Necessary and Sufficient Conditions on the Exponential Stability of Positive Hyperbolic Systems

Liguo Zhang and Christophe Prieur

***Abstract*—In this paper, a strict linear Lyapunov function is developed in order to investigate the exponential stability of a linear hyperbolic partial differential equation with positive boundary conditions. Based on the method of characteristics, some properties of the positive solutions are derived for the hyperbolic initial boundary value problems. The dissipative boundary condition in terms of linear inequalities is proven to be not only sufficient but also necessary under an extra assumption on the velocities of the hyperbolic systems. An application to control of the freeway traffic modeled by the Aw–Rascle traffic flow equation illustrates and motivates the theoretical results. The boundary control strategies are designed by integrating the on-ramp metering with the mainline speed limit. Finally, the proposed feedback laws are tested under simulation, first in the free-flow case and then in the congestion mode, which show adequate performance to stabilize the local freeway traffic.**

***Index Terms*—Aw–Rascle equations, distributed parameter systems, exponential stability, Lyapunov function, positive hyperbolic systems.**

I. Introduction

Many physical processes are represented by a system of 1-D hyperbolic partial differential equations (PDEs). Typical examples include telegrapher equations for electrical circuits of transmission lines [17], shallow water equations for open channels [9], Euler equations for gas flow [12], and the Aw–Rascle traffic flow model for road traffic [2].

The problem of analyzing the exponential stability for linear or quasi-linear hyperbolic systems has been considered for more than 30 years. Main contribution of [14] is the trajectory-based technique via direct estimation of the solutions and their derivatives along the characteristic curves. Recent results have been extended to the case of conservation laws with perturbation source terms [16] and absolutely exponential stability for switched hyperbolic systems [1].

On the other hand, a Lyapunov analysis technique is proposed in [6] for the exponential stability of conservation laws. The sufficient dissipative boundary condition is known to be strictly weaker than the one of [14] and to allow for numerically tractable methods for the design of stabilizing boundary controls [7]. However, the approach by a quadratic Lyapunov function is not always effective to prove stability for hyperbolic systems. It has been shown in [3] that there exist stable 2×2 linear hyperbolic systems, for which there does not exist any quadratic Lyapunov function.

We consider the hyperbolic systems belonging to a class of positive systems whose state variables remain nonnegative. A necessary and sufficient condition for the exponential stability is presented by means of a novel linear Lyapunov function. To the best of our knowledge, this is the first work dealing with such a converse explicit Lyapunov result.

The motivating application is the development of dissipative boundary conditions to regulate traffic states of a freeway link. Choosing boundary feedback control is particularly necessary and natural for the freeway traffic system, since the available control devices often depend on traffic signals for on-ramp metering and variable message signs (VMSs) for regulating vehicle speed, usually located at the boundaries and the cross sections of the freeway link. In this paper, the boundary feedback control is derived and analyzed directly from the Aw–Rascle PDEs without using any model approximation and discretization.

Due to space limitation, some proofs and numerical simulations have been omitted and collected in [19].

This paper is organized as follows. A class of positive linear hyperbolic systems is given in Section II. The well-posedness and the positive properties of solutions are also discussed. In Section III, the necessary and sufficient condition in terms of the linear inequalities is derived for the exponential stability. Finally, an application of boundary feedback control for freeway traffic is presented in Section IV.

Notation: The set of nonnegative integers and reals is $\mathbb{N}$ or $\mathbb{R}_+$, respectively. $\mathbb{R}_+^{n\times n}$ or $\mathbb{R}_+^n$ is the set of n-order square nonnegative matrices or vectors. A matrix (in particular, a vector) A with entries in $\mathbb{R}_+$ is called a nonnegative matrix (vector), and it is denoted as $A \succeq 0$. It is said to be positive ($A \succ 0$), if all its entries are positive. The expression $A \succeq B$ indicates that the difference $A - B$ is nonnegative. The terms nonpositive and negative are defined analogously as $\preceq$ and $\prec$. $\mathrm{row}_i(A)$ denotes the ith row of matrix A. A real symmetric matrix A is positive definite (respectively, semidefinite positive), if all its eigenvalues are positive (respectively, nonnegative); in that case, we will use the respective notation $A > 0$ and $A \geq 0$. Given a function $g : [0,1] \to \mathbb{R}^n$, we define its L^1-norm as $\|g\|_{L^1(0,1)} = \int_0^1 \|g(x)\| dx$, where $\|.\|$ is the Euclidian norm in $\mathbb{R}^n$. We call $L^1(0,1)$ the space of all measurable functions g, for which $\|g\|_{L^1(0,1)} < \infty$.

II. Positive Hyperbolic Systems: Mathematical Preliminaries

Consider a class of positive linear hyperbolic systems described by the following equation:

$$\partial_t \xi(x,t) + \Lambda \partial_x \xi(x,t) = 0,\ t \in \mathbb{R}_+,\ x \in [0,1] \tag{1}$$

Manuscript received February 24, 2016; revised October 25, 2016 and January 20, 2017; accepted January 27, 2017. Date of publication January 31, 2017; date of current version June 26, 2017. This work was partially supported by the National Natural Science Foundation of China (NSFC) under Grant 61374076 and 61533002, and by the International Cooperation and Exchange Program of NSFC under Grant 61111130119. Recommended by Associate Editor D. Dochain.

L. Zhang is with the Key Laboratory of Computational Intelligence and Intelligent Systems and the School of Electronic Information and Control Engineering, Beijing University of Technology, Beijing 100124, China (email: zhangliguo@bjut.edu.cn).

C. Prieur is with CNRS, Gipsa-lab, Université Grenoble Alpes, F-38000, Grenoble, France (e-mail: christophe.prieur@gipsa-lab.fr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2017.2661966

where $\xi : [0,1] \times \mathbb{R}_+ \to \mathbb{R}^n$. Assume that $\Lambda \in \mathbb{R}_+^{n\times n}$ is a diagonal positive-definite matrix such that $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, with $\lambda_i > 0$, for all $i \in \{1, \ldots, n\}$. The boundary condition is written as

$$\xi(0,t) = G\xi(1,t),\ t \in \mathbb{R}_+ \tag{2}$$

where matrix G belongs to $\mathbb{R}_+^{n\times n}$. We shall consider the initial condition given by

$$\xi(x,0) = \xi^0(x),\ x \in [0,1] \tag{3}$$

for a given function $\xi^0 : [0,1] \to \mathbb{R}^n$.

The linear hyperbolic system (1)–(3) is called positive if the trajectories of the system starting from any nonnegative initial conditions remain nonnegative forever. An example ξ could be related to the gas flow in pipelines, as considered in [12]. It could also couple wave equations for dynamics of stimulated Raman scattering; See also [10], where the positivity of the solutions is assumed.

The existence of the solution of the hyperbolic system (1)–(3) in the set $C^0([0,\infty), H^1(0,1)) \cap C^1([0,\infty), L^2(0,1))$ holds for the initial condition ξ^0 in $H^1(0,1)$ satisfying the zero-order compatibility condition $\xi^0(0) = G\xi^0(1)$ (see, e.g., [4, Th. A.1] or [5]).

