A Strict Control Lyapunov Function for a Diffusion Equation With Time-Varying Distributed Coefficients

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***Abstract*—In this paper, a strict Lyapunov function is developed in order to show the exponential stability and input-to-state stability (ISS) properties of a diffusion equation for nonhomogeneous media. Such media can involve rapidly time-varying distributed diffusivity coefficients. Based on this Lyapunov function, a control law is derived to preserve the ISS properties of the system and improve its performance. A robustness analysis with respect to disturbances and estimation errors in the distributed parameters is performed on the system, precisely showing the impact of the controller on the rate of convergence and ISS gains. This is important in light of a possible implementation of the control since, in most cases, diffusion coefficient estimates involve a high degree of uncertainty. An application to the safety factor profile control for the Tore Supra tokamak illustrates and motivates the theoretical results. A constrained control law (incorporating nonlinear shape constraints in the actuation profiles) is designed to behave as close as possible to the unconstrained version, albeit with the equivalent of a variable gain. Finally, the proposed control laws are tested under simulation, first in the nominal case and then using a model of Tore Supra dynamics, where they show adequate performance and robustness with respect to disturbances.**

***Index Terms*—Distributed parameter systems, fusion power generation, Lyapunov method, partial differential equations, time-varying systems, tokamaks.**

I. Introduction

A. Theoretical Contribution

PARABOLIC partial differential equations (PDEs) and, in particular, diffusion or diffusion convection equations, are used to model a wide array of physical phenomena ranging from heat conduction to the distribution of species in biological systems. While the diffusivity coefficients can be assumed to be constant throughout the spatial domain for most applications, spatially distributed coefficients are needed when treating nonhomogeneous or anisotropic (direction-dependent) media. Unfortunately, extending existing results from the homogeneous to the nonhomogeneous case are not straightforward, particularly when the transport coefficients are time-varying.

In this paper, the concept of *input-to-state stability* (ISS) will be the chosen framework to study the stability and robustness of a diffusion equation in a circular domain under a revolution symmetry condition with symmetric initial conditions. The interest of studying such an equation is illustrated and motivated by the proposed application, where a similar equation arises from the averaging of a 2-D physical equation (representing the evolution of the toroidal magnetic flux in a tokamak) over the angle at fixed radius (nested toroidal surfaces). A comprehensive survey of ISS concepts, in the finite-dimensional case, can be found in [30]. ISS essentially implies guaranteeing a bounded gain between disturbances or errors and the state of the system. ISS-like properties in the infinite dimensional framework using a frequency-domain approach can be found, for example, in [15]. Nevertheless, we have favored the use of a Lyapunov-based approach to enable easier treatment of very general disturbances and errors in the system.

Although the use of Lyapunov functions in an infinite dimensional setting is not new, see, for example, [3], it is still an active research topic. Some interesting results for parabolic PDEs can be mentioned: [8], where a Lyapunov function is used to prove the existence of a global solution to the heat equation; [16], where a Lyapunov function is constructed for the heat equation with unknown destabilizing parameters (and subsequent control extensions [28] and [29]). Lyapunov-based approaches are not limited to parabolic PDEs: Lyapunov functions are used in [10] for the stabilization of a rotating beam; in [9], for the stability analysis of quasilinear hyperbolic systems and in [11] for the construction of stabilizing boundary controls for a system of conservation laws. In particular, in [19] and [25], the interest of using a strict-Lyapunov function to obtain ISS-like properties is discussed in the parabolic and hyperbolic cases, respectively. The use of weighted L^2 norms (or similar quadratic expressions with a weight) as Lyapunov functions is not new and a few examples can be found in [23] (for time delay systems) and [12] (where a vanishing weight is also used for the control of the magnetic flux equation in a tokamak but not its gradient).

Some previous works on reaction-diffusion equations in cylindrical 2-D domains are, for instance, [32] and [34] where boundary control laws are developed for the stabilization of thermal convection loops. However, in both of these papers, the domain considered does not include the point $r = 0$, which implies that none of the coefficients in the equation are singular.

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In this paper, we develop a strict Lyapunov function for the diffusion equation for a certain set of diffusivity coefficient profiles. Our main contribution is that the coefficients are allowed to be space and time dependent without imposing any constraints on the rate of variation of the coefficients with respect to time. This is an improvement over other works that consider diffusivity coefficients as being space dependent or time-varying but not both simultaneously. Examples of such approaches are provided by [26], where constant diffusion coefficients and distributed convection coefficients are considered; [27], where the case of nonconstant diffusion coefficients is treated (for continuous, time-invariant coefficients); or [33], where distributed and time-varying convection coefficients are taken into account (with a constant diffusion coefficient). Also, stability and robustness of the system under a simple unconstrained feedback law (that includes the open-loop system as a limiting case) were derived from the Lyapunov function, with results addressing most sources of errors and uncertainties that may be present in a real system. In particular, the following sources of error were considered:

- *state disturbances*: accounting, for example, for unmodeled dynamics;
- *actuation errors*: accounting mainly for errors in the actuator models (similar to the concept of controller fragility);
- *estimation errors in the state and diffusivity coefficients*: accounting, for instance, for discretized measurements or uncertain models as well as measurement noise.

