LMI-based reset $\mathcal{H}_\infty$ design for linear continuous-time plants

Francesco Fichera, Christophe Prieur, Sophie Tarbouriech, Luca Zaccarian

***Abstract*—In this paper, an optimization-based synthesis of a multi-objective reset controller for linear plants is presented. The result allows designing via convex tools with a line-search a multi-objective reset controller optimizing both the exponential converging rate and the $\mathcal{L}_2$ gain. A novel peculiarity of our scheme is that the underlying linear flow dynamics is not necessarily stabilizing. This last property is the consequence of taking into account where the flowing trajectories lay and lead to improved performance, especially in terms of decay rate. Some simulations illustrate the usefulness of our techniques.**

***Index Terms*—$\mathcal{H}_\infty$ performance, Hybrid Lyapunov function, Hybrid control**

I. Introduction

In recent years a lot of attention has been focused on feedback control involving a continuous-time plant interconnected with a controller exhibiting switching or resets (namely, hybrid behavior). The architecture of such controllers introduces a flexibility able to overcome some fundamental limitations of linear control (see [3], [15], [21]) and improve the performance of linear systems (see [7]–[9], [22], [23]). In particular, [2], [7], [24] show how introducing resets on the controller state can significantly decrease the $\mathcal{L}_2$ gain between the perturbation and the performance output. Moreover, [9], [23] show how resets may improve the closed-loop performance in terms of overshoot reduction.

In this context, a fundamental issue studied in recent years concerns the synthesis of a reset controller. [22], [23] provide convex optimization-based syntheses of a hybrid controller that are also compatible with the control schemes in [9], [10]. Nevertheless, these synthesis strategies assume that the continuous-time map of the reset controller be given, so that only the reset part needs to be designed. Note that this approach of augmenting a given flow map with a hybrid loop in the attempt to achieve stability and/or to improve performance has been widely used in reset control since the FORE architecture (see, for instance, [1], [3], [5]).

The problem of simultaneous design of all the components of a hybrid controller (namely, flow and jump sets and flow and jump maps) is challenging. The main difficulty of this synthesis comes from matching the constraints between the Lyapunov function and the controller architecture, in order to obtain convex conditions. At present, besides the result presented here, whose preliminary results were given in [7], the only other attempt of optimization-based synthesis of an $\mathcal{H}_\infty$ reset controller for a linear plant is in [24]. As compared to the results in [7], we include a more general construction where the underlying linear continuous-time dynamics (before resets) may be exponentially unstable.

In this paper, we present synthesis results for a multi-objective $\mathcal{H}_\infty$ reset controller providing an optimality-based trade-off between the decay rate and the $\mathcal{L}_2$ gain for a linear continuous-time plant. We rely on the results in [11] which extend the work in [20], providing relaxed Lyapunov-based conditions to estimate an $\mathcal{L}_2$ gain bound for a class of hybrid control systems. Note that this class is wide and includes several works in the literature (notably [9], [10], [19], [27]). The main idea is to combine the reset controller architecture in [9] and the analysis results in [11] by means of a suitable change of coordinates, in order to obtain convex synthesis conditions. As an improvement of our preliminary results of [7], the hybrid controller architecture in this paper allows us to design an $\mathcal{H}_\infty$ reset controller whose flow map is not necessarily stabilizing in the whole state space. This fact, as already noted in [27], enhances the potential of hybrid control, where divergent trajectories, suitably combined by way of resets can characterize stability and fast convergence. We mention also that [24] provides conditions to design a complete $\mathcal{H}_\infty$ reset controller, although the synthesis is single-objective with respect to $\mathcal{L}_2$ gain.

The paper is structured as follows. Some notation and definitions with references are given in Section II. Section III introduces the hybrid controller architecture and the problem we address. Section IV presents the main synthesis result. The effectiveness of our technique is shown through simulations, comparatively to linear designs, in Section V. The proofs of the main results are gathered in Section VI. Finally some concluding remarks complete the paper.

Work supported by ANR project LimICOS, contract number 12BS0300501 and by grant OptHySYS funded by the University of Trento.

F. Fichera, S. Tarbouriech and L. Zaccarian are with LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France (email: {ffichera,tarbour,zaccarian}@laas.fr)

L. Zaccarian is also with the Dip. di Ingegneria Industriale, University of Trento, Italy.

C. Prieur is with Department of Automatic Control, Gipsa-lab, Department of Automatic Control, Grenoble Campus, 11 rue des Mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France (email: christophe.prieur@gipsa-lab.fr)

II. Notation and preliminaries

The notation is standard. The Euclidean norm of a vector is denoted by $|\cdot|$. If $\mathcal{A}$ is a compact set, the notation $|x|_\mathcal{A} = \min\{|x-y| : y \in \mathcal{A}\}$ indicates the distance of the vector x from the set $\mathcal{A}$. If $\mathcal{A}$ is the origin then $|x|_\mathcal{A} = |x|$. For any $s \in \mathbb{R}$, the function $\mathrm{dz} : \mathbb{R} \to \mathbb{R}$ is defined by $\mathrm{dz}(s) = 0$ if $|s| \leq 1$ and $\mathrm{dz}(s) = \mathrm{sgn}(s)(|s|-1)$ if $|s| \geq 1$. Given a matrix Q, $\mathrm{He}(Q) = Q + Q^\top$. Moreover, $\lambda_{min}(Q)$ (respectively, $\lambda_{max}(Q)$) denotes the minimum (respectively, the maximum) eigenvalue of Q. The symbol $\otimes$ denotes the Kronecker product.

For an introduction of the framework of hybrid systems see [13].

Definition 1:

i. ***(t-decay rate)*** *Given a hybrid system, a compact set $\mathcal{A} \subset \mathbb{R}^n$ is uniformly globally exponentially stable with t-decay rate $\lambda > 0$ if there exists a strictly positive real number k such that each solution x satisfies*

$$|x(t,j)|_\mathcal{A} \leq k \exp(-\lambda t)|x(0,0)|_\mathcal{A}, \ \forall (t,j) \in \mathrm{dom}(x), \quad (1)$$

where $\mathrm{dom}(x)$ denotes the hybrid time domain of the solution x.

ii. ***(t-$\mathcal{L}_2$ norm of a hybrid signal)*** *For a hybrid signal w, with domain $\mathrm{dom}(w) \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, the t-$\mathcal{L}_2$ norm of w is given by*

$$\|w\|_{2t} = \left(\sum_{j \in \mathrm{dom}_j(w)} \int_{t_j}^{t_{j+1}} |w(t,j)|^2 dt \right)^{\frac{1}{2}}, \quad (2)$$

where $\mathrm{dom}_j(w) := \{j \in \mathbb{Z}_{\geq 0} : (t,j) \in \mathrm{dom}(w)$ for some $t \geq 0\}$ and with t_{j+1} possibly being ∞ if $j \in \mathrm{dom}_j(w)$ and $(j+1) \notin \mathrm{dom}_j(w)$.

