Singular perturbation approximation of linear hyperbolic systems of balance laws

Ying Tang, Christophe Prieur, and Antoine Girard

The paper is organized as follows. Section II presents the full system and the reduced subsystem under consideration. The Tikhonov approximation is given in Section III. Section IV shows the statement of the proof of the Tikhonov theorem. In Section V we first use an academic example to illustrate the general main result. Then, a physical application to a gas flow transport model is shown in the same section. The conclusions are given in Section VI. Due to space limitation, some proofs have been omitted and given in [17].

Notation. Given a matrix $A \in \mathbb{R}^{n \times m}$, $A^{-1}$ and $A^T$ represent the inverse and the transpose of matrix $A$. Respectively. The minimum and maximum eigenvalues of the matrix $A$ are denoted by $\lambda_\min(A)$ and $\lambda_\max(A)$. For a positive integer $n$, $I_n$ is the identity matrix in $\mathbb{R}^{n \times n}$. $\| \cdot \|$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\| \cdot \|$ is associated with the matrix norm. $\| \cdot \|_{L^2}$ denotes the associated norm in $L^2(0, 1)$ space, defined by $\|f\|_{L^2} = \int_0^1 |f(x)|^2 dx$ for all functions $f \in L^2(0, 1)$. Similarly, the associated norm in $H^1(0, 1)$ space is denoted by $\| \cdot \|_{H^1}$, defined for all functions $f \in H^1(0, 1)$, by $\|f\|_{H^1} = \sqrt{\int_0^1 (|f(x)|^2 + |f'(x)|^2) dx}$. According to [3], for all matrices $G \in \mathbb{R}^{n \times n}$, $p_1(G) = \inf\{\|\Delta G \Delta^{-1}\|, \Delta \in D_{n,+}\}$, where $D_{n,+}$ denotes the set of diagonal positive matrices in $\mathbb{R}^{n \times n}$.

II. System description

Consider the following linear hyperbolic system of balance laws

$$
y_i(x, t) + \lambda_1(x) y_{x_i}(x, t) = a(x) y(x, t) + b(x) z(x, t), \quad (1a)
y_{x_i}(x, t) + \lambda_2(x) z_{x_i}(x, t) = c(x) y(x, t) + d(x) z(x, t), \quad (1b)
$$

where $x \in [0, 1]$, $t \in [0, +\infty)$. $\lambda_1(x)$ is a diagonal matrix in $\mathbb{R}^{n \times n}$ such that $\lambda_1(x) = \text{diag}(\lambda_1(x), \ldots, \lambda_n(x))$, where the first $l$ elements are positive and the $n-l$ last elements are negative. Similarly $\lambda_2(x)$ is a diagonal matrix in $\mathbb{R}^{m \times m}$, such that $\lambda_2(x) = \text{diag}(\lambda_2(x), \ldots, \lambda_m(x))$, where the first $m-l$ elements are negative and the $m-l$ last elements are positive. The matrices $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are in appropriate dimensions and vanish at $x = 0$.

The boundary condition under consideration is given by

$$
\begin{bmatrix}
y^-(0, t) 
y^+(0, t) 
z^-(0, t) 
z^+(0, t)
\end{bmatrix}
= G(x)
\begin{bmatrix}
y^-(0, t) 
y^+(0, t) 
z^-(0, t) 
z^+(0, t)
\end{bmatrix}, \quad t \in [0, +\infty),
$$

where $y^-$ and $y^+$ denote the transport in the left and right direction, respectively, $z^-$ and $z^+$ denote the conservation in the left and right direction, respectively. The other elements are defined in a similar way.
where \( G(\epsilon) = \begin{pmatrix} G_{11}(\epsilon) & G_{12}(\epsilon) \\ G_{21}(\epsilon) & G_{22}(\epsilon) \end{pmatrix} \) is a matrix in \( \mathbb{R}^{(n+m) \times (n+m)} \) with the matrices \( G_{11}(\epsilon) \in \mathbb{R}^{n \times n}, G_{12}(\epsilon) \in \mathbb{R}^{n \times m}, G_{21}(\epsilon) \) in \( \mathbb{R}^{m \times n}, G_{22}(\epsilon) \in \mathbb{R}^{m \times m} \). Given two functions \( \tilde{y}^0 : [0, 1] \to \mathbb{R}^n \) and \( z^0 : [0, 1] \to \mathbb{R}^m \), the initial condition is

