < s < t$ then (48) is equivalent to

$$\begin{aligned}\Phi_D(t,s) &= e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(D,t)}(s) + \int_{s}^{t} e^{(t-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau,s)\, d\tau \\ &= e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(D,t)}(s) + \int_{0}^{t-s} e^{(t-s-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau+s,s)\, d\tau\end{aligned}$$

and then setting $r = t - s$ (note that $0 < r < 2D$) we search $\Phi_D(t,s) = f(r)$ with

$$\begin{aligned}f(r) &= e^{(r-D)A_1} B_1 K_1(D) + \textstyle\int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) f(\tau)\, d\tau \\ &= f_0(r) + (\tilde{T}_D f)(r)\end{aligned} \quad (49)$$

with $f_0(r) = e^{(r-D)A_1} B_1 K_1(D)$ and $(\tilde{T}_D f)(r) = \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) f(\tau)\, d\tau$. Formally, we set $f(r) = 0$ whenever $r < 0$ and $r > 2D$, and

$$\begin{aligned}f(r) &= \sum_{j=0}^{+\infty} (\tilde{T}_D^j f_0)(r) \\ &= e^{(r-D)A_1} B_1 K_1(D) \\ &\quad + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) e^{(\tau-D)A_1} B_1 K_1(D)\, d\tau \\ &\quad + \int_0^r e^{(r-\tau-D)A_1} B_1 K_1(D) \\ &\quad \times \int_0^\tau e^{(\tau-s-D)A_1} B_1 K_1(D) e^{(s-D)A_1} B_1 K_1(D)\, ds\, d\tau \\ &\quad + \cdots\end{aligned}$$

for every $r \in (0, 2D)$. The convergence of the series follows from the estimate

$$|(\tilde{T}_D^j f_0)(r)| \leqslant \|B_1\|_{\mathbb{R}^{n+1}}^{j+1} \|K_1(D)\|_{\mathbb{R}^{n+1}}^{j+1} \frac{r^j}{j!} e^{(r-jD)\|A_1\|},$$

which is immediate to establish by induction. It yields (49) and thus (48) in the case $D < t < 2D$.

If $t > 2D$ then $\max(t-D,D) = t-D$ and (48) is equivalent to

$$\Phi_D(t,s) = e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(t-D,t)}(s) + \int_{\max(s,t-D)}^{t} e^{(t-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau,s)\, d\tau \quad (50)$$

There are two subcases.

If $s < t - D$ or if $s > t$ then clearly $\Phi_D(t,s) = 0$ is a solution of (50).

If $t - D < s < t$ then

$$\begin{aligned}\Phi_D(t,s) &= e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(t-D,t)}(s) + \int_{s}^{t} e^{(t-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau,s)\, d\tau \\ &= e^{(t-s-D)A_1} B_1 K_1(D) \chi_{(t-D,t)}(s) + \int_{0}^{t-s} e^{(t-s-\tau-D)A_1} B_1 K_1(D) \Phi_D(\tau+s,s)\, d\tau\end{aligned}$$

and then setting $r = t - s$ (note that, then, $0 < r < D$), similarly as above, we search $\Phi_D(t,s) = f(r)$ with $f(r) = f_0(r) + (\tilde{T}_D f)(r)$ for every $r \in (0,D)$. Formally, we get $f(r) = 0$ whenever $r < 0$ and $r > D$, and $f(r) = \sum_{j=0}^{+\infty}(\bar{T}_D^j f_0)(r)$ for every $r \in (0,D)$. The convergence is established as previously. Thus (50) holds in the case $t > 2D$, and (48) holds in all cases. This concludes the proof of Lemma 4.

V. Conclusion

For a reaction-diffusion equation with delay boundary control, a new constructive design method has been proposed, based on an explicit form of the classical Artstein transformation for the finite-dimensional unstable part of the delay system. By an appropriate Lyapunov function, it has been shown that the designed boundary delay control stabilizes the whole reaction-diffusion partial differential equation.

Some issues are in order. First, by exploiting the observability property of (1) when defining the output as e.g., a boundary Neumann measure, it may be fruitful to combine an observer with the present finite-dimensional feedback control, in the spirit of [20]. Secondly, by noting that the studied system is equivalent to a scalar parabolic equation coupled with a scalar transport equation, it is natural to wonder whether it is possible to adapt this design method to a system composed of several parabolic PDEs coupled with a hyperbolic system, coupled at the boundary (or inside by internal terms) and controlled by means of a delay controller. Finally, a degenerate reaction-diffusion system system has been studied in [7] for an approximate controllability problem. It would be interesting to investigate the stabilization problem of this PDE by means of a boundary delay control.

Acknowledgment

The authors thank the anonymous referees for their valuable suggestions.

References

[1] Z. Artstein. Linear systems with delayed controls: a reduction. *IEEE Transactions on Automatic Control*, 27(4):869–879, 1982.