B. Application to the Control of a Tokamak Safety Factor

The motivating application for the theoretical results presented in this paper is the development of a strict Lyapunov function and control laws for the poloidal magnetic flux profile in the Tore Supra tokamak. This application is particularly interesting for the method developed in this paper since the diffusivity coefficient profiles depend mainly on the temperature profile inside the plasma, which is rapidly time-varying (more than ten times faster than the magnetic flux dynamics), and on other physical quantities (like particle density and effective electric charge) that induce large model uncertainties and unmeasured disturbances. Furthermore, neglected inputs and unmodeled dynamics provide other sources of disturbances. The robustness results obtained from our theoretical contribution allow us to construct a constrained control law that will preserve ISS properties while taking into account strong nonlinear shape constraints in the distributed control action.

A tokamak is a toroidal chamber lined with magnetic coils that generate a very strong magnetic field with a toroidal and a poloidal component. In this chamber, a plasma (generally constituted of Hydrogen isotopes) is confined by strong magnetic fields so that the fusion reaction can take place. The relation between the two components of the associated magnetic flux determines what is known as the *safety factor profile* or *q-profile*. This important physical quantity has been found to be related to several phenomena in the plasma, in particular *magnetohydrodynamic* (MHD) instabilities. Having an adequate safety factor profile is particularly important to achieve advanced tokamak operation, providing high confinement and MHD stability. A detailed account of tokamak physics can be found in [36]. An overview of challenges of tokamak plasma control is given in [24] and [35].

The problem of poloidal magnetic flux profile control is closely related, via the Maxwell equations, to the control of current profiles in the plasma. Some previous results in this areas can be found in [17], where experimentally identified linear models based on a Galerkin projection are used to control multiple profiles in JET; [20], where a reduced-order linear model is used to control some points in the safety factor profile; and in [22] among other papers, where an infinite-dimensional model is used to construct an optimal feedback controller for the current profile, albeit considering a fixed form profile for the current deposit from the actuators and a good knowledge of the diffusivity profile.

In particular, some works related to Tore Supra are [21], with an overview of control achievements [6], where a polytopic LPV approach is used to build a common Lyapunov function guaranteeing stability of the discretized system with time-varying coefficients; [12], where the sum-of-square polynomials are used to construct a Lyapunov function considering constant diffusivity coefficients and [13], where a sliding-mode controller is designed on the infinite-dimensional system, considering constant diffusivity coefficients.

For the application, the method proposed in this paper has the advantage of not only considering the diffusivity coefficients as uncertain, but also of not bounding their rate of time variation, thus reflecting the actual plasma physics where the temperature evolves in a much faster timescale than that of the flux diffusion. A deep robustness analysis has been carried out with respect to different sources of error that have a prime importance in the physical system. Finally, nonlinear constraints in the actuators (representing the complex coupling between the plasma and the input wave generated by the radio-frequency antenna) that do not assume a constant Lower Hybrid current source deposit profile (contrarily to previous works) are introduced.

This paper is organized as follows. In Section II, the reference diffusion equation is presented and the existence and uniqueness of sufficiently regular solutions with time-varying coefficients is stated. Next, in Section III, the main result is presented, namely, the strict Lyapunov function and sufficient conditions for exponential stability of the system in an L^2 sense. In Section IV, results are obtained regarding the robustness of the system with respect to several sources of errors and disturbances. In Section V, the results are applied to the control of the poloidal magnetic flux profile in the Tore Supra tokamak, and actuator constraints are added in such a way as to preserve ISS properties and to maximize the convergence rate of the system within some admissible limits.

II. Problem Statement and Existence Results

The diffusion equation considered in this paper, in its polar representation with a revolution symmetry (angle independence) constraint is:[1]

$$\zeta_t(r,t) = \frac{\eta(r,t)}{r}\left[r\zeta_r(r,t)\right]_r + \eta(r,t)u(r,t), \forall\,(r,t) \in (0,1)\times[0,T) \quad (1)$$

[1]In this paper, for any function ξ depending on r and/or t, ξ_r and ξ_{rr} are used to denote $(\partial/\partial r)\xi$ and $(\partial^2/\partial r^2)\xi$, respectively; $\dot{\xi}$ represents $(d/dt)\xi$ and ξ' represents $(d/dr)\xi$.