\[
\begin{pmatrix} y(x,0) \\ z(x,0) \end{pmatrix} = \begin{pmatrix} \tilde{y}^0(x) \\ \tilde{z}^0(x) \end{pmatrix}, \quad x \in [0, 1]. \tag{3}
\]

Replacing \( y(x,t) \) by \( \begin{pmatrix} y^-(1-x,t) \\ y^+(x,t) \end{pmatrix} \) and \( z(x,t) \) by \( \begin{pmatrix} z^-(1-x,t) \\ z^+(x,t) \end{pmatrix} \), it may be assumed, without loss of generality, that the matrices \( \Lambda_1(\epsilon) \) and \( \Lambda_2(\epsilon) \) are diagonal positive. The full system (1) can then be rewritten under the form

\[
\begin{align*}
y_t(x,t) + \Lambda_1(\epsilon)y_x(x,t) &= a^+(\epsilon)y(x,t) + a^- \epsilon y(1-x,t) + b^+(\epsilon)z(x,t) + b^- \epsilon z(1-x,t), \\
\epsilon z_t(x,t) + \Lambda_2(\epsilon)z_x(x,t) &= c^+(\epsilon)y(x,t) + c^- \epsilon y(1-x,t) + d^+(\epsilon)z(x,t) + d^- \epsilon z(1-x,t). 
\end{align*}
\tag{4a}
\]

Then the boundary condition (2) becomes

\[
\begin{pmatrix} y(0,t) \\ z(0,t) \end{pmatrix} = G(\epsilon) \begin{pmatrix} y(1,t) \\ z(1,t) \end{pmatrix}, \quad t \in (0, +\infty). \tag{5}
\]

Adapting the approach in [13], [8] to infinite dimensional systems, the reduced subsystem for (4) and (5) is formally computed as follows. By setting \( \epsilon = 0 \) in (4b), we get \( z_x(x,t) = 0 \), which implies \( z(\cdot,t) = z(1,t) \). Substituting it into the second line of the boundary condition (5) and assuming \( (I_m - G_{22}(0))^{-1} \) invertible, we obtain \( z(\cdot,t) = (I_m - G_{22}(0))^{-1} G_{21}(0) y(1,t) \) and \( y(0,t) = (G_{11}(0) + G_{12}(0)(I_m - G_{22}(0))^{-1} G_{21}(0)) y(1,t) \). The reduced system is thus written as

\[
y_t(x,t) + \Lambda_1(0)\tilde{y}_x(x,t) = 0, \quad x \in [0, 1], \quad t \in [0, +\infty), \tag{6}
\]

with the boundary condition

\[
y(0,t) = G_r \tilde{y}(1,t), \quad t \in [0, +\infty), \tag{7}
\]

where \( G_r = G_{11}(0) + G_{12}(0)(I_m - G_{22}(0))^{-1} G_{21}(0) \), whereas the initial condition is given as the same as for the full system

\[
\tilde{y}(x,0) = \tilde{y}^0(x) = y^0(x), \quad x \in [0, 1]. \tag{8}
\]

The compatibility conditions for the existence of solutions of (6)-(8) in \( H^2 \)-norm are given as follows

\[
\begin{align*}
\tilde{y}_x^0(0) &= G_r \tilde{y}^0(1), \\
\tilde{y}_x^0(0) &= \Lambda_1^{-1}(0) G_r \Lambda_1(0) \tilde{y}_x^0(1). \tag{9}
\end{align*}
\]

Remark 1. Compared with [19], the transport velocities of the full system in the present work depend on \( \epsilon \) as well as the boundary conditions. Moreover, we consider an additional source term which is also dependent on \( \epsilon \). Due to the presence of \( \epsilon \) in both dynamics and boundary conditions, the full system becomes more complex. The assumptions on the continuity for such terms with respect to \( \epsilon \) should be used to ensure that the Tikhonov approximation is valid for \( \epsilon \) sufficiently small. The proof of the main result is then more sophisticated and is a non trivial extension.