iii. ***($w \in$ t-$\mathcal{L}_2$)*** *For a hybrid signal w, with domain $\mathrm{dom}(w) \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, we say $w \in$ t-$\mathcal{L}_2$ whenever $\|w\|_{2t} < \infty$. Moreover, for any pair $t_1 \geq t_2$ such that $t_1, t_2 \in \mathrm{dom}_t(w)$, we use $\|w\,[t_1,t_2]\|_2$ to denote the restriction of (2) to the corresponding subdomain.*

$\Diamond$

III. Problem statement

Consider a linear continuous-time plant $\mathcal{P}$, represented by

$$\begin{array}{rcl} \dot{x}_p & = & \bar{A}_p x_p + \bar{B}_p u + \bar{B}_w w \\ z & = & \bar{C}_z x_p + \bar{D}_z u + \bar{D}_{zw} w \\ y & = & \bar{C}_p x_p + \bar{D}_p u + \bar{D}_w w \end{array} \tag{3}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state of the plant, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output (used for the feedback), $w \in \mathbb{R}^{n_w}$ is an exogenous input (comprising disturbances and references) and $z \in \mathbb{R}^{n_z}$ is the performance output.

To keep the presentation simple, we avoid algebraic loops by making the following typical assumption.

Assumption 1: Plant (3) is strictly proper from u to y, namely $\bar{D}_p = 0$. ∘

The reset controller architecture $\mathcal{H}_c$ that we propose is given by

$$\begin{array}{l} \left\{ \begin{array}{rcl} \dot{x}_c & = & \bar{A}_c x_c + \bar{B}_c y \\ \dot{\tau} & = & 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) \end{array} \right. \quad (x,\tau) \in \mathcal{C}, \\ \left\{ \begin{array}{rcl} x_c^+ & = & K_p x_p \\ \tau^+ & = & 0 \end{array} \right. \quad (x,\tau) \in \mathcal{D}, \\ \quad u = \bar{C}_c x_c + \bar{D}_c y, \end{array} \tag{4a}$$

where $x_c \in \mathbb{R}^{n_c}$ and $\tau \in [0, 2\rho]$ is the dwell-time logic and the flow and jump sets $\mathcal{C}$ and $\mathcal{D}$ are respectively, defined as

$$\mathcal{C} := \{(x,\tau) : x \in \mathcal{F}\} \cup \{(x,\tau) : \tau \in [0,\rho]\}, \tag{4b}$$

$$\mathcal{D} := \{(x,\tau) : x \in \mathcal{J}\} \cap \{(x,\tau) : \tau \in [\rho, 2\rho]\}, \tag{4c}$$

with $\mathcal{F}$ and $\mathcal{J}$ symmetric cones defined by

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n : x^\top M x \le 0 \right\}, \tag{4d}$$

$$\mathcal{J} := \left\{ x \in \mathbb{R}^n : x^\top M x \ge 0 \right\} \tag{4e}$$

where $M = M^\top \in \mathbb{R}^{n \times n}$ is defined as

$$M := \mathrm{He}\left(PA + \frac{\tilde{\alpha}}{2} P \right), \tag{4f}$$

with A, $\tilde{\alpha}$ and P to be defined. The basic idea behind the dwell-time logic is to separate jumps by at least ρ amount of time (since $\dot{\tau} = 1$ if $\tau \in [0,\rho]$, $\tau = 0$ after a jump, and $\tau \ge \rho$ before a jump). See [13, Section 2.4] for further properties on dwell-time logic.

The feedback interconnection between $\mathcal{H}_c$ and $\mathcal{P}$ gives the following hybrid closed-loop system

$$\begin{array}{l} \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bw \\ \dot{\tau} & = & 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) \end{array} \right. \quad (x,\tau) \in \mathcal{C} \\ \left\{ \begin{array}{rcl} x^+ & = & Gx \\ \tau^+ & = & 0 \end{array} \right. \quad (x,\tau) \in \mathcal{D} \\ \quad z = C_z x + D_{zw} w \\ \quad y = C_p x + D_{pw} w \end{array} \tag{5a}$$

with $x = [x_p^\top \; x_c^\top]^\top \in \mathbb{R}^{n = n_p + n_c}$, and with and the selections:

$$\left(\begin{array}{c|c} A & B \\ \hline G & - \\ \hline - & M \\ \hline C_z & D_{zw} \\ \hline C_p & D_{pw} \end{array} \right) = \left(\begin{array}{cc|c} A_p & B_p & B_{pw} \\ B_c & A_c & B_{cw} \\ \hline \multicolumn{2}{c|}{G} & - \\ \hline \multicolumn{2}{c|}{-} & M \\ \hline \multicolumn{2}{c|}{C_z} & D_{zw} \\ \hline \multicolumn{2}{c|}{C_p} & D_{pw} \end{array} \right)$$

$$= \left(\begin{array}{cc|c} \bar{A}_p + \bar{B}_p \bar{D}_c \bar{C}_p & \bar{B}_p \bar{C}_c & \bar{B}_w + \bar{B}_p \bar{D}_c \bar{D}_w \\ \bar{B}_c \bar{C}_p & \bar{A}_c & \bar{B}_c \bar{D}_w \\ \hline I & 0 & - \\ K_p & 0 & - \\ \hline - & - & \mathrm{He}\left(PA + \frac{\tilde{\alpha}}{2} P\right) \\ \hline \bar{C}_z + \bar{D}_z \bar{D}_c \bar{C}_p & \bar{D}_z \bar{C}_c & \bar{D}_{zw} + \bar{D}_z \bar{D}_c \bar{D}_w \\ \hline \bar{C}_p & 0 & \bar{D}_w \end{array} \right). \tag{5b}$$

Notice that the parameters to design are the matrices $\bar{A}_c$, $\bar{B}_c$, $\bar{C}_c$, $\bar{D}_c$, K_p, P and the positive scalars $\tilde{\alpha}$ and ρ. As compared to [9], we use the same reset controller architecture to propose a multi-objective simultaneous synthesis optimizing the t-decay rate and the t-$\mathcal{L}_2$ performance introduced in Definition 1. Note that in [9], the proposed optimization-based synthesis for overshoot reduction only concerned the design of the reset loop. In other words for any given flow map (namely, matrix A is given), a solution was proposed to design flow and jump sets and the jump map (namely, M and K_p in (4)) to achieve global exponential stability of the origin, guaranteeing overshoot reduction. However, controller matrices $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$ were not part of the design. Here instead, similar to [7], [24], we propose an optimization-based synthesis to completely design the $\mathcal{H}_\infty$ reset controller, that is, flow and jump maps and flow and jump sets altogether.