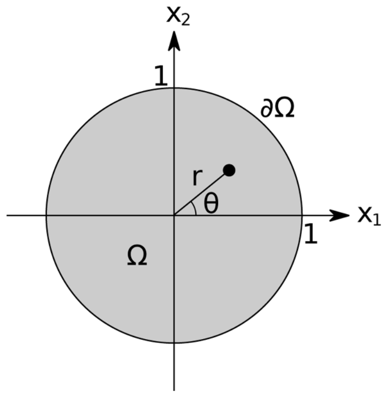

Fig. 1. Coordinates (x_1, x_2), (r, θ) and domain Ω used to define the diffusion equation.

with Neumann boundary conditions

$$\begin{aligned}\zeta_r(0,t) &= 0,\ \forall\, t \in [0,T)\\ \zeta_r(1,t) &= 0,\ \forall\, t \in [0,T)\end{aligned} \tag{2}$$

and initial condition

$$\zeta(r,0) = \zeta_0(r),\ \forall\, r \in (0,1) \tag{3}$$

where η stands for the diffusivity coefficient; $\zeta(\cdot,t)$ is the state of the system at time t; $u(\cdot,t)$ is a distributed input which can be either a control, a disturbance, or the sum of both; and $0 < T \leq +\infty$ is the time horizon. Hereafter, the dependence of ζ, u, and η on (r,t) will be implicit and omitted in most equations.

The following properties are assumed to hold in (1):

$\mathbf{P_1}$: $\eta(r,t) \geq k > 0$ for all $(r,t) \in [0,1] \times [0,T)$.

$\mathbf{P_2}$: The 2-D Cartesian representations of η and u are in[2] $C^{1+\alpha_c,\alpha_c/2}(\overline{\Omega} \times [0,T])$, $0 < \alpha_c < 1$, where $\Omega \doteq \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\right\}$ as shown in Fig. 1.

The set of equilibria of (1) and (2) is given by $\mathcal{E} = \left\{\overline{\zeta}(r) = K | K \in \mathbb{R}\right\}$ (the origin plus a constant). Since we are only interested in the convergence of the solutions toward $\mathcal{E}$, we will consider hereafter the evolution of the variable $z \doteq \nabla\zeta \cdot \overrightarrow{\rho}$ (where $\overrightarrow{\rho}$ is the unit vector in the radial direction and ∇ the gradient operator), as defined by

$$z_t = \left[\frac{\eta}{r}\left[rz\right]_r\right]_r + \left[\eta u\right]_r,\ \forall\, (r,t) \in (0,1) \times [0,T) \tag{4}$$

with Dirichlet boundary conditions

$$\begin{aligned}z(0,t) &= 0,\ \forall\, t \in [0,T)\\ z(1,t) &= 0,\ \forall\, t \in [0,T)\end{aligned} \tag{5}$$

and initial condition

$$z(r,0) = z_0(r),\ \forall\, r \in (0,1) \tag{6}$$

where $z_0 \doteq \nabla\zeta_0 \cdot \overrightarrow{\rho}$.

[2]Here, $C^{\alpha_c,\beta_c}(\overline{\Omega} \times [0,T])$ denotes the space of functions which are α_c-Hölder continuous in $\overline{\Omega}$, β_c-Hölder continuous in $[0,T]$. P_2 can be strengthened by assuming that η and u are in $C^{2,1}(\overline{\Omega} \times [0,T])$, which is the case for the physical application in Section V.

The objectives of this paper are as follows.

- To guarantee the exponential stability, in the topology of the L^2 norm,[3] of solutions of (4) to zero, both in open loop (with $u = 0$) and by closing the loop with a controlled input $u(\cdot,t)$;
- to be able to adjust (in particular, to accelerate) the rate of convergence of the system using the controlled input;
- to determine the impact of a controller in the ISS gain in the presence of a large class of errors; in particular, actuation errors, estimation/measurement errors, and state disturbances are considered.

To tackle this problem, a strict Lyapunov function will be defined in Section III. Let us first state an existing result assuming properties P_1 and P_2.

Theorem 2.1: For every $z_0 : [0,1] \to \mathbb{R}$ in $C^{2+\alpha_c}([0,1])$, $0 < \alpha_c < 1$, so that $z_0(0) = z_0(1) = 0$, the evolution (4)–(6) have a unique solution $z \in C^{1+\alpha_c,1+\alpha_c/2}\ ([0,1] \times [0,T]) \cap C^{2+\alpha_c,1+\alpha_c/2}([0,1] \times [0,T])$.

The proof of this result is given in Appendix A.

III. Control Lyapunov Function and Nominal Stability

In this section, the input u is considered to be perfectly controlled (without constraints) and a strict control Lyapunov function is developed, allowing us to construct a feedback law that ensures exponential convergence to the origin, at any desired rate, of the solutions of (4)–(6) in an L^2 sense.