III. TIKHONOV APPROXIMATION OF LINEAR HYPERBOLIC SYSTEMS OF BALANCE LAWS

In this section, the approximation of the solutions to the full system by that to the reduced subsystem is established by Lyapunov techniques. First let us consider the following assumptions.

Assumption 1. The functions \( \Lambda_1 \) and \( \Lambda_2 \) are Lipschitz continuous at 0, that is there exist positive constants \( R_1 \) and \( \epsilon \) such that for all \( 0 < \epsilon < \epsilon_0 \),

\[
\| \Lambda_1(\epsilon) - \Lambda_1(0) \| \leq \epsilon R_1, \quad \| \Lambda_2(\epsilon) - \Lambda_2(0) \| \leq \epsilon R_1.
\]

Assumption 2. Let \( \epsilon \) as in Assumption 1, the functions \( a, b, c \) and \( d \) are Lipschitz continuous at 0, that is there exists a positive constant \( R_2 \), such that for all \( 0 < \epsilon < \epsilon_0 \),

\[
\| a(\epsilon) \| \leq \epsilon R_2, \quad \| b(\epsilon) \| \leq \epsilon R_2, \quad \| c(\epsilon) \| \leq \epsilon R_2, \quad \| d(\epsilon) \| \leq \epsilon R_2.
\]

Assumption 3. Let \( \epsilon \) as in Assumption 1, the functions \( G_{11}, G_{12}, G_{21} \) and \( G_{22} \) are Lipschitz continuous at 0, that is there exists a positive value \( R_3 \), such that for all \( 0 < \epsilon < \epsilon_0 \),

\begin{align*}
\| G_{11}(\epsilon) - G_{11}(0) \| &\leq \epsilon R_3, \\
\| G_{12}(\epsilon) - G_{12}(0) \| &\leq \epsilon R_3, \\
\| G_{21}(\epsilon) - G_{21}(0) \| &\leq \epsilon R_3, \\
\| G_{22}(\epsilon) - G_{22}(0) \| &\leq \epsilon R_3.
\end{align*}

We are ready to state the main result in the following theorem.

Theorem 1. Consider the linear hyperbolic system (4)-(5), under Assumptions 1-3, if \( \rho_1(G(0)) < 1 \), there exist positive values \( C_1, C_2, \theta, \epsilon^* \) such that for all \( 0 < \epsilon < \epsilon^* \), for any initial condition \( y^0 \in H^2(0,1) \) satisfying compatibility conditions (9) with \( \tilde{y}^0 = y^0 \), and \( z^0 \in L^2(0,1) \), it holds for all \( t \geq 0 \)

\[
\| y(\cdot,t) - \tilde{y}(\cdot,t) \|_{L^2}^2 \leq C_1 e^{\theta t} \left( \| y^0 \|_{H^2}^2 + \int_0^t \| z(\cdot,t) - (I_m - G_{22}(0))^{-1} G_{21}(0) \tilde{y}(1,t) \|_{L^2}^2 dt \right) \tag{10a}
\]

\[
+ \| z^0 \|_{L^2}^2 + \| z(\cdot,t) - (I_m - G_{22}(0))^{-1} G_{21}(0) \tilde{y}(1,t) \|_{L^2}^2 dt \right) \tag{10b}
\]

Corollary 1. If \( \rho_1(G(0)) < 1 \), under Assumptions 1-3, the full system (4) with the boundary condition (5) is exponentially stable in \( L^2 \)-norm for all \( 0 < \epsilon < \epsilon^* \).