IV. Main results

The following theorem states sufficient conditions for an optimization-based design of the $\mathcal{H}_\infty$ reset controller (4) with respect to the t-decay rate $\tilde{\alpha}$ and the t-$\mathcal{L}_2$ gain γ, introduced in Definition 1. In particular, the theorem provides an almost convex procedure to design a plant-order $\mathcal{H}_\infty$ reset controller. The result is proved by merging the exponential stability results in [9], from which a t-decay rate can be inferred, and the t-$\mathcal{L}_2$ analysis in [11, Proposition 1]. The details of the proof are reported in Section VI.

Theorem 1: Given plant (3) satisfying Assumption 1 and any set of matrices $Y = Y^\top \in \mathbb{R}^{n_p \times n_p}$, $W = W^\top \in \mathbb{R}^{n_p \times n_p}$, $\hat{A} \in \mathbb{R}^{n_p \times n_p}$, $\hat{B} \in \mathbb{R}^{n_p \times n_y}$, $\hat{C} \in \mathbb{R}^{n_u \times n_p}$, $\hat{D} \in \mathbb{R}^{n_u \times n_y}$, positive scalars $\bar{\gamma}$, α and a nonnegative scalar $\tau_S \ge 0$ satisfying

$$\begin{bmatrix} Y & I \\ I & W \end{bmatrix} > 0, \tag{6}$$

$$\mathrm{He}\left(\bar{A}_p Y + \bar{B}_p \hat{C} \right) + \alpha Y < 0, \tag{7}$$

and (10) (at the top of next page) for some $\tilde{\alpha} \in (0, \alpha]$, select the controller parameters as:

$$\begin{array}{rcl} P & = & \begin{bmatrix} W & -W \\ -W & W + (Y - W^{-1})^{-1} \end{bmatrix}, \\ K_p & = & (Y - W^{-1}) Y^{-1}, \\ \bar{D}_c & = & \hat{D}, \\ \bar{C}_c & = & (\hat{C} - \bar{D}_c \bar{C}_p Y)(Y - W^{-1})^{-1}, \\ \bar{B}_c & = & -W^{-1} \hat{B} + \bar{B}_p \bar{D}_c, \\ \bar{A}_c & = & -W^{-1}(\hat{A} + W \bar{B}_c \bar{C}_p Y - W \bar{B}_p \bar{C}_c (Y - W^{-1}) \\ & & -W(\bar{A}_p + \bar{B}_p \bar{D}_c \bar{C}_p) Y)(Y - W^{-1})^{-1}. \end{array} \tag{8}$$

Then, there exists $\bar{\rho} > 0$ such that for any $\rho \in (0, \bar{\rho})$:

- **t-decay rate:** the set
$$\mathcal{A} = \{0\} \times [0, 2\rho] \subset \mathbb{R}^n \times [0, 2\rho].$$
is globally exponentially stable for the hybrid closed-loop system (5) with $w = 0$, and the t-decay rate is $\tilde{\alpha}/2$;
- **$\mathcal{H}_\infty$ specification:** for any $w \in t$-$\mathcal{L}_2$, the t-$\mathcal{L}_2$ gain from w to z is smaller than or equal to
$$\gamma = \min\{\bar{\gamma}, \sqrt{2}|D_{zw}|\}. \tag{9}$$

□

Theorem 1 gives an LMI-based convex procedure with a line-search on $\tau_S \ge 0$ to design an $\mathcal{H}_\infty$ reset controller. Note that the line-search must be carried out to get convexity of the optimization. Indeed, (10) becomes an LMI after fixing τ_S. The $(\alpha, \bar{\gamma})$ trade-off in our design can be addressed by fixing $\tilde{\alpha} = \alpha > 0$ and solving an eigenvalue problem minimizing $\bar{\gamma}$. Then the t-decay rate is fixed to be $\tilde{\alpha}/2$

$$\mathrm{He}\left(\left[\begin{array}{cc|cc} (1-\tau_S)(\bar{A}_pY+\bar{B}_p\hat{C})-\frac{\tau_S\tilde{\alpha}}{2}Y & (1-\tau_S)(\bar{A}_p+\bar{B}_p\hat{D}\bar{C}_p)-\frac{\tau_S\tilde{\alpha}}{2}I & \bar{B}_w+\bar{B}_p\hat{D}\bar{D}_w & 0 \\ (1-\tau_S)\hat{A}-\frac{\tau_S\tilde{\alpha}}{2}I & (1-\tau_S)(W\bar{A}_p+\hat{B}\bar{C}_p)-\frac{\tau_S\tilde{\alpha}}{2}W & W\bar{B}_w+\hat{B}\bar{D}_w & 0 \\ \hline 0 & 0 & -\frac{\bar{\gamma}}{2}I & 0 \\ \bar{C}_zY+\bar{D}_z\hat{C} & \bar{C}_z+\bar{D}_z\hat{D}\bar{C}_p & \bar{D}_{zw}+\bar{D}_z\hat{D}\bar{D}_w & -\frac{\bar{\gamma}}{2}I \end{array}\right]\right) < 0. \tag{10}$$

and the t-$\mathcal{L}_2$ gain can be minimized. It may sometimes be desirable to pick $\tilde{\alpha}$ smaller than α to induce longer times between pairs of consecutive resets.

Note that whenever $\tau_S = 0$, the flow set does not appear in (10) so that the second term in (9) is not needed, and the t-$\mathcal{L}_2$ gain is $\gamma = \bar{\gamma}$. This is the approach that we followed in our preliminary work [7]. The choice of $\tau_S = 0$ is, however, conservative because, in this case, it is shown that the Lyapunov flow condition holds in all the state space. Whenever $\tau_S > 0$, condition (9) needs to be satisfied. Furthermore several different scenario can be characterized, according to the value of τ_S in (10) and $\tilde{\alpha} = 0$:

- $0 \le \tau_S < 1$ implies that A in (5b) is Hurwitz, namely the linear dynamics before resets is exponentially stable (that is, the flow map of the $\mathcal{H}_\infty$ reset controller stabilizes the continuous-time loop);
- $\tau_S = 1$ implies that A is not necessarily Hurwitz;
- $\tau_S > 1$ implies that A is non Hurwitz and interesting closed-loop responses exhibiting exponentially diverging branches might be observed (see [27]), because the linear dynamics before resets is exponentially unstable.

An important weakness of the proposed reset $\mathcal{H}_\infty$ architecture (4) is that despite the typical output feedback structure of the flow dynamics, both the jump/flow sets and the jump rule depend on the availability of the complete plant state x_p. This drawback was already recognized in [10] where a generic continuous-time Luenberger observer structure is shown to satisfy a separation principle for an output feedback implementation of (4) under a less general structure than that of closed loop (5). A parallel result to that of [10, Thms 1 & 2] is given by the following proposition, whose proof uses a different approach from [10] more similar in nature to the results in [26].