A. Candidate Control Lyapunov Function

Given $f : [0,1] \to (0,\infty)$, a positive function with bounded second derivative, let us consider a candidate control Lyapunov function for the system (4) with boundary condition (5) and initial condition (6) defined, for all z in $L^2([0,1])$, by

$$V(z(\cdot)) = \frac{1}{2}\int_0^1 f(r)z^2(r)dr. \tag{7}$$

Remark 3.1: Since $f(r)$ is positive and continuous on $[0,1]$, the weighted norm $\|z(\cdot)\|_f \doteq \sqrt{V(z(\cdot))}$ is equivalent to the usual L^2 norm. In particular, it verifies

$$\sqrt{\frac{f_{\min}}{2}}\|z(\cdot)\|_{L^2} \leq \|z(\cdot)\|_f \leq \sqrt{\frac{f_{\max}}{2}}\|z(\cdot)\|_{L^2} \tag{8}$$

where $f_{\max} \doteq \max_{r\in[0,1]} f(r)$ and $f_{\min} \doteq \min_{r\in[0,1]} f(r)$.

Theorem 3.2: If a positive function $f : [0,1] \to (0,\infty)$ exists with a bounded second derivative, and a positive constant α so that the following inequality is verified:

$$\eta f''(r) + r\left[\frac{\eta}{r}\right]_r f'(r) + \left[\frac{\eta}{r}\right]_r f(r) \leq -\alpha f(r), \forall\, (r,t) \in [0,1] \times [0,T) \tag{9}$$

then the time derivative $\dot{V}$ of the function V defined by (7) verifies

$$\dot{V} \leq -\alpha V(z(\cdot,t)) + \int_0^1 f(r)\left[\eta u\right]_r z(r,t)dr,\ \forall\, t \in [0,T) \tag{10}$$

along the solutions of (4)–(6).

[3]The L^p norm of ξ on a domain Ω, will be denoted as $\|\xi\|_{L^p} \doteq \left(\int_\Omega \xi^p d\Omega\right)^{1/p}$ for $1 < p < \infty$.

Proof: Since Theorem 2.1 guarantees the existence of solutions to (4) such that $V(z(\cdot,t))$ is differentiable with respect to time, the derivative of V along those trajectories is

$$\begin{aligned}\dot V &= \int_0^1 f(r) z z_t dr \\ &= T_1 + T_2 + T_3 \end{aligned} \tag{11}$$

with

$$\begin{aligned} T_1 &= \int_0^1 f(r)\left[\eta_r u + \eta u_r\right] z dr \\ T_2 &= \int_0^1 f(r)\left(\eta_r\left[z_r + \frac{1}{r}z\right]z + \eta\left[\frac{1}{r}z_r - \frac{1}{r^2}z\right]z\right)dr \\ T_3 &= \int_0^1 f(r)\eta z z_{rr} dr. \end{aligned}$$

Term T_2 can be rewritten as

$$T_2 = \int_0^1 f(r)\left[\frac{1}{r}\eta z\right]_r z dr + \int_0^1 f(r)\eta_r z z_r dr.$$

Integrating by parts, we get

$$\begin{aligned} T_2 = \frac{1}{r} f(r)\eta z^2\Big|_0^1 &- \int_0^1 f'(r)\eta\frac{1}{r}z^2 dr \\ &- \int_0^1 f(r)\eta\frac{1}{r}z z_r dr + \int_0^1 f(r)\eta_r z z_r dr \end{aligned}$$

and, using the boundary conditions, (5) implies

$$\begin{aligned} T_2 = &-\int_0^1 f'(r)\eta\frac{1}{r}z^2 dr \\ &- \int_0^1 f(r)\eta\frac{1}{r}z z_r dr + \int_0^1 f(r)\eta_r z z_r dr. \end{aligned} \tag{12}$$

Integrating by parts T_3, the following equation is obtained:

$$\begin{aligned} T_3 = f(r)\eta z z_r|_0^1 &- \int_0^1 \left(f'(r)\eta + f(r)\eta_r\right) z z_r dr \\ &- \int_0^1 f(r)\eta z_r^2 dr \end{aligned}$$

which, considering again the boundary conditions (5), becomes

$$T_3 = -\int_0^1 \left(f'(r)\eta + f(r)\eta_r\right) z z_r dr - \int_0^1 f(r)\eta z_r^2 dr. \tag{13}$$

From (12) and (13), (11) can thus be written as

$$\dot V = T_1 + T_4 - \int_0^1 f'(r)\eta\frac{1}{r}z^2 dr - \int_0^1 f(r)\eta z_r^2 dr \tag{14}$$

with

$$T_4 = \int_0^1 \left[-f(r)\eta\frac{1}{r} - f'(r)\eta\right] z z_r dr.$$

Integrating by parts T_4, the following equation is obtained:

$$\begin{aligned} T_4 = &\frac{1}{2}\left(-f(r)\eta\frac{1}{r} - f'(r)\eta\right) z^2\Big|_0^1 \\ &- \frac{1}{2}\int_0^1 \Big(- f'(r)\eta\frac{1}{r} - f(r)\eta_r\frac{1}{r} + f(r)\eta\frac{1}{r^2} \\ &- f''(r)\eta - f'(r)\eta_r\Big) z^2 dr \end{aligned}$$

and the boundary conditions (5) imply that

$$\begin{aligned} T_4 = \frac{1}{2}\int_0^1 \Big(& f'(r)\eta\frac{1}{r} + f(r)\eta_r\frac{1}{r} \\ & - f(r)\eta\frac{1}{r^2} + f''(r)\eta + f'(r)\eta_r\Big) z^2 dr. \end{aligned} \tag{15}$$