We have the proofs of Theorem 1 and Corollary 1 in the following section.

IV. PROOF OF THEOREM 1 AND COROLLARY 1

Proof of Theorem 1: In the following we will use three steps to prove Theorem 1.

Step 1) Let us perform the following change of variables,

\[
y(x,t) = y(x,t) - \tilde{y}(x,t), \tag{10a}
\]

\[
\delta(x,t) = z(x,t) - (I_m - G_{22}(0))^{-1} G_{21}(0) \tilde{y}(1,t), \tag{10b}
\]

where \( \eta \) stands for the error between the slow dynamics \( y \) in (4) and \( \tilde{y} \) in (6), and \( \delta \) is the error between the fast dynamics.
Similarly, we compute the time derivative of $V_1$ along (11a) to yield
\begin{align*}
\dot{V}_1 &= -\left[e^{-\mu x} \eta^\top (x) Q \eta_1(x) \right]_{x=0}^1 \nonumber \\
&\quad - \int_0^1 e^{-\mu x} \eta^\top (x) \left( \mu A_1(\epsilon) - 2Qa(\epsilon) \right) \eta(x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \eta^\top (x) Qa(\epsilon) \eta(x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \eta^\top (x) Qb(\epsilon) \delta(x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \eta^\top (x) Qb(\epsilon) \delta(1-x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \eta^\top (x) Qa(\epsilon) \eta(x, t) \, dx \\
&\quad - 2 \int_0^1 e^{-\mu x} \eta^\top (x) Q \left( \Lambda_1(\epsilon) - \Lambda_1(0) \right) \eta_1(x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \eta^\top (x) Qb(\epsilon) \left( I_m - G_{22}(0) \right)^{-1} G_{21}(0) \eta_1(1, t) \, dx.
\end{align*}

Similarly, we compute the time derivative of $V_2$ along (11b) yield
\begin{align*}
\dot{V}_2 &= -\left[e^{-\mu x} \delta^\top (x) P \Lambda_2(\epsilon) \delta(x) \right]_{x=0}^1 \\
&\quad - \int_0^1 e^{-\mu x} \delta^\top (x) \left( \mu P A_2(\epsilon) - 2Pd(\epsilon) \right) \delta(x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \delta^\top (x) Pd(\epsilon) \delta(1-x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \delta^\top (x) Pd(\epsilon) \delta(1-x, t) \, dx \\
&\quad + 2 \int_0^1 e^{-\mu x} \delta^\top (x) P \left( I_m - G_{22}(0) \right)^{-1} G_{21}(0) \eta_1(1, t) \, dx.
\end{align*}
and for all constants $\rho$ exist positive values and $\varpi < C V - T \varpi t (\epsilon C \eta)$, we consider the following academic example which illustrates the full generality of our result. Consider system (4) with $A_1(\epsilon) = 1 + \epsilon, A_2(\epsilon) = \epsilon - 1, a(\epsilon) = 0.1 \epsilon, b(\epsilon) = 0.2 \epsilon, c(\epsilon) = 0.05 \epsilon$ and $\delta(\epsilon) = 0.4 \epsilon$, which satisfies Assumptions 1 and 2. The boundary condition (5) is given by $G(\epsilon) = (0.5 + \epsilon, 0.25 + \epsilon / 2, -0.5 + \epsilon)$, thus Assumption 3 holds. Considering a diagonal positive matrix $\Delta = (0.5, 0.7, 0.8)$, it holds $\Delta G(\epsilon) / \Delta^{-1} < 1$. Thus $\rho_1(G(\epsilon)) < 1$ is satisfied. Theorem 1 applies. To numerically compute the solutions of the discretized system (13) by using a two-step variant of the Lax-Wendroff method (see [14] and [15]). Precisely, the space domain $[0, 1]$ is divided into 100 intervals of identical length, the final time is chosen as 30. We take a time-step $dt = (0.9 \epsilon / (\epsilon + 1)) dx$ that satisfies the CFL condition and select the initial conditions $y^0(x) = 1 - \cos(4\pi x), \varphi^0(x) = \sin(2\pi x)$, for all $x \in [0, 1]$, such that the compatibility condition is satisfied. The evolutions of $\|\eta(\cdot, t)\|_{L^2}^2$ and of $\int_0^t \|\delta(\cdot, t)\|_{L^2}^2 dt$ for different $\epsilon$ are given by Table I. The values are close to zero and decrease as $\epsilon$ decreases.
TABLE I: Evolutions of square of $L^2$-norm of $\eta$ and of time integral of square of $L^2$-norm of $\delta$ for different $\epsilon$