Proposition 1: Consider plant (3) in feedback from a hybrid controller of the form (4), where state x appearing in (4b)–(4e) is replaced by $x = \begin{bmatrix}\hat{x}_p \\ x_c\end{bmatrix}$ and the plant state estimate $\hat{x}_p$ arises from the continuous-time Luenberger observer having flow equation:

$$\dot{\hat{x}}_p = (\bar{A}_p - L\bar{C}_p)\hat{x}_p + (\bar{B}_p - L\bar{D}_p)u + Ly, \tag{11}$$

and jump equation $\hat{x}_p^+ = \hat{x}_p$. If matrix $\bar{A}_p - L\bar{C}_p$ is Hurwitz, then the corresponding output feedback closed loop guarantees global exponential stability of the attractor $\mathcal{A}_e := \{(x_p, x_c, \hat{x}_p, \tau) : (\hat{x}_p, x_c, \tau) \in \mathcal{A} \text{ and } x_p - \hat{x}_p = 0\}$. □

Note that the result of Proposition 1 does not allow to carry forward the performance properties established in Theorem 1 from the state-feedback case to the output feedback case. Nevertheless, similar results to those in [10, Prop. 1 & 2] could be proven for our case too, thus establishing some tight relation between the solutions to the state feedback and output feedback cases. Extensions to a convex observer-free reset $\mathcal{H}_\infty$ full output feedback design is regarded as future work.

V. Simulations

Let us consider the example of a DC motor compared to the corresponding linear classical multi-objective case [25].

In order to avoid fast exponential branches that may damage the actuator or require excessive bandwidth in our control systems, we exploit the advantages of our LMI formulation by adding the following extra constraints to our syntheses (and also to the corresponding linear designs):

$$-2\beta_1 \otimes X - \mathrm{He}\,(AX) < 0, \tag{12a}$$

$$-2\beta_2 \sin(\theta)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \sin(\theta)I & \cos(\theta)I \\ -\cos(\theta)I & \sin(\theta)I \end{bmatrix} \otimes AX + \begin{bmatrix} \sin(\theta)I & -\cos(\theta)I \\ \cos(\theta)I & \sin(\theta)I \end{bmatrix} \otimes (AX)^\top < 0, \tag{12b}$$

where X and AX in our change of coordinates (see the proof of Theorem 1 in Section VI and also [6], [25]) are given by:

$$X := \begin{bmatrix} Y & I \\ I & W \end{bmatrix}, \; AX := \begin{bmatrix} \bar{A}_pY + \bar{B}_p\hat{C} & \bar{A}_p + \bar{B}_p\hat{D}\bar{C}_p \\ \hat{A} & W\bar{A}_p + \hat{B}\bar{C}_p \end{bmatrix},$$

and correspond to the Lyapunov matrix and the closed-loop dynamical matrix, respectively. Enforcing the constraints in (12) guarantees that the poles of the closed-loop feedback (or the continuous-part of the feedback for the reset case) lay in a polynomial region [11]. We consider $\beta_1 = 50$, $\beta_2 = 25$ and $\theta = \pi/30$ as reasonable values. All the design is performed by means of *YALMIP* [16].

In this example, already used in [7], we want to present the synthesis of a multi-objective $\mathcal{H}_\infty$ reset controller compared to a multi-objective linear $\mathcal{H}_\infty$ controller. Moreover according to [10], we augment the $\mathcal{H}_\infty$ reset controller with an observer in order to have a complete output feedback (where also the resets depend on an estimate of the plant state provided by the observer) and we will use the t-$\mathcal{L}_2$ analysis in [11] to estimate the new t-$\mathcal{L}_2$ gain for the arising hybrid closed loop comprising the $\mathcal{H}_\infty$ reset controller and the observer.

According to (3), let us first introduce the plant:

$$\left[\begin{array}{c|c|c} \bar{A}_p & \bar{B}_p & \bar{B}_w \\ \hline \bar{C}_z & \bar{D}_z & \bar{D}_{zw} \\ \hline \bar{C}_p & \bar{D}_p & \bar{D}_w \end{array}\right] = \left[\begin{array}{cc|c|c} -2.4 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \\ \hline 0 & 1 & 10 & 0 \\ \hline 0 & 1 & 0 & 5 \end{array}\right].$$

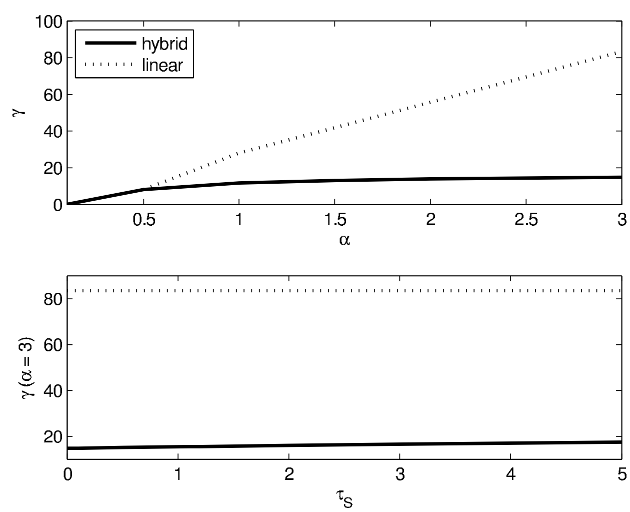
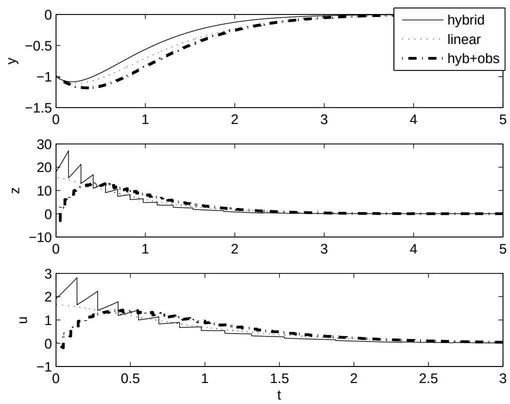
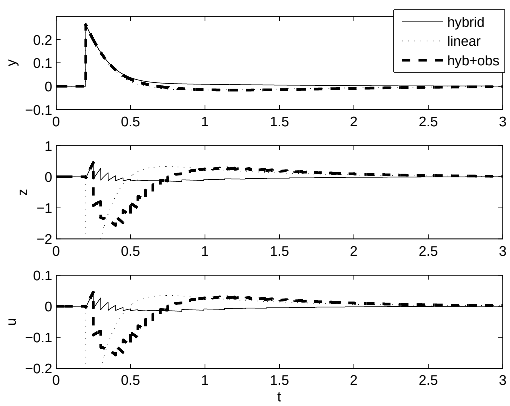


Figure 1. Comparison reset $\mathcal{H}_\infty$ and linear $\mathcal{H}_\infty$ control feedback for the DC motor.