Using (15), (14) is equivalent to

$$\begin{aligned} \dot V = &\int_0^1 f(r)\left[\eta_r u + \eta u_r\right] z dr - \int_0^1 f(r)\eta z_r^2 dr \\ &+ \frac{1}{2}\int_0^1 \Big(- f'(r)\eta\frac{1}{r} + f(r)\eta_r\frac{1}{r} - f(r)\eta\frac{1}{r^2} \\ &+ f''(r)\eta + f'(r)\eta_r\Big) z^2 dr. \end{aligned} \tag{16}$$

From (9) and the definition of the Lyapunov candidate function, (16) provides the inequality

$$\begin{aligned} \dot V \le -\alpha V(z(\cdot,t)) &+ \int_0^1 f(r)\left[\eta u\right]_r z dr \\ &- \int_0^1 f(r)\eta z_r^2 dr, \forall\, t \in [0,T) \end{aligned} \tag{17}$$

which implies the inequality (10), thus concluding the proof of Theorem 3.2. ∎

Remark 3.3: The last term in (17) can be bounded in order to obtain exponential stability of the system with a rate $\alpha+\epsilon$, where ϵ is a positive constant given by the application of Poincaré's inequality, the lower bound of η, and some bounds on f. However, for the physical application described in Section V, the rate of convergence obtained by adding this term is almost the same as the value of α that can be obtained by adequately solving the differential inequality in Theorem 3.2.

Remark 3.4: For a large class of diffusivity profiles, the differential inequality in Theorem 3.2 has easily computable solutions: whenever $\eta_r(1/r) - \eta(1/r^2) \le -k$ for some $k > 0$ and all $(r,t) \in [0,1] \times [0,T)$ (for example, if the spatial derivative of the diffusivity coefficient remains nonpositive), a constant f satisfies (9). For our motivating application, however, this condition is not satisfied. Section V presents a suitable numerically computed weight satisfying (10) for the application. A heuristic method to compute such weights for the particular case of exponential diffusivity coefficient profiles is provided in [7].

B. Some Implications

Corollary 3.5: Global Exponential Stability: If the conditions of Theorem 3.2 are verified, and if $u(r,t) = 0$ for all (r,t) in $[0,1] \times [0,T)$, then the origin of the system (4) with boundary conditions (5) and initial condition (6) is globally exponentially stable. The rate of convergence is $-\alpha/2$ in the topology of the norm L^2 (i.e., $\|z(\cdot,t)\|_{L^2} \le ce^{-(\alpha/2)t}\|z_0\|_{L^2}$ for a positive constant $c \doteq \sqrt{f_{\max}/f_{\min}}$), where $f_{\max}$ and $f_{\min}$ are defined as in Remark 3.1, and for all $t \in [0,T)$.

Proof: From Theorem 3.2 and setting $u(r,t) = 0$ for all (r,t) in $[0,1] \times [0,T)$, the following inequality is obtained:

$$\dot V \le -\alpha V(z(\cdot,t)),\ \forall\, t \in [0,T).$$

Therefore, considering the function $t \mapsto V(z(\cdot,t))$ and integrating the previous inequality over time implies that

$$V(z(\cdot,t)) \leq e^{-\alpha t} V(z_0(r)),\ \forall\, t \in [0,T)$$

and consequently

$$\|z(\cdot,t)\|_f \leq e^{-(\alpha/2)t} \|z_0\|_f,\ \forall\, t \in [0,T).$$

Since the norm $\|\cdot\|_f$ is equivalent to the usual L^2 norm[4] as shown in Remark 3.1, Corollary 3.5 as follows. ■

Corollary 3.6: Convergence Rate Control: If the conditions of Theorem 3.2 are verified, and considering $u \doteq u_{ctrl}$, where u_{ctrl} is chosen, for all $(r,t) \in (0,1) \times [0,T)$, as

$$u_{ctrl}(r,t) = -\frac{\gamma}{\eta} \int_0^r z(\rho,t) d\rho \tag{18}$$

with $\gamma \geq 0$ being a tuning parameter, then the system (4) with boundary conditions (5) and initial condition (6) is globally exponentially stable. Its convergence rate is $-\beta/2 \doteq -(\alpha+\gamma)/2$, in the topology of the norm L^2.

The proof of this corollary is similar to that of Corollary 3.5, using Theorem 3.2 and the fact that $[\eta u_{ctrl}]_r = -\gamma z$ for all $(r,t) \in [0,1] \times [0,T)$.