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$|\eta(t, t = 3)|_{L^2}$</th>
<th>$|\delta(t, t = 3)|_{L^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>$3 \times 10^{-3}$</td>
<td>$7 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.01</td>
<td>$2 \times 10^{-2}$</td>
<td>$2.6 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.015</td>
<td>$2 \times 10^{-2}$</td>
<td>$5.7 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Remark 4. The simulation cost is lower when we simulate the reduced subsystem with a time-step which does not depend on $\epsilon$ and satisfies the CFL condition $\mathcal{L}(\lambda_1(0))dt < dx$ than simulating the full system by using a smaller time-step satisfying CFL condition $\mathcal{L}(\lambda_2(\epsilon))dt < edx$.

B. Physical application

a) System description: The gas dynamics through a constant cross section tube, where all the friction losses and heat transfers are neglected, can be modeled by the following Euler equations as considered in [20, Chapter 2], by considering a tube of length equals to 1.

\[
\begin{pmatrix}
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( u \rho \right) \\
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \left( \rho u \right)
\end{pmatrix} = 0,
\]

where $u = u(x,t)$ stands for the gas velocity at location $x$ in $[0,1]$ and at time $t$; $\rho = \rho(x,t)$ represents the gas density; $p = p(x,t)$ is the gas pressure; $a$ is sound speed in ideal gas. System (19) admits a constant in space steady-state $(u^*, \rho^*, p^*)$. The deviations of the state $(u, \rho, p)$ around the steady-state are defined as $\pi = u - u^*$, $\varphi = \rho - \rho^*$, $\varphi = p - p^*$. Then the linearization of system (19) at this equilibrium is given by

\[
\begin{pmatrix}
\frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x} \left( \pi \rho \right) \\
\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \left( \varphi \rho \right)
\end{pmatrix} = 0.
\]

Performing a change of variable in Riemann coordinates and assuming that the propagation speed of gas is much slower than the sound speed, i.e. $u << a$, we define $\epsilon = \frac{u}{a}$, then (20) can be written as a singularly perturbed system

\[
\begin{pmatrix}
\frac{M_1}{\epsilon M_2} \\
\frac{M_1}{\epsilon M_2}
\end{pmatrix} \left( \begin{array}{c}
\frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x} \left( \varphi \rho \right) \\
\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \left( \pi \rho \right)
\end{array} \right) = 0,
\]

with $M = \left( \begin{array}{cc}
M_1 & M_2 \\
M_3 & M_4
\end{array} \right)^{-1} \left( \begin{array}{c}
\frac{\partial \pi}{\partial t} + \frac{\partial}{\partial x} \left( \varphi \rho \right) \\
\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \left( \pi \rho \right)
\end{array} \right)$.

b) Boundary conditions: The setup is provided with fans which are located at the two extremities of the tube. The rotation speed is considered as the control action. We consider the following three boundary conditions for system (19).

1. The first boundary condition describes the operation of the inflow fan (see the fan specification map in [21]),

\[
u(0,t)s = \sigma c_0(t)(p(0,t) - p_{in}),
\]

where $s$ stands for the tube’s constant cross section, $\sigma$ is a constant coefficient, the control input is denoted by $c_0(t)$ and $p_{in}$ is a constant pressure before the inflow fan.