In the context of positive hyperbolic systems, we are able to state the following.

Proposition 1: Assume the matrix $G \in \mathbb{R}_+^{n\times n}$ in the boundary condition (2) is nonnegative, i.e., $G \succeq 0$. Then, for any function $\xi^0 : [0,1] \to \mathbb{R}^n$ in $H^1(0,1)$ satisfying the zero-order compatibility condition, there exists a unique solution $\xi : [0,1] \times [0,\infty)$ to the Cauchy problem (1)–(3).

Moreover, the solution satisfies the following.

1) *If $\xi^0(0) \in \mathbb{R}_+^n$ for all $x \in [0,1]$, then $\xi(x,t) \succeq 0$, for all $(x,t) \in [0,1] \times \mathbb{R}_+$.*
2) *If for all $i = 1, \ldots, n$, row$_i(G) \neq 0$, and $\xi^0(x) \succ 0$ for all $x \in [0,1]$, then $\xi(x,t) \succ 0$, for all $(x,t) \in [0,1] \times \mathbb{R}_+$.*

Roughly speaking, besides the existence and uniqueness result of the Cauchy problem (1)–(3), the previous result states the positivity of hyperbolic systems, and that the positivity of solutions is preserved for a special kind of boundary condition matrices.

Proof: The well-posedness follows from a classical application of the Lumer–Phillips theorem, as done, e.g., in [4, Th. A.1].

For system (1), we can calculate the propagating period of each state component $\xi_i(x,t)$ from the boundary $x = 0$ to $x = 1$ as

$$\tau_i = \frac{1}{\lambda_i},\ \ i = 1, \ldots, n. \tag{4}$$

Denote $\underline{\tau}$ be the minimum time of all τ_i, i.e., $\underline{\tau} = \min_{i=1,\ldots,n} \tau_i$, and for $p \in \mathbb{N}$, let $\Delta_p \subset \mathbb{R}_+$ be defined by

$$\Delta_p = [p\underline{\tau}, (p+1)\underline{\tau}]. \tag{5}$$

We now proceed by induction over the time interval Δ_p to prove Item 1 of Proposition 1. The first step consists to prove $\xi(x,t) \succeq 0$, for all $t \in \Delta_0$, $x \in [0,1]$.

For an initial state $\xi^0 \in \mathbb{R}_+^n$ in $H^1(0,1)$, by the method of characteristics, the state component to the Cauchy problem (1)–(3) on $[0,1] \times \Delta_0$ is given as, for all $i = 1, \ldots, n$,

$$\xi_i(x,t) = \begin{cases} \sum_{j=1}^n g_{\mathrm{ij}} \xi_j^0 \left[1 - \lambda_j \left(t - \dfrac{x}{\lambda_i}\right)\right], & \text{for } \lambda_i t > x \\ \xi_i^0(x - \lambda_i t), & \text{for } \lambda_i t \le x \end{cases} \tag{6}$$

where g_{ij} is the entry (i,j) of the boundary condition matrix G.

Since $\xi^0(x) \in \mathbb{R}_+^n$, for all $x \in [0,1]$, and $g_{\mathrm{ij}} \ge 0$, it follows from (6) that $\xi(x,t) \succeq 0$, for all $t \in \Delta_0$. This concludes the initial step of the induction.

Suppose that for $p \ge 0$, $\xi(x,t) \succeq 0$, for all $t \in \Delta_p$. Taking $\xi(\cdot, pT)$ as the initial condition of the system and applying the same argument as above, we get that the solution is uniquely defined, and $\xi(x,t) \succeq 0$ for all $t \in \Delta_{p+1}$.

Therefore, the solution satisfies $\xi(x,t) \succeq 0$, $t \in \mathbb{R}_+$. It concludes the proof of Item 1 in Proposition 1.

Let us prove Item 2, using again an induction argument. First, from (6), noting that all the terms in the sum of the first line of (6) are nonnegative and there exist at least one positive term g_{ij} of the sum in (6), we have $\xi_i(x,t) > 0$, for all $(x,t) \in [0,1] \times \Delta_0$. The induction is shown in a similar way taking $\xi(\cdot, pT)$ as an initial condition.

This concludes the proof of Proposition 1. ■

Before stating the general results on the solutions of positive linear hyperbolic systems (1)–(3), let us construct a sequence of the index set according to the structure of matrix G.

Let $\Omega_0 = \{1, \ldots, n\}$, and $\mathcal{I}_1 = \{i : \sum_{j\in\Omega_0} g_{\mathrm{ij}} = 0\}$, which includes all indices of the zero rows of matrix G, i.e., row$_i(G) = 0$, $i = 1, \ldots, n$.

We denote the following index sets iteratively, for all $k \in \mathbb{N}$, $k \ge 2$:

$$\Omega_{k-1} = \{1, \ldots, n\} \setminus \bigcup_{l=1}^{k-1} \mathcal{I}_l \tag{7}$$

and

$$\mathcal{I}_k = \left\{ i : i \in \Omega_{k-1}, \text{ and} \sum\nolimits_{j\in\Omega_{k-1}} g_{\mathrm{ij}} = 0 \right\}. \tag{8}$$

Then, $\mathcal{I}_k$ is the largest subset in Ω_{k-1} such that $\sum_{j\in\Omega_{k-1}} g_{\mathrm{ij}} = 0$ for all $i \in \mathcal{I}_k$, and for all indices $i \notin \mathcal{I}_k$, it holds $\sum_{j\in\Omega_{k-1}} g_{\mathrm{ij}} \neq 0$.

Since G is an n-dimensional matrix, the construction process (7), (8) could be finished in a finite number of iterative steps, that is, the index sequence $\mathcal{I}_k$, $k = 1, \ldots$, is a finite set and includes at most n subsets. Let K be such that the last subset $\mathcal{I}_K$ is empty, $\mathcal{I}_K = \emptyset$, and other sets $\mathcal{I}_1, \ldots, \mathcal{I}_{K-1}$ are nonempty.

Item 2 in Proposition 1 indicates that the positive linear hyperbolic systems (1)–(3) might experience a positive solution as the boundary condition matrix G does not include nonzero rows. Given a submatrix G_{Ω_k} (the subscripts instruct the rows and columns of G that make up the square submatrix), $k = 0, 1, \ldots, K-1$, we prove that the state components ξ_i, whose index i belongs to the subset Ω_k, remain positive as the submatrix G_{Ω_k} does not have nonzero rows. To be more specific, the following proposition holds.