IV. Input-to-State Stability and Robustness

Let us first consider the effect of disturbing (4) by including a term w as follows:

$$z_t = \left[\frac{\eta}{r}[rz]_r\right]_r + [\eta u]_r + w,\ \forall\,(r,t) \in (0,1) \times [0,T) \tag{19}$$

where w is a function of (r,t) and the following property is assumed to hold:

$\mathbf{P_3}$: The 2-D Cartesian representation of w belongs to $C^{\alpha_c,\alpha_c/2}(\overline{\Omega} \times [0,T])$,
$0 < \alpha_c < 1$.

Proposition 4.1: Disturbed Version of Theorem 3.2: Let the conditions of Theorem 3.2 hold. Then, along the solution to (19), (5), (6), the following inequality holds:

$$\dot{V} \leq -\alpha V(z(\cdot,t)) + \int_0^1 f(r)\,[\eta u]_r\, z dr + \int_0^1 f(r) w z dr, \forall\, t \in [0,T). \tag{20}$$

This fact follows from Theorem 3.2, by using (7) and noting that $\dot{V}_{|(19)} = \dot{V}_{|(4)} + \int_0^1 f(r) w z dr$, where $\dot{V}_{|(19)}$ and $\dot{V}_{|(4)}$ stand for the derivative of V along the solution of (19) and (4), respectively, with boundary conditions (5) and initial conditions (6).

Theorem 4.2: ISS: Let the conditions of Proposition 4.1 be verified and consider $u \doteq u_{ctrl}$ as defined in Corollary 3.6. The following inequality holds for the evolution of the system (19) with boundary condition (5) and initial condition (6), for all $t \in [0,T)$

$$\|z(\cdot,t)\|_{L^2} \leq c e^{-(\beta/2)t} \|z_0\|_{L^2} + c \int_0^t e^{-(\beta/2)(t-\tau)} \|w(\cdot,\tau)\|_{L^2} d\tau \tag{21}$$

with $c = \sqrt{f_{\max}/f_{\min}}$, $f_{\max} \doteq \max_{r\in[0,1]} f(r)$ and $f_{\min} \doteq \min_{r\in[0,1]} f(r)$.

Proof: From Proposition 4.1 and Corollary 3.6, along the solution of (19), (5), and (6), we have

$$\dot{V} \leq -\beta V(z(\cdot,t)) + \int_0^1 |f(r) w(r,t) z(r,t)|\, dr,\ \forall\, t \in [0,T).$$

The function f is positive and using the Cauchy–Schwarz inequality, the following upper bound is obtained:

$$\begin{aligned}\dot{V} \leq & -\beta V(z(\cdot,t)) + \|\sqrt{f} z(\cdot,t)\|_{L^2} \|\sqrt{f} w(\cdot,t)\|_{L^2} \\ = & -\beta V(z(\cdot,t)) + 2\|z(\cdot,t)\|_f \|w(\cdot,t)\|_f,\ \forall\, t \in [0,T)\end{aligned}$$

Defining $X(z(\cdot,t)) \doteq \sqrt{V(z(\cdot,t))} = \|z(\cdot,t)\|_f \geq 0$, this inequality implies

$$2X(z(\cdot,t))\dot{X} \leq -\beta X^2(z(\cdot,t)) + 2X(z(\cdot,t))\|w(\cdot,t)\|_f, \forall\, t \in [0,T)$$

where $\dot{X} = (d/dt)X(z(\cdot,t))$.

If $X(z) = 0$, then $V(z) = 0$ and $\dot{V} = 0$. Otherwise, we may divide both sides of the previous inequality by $2X(z(\cdot,t))$ to obtain

$$\dot{X} \leq -\frac{\beta}{2} X(z(\cdot,t)) + \|w(\cdot,t)\|_f,\ \forall\, t \in [0,T).$$

From the last equation, by easy calculations, we obtain

$$\|z(\cdot,t)\|_f \leq e^{-(\beta/2)t} \|z_0\|_f + \int_0^t e^{-(\beta/2)(t-\tau)} \|w(\cdot,\tau)\|_f d\tau \tag{22}$$

which, in turn, implies the desired result. ■

Corollary 4.3: Actuation Errors: In addition to the conditions in Theorem 3.2, we consider $u \doteq u_{ctrl} - \varepsilon^u(r,t)$, with u_{ctrl} as defined in Corollary 3.6 and $\varepsilon^u(r,t)$ a distributed actuation error verifying the regularity conditions stated in P_2. Then, with $w \doteq 0$, the following inequality holds:[5]

$$\|z(\cdot,t)\|_{L^2} \leq c e^{-(\beta/2)t} \|z_0\|_{L^2} + cM \int_0^t e^{-(\beta/2)(t-\tau)} \|\varepsilon^u(\cdot,\tau)\|_{H^1} d\tau, \quad \forall\, t \in [0,T) \tag{23}$$

with $M \doteq \max\{\eta_{\max}, \eta_{r,\max}\}$, $\eta_{\max} \doteq \sup_{(r,t)\in[0,1]\times[0,T)} |\eta|$, and $\eta_{r,\max} \doteq \sup_{(r,t)\in[0,1]\times[0,T)} |\eta_r|$.