2. Similarly, the second boundary condition is given by the outflow fan,

\[
u(1,t)s = \sigma c_1(t)(p_{out} - p(1,t)),
\]

the control input is denoted by $c_1(t)$ and $p_{out}$ is a constant pressure behind the outflow fan.

3. The third boundary condition is a physical constraint. Precisely, the gas pressure at the inflow fan is close to the atmospheric pressure (see [2]),

\[
\rho(0,t) = \tilde{\rho}
\]

where $\tilde{\rho}$ is constant.

The boundary conditions for system (20) are obtained by linearizing the above three boundary conditions,

\[
\begin{array}{l}
\pi(0,t)s = \sigma \left[ c_0(t)(p^* - p_{in}) + c_0^* \pi(0,t) \right], \\
\varphi(1,t)s = \sigma \left[ c_1(t)(p_{out} - p^*) - c_1^* \varphi(1,t) \right], \\
\varphi(0,t) = 0,
\end{array}
\]

where $c_0^*$, $c_1^*$ are the constant control actions at the steady-state $(u^*, \rho^*, p^*)$.

Proposition 1. For any values $K_{23}$ and $K_{32}$ in $\mathbb{R}$ such that $K_{23} \neq 1$ and $K_{32} \neq 1$, defining control actions by

\[
c_0(t) = c_0^* + \frac{s(a^*(1-K_{32}^-) - 1)}{\sigma a^* p^*(1-K_{32}^-)} \varphi(0,t),
\]

\[
c_1(t) = c_1^* + \frac{s(a^*(1-K_{32}^-) - 2p^* a^* K_{23}^2)}{\sigma a^* p^*(1-K_{32}^-)} \varphi(1,t) + \frac{2s a^* K_{23}^2}{\sigma a^* p^*(1-K_{32}^-)} \varphi(1,t),
\]

the following conditions are equivalent to (25)-(27),

\[
\begin{pmatrix}
M_1(0,t) \\
M_2(1,t) \\
M_3(0,t) \\
M_4(1,t)
\end{pmatrix} = \begin{pmatrix}
0 & K_{23} & 0 & 0 \\
K_{21} & 0 & K_{23} & 0 \\
0 & K_{32} & 0 & 0 \\
M_4(1,t) & M_3(0,t) & M_2(1,t) & M_1(0,t)
\end{pmatrix},
\]

where $K_{12} = f(K_{32}) = \frac{a^*(1-K_{32}^-)}{a^*}$.

The interest of the feedback laws $c_0(t)$ and $c_1(t)$ leads in the equivalent form (28) in Riemann coordinates, for which the stability analysis could be studied by applying our main result.

Checking the assumptions of Theorem 1 allows to compute suitable tuning parameters $K_{21}$, $K_{23}$ and $K_{32}$. Moreover note that the controllers $c_0(t)$ and $c_1(t)$ do not depend on all the state $(\pi, \varphi, \varphi)^T$, but depend on some boundary values, namely $\pi(0,t)$, $\varphi(1,t)$ and $\varphi(1,t)$. The proof of Proposition 1 is available in [17].

C. Boundary condition synthesis based on singular perturbation method

According to Section II, the reduced subsystem for gas transport system is computed as follows,

\[
\dot{M}_{14} + u^* M_{1x} = 0,
\]

with the boundary condition

\[
\dot{M}_{14}(0,t) = K_r M_{14}(1,t),
\]

where $K_r = \frac{a^*(1-K_{32}^-) K_{21}}{a^* (1-K_{32}^-)}$.