Proposition 2: Assume that the matrix G includes zero rows, i.e., row$_i(G) = 0$, for some $i \in \{1, \ldots, n\}$, and the sequence of index sets $\mathcal{I}_k$, $k = 1, \ldots, K$, is defined as (8), $\mathcal{I}_K = \emptyset$. Then, for any initial condition ξ^0 in $H^1(0,1)$ satisfying the zero-order compatibility condition and $\xi^0(x) \succ 0$, $x \in [0,1]$, the solution to the system (1)–(3) satisfies the following:

1) $\forall i \in \mathcal{I}_{K_0}$, $K_0 = 1, \ldots, K-1$,
$$\xi_i(x,t) = 0 \tag{9}$$
for all $(x,t) \in [0,1] \times (\sum_{k=1}^{K_0} \max_{i\in\mathcal{I}_k}(\tau_i), +\infty)$, *where* τ_i *is defined by (4);*
2) $\forall j \in \Omega_{K-1}$,
$$\xi_j(x,t) > 0 \tag{10}$$
for all $(x,t) \in [0,1] \times \mathbb{R}_+$.

The complete proof of this proposition is detailed in [19] and is based on the induction that the state component $\xi_i(x,t)$, $i \in \mathcal{I}_{K_0}$, $K_0 = 1, \ldots, K-1$, will become zero in a finite time $\sum_{k=1}^{K_0} \max_{i\in\mathcal{I}_k}(\tau_i)$, while the others ξ_j keep positive as $t \in [0, \sum_{k=0}^{K_0} \min_{j\in\Omega_k}(\tau_j)]$.

Example 1: Let us consider the following nonnegative boundary condition matrix:

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We have $\mathcal{I}_1 = \{1\}$, $\mathcal{I}_2 = \{4\}$, $\mathcal{I}_3 = \{2\}$, and $\mathcal{I}_4 = \emptyset$. Thus, $\{1, 2, 3, 4\} - \bigcup_{k=1}^{3} \mathcal{I}_k = \{3\}$. For any initial condition $\xi^0(x) \succ 0$, the third state component satisfies $\xi_3(x,t) \succ 0$ for all $(x,t) \in [0,1] \times \mathbb{R}_+$. For any $x \in [0,1]$, other state components become zeros in finite time. More precisely, $\xi_1(x,t) = 0$ when $t \in [\tau_1, \infty)$, $\xi_4(x,t) = 0$ when $t \in [\tau_1 + \tau_4, \infty)$, and $\xi_2(x,t) = 0$ when $t \in [\tau_1 + \tau_4 + \tau_2, \infty)$. ⋆

Remark 1: We can consider more general positive linear hyperbolic systems with both negative and positive velocities for Λ, such as with $\lambda_i < 0$ for $i \in \{1, \dots, m\}$, and $\lambda_i > 0$ for $i \in \{m+1, \dots, n\}$. By defining the state description $\xi = (\xi^-, \xi^+)^\top$, where $\xi^- \in \mathbb{R}^m$ and $\xi^+ \in \mathbb{R}^{n-m}$, and the change of variable $w(x,t) = [\xi^-(1-x,t), \xi^+(x,t)]^\top$, we can obtain a new hyperbolic system in the same form as (1), (2). ◦

III. Necessary and Sufficient Conditions on the Exponential Stability

Let us start this section by defining the notion of the exponential stability under consideration in this paper.

Definition 1: The linear hyperbolic system (1), (2) is said to be exponentially stable in L^1-norm, if there exist $\nu > 0$ and $C > 0$ such that, for every initial condition $\xi(\cdot, 0)$ in $H^1(0,1)$ satisfying the zero-order compatibility condition, the solution ξ to the Cauchy problem (1)–(3) satisfies, for all $t \in \mathbb{R}_+$,

$$\|\xi(.,t)\|_{L^1(0,1)} \le C e^{-\nu t} \|\xi(.,0)\|_{L^1(0,1)}. \tag{11}$$

A. Conditions for the Existence of a Linear Lyapunov Function

Let us first deal with a sufficient condition for the existence of a linear Lyapunov function yielding an exponential stability of positive linear hyperbolic systems. This sufficient condition is written in terms of a dissipative boundary condition and is also necessary under an additional assumption, as written in the following first main result.

Theorem 1: Consider the positive linear hyperbolic system (1)–(3) with matrix $G \in \mathbb{R}_+^{n\times n}$. Let $\theta \in \mathbb{R}_+^n$ be a positive vector, i.e., $\theta \succ 0$, and $\mu > 0$ be a constant such that

$$[G^\top - e^{-\mu} I_n]\theta \preceq 0. \tag{12}$$

1) *The system (1)–(3) is exponentially stable if condition (12) holds, and moreover, a Lyapunov function is given by, for all ξ in $L^1(0,1)$,*

$$V(\xi) = \int_0^1 \theta^\top \Lambda^{-1} \xi e^{-\mu x} dx. \tag{13}$$

2) *Conversely, if the system (1)–(3) is exponentially stable and the inverse of velocities, λ_i^{-1}, is commensurable, then there exist $\theta \succ 0$ and $\mu > 0$ such that (12) holds.*

Remark 2: Before proving this theorem, let us note that the Lyapunov function candidate V defined in (13) is inspired by [6] among other studies, where the same weight $x \mapsto \exp(-\mu x)$ is used combined with a L^2 integral norm (see also [8]). It is also inspired by [15], where the Lyapunov function is linear, which implies a prior requirement that all state variables are nonnegative. This linear Lyapunov function candidate is not appropriate for general linear hyperbolic systems. ◦

Proof: (Sufficiency) Using an integration by part, the time derivative of the Lyapunov function (13) along the solutions to (1) is computed as $\dot V = -\int_0^1 \theta^\top \partial_x \xi e^{-\mu x} dx = -[\theta^\top \xi e^{-\mu x}]_0^1 - \mu \int_0^1 \theta^\top \xi e^{-\mu x} dx$. Now, under the boundary condition (2), it holds $\dot V = \xi^\top(1,t)[G^\top - e^{-\mu} I_n]\theta - \mu \int_0^1 \theta^\top \xi e^{-\mu x} dx$.

Moreover, using G in $\mathbb{R}_+^n$ and Proposition 1, we have $\xi(1,t) \in \mathbb{R}_+^n$ for all $t \ge 0$, and thus, $\dot V \le -\mu \underline{\lambda} V(\xi)$, where $\underline{\lambda} = \min\{\lambda_i, i = 1, \dots, n\}$. By remarking that there exist positive values C_1 and C_2 (depending on θ, Λ, and μ) such that, for all nonnegative functions ξ, it holds $C_1 \|\xi\|_{L^1(0,1)} \le V(\xi) \le C_2 \|\xi\|_{L^1(0,1)}$, we may deduce that the solutions of the positive linear hyperbolic system (1) exponentially converge to 0 in L^1-norm.

This complete the proof of sufficiency.

(Necessity) Since the inverse of the velocities, λ_i^{-1}, is commensurable, there exists a sufficiently small time interval $\Delta t \in \mathbb{R}_+$, such that the time periods satisfy $\tau_j = d_j \Delta t$, for suitable integers $d_j \in \mathbb{N}$, $j = 1, \dots, n$.

Then, we could find a common time sample, at which the dynamics of each state component $\xi_i(x,t)$ project to its left boundary. Roughly speaking, the necessary condition is later derived by considering the time evolution of $\xi_i(0,t)$, for all $t \ge 0$.