(a) Free response $x_p(0,0) = (-1, -1)$ (*i.e.*, $w = 0$)



(b) Response with noise $w \in t\text{-}\mathcal{L}_2$.

Figure 2. Simulations for the DC motor.

The top of Figure 1 shows the t-$\mathcal{L}_2$ gain obtained for the reset and linear case for a given decay-rate α. Similar to [7], the reset controller guarantees lower t-$\mathcal{L}_2$ gains than the linear case, as the the decay-rate increases. Unlike [7], the design strategy in Section IV allows us to design an $\mathcal{H}_\infty$ reset controller through a line-search on $\tau_S \geq 0$ (see (10)). The bottom of Figure 1 shows the t-$\mathcal{L}_2$ gains obtained with the hybrid synthesis for $\alpha = 3$ and for $\tau_S \in [0, 5]$. Although the t-$\mathcal{L}_2$ gain increases with τ_S, for $\tau_S \geq 1$ we have $\mathcal{H}_\infty$ reset controllers with nonstabilizing continuous-time part. Indeed, Figure 2(a) shows the behavior of the closed-loop system with the $\mathcal{H}_\infty$ reset controller obtained with $\alpha = \tilde{\alpha} = 3$, $\tau_S = 5$ and $\rho = 5 \cdot 10^{-2}$. Since both the linear and the reset synthesis are designed imposing $\alpha = 3$, we do not have any guarantee that the $\mathcal{H}_\infty$ reset controller leads to faster responses. Nevertheless, the fact that the flow map is unstable requires the action of the reset part to mitigate the unstable modes and to induce exponential stability, with the interesting effects of showing a faster decay rate than the linear case even though the same speed of convergence was imposed by design. In particular according to (4), the synthesis returns the following controller (where M has been divided by its determinant)

$$\left[\begin{array}{c|c} \bar{A}_c & \bar{B}_c \\ \hline K_p & - \\ \hline \bar{C}_c & \bar{D}_c \end{array}\right] = \left[\begin{array}{cc|c} 1.51871 & -1.82471 & 2.17031 \\ 0.89613 & 0.67999 & -0.75037 \\ \hline -0.27132 & -0.87136 & - \\ 0.60395 & 1.41394 & - \\ \hline 1.85009 & 0.11383 & -0.01509 \end{array}\right],$$

$$M = \begin{bmatrix} 0.00579 & 0.01221 & -0.00579 & -0.01221 \\ 0.01221 & 0.02569 & -0.01221 & -0.02569 \\ -0.00579 & -0.01221 & 0.00579 & 0.01221 \\ -0.01221 & -0.02569 & 0.01221 & 0.02569 \end{bmatrix}.$$

Figure 2 contains also the hybrid output feedback case obtained by applying the Luenberger observer (11) according to Proposition 1. The scheme corresponds to replacing x_p by $\hat{x}_p$ in (4) (flow and jump sets included), where $\hat{x}_p$ is the estimated state coming from (11), where the observer gain $L = \begin{bmatrix} 1.5 & 5.7 \end{bmatrix}^\top$ has been selected to guarantee a suitable convergence rate of the estimation error.

Notice that the multi-objective nature of the synthesis is lost once the observer is introduced because the t-decay rate is no longer guaranteed. Nevertheless we can use the analysis in [11] to estimate the t-$\mathcal{L}_2$ gain of the new hybrid system. By applying [11, Proposition 1], we obtain that the new t-$\mathcal{L}_2$ gain for the hybrid output feedback is $\gamma = 102.86$. Clearly, an increase of the t-$\mathcal{L}_2$ gain is to be expected, as compared to the state feedback case, nevertheless through [11, Proposition 1] we are able to still establish an upper bound. Figure 2 shows a desirable behavior of the $\mathcal{H}_\infty$ reset controller, although the case with the observer (bold dashed dot line) is closer to the linear response. The external disturbance w is chosen as $w(t) = \exp(-10t)\sin(2t)$, for all $t \geq 0.2$ and zero otherwise.

VI. Proofs

The next lemma is useful for the proof of Theorem 1 and is a straightforward generalization of [9, Theorem 1].

Lemma 1: Consider plant (3) under Assumption 1, the reset controller (4) and their interconnection (5). If there exists a matrix $P = P^\top = \begin{bmatrix} P_p & P_{pc} \\ P_{pc}^\top & P_c \end{bmatrix} > 0$ such that

$$\text{He}\left(\bar{P}_p(A_p + B_p K_p) + \frac{\alpha}{2}\bar{P}_p\right) < 0, \tag{13a}$$

$$\bar{P}_p = P_p - P_{pc}P_c^{-1}P_{pc}^\top > 0, \qquad K_p = -P_c^{-1}P_{pc}^\top, \tag{13b}$$

for some $\alpha > 0$, then for any $\tilde{\alpha} \in (0, \alpha]$ there exists $\overline{\rho} > 0$ such that for all $\rho \in (0, \overline{\rho})$, the Lyapunov function $x \mapsto V(x) = x^\top P x$ satisfies the following properties:

$$\Delta V(x) \leq 0, \quad \forall x \in \mathbb{R}^n, \tag{14a}$$

$$Gx \in \tilde{F} \subset \mathcal{F}, \quad \forall x \in \mathbb{R}^n, \tag{14b}$$

where $\tilde{\mathcal{F}} := \left\{x : x^\top(M + \epsilon I)x \leq 0\right\}$, with M in (4f) (and defining set $\mathcal{F}$), $\epsilon = -\frac{\lambda_{max}(\Xi)}{|I + K_p^\top K_p|}$, with $\Xi := \text{He}\left(\bar{P}_p(A_p + B_p K_p) + \frac{\tilde{\alpha}}{2}\bar{P}_p\right)$. Moreover there exists $K > 0$ such that for all $\xi(t_0, 0) = (x(t_0, 0), \tau(t_0, 0)) \in \mathbb{R}^n \times [0, 2\rho]$, we have

$$V(x(t,j)) \leq \frac{a_1}{a_2}K^2 \exp(-\tilde{\alpha}(t - t_0))V(x(t_0, 0)), \tag{15}$$

for all $(t, j) \in \text{dom}(\xi)$, where $a_1 := \lambda_{min}(P)$ and $a_2 := \lambda_{max}(P)$. $\Diamond$

Proof of Lemma 1. First notice that

$$\lambda_{min}(P)|x|^2 \leq V(x) \leq \lambda_{max}(P)|x|^2, \quad \forall x \in \mathbb{R}^n.$$

Furthermore, we have $\langle \nabla V(x), Ax \rangle = x^\top \text{He}(PA)x$, thus the flow and jump sets defined in (4d) and (4e) by means of M in (4f) can be rewritten as

$$\mathcal{F} = \{x \in \mathbb{R}^n : \langle \nabla V(x), Ax \rangle + \tilde{\alpha}V(x) \leq 0\}, \tag{16a}$$

$$\mathcal{J} = \{x \in \mathbb{R}^n : \langle \nabla V(x), Ax \rangle + \tilde{\alpha}V(x) \geq 0\}. \tag{16b}$$