Due to Proposition 1 in [18], the reduced subsystem (29) and (30) is convergent in finite time $T$ if the boundary condition $K_r = 0$. Assuming $1 - K_{23} K_{32} \neq 0$, since $K_{32} \neq 1$ in Proposition 1, it holds $K_r = 0$ as soon as $K_{21} = 0$. The boundary condition matrix $K$ in (28) becomes $K = \begin{pmatrix}
0 & \frac{s a^*(1-K_{32}^-)}{\sigma a^* p^*(1-K_{32}^-)} & 0 \\
0 & 0 & K_{32}
\end{pmatrix}$. 

\[
\rho(0,t) = \tilde{\rho}
\]
To ensure $\rho_1(K) < 1$, it is sufficient to choose $\|K\| < 1$. In order to decrease the control cost, we can minimize $\|K\|$ that is equivalent to minimize $K_{32}^2 +\left(\frac{\rho^*(1-K_{32})}{\rho^*}\right)^2 + K_{23}$. Let $K_{23}$ be zero. Computing the derivative of $K_{32}^2 +\left(\frac{\rho^*(1-K_{32})}{\rho^*}\right)^2$ with respect to $K_{32}$, we obtain $K_{32} = \frac{\rho^2}{\rho^2+\rho_1^2}$. Therefore the control actions become $c_0(t) = c_0^* - \frac{\rho_1^2}{\rho^2+\rho_1^2} \rho(0,t)$ and $c_1(t) = c_1^* + \frac{\rho_1^2}{\rho^2+\rho_1^2} \rho(1,t)$.

c) Numerical results: Let us consider the following values for numerical simulation: $\alpha^* = (200, 150, 100)$, $\alpha^* = 10$, $\rho^* = 2$, $K = 10^{-5}$ \(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\). The time evolution of the solution $\bar{M}_1$ for the reduced subsystem (29) with $k_r = 0$ is shown in Figure 1a. It is observed that $\bar{M}_1$ converges to the origin in finite time. Time evolution of $\eta$ in Figure 1b shows that the error between the full system and the reduced subsystem is close to 0 as time increases. Table II gives the evolutions of $\|\eta(\cdot, t = 0.1)\|_2^2$ and of $\int_0^{t_1} \|\delta(\cdot, t)\|_2^2 dt$. It is found that the values are near zero and increase when $\epsilon$ increases, as expected from Theorem 1.

![Fig. 1: Time evolutions of $M_1$ and $\eta$](image)

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\frac{\alpha^<em>}{\alpha^</em>}$</th>
<th>$\frac{|\eta(\cdot, t = 0.1)|_2^2}{10^8}$</th>
<th>$\frac{|\delta_1(\cdot, t)|_2^2 dt}{10^{-10}}$</th>
<th>$\frac{|\delta_2(\cdot, t)|_2^2 dt}{10^{-10}}$</th>
</tr>
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<tbody>
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<td>$\frac{\alpha^<em>}{\alpha^</em>}$</td>
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<td>$\frac{|\eta(\cdot, t = 0.1)|_2^2}{10^8}$</td>
<td>$\frac{|\delta_1(\cdot, t)|_2^2 dt}{10^{-10}}$</td>
<td>$\frac{|\delta_2(\cdot, t)|_2^2 dt}{10^{-10}}$</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>$4.0 \times 10^{-7}$</td>
<td>$9.9 \times 10^{-11}$</td>
<td>$2.6 \times 10^{-10}$</td>
<td>$3.7 \times 10^{-13}$</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>$3.1 \times 10^{-11}$</td>
<td>$1.1 \times 10^{-10}$</td>
<td>$3.4 \times 10^{-12}$</td>
<td>$1.1 \times 10^{-14}$</td>
</tr>
</tbody>
</table>

VI. CONCLUSION

This paper is concerned with a class of singularly perturbed linear hyperbolic systems with source term which depends on the perturbation parameter. The hetero-directional transport velocities depend on $\epsilon$ as well as the boundary conditions. Under some assumptions and the condition $\rho_1(G(0)) < 1$, the approximation of the solution of the full system by that of the reduced subsystem has been established in Theorem 1. An academic example has been used to illustrate the main result. Furthermore, a new boundary control synthesis has been given with an application of gas flow transport model where the slow dynamics is convergent in finite time. For the future work, it would be interesting to study a physical application with small source term which vanishes when the perturbation parameter tends to zero.

REFERENCES