Let $\overline{\tau}$ be the maximum time of all τ_i, i.e., $\overline{\tau} = \max_{i=1,\dots,n} \tau_i$. Using that state ξ_i is constant along the characteristic curves, the values of every state component at the right boundary at time t are equivalent to those at time $t - \tau_i$, that is,

$$\xi_i(1,t) = \xi_i(0, t - \tau_i) \tag{14}$$

for all $t \ge \tau_i$, $i = 1, \dots, n$.

Combined with the boundary condition (2), for $t \ge \overline{\tau}$, the dynamics of system (1) at the left boundary $x = 0$ can be represented as

$$\xi(0,t) = \sum_{j=1}^{n} G[j] \xi(0, t - \tau_j) \tag{15}$$

where $G[j] = [0, \dots, \mathrm{col}_j(G), \dots, 0]$ is an n-dimensional square matrix, and $\mathrm{col}_j(G)$ is the jth column of the matrix G. We find $\sum_{j=1}^{n} G[j] = G$. Using the notation $l = \frac{\overline{\tau}}{\Delta t}$, we note that $d_j \le l$ for all $j = 1, \dots, n$, and thus, we get the following time-delay equation from (15):

$$\xi(0, s\Delta t) = \sum_{j=1}^{n} G[j] \xi(0, s\Delta t - d_j \Delta t) \tag{16}$$

for $s = l, l+1, \dots$.

Simply denote $y_s = \xi(0, s\Delta t)$. By reorganizing the terms in (16), we get the following time-delay system, for $s = l, l+1, \dots$:

$$y_s = \sum_{d=1}^{l} \mathcal{G}_d y_{s-d} \tag{17}$$

where

$$\mathcal{G}_d = \begin{cases} \sum_{d_j = d} G[j], & \text{there exists } j \text{ such that } d = d_j \\ 0, & \text{otherwise} \end{cases}$$

and initial values $y_{l-d} = \xi(0, (l-d)\Delta t)$, for all $d = 1, \dots, l$, is uniquely defined with Proposition 1. Furthermore, with Proposition 2, the state components $\xi_i(0, s\Delta t) > 0$, as $i \in \Omega_{K-1}$, and the others are kept nonnegative.

Now, we consider the discrete-time instant

$$h = \frac{1}{\Delta t} \sum_{i=1}^{n} \tau_i. \tag{18}$$

From (17), for all $s = h+2, \dots, h+p$, $(p \in \mathbb{N})$, we have, respectively, $y_{h+2} = \sum_{d=1}^{l} \mathcal{G}_d y_{h+2-d}$, ..., $y_{h+p} = \sum_{d=1}^{l} \mathcal{G}_d y_{h+p-d}$. Summing the above equalities, one obtains

$$\sum_{s=h+2}^{h+p} y_s = \sum_{s=h+2}^{h+p} \sum_{d=1}^{l} \mathcal{G}_d y_{s-d} = \sum_{d=1}^{l} \mathcal{G}_d \sum_{s=h+2}^{h+p} y_{s-d}$$
$$= \sum_{d=1}^{l} \mathcal{G}_d \left[\sum_{s=h+2}^{h+p-d} y_s + \sum_{s=h+2-d}^{h+1} y_s \right]. \quad (19)$$

Since the system (1)–(3) is exponentially stable in L^1-norm, then for all $x \in (0,1)$, except on a set of measure zero, $\xi(x,t) \to 0$, as $t \to \infty$, exponentially fast. Since the solution is continuous with respect to x, this implies the exponential pointwise convergence of $\xi(x,t) \to 0$, as $t \to \infty$, for all x in $(0,1)$. In particular, $\xi(0,t) \to 0$, exponentially fast, as $t \to \infty$. For the corresponding discrete system (17), $\sum_{s=h+2}^{\infty} y_k < \infty$. It follows from (19) that

$$-\sum_{d=1}^{l} \mathcal{G}_d \sum_{s=h+2-d}^{h+1} y_s = \left[\sum_{d=1}^{l} \mathcal{G}_d - I_n \right] \sum_{s=h+2}^{\infty} y_s$$
$$= [G - I_n] \sum_{s=h+2}^{\infty} y_s. \quad (20)$$

Moreover, due to Item 1 of Proposition 1, all the terms on the left-hand side of equality (20) are nonpositive, we have

$$[G - I_n] \sum_{s=h+2}^{\infty} y_s \preceq -\sum_{d=1}^{l} \mathcal{G}_d y_{h+1}$$
$$\preceq -G y_{h+1}. \quad (21)$$

To conclude the necessity part of the proof and show that inequality (12) holds for suitable $\theta \succ 0$ and $\mu > 0$, two cases about matrix G may occur.

Case 1: $\text{row}_i(G) \neq 0$, for all $i = 1, \dots, n$.

According to Item 2 in Proposition 1, the solution of the Cauchy problem (1)–(3) with an positive initial condition $\xi^0(x) \succ 0$ satisfies $\xi(x,t) \succ 0$ for all $(x,t) \in [0,1] \times \mathbb{R}_+$.

In particular, $y_s \succ 0$, for all $s \geq h+1$. Thus, $-G y_{h+1} \prec 0$. With (21), it holds

$$[G - I_n] \, b \prec 0 \quad (22)$$

by letting $b = \sum_{s=h+2}^{\infty} y_s \succ 0$.

Moreover, there exists $\mu > 0$ such that

$$[G - e^{-\mu} I_n] \, b \preceq 0. \quad (23)$$

Because $-[G - e^{-\mu} I_n]$ is an M-matrix and $b \succ 0$ satisfying (23), there exists a positive vector $\theta \in \mathbb{R}_+^n$, $\theta \succ 0$, such that $[G^\top - e^{-\mu} I_n]\theta \preceq 0$.

Case 2: There exists at least one row of indices $i \in \{1, \dots, n\}$, such that $\text{row}_i(G) = 0$.

Assume that the sequence of index set $\mathcal{I}_k$ as (8) is constructed, $k = 1, \dots, K-1$, with an initial condition to (1)–(3) starting from $\xi^0(x) \succ 0$ for all $x \in [0,1]$.

Since $h = \frac{1}{\Delta t} \sum_{i=1}^{n} \tau_i$, we have

$$h \geq \frac{1}{\Delta t} \sum_{k=1}^{K-1} \max_{i \in \mathcal{I}_k}(\tau_i). \quad (24)$$

Therefore, according to Proposition 2, for all $s \geq h+1$, it follows, $(y_s)_{\mathcal{I}_k} = 0$, for all $k = 1, \dots, K-1$, and $(y_s)_{\Omega_{K-1}} \succ 0$.

Because again $b = \sum_{s=h+2}^{\infty} y_s$ is constant, it follows $b_{\mathcal{I}_k} = 0$, for all $k = 1, \dots, K-1$, and $b_{\Omega_{K-1}} \succ 0$.