The proof of Corollary 4.3 is directly obtained by replacing w by $[\eta \varepsilon^u]_r$ in Theorem 4.2.

[4]For generality purposes, results in this paper are stated in terms of usual norms. It should be noted, however, that the results stated in $\|\cdot\|_f$ norm are less conservative.

[5]The H^1 norm of ξ on $[0,1]$, will be denoted as $\|\xi\|_{H^1} \doteq \|\xi\|_{L^2} + \|\partial\xi/\partial r\|_{L^2}$

Corollary 4.4: Estimation Errors in the z Profile: Assume that the conditions of Theorem 3.2 are verified and consider the control defined in Corollary 3.6 but substituting z by an estimate, $\hat{z}(r,t) \doteq z(r,t) - \varepsilon^z(r,t)$ for all $(r,t) \in [0,1] \times [0,T)$, with $\varepsilon^z(r,t)$ being a distributed estimation error verifying the regularity conditions stated in P_3. The following inequality is then verified:

$$\begin{aligned}\|z(\cdot,t)\|_{L^2} &\leq c e^{-(\beta/2)t}\|z_0\|_{L^2} \\ &\quad + \gamma c \int_0^t e^{-(\beta/2)(t-\tau)}\|\varepsilon^z(\cdot,\tau)\|_{L^2} d\tau, \\ &\quad \forall\ t \in [0,T).\end{aligned} \tag{24}$$

Corollary 4.4 follows readily by replacing w by $\gamma\varepsilon^z$ in Theorem 4.2.

Proposition 4.5: Estimation Errors in the η Profile: Assume that the conditions of Theorem 3.2 are verified and consider the control defined in Corollary 3.6 but substituting η by an estimate $\hat{\eta}(r,t) \doteq \eta(r,t) - \varepsilon^\eta(r,t)$ for all $(r,t) \in [0,1]\times[0,T)$, with $\varepsilon^\eta(r,t)$ being a distributed estimation error verifying the regularity conditions stated in P_2. The following inequality is then verified:

$$\|z(\cdot,t)\|_{L^2} \leq c e^{-(\beta'/2)t}\|z_0\|_{L^2},\ \forall\, t \in [0,T) \tag{25}$$

where $\beta' \doteq \beta + \gamma \inf_{(r,t)\in[0,1]\times[0,T)} (\varepsilon^\eta/\hat{\eta}) - 2\gamma c \sup_{t\in[0,T)} \|[\varepsilon^\eta/\hat{\eta}]_r\|_{L^2}$.

Proof: Since the conditions of Theorem 3.2 are assumed to be verified to apply Corollary 3.6, inequality (10) holds. The control u in Corollary 3.6 with $\hat{\eta}$ becomes

$$u = -\frac{\gamma}{\hat{\eta}} \int_0^r z(\rho,t) d\rho.$$

This implies

$$\begin{aligned}\eta u &= -\gamma \frac{\eta}{\hat{\eta}} \int_0^r z(\rho,t)d\rho \\ &= -\gamma \frac{\hat{\eta}+\varepsilon^\eta}{\hat{\eta}} \int_0^r z(\rho,t)d\rho \\ &= -\gamma \int_0^r z(\rho,t)d\rho - \gamma\frac{\varepsilon^\eta}{\hat{\eta}} \int_0^r z(\rho,t)d\rho.\end{aligned}$$

Differentiating with respect to the spatial variable

$$[\eta u]_r = -\gamma z - \gamma \frac{\varepsilon^\eta}{\hat{\eta}} z - \gamma \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \int_0^r z(\rho,t)d\rho. \tag{26}$$

Substituting (26) in (10), the following inequalities are obtained for all $t \in [0,T)$:

$$\begin{aligned}\dot{V} &\leq -\alpha V(z) - \gamma \int_0^1 f(r) z^2 dr - \gamma \int_0^1 f(r)\frac{\varepsilon^\eta}{\hat{\eta}} z^2 dr \\ &\quad - \gamma \int_0^1 f(r) \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \left(\int_0^r z(\rho,t)d\rho\right) z dr \\ &\leq -(\alpha+\gamma)V(z) \\ &\quad - \gamma \left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right] \int_0^1 f(r) z^2 dr \\ &\quad - \gamma \int_0^1 f(r) \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \left(\int_0^r z(\rho,t)d\rho\right) z dr \\ &\leq -\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right) V(z) \\ &\quad + \gamma \int_0^1 \left| f(r) \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \left(\int_0^r z(\rho,t)d\rho\right) z \right| dr \\ &\leq -\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right) V(z) \\ &\quad + \gamma \int_0^1 \left| f(r) \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \left(\int_0^1 \mid z(\rho,t) \mid d\rho\right) z \right| dr \\ &\leq -\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right) V(z) \\ &\quad + \gamma \|z\|_{L^1} \int_0^1 \left| f(r) \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r z \right| dr.\end{aligned}$$