Define $V_p(x_p) = x_p^\top \bar{P}_p x_p$, then from the definitions of P, G (see (5b)) and (13b), we have $\bar{P}_p = \bar{P}_p^\top > 0$ and

$$V(x^+) = V(Gx) = x^\top G^\top P G x$$

$$= x^\top \begin{bmatrix} I & K_p^\top \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_p & P_{pc} \\ P_{pc}^\top & P_c \end{bmatrix} \begin{bmatrix} I & 0 \\ K_p & 0 \end{bmatrix} x$$
$$= x_p^\top (P_p + K_p^\top P_{pc}^\top + P_{pc}K_p + K_p^\top P_c K_p) x_p$$
$$= x_p^\top \bar{P}_p x_p = V_p(x_p), \tag{17}$$

for all $x \in \mathbb{R}^n$.

Consider the Lyapunov function $x \mapsto V(x)$ at jumps. By using (13b) and (17), we have

$$\Delta V(x) = V(x^+) - V(x)$$
$$= x_p^\top \bar{P}_p x_p - x_p^\top P_p x_p - 2x_p^\top P_{pc} x_c - x_c^\top P_c x_c$$
$$= \begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top \begin{bmatrix} -P_{pc}P_c^{-1}P_{pc}^\top & -P_{pc} \\ -P_{pc}^\top & -P_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} \leq 0,$$

for all $x \in \mathbb{R}^n$, where last inequality is obtained by applying a Schur complement (see [4, pag. 28]) and implies (14a).

Now recall that $x = (x_p, x_c)$ and $x^+ = (x_p, K_p x_p)$ and notice that $|Gx|^2 = x_p^\top (I + K_p^\top K_p) x_p \leq |I + K_p^\top K_p| |x_p|^2$. From the definition of $\tilde{\mathcal{F}}$ in the statement, we have $\tilde{M} = M + \epsilon I = \mathrm{He}\left(PA + \frac{\tilde{\alpha}}{2}P\right) + \epsilon I$ and by noticing that $P_p + K_p^\top P_{pc}^\top = \bar{P}_p$ and $P_{pc} + K_p^\top P_c = 0$, we have

$$x^\top G^\top \tilde{M} G x = x^\top G^\top (M + \epsilon I) G x$$
$$= x^\top \left(\mathrm{He}\left(\begin{bmatrix} \bar{P}_p(A_p + B_p K_p) + \frac{\tilde{\alpha}}{2}\bar{P}_p & 0 \\ 0 & 0 \end{bmatrix} \right) + \epsilon \begin{bmatrix} I + K_p^\top K_p & 0 \\ 0 & 0 \end{bmatrix} \right) x$$
$$= \begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top \begin{bmatrix} \Xi + \epsilon(I + K_p^\top K_p) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix}$$
$$\leq \begin{bmatrix} x_p \\ x_c \end{bmatrix}^\top \begin{bmatrix} (\lambda_{max}(\Xi) + \epsilon |I + K_p^\top K_p|) I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix}$$
$$= 0, \quad \forall x \in \mathbb{R}^n, \tag{18}$$

which implies $Gx \in \tilde{\mathcal{F}}$, for all $x \in \mathcal{J}$ and $\tilde{\mathcal{F}} \subset \mathcal{F}$, whenever $\epsilon = -\frac{\lambda_{max}(\Xi)}{|I + K_p^\top K_p|} > 0$. Therefore also condition (14b) is satisfied.

Consider the Lyapunov function $x \mapsto V(x)$ during flow. It is straightforward from (16a) to have

$$\langle \nabla V(x), Ax \rangle \leq -\tilde{\alpha} V(x), \quad \forall x \in \mathcal{F}. \tag{19}$$

Now consider a generic solution ξ with its hybrid time domain $(t, j) \in \mathrm{dom}(\xi)$. Notice that due to the dwell time, we have $t_{i+1} - t_i \geq \rho > 0$, for all $i \in \mathbb{Z}_{\geq 1}$.

By applying [11, Claim 1] there exists $\overline{\rho} > 0$ such that for all $\rho \in (0, \overline{\rho})$, $x(t, i) \in \mathcal{F}$ for all $t \in [t_i, t_{i+1}]$, $i \in \mathbb{Z}_{\geq 1}$. Therefore from (19), we have

$$V(x(t, i)) \leq \exp(-\tilde{\alpha}(t - t_i)) V(x(t_i, i)), \tag{20}$$

for all $t \in [t_i, t_{i+1}]$, $i \in \mathbb{Z}_{\geq 1}$.

Regarding the interval $[t_0, t_1]$, we consider two subcases: $t \in [t_0, t_0 + \rho]$ and $t \in (t_0 + \rho, t_1]$.

Case i: $t \in [t_0, t_0 + \rho]$. From $|\dot{x}| \leq |A||x|$, one has $|x(t, i)|^2 \leq \exp(2|A|(t - t_i))|x(t_i, i)|^2$ for all $t \in [t_i, t_{i+1}]$, $i \in \mathbb{Z}_{\geq 0}$ and so also in the interval of interest. Therefore we get

$$V(x(t, 0)) \leq \frac{a_2}{a_1} \exp(2|A|(t - t_0)) V(x(t_0, 0)),$$

where $a_1 := \lambda_{min}(P)$ and $a_2 := \lambda_{max}(P)$.

Case ii: $t \in (t_0 + \rho, t_1]$. Because the system is flowing, we have $x(t, 0) \in \mathcal{F}$ for all $t \in (t_0 + \rho, t_1]$, therefore also (20) holds.

By combining the two subcases, one has

$$V(x(t, 0)) \leq \exp(-\tilde{\alpha}(t - t_0 - \rho)) V(x(t_0 + \rho, 0))$$
$$\leq \frac{a_2}{a_1} \exp(2|A|\rho) \exp(-\tilde{\alpha}(t - t_0 - \rho)) V(x(t_0, 0))$$
$$= \frac{a_2}{a_1} \exp((2|A| + \tilde{\alpha})\rho) \exp(-\tilde{\alpha}(t - t_0)) V(x(t_0, 0))$$
$$= \frac{a_1}{a_2} K^2 \exp(-\tilde{\alpha}(t - t_0)) V(x(t_0, 0)), \tag{22}$$

for all $t \in [t_0, t_1]$.

Finally, by combining (14a), (20) and (22), we have (15). This concludes the proof of Lemma 1. ■

Proof of Theorem 1. The proof is carried out by showing that conditions (10) and definitions (8) imply all the conditions of Lemma 1 and [11, Proposition 1], by using the same Lyapunov function $V(x) = x^\top P x$, with $P = P^\top > 0$.