Furthermore, under the structure of matrix G, the (i,i) entry of $[G - I_n]_{\mathcal{I}_k}$ (a submatrix of $[G - I_n]$ whose row indices belong to $\mathcal{I}_k$) is -1, $i \in \mathcal{I}_k$, and the (i,j) entry of $[G - I_n]_{\mathcal{I}_k}$ is 0, for $i \in \mathcal{I}_k, j \in \Omega_k$, $k = 1, \dots, K-1$. Hence, we have, for all $k = 1, \dots, K-1$,

$$([G - I_n] \, b)_{\mathcal{I}_k} = 0. \quad (25)$$

On the other hand, from inequality (21), we have

$$([G - I_n] \, b)_{\Omega_{K-1}} \preceq (-G y_{h+1})_{\Omega_{K-1}} \prec 0. \quad (26)$$

Let vector $\tilde{b}$ in $\mathbb{R}^n$ such that $\tilde{b}_{\mathcal{I}_k} = \epsilon_k$, for all $k = 1, \dots, K-1$, and $\tilde{b}_{\Omega_{K-1}} = 0$ for the remaining entries of $\tilde{b}$, ϵ_k are sufficiently small positive values satisfying

$$\epsilon_k > \max_{i \in \mathcal{I}_k} \left\{ \sum_{r=1}^{k} \sum_{j \in \mathcal{I}_r} g_{\mathrm{ij}} \epsilon_r \right\}. \quad (27)$$

Hence, $b + \tilde{b} \succ 0$, and, due to (25) and (26), $[G - I_n] \, (b + \tilde{b}) \prec 0$ holds. The remaining proof of this case is as in the first case.

It concludes the proof of Theorem 1. ∎

B. Conditions for the Existence of a Quadratic Lyapunov Function

Before commenting on the second main result, let us recall the following sufficient condition for the existence of a quadratic Lyapunov function.

Proposition 3: (see [7]) Let $P \in \mathbb{R}^{n \times n}$ be a diagonal positive-definite matrix, $\mu > 0$ be a constant, and the function

$$V(\xi) = \int_0^1 \xi^\top \Lambda^{-1} P \xi e^{-2\mu x} \, dx \quad (28)$$

is a quadratic Lyapunov function for system (1)–(3), that is, $\dot{V} \leq -\frac{2\mu}{(\min_i \lambda_i)^{-1}} V$ along the solutions to (1)–(3), if the matrix inequality

$$e^{-2\mu} P - G^\top P G \geq 0 \quad (29)$$

holds.

The second main result, namely Theorem 2, proves that the sufficient condition for the existence of a linear Lyapunov function as written in item 1 of Theorem 1 is equivalent to the dissipative boundary condition (29) considered in Proposition 3. In detail, Theorem 2 gives three characterizations for the matrix inequality (29) to hold, each of them implying the existence of a quadratic Lyapunov function for system (1)–(3).

Theorem 2: Given a positive linear hyperbolic system (1)–(3), the following are equivalent.

1) *There exist $\mu > 0$, and a vector $\theta \succ 0$, such that condition (12) holds.*
2) *There exist $\mu > 0$ and a diagonal positive-definite matrix $P > 0$, such that inequality (29) holds.*
3) *$\rho(G) < 1$, where $\rho(G)$ is the spectral radius of the matrix G.*

The proof of Theorem 2 is detailed in [19] and mainly follows from the Perron–Frobenius theorem.

Example 2: Consider a positive linear hyperbolic system (1), (2) with the characteristic and the boundary condition matrices given, respectively, by $\Lambda = \text{diag}\{1, -1, 2\}$, and

$$G = \begin{bmatrix} 0.2 & 0.4 & 0.2 \\ 0.8 & 0.2 & 0.1 \\ 0.4 & 0 & 0.2 \end{bmatrix}. \quad (30)$$

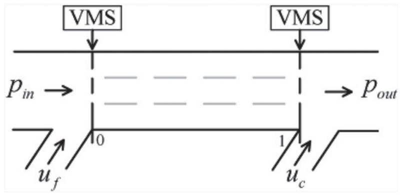

Fig. 1. Freeway link controlled by local on-ramp metering and variable speed sign.

The spectral radius of the nonnegative matrix G is $\rho(G) = 0.8990 < 1$. After solving the linear inequality condition (12), one can easily construct a linear Lyapunov function (13) with $\theta = (6, 4, 3)^\top \succ 0$ and $\mu = 0.01 > 0$. Following the results of Theorem 2, a quadratic Lyapunov function (28) can be further constructed with $b = (4, 5, 5)^\top \succ 0$ and $P = \mathrm{diag}\{1.5, 0.8, 0.6\} > 0$. ⋆

IV. Application to Freeway Traffic Control

A homogeneous freeway section between two successive on-ramps is sketched in Fig. 1, where u_f and u_c are two boundary feedback laws that will be defined in this section. We assume that the upstream and the downstream boundaries of the freeway section are provided with the on-ramp metering to regulate the flow-rate of driving-in vehicles, and with the VMSs to limit the driving speed of the mainline traffic.

The control of the freeway traffic is defined from upstream to downstream and regulates the traffic limit of the freeway section (see Section IV-C for more details). Usually, the transmission speed of information from upstream to downstream is faster than the speed of the vehicles in the freeway. Therefore, for this application, the transmission delay will be neglected.

A. Aw–Rascle Traffic Flow model

The traffic dynamics of the freeway link are described by a system of two laws of conservation, the so-called Aw–Rascle traffic flow model [2]. It is

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t (v + p(\rho)) + v \partial_x (v + p(\rho)) = 0 \end{cases} \tag{31}$$

where $\rho(x, t)$ is the vehicle density, $v(x, t)$ is the average speed, $x \in [0, 1]$, $t \geq 0$, and $p(\rho)$ means the traffic pressure term, which is supposed to increase over the vehicle density.

In [18], a special pressure function $p(\rho)$ is given as

$$p(\rho) = v_f - V(\rho) \tag{32}$$

where v_f is the free (maximal) speed, and $V(\rho)$ is the speed–density fundamental diagram. Typically, with the Greenshields fundamental diagram [11]

$$V(\rho) = v_f \left(1 - \frac{\rho}{\rho_m}\right) \tag{33}$$

where ρ_m is the maximal density. Thus, we have $p(\rho) = a\rho$, and $a = \frac{v_f}{\rho_m}$.

Let $w = v + a\rho, z = v$. The nonlinear hyperbolic equation (31) may be written in the characteristic Riemann coordinates as

$$\begin{cases} \partial_t w + z \partial_x w = 0 \\ \partial_t z + (2z - w)\partial_x z = 0. \end{cases} \tag{34}$$

In (34), the first velocity is assumed to be positive $z > 0$, and the second velocity $2z - w$ is assumed to be nonzero and its sign does not change. The sign of the second velocity indicates the transfer direction of the vehicle speed from the freeway upstream section to the downstream, or inverse. It is usually used as the feature to determine the freeway traffic lies in the free-flow mode or in the congestion mode [13]. In practice, it also determines how to regulate the freeway traffic by using the upstream or the downstream traffic measurements (see Section V-C).