[2] D. Bresch-Pietri and M. Krstic. Delay-adaptive control for nonlinear systems. *IEEE Transactions on Automatic Control*, 59(5):1203–1218, 2014.

[3] D. Bresch-Pietri and M. Krstic. Output-feedback adaptive control of a wave PDE with boundary anti-damping. *Automatica*, 50(5):1407–1415, 2014.

[4] D. Bresch-Pietri, C. Prieur, and E. Trélat. New formulation of predictors for finite-dimensional linear control systems with input delay. *Systems Control Lett.*, 113:9–16, 2018.

[5] J.-M. Coron and E. Trélat. Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM Journal on Control and Optimization*, 43(2):549–569, 2004.

[6] J.-M. Coron and E. Trélat. Global steady-state stabilization and controllability of 1D semilinear wave equations. *Commun. Contemp. Math.*, 8(4):535–567, 2006.

[7] E. Crépeau and C. Prieur. Approximate controllability of a reaction-diffusion system. *Systems Control Lett.*, 57(12):1048–1057, 2008.

[8] E. Fridman, S. Nicaise, and J. Valein. Stabilization of second order evolution equations with unbounded feedback with time-dependent delay. *SIAM Journal on Control and Optimization*, 48(8):5028–5052, 2010.

[9] E. Fridman and Y. Orlov. Exponential stability of linear distributed parameter systems with time-varying delays. *Automatica*, 45(1):194–201, 2009.

[10] H.K. Khalil. *Nonlinear Systems*. Prentice-Hall, 3rd edition, 2002.

[11] M. Krstic. On compensating long actuator delays in nonlinear control. *IEEE Transactions on Automatic Control*, 53(7):1684–1688, 2008.

[12] M. Krstic. Control of an unstable reaction-diffusion PDE with long input delay. *Systems & Control Letters*, 58(10-11):773–782, 2009.

[13] A. Manitius and A.W. Olbrot. Finite spectrum assignment problem for systems with delays. *IEEE Transactions on Automatic Control*, 24(4):541–552, 1979.

[14] S. Nicaise and C. Pignotti. Stabilization of the wave equation with boundary or internal distributed delay. *Differential and Integral Equations*, 21(9-10):935–958, 2008.

[15] S. Nicaise and J. Valein. Stabilization of the wave equation on 1-d networks with a delay term in the nodal feedbacks. *Networks and Heterogeneous Media*, 2(3):425–479, 2007.

[16] S. Nicaise, J. Valein, and E. Fridman. Stability of the heat and wave equations with boundary time-varying delays. *Discrete and Continuous Dynamical Systems*, 2:559–581, 2009.

[17] J.-P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[18] D. L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Review*, 20(4):639–739, 1978.

[19] M. Schmidt and E. Trélat. Controllability of Couette flows. *Commun. Pure Appl. Anal.*, 5(1):201–211, 2006.

[20] A. Smyshlyaev and M. Krstic. *Adaptive control of parabolic PDEs*. Princeton University Press, Princeton, NJ, 2010.

[21] O. Solomon and E. Fridman. Stability and passivity analysis of semilinear diffusion PDEs with time-delays. *International Journal of Control*, 88(1):180–192, 2015.

[22] E. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer, New York, second edition, 1998.

[23] E Trélat. *Contrôle optimal (French) [Optimal control], Théorie & applications [Theory and applications]*. Math. Concretes [Concrete Mathematics], Vuibert, Paris, 2005.

Christophe Prieur was born in 1974. He is currently a senior researcher of the CNRS at the Gipsa-lab, Grenoble, France. He was the Program Chair of the 9th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2013) and of the 14th European Control Conference (ECC 2015). He was a member of the CSS-CEB, and has been an associate editor for the IEEE Trans. on Automatic Control, and IMA J. Mathematical Control and Information. He is currently a member of the EUCA-CEB, an associate editor for the European J. of Control, and IEEE Trans. on Control Systems Technology, a senior editor for the IEEE Control Systems Letters, and an editor for the IMA Journal of Mathematical Control and Information. His current research interests include nonlinear control theory, hybrid systems, and control of partial differential equations.

Emmanuel Trélat was born in 1974. He is currently full professor at Sorbonne University (Paris 6). He is the director of the Fondation Sciences Mathématiques de Paris. He is editor in chief of the journal ESAIM: Control Calculus of Variations and Optimization and is associate editor of many other journals. He has been awarded the SIAM Outstanding Paper Prize (2006), Maurice Audin Prize (2010), Felix Klein Prize (European Math. Society, 2012), Blaise Pascal Prize (french Academy of Science, 2014), Big Prize Victor Noury (french Academy of Science, 2016). His research interests range over control theory in finite and infinite dimension, optimal control, stabilization, geometry, numerical analysis.