In particular, consider the following partitioned matrix $P = P^\top = \left[\begin{smallmatrix} Y & Z \\ Z & Z \end{smallmatrix}\right]^{-1} > 0$. By applying the matrix inversion lemma in [14], we get

$$P = \begin{bmatrix} Y & Z \\ Z & Z \end{bmatrix}^{-1} = \begin{bmatrix} (Y - Z)^{-1} & -(Y - Z)^{-1} \\ -(Y - Z)^{-1} & Z^{-1} + (Y - Z)^{-1} \end{bmatrix}$$
$$:= \begin{bmatrix} W & -W \\ -W & W + Z^{-1} \end{bmatrix}, \tag{23}$$

which corresponds to the first of (8) (notice that since $W = (Y - Z)^{-1}$, we have also $Z = Y - W^{-1}$ or equivalently $Y = Z + W^{-1}$). Similarly to [17], [25], by pre- and post-multiplying (23) by $\Pi := \left[\begin{smallmatrix} Y & Z \\ I & 0 \end{smallmatrix}\right]$ (note that $\Pi P = \left[\begin{smallmatrix} I & 0 \\ W & -W \end{smallmatrix}\right]$), and its transpose, we get (6), which implies $P = P^\top > 0$.

By defining $R = W + Z^{-1}$ and by applying again the matrix inversion lemma (see [14]), we can establish the following useful identities after cumberstone derivations

$$R^{-1} = Z Y^{-1} (Y - Z). \tag{24}$$

Consider also the following definitions

$$\begin{aligned} \hat{A} &:= W(-\bar{A}_c Z - \bar{B}_c \bar{C}_p Y + \bar{B}_p \bar{C}_c Z + (\bar{A}_p + \bar{B}_p \bar{D}_c \bar{C}_p) Y), \\ \hat{B} &:= W(-\bar{B}_c + \bar{B}_p \bar{D}_c), \\ \hat{C} &:= \bar{C}_c Z + \bar{D}_c \bar{C}_p Y, \\ \hat{D} &:= \bar{D}_c, \end{aligned} \tag{25}$$

and notice that we retrieve (8) from (25) and vice versa.

Let us now show that all the conditions of Lemma 1 are satisfied. By imposing $P = \left[\begin{smallmatrix} P_p & P_{pc} \\ P_{pc}^\top & P_c \end{smallmatrix}\right] = \left[\begin{smallmatrix} W & -W \\ -W & W + Z^{-1} \end{smallmatrix}\right]$ and using (23) and last one in (24), we have $\bar{P}_p = P_p - P_{pc} P_c^{-1} P_{pc}^\top = W - W(W + Z^{-1})^{-1} W = W W^{-1} Y^{-1} = Y^{-1}$, and $K_p = -P_c^{-1} P_{pc}^\top = (Y - W^{-1}) Y^{-1}$, which imply both definitions in (13b) and the second definition of (8). Furthermore, by multiplying (7) on both sides by $\bar{P}_p = Y^{-1}$ and using (25) and (23), we get

$$\begin{aligned} \mathrm{He}(Y^{-1}(\bar{A}_p + \bar{B}_p \hat{C} Y^{-1})) + \alpha Y^{-1} &= \mathrm{He}(Y^{-1}(\bar{A}_p \\ &\quad + \bar{B}_p(\bar{C}_c(Y - W^{-1}) + \bar{D}_c \bar{C}_p Y) Y^{-1})) + \alpha Y^{-1} \\ &= \mathrm{He}(\bar{P}_p(A_p + B_p K_p)) + \alpha \bar{P}_p, \end{aligned} \tag{26}$$

from which we get (13a) and therefore Lemma 1 holds and hence also (14).

Notice that from (15) and the fact that $a_1|x|^2 \leq V(x) \leq a_2|x|^2$ with $a_1 := \lambda_{min}(P)$ and $a_2 := \lambda_{max}(P)$, we get

$$|x(t, j)|^2 \leq K \exp(-\frac{\tilde{\alpha}}{2}(t - t_0)) |x(t_0, 0)|^2, \quad (t, j) \in \mathrm{dom}(\xi),$$

which returns item i of Definition 1. This completes the proof of the first item of Theorem 1.

We want to prove now that (8) and (10) imply [11, Proposition 1]. First let us select $M = \tilde{M} - \epsilon I$ and notice that [11, (13c) in Proposition 1] is directly satisfied with $\tau_F = 1$. Moreover, conditions

$$\mathrm{He}\left(\left[\begin{array}{cc|cc} (1-\tau_S)(\bar{A}_pY+\bar{B}_p\hat{C})-\frac{\tau_S\tilde{\alpha}-a_3}{2}Y & (1-\tau_S)(\bar{A}_p+\bar{B}_p\hat{D}\bar{C}_p)-\frac{\tau_S\tilde{\alpha}-a_3}{2}I & \bar{B}_w+\bar{B}_p\hat{D}\bar{D}_w & 0 \\ (1-\tau_S)\hat{A}-\frac{\tau_S\tilde{\alpha}-a_3}{2}I & (1-\tau_S)(W\bar{A}_p+\hat{B}\bar{C}_p)-\frac{\tau_S\tilde{\alpha}-a_3}{2}W & W\bar{B}_w+\hat{B}\bar{D}_w & 0 \\ \hline 0 & 0 & -\frac{\gamma}{2}I & 0 \\ \bar{C}_zY+\bar{D}_z\hat{C} & \bar{C}_z+\bar{D}_z\hat{D}\bar{C}_p & \bar{D}_{zw}+\bar{D}_z\hat{D}\bar{D}_w & -\frac{\gamma}{2}I \end{array}\right]\right) < 0, \qquad (21)$$

(14) imply [11, (13b) in Proposition 1] with $\underline{\rho} = \tau_R = 0$ and [11, (13d) in Proposition 1] with $\tau_C = 0$, respectively.