B. Steady State and Linearization

A steady state of the freeway traffic is a constant traffic state (ρ^*, v^*), which satisfies one of the following relation:

$$\begin{aligned} p_{\text{in}} + r_f &= \rho^* v^* \qquad \text{or} \\ \rho^* v^* &= p_{\text{out}} - r_c \end{aligned} \tag{35}$$

where p_{in} and r_f are constant flow rates of the driving-in vehicles through the upstream mainline and the upstream on-ramp of the freeway section, and p_{out} and r_c are constant driving-out flow rates of the downstream mainline and on-ramp.

In order to linearize the model (31), we define the deviations of the states $\rho(x, t)$ and $v(x, t)$ with respect to the steady state as, respectively:

$$\begin{aligned} \hat{\rho} &= \rho - \rho^* \\ \hat{v} &= v - v^*. \end{aligned} \tag{36}$$

The linearization of the system (34) with the steady state (w^*, z^*) is written as

$$\begin{cases} \partial_t \hat{w} + (z^* + \hat{z})\partial_x \hat{w} = 0 \\ \partial_t \hat{z} + (2z^* - w^* + 2\hat{z} - \hat{w})\partial_x \hat{z} = 0. \end{cases} \tag{37}$$

In above Riemann coordinates, the deviations are $\hat{w} = w - w^*$ and $\hat{z} = z - z^*$, with the associated steady state given as $w^* = v^* + a\rho^*$, $z^* = v^*$.

For the traffic steady state, we only assume that the product of vehicle density and speed, that is the flow rate, is constant. Then, there exists a set of the steady states (ρ^*, v^*) satisfying condition (35). In fact, since

$$\lambda_2^* = 2z^* - w^* = v^* - a\rho^* \tag{38}$$

we regard just as two categories of the traffic steady states. While the steady state (ρ^*, v^*) satisfies $v^* - a\rho^* > 0$, we call it the free-flow steady state, while $v^* - a\rho^* < 0$, the congestion steady state.

C. Boundary Feedback Control

As mentioned above, the regulation strategies (using different controller and measurement) are designed depending on the steady state of the freeway traffic lying in the free-flow or congestion traffic modes.

Two cases are discussed separately.

Case 1: $\lambda_2^* = 2z^* - w^* > 0$; the steady state (ρ^*, v^*) lies in the free-flow mode.

In this case, the velocity information (z or v) is propagating from upstream to downstream, and thus, it is natural to control u_f and $v(0, t)$. We assume that the vehicle density $\rho(1, t)$ and the average speed $v(1, t)$ at the downstream boundary are measured, and the control units are the upstream on-ramp metering and the driving-in speed limit of the freeway section.

Precisely, we introduce the boundary feedback law:

$$\begin{cases} u_f(t) = r_f + k_\rho^f (\rho(1, t) - \rho^*) \\ v(0, t) = v^* + k_v^f (v(1, t) - v^*) \end{cases} \tag{39}$$

with feedback gains $k_\rho^f > 0$ and $k_v^f > 0$. The previous controllers are proportional controllers that do not need the knowledge of the entire state, but only the measurement of the vehicle density $\rho(1,t)$ and the average speed $v(1,t)$ at the downstream boundary.

At the upstream boundary of the freeway section, i.e., $x = 0$, the driving-in flow-rate conservation equation holds

$$u_f(t) + p_{\text{in}} = \rho(0,t)v(0,t). \tag{40}$$

After the linearization of the boundary condition (40) with integrating the feedback control law (39), we have the following boundary condition:

$$\hat{\rho}(0,t) = \frac{k_\rho^f}{v^*}\hat{\rho}(1,t) - \rho^* \frac{k_v^f}{v^*}\hat{v}(1,t). \tag{41}$$

Since $\hat{v}(1,t) = \hat{z}(1,t)$ and $\hat{\rho}(x,t) = (\hat{w}(x,t) - \hat{v}(x,t))/a$, as $x = 0, 1$, we could rewrite condition (41) in the Riemann coordinates for the system (37) as

$$\begin{aligned}\hat{w}(0,t) = \;& \frac{k_\rho^f}{v^*}\hat{w}(1,t)\\ &+ \left[k_v^f - a\rho^* \frac{k_v^f}{v^*} - \frac{k_\rho^f}{v^*}\right]\hat{z}(1,t).\end{aligned} \tag{42}$$

Then, the boundary condition that needs to be imposed for the system (37) is written as

$$\begin{bmatrix}\hat{w}(0,t)\\ \hat{z}(0,t)\end{bmatrix} = G_f \begin{bmatrix}\hat{w}(1,t)\\ \hat{z}(1,t)\end{bmatrix} \tag{43}$$

where

$$G_f = \begin{bmatrix}\frac{k_\rho^f}{v^*} & k_v^f - a\rho^* \frac{k_v^f}{v^*} - \frac{k_\rho^f}{v^*}\\ 0 & k_v^f\end{bmatrix}. \tag{44}$$

In the boundary condition matrix G_f,

$$k_v^f - a\rho^* \frac{k_v^f}{v^*} - \frac{k_\rho^f}{v^*} = \frac{\lambda_2^* k_v^f - k_\rho^f}{v^*}. \tag{45}$$

Let us choose $\lambda_2^* k_v^f - k_\rho^f \geq 0$; the boundary condition matrix G_f is nonnegative, i.e., $G_f \succeq 0$. Therefore, applying Proposition 1, system (37) with the boundary condition (43) is a positive linear hyperbolic system.

Straightforward calculations show that Theorem 1 holds for the matrix G_f, if and only if k_ρ^f and k_v^f of the feedback control law (39) can be selected as $0 < k_\rho^f < v^*$, $0 < k_v^f < 1$, and $\lambda_2^* k_v^f - k_\rho^f \geq 0$.

Case 2: $\lambda_2^* = 2z^* - w^* < 0$; the steady state (ρ^*, v^*) lies in the congestion mode.

In this case, velocity information is propagating from upstream to downstream, and thus, it is natural to control u_c and $v(1,t)$. Thus, conversely, the measurements are the vehicle density $\rho(0,t)$ and the average speed $v(0,t)$ at the upstream boundary, and the control units are the downstream on-ramp metering and the driving-out speed limit of the freeway section.

Then, we introduce the other boundary feedback law:

$$\begin{cases} u_c(t) = r_c - k_\rho^c(\rho(0,t) - \rho^*)\\ v(1,t) = v^* + k_v^c(v(0,t) - v^*)\end{cases} \tag{46}$$

with feedback gains $k_\rho^c > 0$ and $k_v^c > 0$. The previous controllers are again proportional controllers that need only the measurements of the vehicle density $\rho(0,t)$ and the average speed $v(0,t)$ at the upstream boundary.

At the right boundary of the freeway link, i.e., $x = 1$, the driving-out flow-rate conservation equation holds

$$p_{\text{out}} - u_c(t) = \rho(1,t)v(1,t). \tag{47}$$

Combining the boundary condition (47) with the feedback control law (46), we have

$$\hat{\rho}(0,t) = \frac{v^*}{k_\rho^c}\hat{\rho}(1,t) + \rho^* \frac{k_v^c}{k_\rho^c}\hat{v}(0,t). \tag{48}$$

Under the Riemann coordinates, since $\hat{v}(0,t) = \hat{z}(0,t)$ and $\hat{\rho}(x,t) = (\hat{w}(x,t) - \hat{v}(x,t))/a$, as $x = 0, 1$, then it holds

$$\begin{aligned}\hat{w}(0,t) = \;& \frac{v^*}{k_\rho^c}\hat{w}(1,t)\\ &+ \left[1 + a\rho^* \frac{k_v^c}{k_\rho^c} - v^* \frac{k_v^c}{k_\rho^c}\right]\hat{z}(0,t).\end{aligned} \tag{49}$$

The corresponding boundary condition for the system (37) is given as

$$\begin{bmatrix}\hat{w}(0,t)\\ \hat{z}(1,t)\end{bmatrix} = G_c \begin{bmatrix}\hat{w}(1,t)\\ \hat{z}(0,t)\end{bmatrix} \tag{50}$$

where

$$G_c = \begin{bmatrix}\frac{v^*}{k_\rho^c} & 1 + a\rho^* \frac{k_v^c}{k_\rho^c} - v^* \frac{k_v^c}{k_\rho^c}\\ 0 & k_v^c\end{bmatrix}. \tag{51}$$

Because

$$1 + a\rho^* \frac{k_v^c}{k_\rho^c} - v^* \frac{k_v^c}{k_\rho^c} = 1 - \lambda_2^* \frac{k_v^c}{k_\rho^c} > 0 \tag{52}$$

the boundary condition matrix G_c is nonnegative, i.e., $G_c \succeq 0$, and the solution to system (37) is nonnegative for all the time, if only $k_\rho^c > 0$ and $k_v^c > 0$, and if the initial condition is nonnegative.

Moreover, as the freeway traffic is lying in the congestion mode, the system (37) with the boundary condition matrix G_c is locally exponentially stable if and only if the control gains k_ρ^c and k_v^c of the boundary feedback strategy (46) satisfy $k_\rho^c > v^*$, $0 < k_v^c < 1$.

Remark 3: Our designed boundary feedback strategies (39) and (46) depend on the sign of the second velocity, which keeps either positive or negative in the whole spatial domain. For the spatial-varying steady states containing phase transition between the free-flow mode and the congestion mode, the boundary feedback stabilization is not discussed in the present note. ○

D. Simulation

The developed boundary feedback strategies (39) and (46) are now tested with some numerical simulations in the two traffic modes presented above. To this end, we consider a local freeway section with parameters $\rho_m = 200$ veh./km, $v_f = 150$ km/h, $a = 0.75$, $p_{\text{in}} = 6000$ veh./h, $r_f = r_c = 1000$ veh./h, $p_{\text{out}} = 8000$ veh./h, and the total road length is 1 km, i.e., $x \in [0, 1]$.

First, for the steady state $(\rho_1^*, v_1^*) = (70, 100)$, we have $\lambda_2^* = 47.5 > 0$. In the neighborhood of this steady state, the freeway traffic lies in the free-flow mode. Thus, we apply the boundary feedback strategy (39), and the feedback gains are chosen as $k_\rho^f = 20$ and $k_v^f = 0.5$. The associated boundary condition matrix G_f in (44) is given as

$$G_f = \begin{bmatrix}0.2 & 0.0375\\ 0 & 0.5\end{bmatrix}. \tag{53}$$

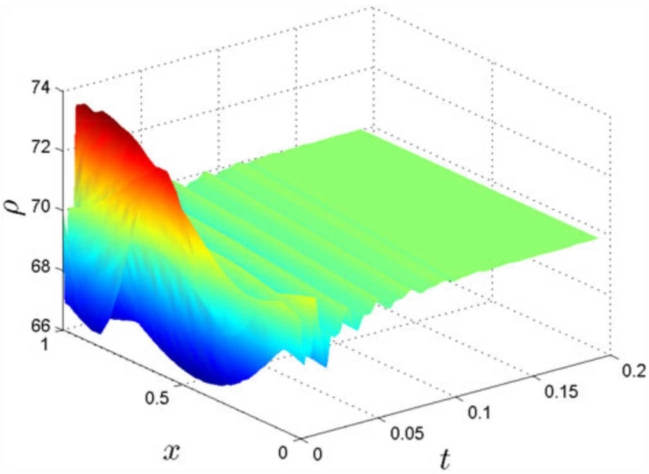

Fig. 2. Evolution of the density ρ of the Aw–Rascle traffic flow equation (31) with the steady state (ρ_1^*, v_1^*).

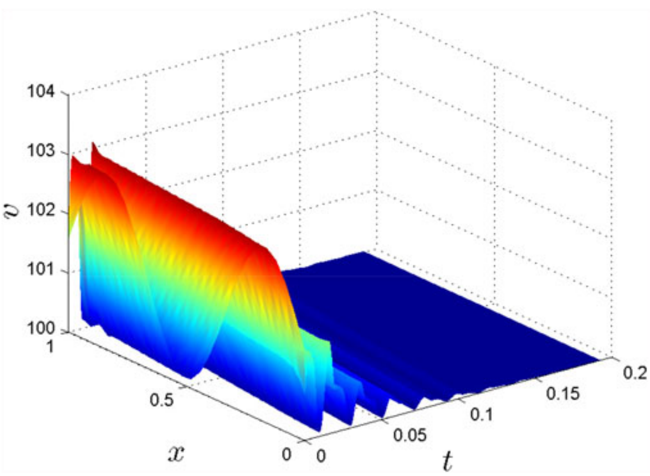

Fig. 3. Evolution of average speed v of the Aw–Rascle traffic flow equation (31) with the steady state (ρ_1^*, v_1^*).

The initial deviations for the Aw–Rascle traffic flow equation (31) are chosen as

$$\begin{cases} \rho(x,0) = \rho^* + \sqrt{2}\sin(4\pi x) + 1.45 \\ v(x,0) = v^* + \sqrt{2}\sin(3\pi x) + 1.45. \end{cases} \quad (54)$$

Figs. 2 and 3 show the solutions $\rho(x,t)$ and $v(x,t)$ to the Aw–Rascle equation (31). The states clearly converge to their steady state $\rho_1^* = 70$ veh./km and $v_1^* = 100$ km/h, respectively, in this simulation.

In the other simulation, we assume freeway traffic lies in the congestion mode and consider the steady state as $(\rho_2^*, v_2^*) = (100, 70)$. In this case, the second eigenvalue $\lambda_2^* = -5 < 0$. We apply the boundary feedback strategy (46), and the feedback gains are chosen as $k_\rho^c = 400$ and $k_v^c = 0.5$.

Then, the boundary condition matrix G_c in (51) is calculated as

$$G_c = \begin{bmatrix} 0.175 & 1.0025 \\ 0 & 0.2 \end{bmatrix}. \quad (55)$$

Figs. 4 and 5 show that such a strategy is effective to stabilize the congestion freeway traffic, by simulating the Aw–Rascle equation (31).