Consider now condition [11, (13a) in Proposition 1], with P in (23), $\tilde{M} - \epsilon I = M = \mathrm{He}(PA) + \tilde{\alpha}P$. By pre- and post-multiplying [11, (13a) in Proposition 1] by $T := \left[\begin{smallmatrix} \Pi & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{smallmatrix}\right]$ and its transpose, and by using (25), condition [11, (13a) in Proposition 1] is equivalent to (21), which is implied by (10), since $a_3 > 0$ can be selected a posteriori due to the strict inequality. Therefore [11, Proposition 1] holds and this completes the proof of item ii and hence of the theorem. ■

Proof of Proposition 1. The proof is best carried out by looking at the error coordinates $e = x_p - \hat{x}_p$ and realizing that the dwell-time properties of solutions (arising from timer τ), together with the fact that $\bar{A}_p - L\bar{C}_p$ is Hurwitz, ensures that the closed set $\mathcal{M} := \{(x_p, x_c, \hat{x}_p, \tau) : \ x_p - \hat{x}_p = 0\} \supset \mathcal{A}_e$ is uniformly globally asymptotically stable. Such a property is certified by the Lyapunov function $e^T P_e e$, where P_e solves a Lyapunov equation for Hurwitz matrix $\bar{A}_p - L\bar{C}_p$. Then Theorem 1 can be applied to obtain global asymptotic stability of $\mathcal{A}_e$, relative to initial conditions in $\mathcal{M}$. These two nested stability properties can be used following a nested version of the reasoning after [12, Corollary 19], implying local asymptotic stability of $\mathcal{A}_e$. Finally, note that the homogeneity of the closed loop enables applying the result in [18, Thm 2] establishing that local asymptotic stability of $\mathcal{A}_e$ is equivalent to its global exponential stability. ■

VII. Conclusions

The synthesis of multi-objective hybrid system has been presented. The result exploits the property of a new reset controller presented in [9], and presents the advantage to preserve convexity (although with a line-search) whenever the flow and jump sets have to be taken into account. This method seems to be much more flexible than the optimization-based synthesis in [7] where a different $\mathcal{H}_\infty$ reset controller architecture was used. The new synthesis allows the design of an $\mathcal{H}_\infty$ reset controller whose continuous-time part does not stabilize the feedback. At the same time the numerical examples seem to suggest that t-$\mathcal{L}_2$ stability is better preserved whenever a hybrid controller with stabilizing continuous-time part is selected.

Further developments might interest the use of this new reset controller architecture to perform optimization-based synthesis with respect to other performance indexes.

References

[1] A. Banos, J. Carrasco, and A. Barreiro. Reset Times-Dependent Stability of Reset Control Systems. *IEEE Transactions on Automatic Control*, 56(1):217–223, 2011.

[2] O. Beker, C Hollot, and Y. Chait. Fundamental Properties of Reset Control Systems. *Automatica*, 40(6):905–915, 2004.

[3] O. Beker, C.V. Hollot, and Y. Chait. Plant with an Integrator: an Example of Reset Control Overcoming Limitations of Linear Feedback. *IEEE Transactions on Automatic Control*, 46(11):1797–1799, 2001.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* Society for Industrial and Applied Mathematics, 1994.

[5] Q. Chen, Y. Chait, and C.V. Hollot. Analysis of Reset Control Systems Consisting of a FORE and Second Order Loop. *J. Dynamic Systems, Measurement and Control*, 123:279–283, 2001.

[6] M. Chilali and P. Gahinet. H_∞ Design with Pole Placement Constraints: an LMI Approach. *IEEE Transactions on Automatic Control*, 41:358–367, 1996.

[7] F. Fichera, C. Prieur, S. Tarbouriech, and L. Zaccarian. A Convex Hybrid $\mathcal{H}_\infty$ Synthesis with Guaranteed Convergence Rate. In *Proceedings of the 51st Conference on Decision and Control*, pages 4217–4222, Maui (HI), USA, 2012.

[8] F. Fichera, C. Prieur, S. Tarbouriech, and L. Zaccarian. Improving the Performance of Linear Systems by Adding a Hybrid Loop: the Output Feedback Case. In *Proceedings of the 2012 American Control Conference*, pages 3192–3197, Montreal, Canada, 2012.

[9] F. Fichera, C. Prieur, S. Tarbouriech, and L. Zaccarian. On Hybrid State-feedback Loops Based on a Dwell-time Logic. In *4th IFAC Conference on Analysis and Design of Hybrid Systems*, pages 388–393, Eindhoven, The Netherlands, 2012.

[10] F. Fichera, C. Prieur, S. Tarbouriech, and L. Zaccarian. Using Luenberger Observers and Dwell-time Logic for Feedback Hybrid Loops in Continuous-time Control Systems. *International Journal of Robust and Nonlinear Control*, 23:1065–1086, 2013.

[11] F. Fichera, C. Prieur, S. Tarbouriech, and L. Zaccarian. LMI-based Reset $\mathcal{H}_\infty$ Analysis and Design for Linear Continuous-time Plants. http://hal.archives-ouvertes.fr/hal-01236972, 2015.

[12] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid Dynamical Systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.

[13] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability and Robustness.* Princeton University Press, 2012.

[14] G.H. Golub and C.F. Van Loan. *Matrix Computations*, volume 3. Johns Hopkins University Press, 2012.

[15] J.P. Hespanha, D. Liberzon, and A.S. Morse. Hysteresis-based Switching Algorithms for Supervisory. *Automatica*, 39(2):263–272, 2004.

[16] J. Löfberg. YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[17] I. Masubuchi, A. Ohara, and N. Suda. LMI-based Controller Synthesis: a Unified Formulation and Solution. *International Journal of Robust and Nonlinear Control*, 8(8):669–686, 1998.

[18] D. Nešić, A.R. Teel, G. Valmórbida, and L. Zaccarian. Finite gain $\mathcal{L}_p$ stability for hybrid dynamical systems. *Automatica*, 49(8):2384–2396, 2013.

[19] D. Nešić, A.R. Teel, and L. Zaccarian. Stability and Performance of SISO Control Systems with First Order Reset Elements. *IEEE Transactions on Automatic Control*, 56(11):2567–2582, 2011.

[20] D. Nešić, L. Zaccarian, and A.R. Teel. Stability Properties of Reset Systems. *Automatica*, 44(8):2019–2026, 2008.

[21] C. Prieur and A. Astolfi. Robust Stabilization of Chained Systems via Hybrid Control. *IEEE Transactions on Automatic Control*, 48(10):1768–1772, 2003.

[22] C. Prieur, S. Tarbouriech, and L. Zaccarian. Improving the Performance of Linear Systems by adding a Hybrid Loop. In *18th IFAC World Congress*, pages 6301–6306, Milano, Italy, September 2011.

[23] C. Prieur, S. Tarbouriech, and L. Zaccarian. Lyapunov-based hybrid loops for stability and performance of continuous-time control systems. *Automatica*, 49(2):577–584, 2013.

[24] A. Satoh. State Feedback Synthesis of Linear Reset Control with $\mathcal{L}_2$ Performance Bound via LMI Approach. In *18th IFAC World Congress*, pages 5860–5865, Milan, Italy, 2011.

[25] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective Output-feedback Control via LMI Optimization. *IEEE Transactions on Automatic Control*, 42(7):896–911, 1997.

[26] A.R. Teel. Observer-based hybrid feedback: a local separation principle. In *Proc. of the American Control Conference*, pages 898–903, June 2010.

[27] L. Zaccarian, D. Nešić, and A.R. Teel. Analytical and Numerical Lyapunov Functions for SISO Linear Control Systems with First-order Reset Elements. *International Journal of Robust and Nonlinear Control*, 21:1134–1158, 2011.