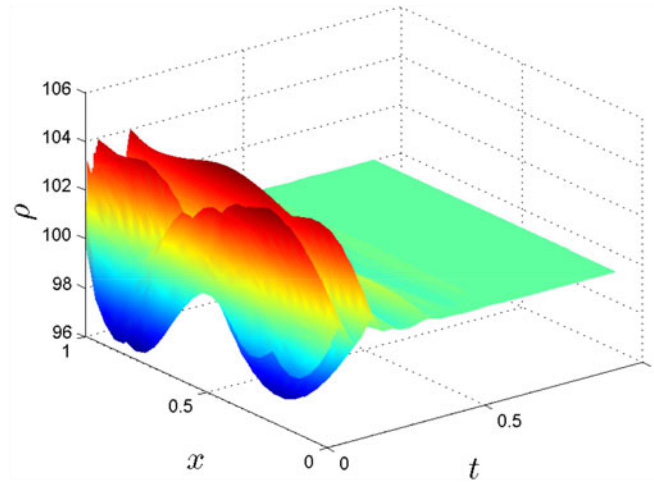

Fig. 4. Evolution of the density ρ of the Aw–Rascle traffic flow equation (31) with the steady state (ρ_2^*, v_2^*).

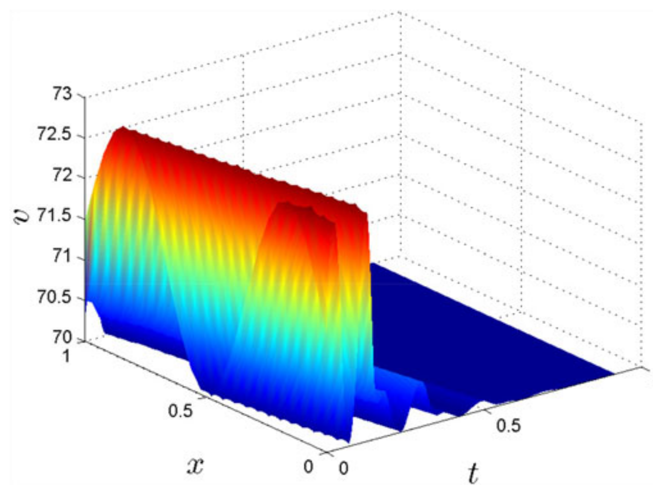

Fig. 5. Evolution of average speed v of the Aw–Rascle traffic flow equation (31) with the steady state (ρ_2^*, v_2^*).

V. Conclusion

In this paper, an important contribution was the necessary and sufficient conditions for the exponential stability of positive linear hyperbolic systems using a constructive linear Lyapunov function. This condition is written in terms of linear inequalities, which are numerically tractable. The theoretical contribution was applied to stabilize freeway traffic represented by the Aw–Rascle traffic flow model. The control strategies combine the on-ramp metering with the speed limit in the distributed action.

Future work shall extend the above theoretical results, such as Propositions 1, 2, and Theorem 1 with space-varying coefficients in the velocity matrix and with the discontinuous selection of boundary conditions. Some effort will also be devoted to the connection between the positivity of hyperbolic systems and the positivity of the associated Riemann coordinates.

References

[1] S. Amin, F. M. Hante, and A. M. Bayen, "Exponential stability of switched linear hyperbolic initial-boundary value problems," *IEEE Trans. Autom. Control*, vol. 57, no. 2, pp. 291–301, Feb. 2012.

[2] A. Aw and M. Rascle, "Resurrection of "second order" models of traffic flow," *SIAM J. Appl. Math.*, vol. 60, no. 3, pp. 916–938, 2000.

[3] G. Bastin and J.-M. Coron, "On boundary feedback stabilization of non-uniform linear 2×2 hyperbolic systems over a bounded interval," *Syst. Control Lett.*, vol. 60, no. 11, pp. 190–196, 2011.

[4] G. Bastin and J.-M. Coron, "Stability and boundary stabilization of 1-D hyperbolic systems," in *Progress in Nonlinear Differential Equations and Their Applications*. New York, NY, USA: Springer, 2016.

[5] J.-M. Coron, G. Bastin, and B. d'Andréa Novel, "Dissipative boundary conditions for one-dimensional nonlinear hyperbolic systems," *SIAM J. Control Optim.*, vol. 47, no. 3, pp. 1460–1498, 2008.

[6] J.-M. Coron, B. d'Andréa Novel, and G. Bastin, "A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws," *IEEE Trans. Autom. Control*, vol. 52, no. 1, pp. 2–11, Jan. 2007.

[7] A. Diagne, G. Bastin, and J.-M. Coron, "Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws," *Automatica*, vol. 48, no. 1, pp. 109–114, 2012.

[8] M. Dick, M. Gugat, and G. Leugering, "Classical solutions and feedback stabilization for the gas flow in a sequence of pipes," *Netw. Heterogeneous Media*, vol. 5, no. 4, pp. 691–709, 2010.

[9] V. Dos Santos and C. Prieur, "Boundary control of open channels with numerical and experimental validations," *IEEE Trans. Control Syst. Technol.*, vol. 16, no. 6, pp. 1252–1264, Nov. 2008.

[10] P. W. Dower and P. M. Farrel, "On linear control of backward pumped Raman amplifiers," in *Proc. IFAC Symp. Syst. Identif.*, Newcastle, Australia, 2006, pp. 547–552.

[11] B. D. Greenshields, "A study of traffic capacity," in *Proc. Highway Res. Board*, 1935, pp. 448–477.

[12] M. Gugat, M. Dick, and G. Leugering, "Gas flow in fan-shaped networks: Classical solutions and feedback stabilization," *SIAM J. Control Optim.*, vol. 49, no. 5, pp. 2101–2117, 2011.

[13] B. S. Kerner. *Introduction to Modern Traffic Flow Theory and Control*. Berlin, Germany: Springer, 2009.

[14] T.-T. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems (Wiley-Masson Series Research in Applied Mathematics)*. Hoboken, NJ, USA: Wiley, 1994.

[15] O. Mason and R. N. Shorten, "On linear copositive Lyapunov functions and the stability of switched positive linear systems," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1346–1349, Jul. 2007.

[16] C. Prieur and F. Mazenc, "ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws," *Math. Control, Signals, Syst.*, vol. 24, no. 1, pp. 111–134, 2012.

[17] M. Suzukiand and N. Sakamoto, "Controlling ideal turbulence in time-delayed Chua's circuit: Stabilization and synchronization," *Int. J. Bifurcation Chaos*, vol. 20, no. 05, pp. 1351–1363, 2010.

[18] H. M. Zhang, "A non-equilibrium traffic model devoid of gas-like behavior," *Transp. Res. B*, vol. 36, no. 3, pp. 275–290, 2002.

[19] L. Zhang and C. Prieur, "Necessary and sufficient conditions on the exponential stability of positive hyperbolic systems (full version)," Tech. Rep. HAL-01385745, 2016. [Online]. Available: https://hal.archives-ouvertes.fr/hal-01385745