Applying the Cauchy–Schwarz inequality on the integral term and on the L^1 norm of z, it implies that for all $t \in [0,T)$

$$\begin{aligned}\dot{V} \leq &-\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right) V(z(\cdot,t)) \\ &+ 2\gamma \|z(\cdot,t)\|_{L^2} \left\| \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \right\|_f \|z(\cdot,t)\|_f.\end{aligned}$$

Using the equivalence between $\|\cdot\|_f$ and the usual L^2 norm, the previous inequality can be rewritten as

$$\begin{aligned}\dot{V} \leq &-\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right) V(z(\cdot,t)) \\ &+ \frac{2\sqrt{2}\gamma}{\sqrt{f_{\min}}} \|z(\cdot,t)\|_f^2 \left\| \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \right\|_f\end{aligned}$$

which, in turn, implies

$$\begin{aligned}\dot{V} \leq &-\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right) V(z(\cdot,t)) \\ &+ 2\gamma c \|z(\cdot,t)\|_f^2 \left\| \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \right\|_{L^2}\end{aligned}$$

with c as defined in Theorem 4.2. Consequently

$$\begin{aligned}\dot{V} &\leq -\left(\beta + \gamma\left[\inf_{(r,t)\in[0,1]\times[0,T)} \left(\frac{\varepsilon^\eta}{\hat{\eta}}\right)\right]\right. \\ &\qquad \left. - 2\gamma c \sup_{t\in[0,T)} \left\| \left[\frac{\varepsilon^\eta}{\hat{\eta}}\right]_r \right\|_{L^2}\right) V(z(\cdot,t)),\ \forall\, t \in [0,T) \\ &\leq -\beta' V(z(\cdot,t)),\ \forall\, t\in[0,T)\end{aligned}$$

and using the same arguments as in the proof of Corollary 3.5, it implies the desired result. ■

Remark 4.6: Although finding a stabilizing control law for system (4)–(6) considering unconstrained in-domain actuation is quite simple, the main interest of Sections III and IV lies in the fact that the stability of the open-loop system is guaranteed while giving a precise characterization of the impact of the control action in the closed-loop behavior of the system, both in terms of rate of convergence and ISS gains. Furthermore, the fact that the ISS inequalities hold for the open-loop system is crucial for the application presented in Section V, since it also implies that stabilizing control laws can be found *despite strong shape constraints on the admissible control action* imposed by

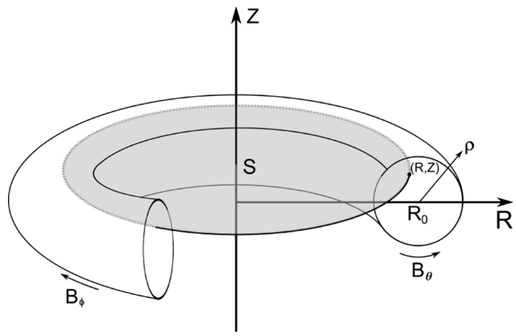

Fig. 2. Coordinates (R, Z) and surface $\mathbf{S}$ used to define the poloidal magnetic flux $\psi(R, Z)$.

the physical actuators (represented in Section V-D by a nonlinear function of the two available engineering parameters in the LH antennas that can only take values in bounded sets).

V. Application to the Control of the Poloidal Magnetic Flux Profile in a Tokamak Plasma

A. Physical Model

Inside the toroidal chamber of a tokamak, the poloidal magnetic flux in the plasma, denoted as $\psi(R, Z)$, is defined as the flux per radian of the magnetic field $\mathbf{B}(R, Z)$ through a disc centered on the toroidal axis at height Z, having a radius R and surface $\mathbf{S}$, as depicted in Fig. 2. Since the safety factor scales basically as the ratio of the normalized radius to poloidal magnetic gradient, controlling the latter allows controlling the safety factor profile, which is an important physical heuristic that relates to the plasma magnetohydrodynamic (MHD) stability and possible enhanced energy confinement. For a discussion on advanced tokamak scenarios, refer, for instance, to [14], [31], and [38].

In order to apply our analytical results, a simplified model for the magnetic flux profile ψ in its 1-D representation is considered. Its dynamics are given by the following equation [4]:

$$\psi_t = \frac{\eta_{\|} C_2}{\mu_0 C_3} \psi_{\rho\rho} + \frac{\eta_{\|}\rho}{\mu_0 C_3^2} \left(\frac{C_2 C_3}{\rho} \right)_\rho \psi_\rho + \frac{\eta_{\|} V_\rho B_{\phi_0}}{F C_3} j_{ni} \quad (27)$$

where $\rho \doteq \sqrt{\phi / \pi B_{\phi_0}}$ (ϕ is the toroidal magnetic flux and B_{ϕ_0} is the toroidal magnetic field at the center of the vacuum vessel) is an equivalent radius indexing the magnetic surfaces, $\eta_{\|}$ is the parallel resistivity of the plasma, the source term j_{ni} represents the current density profile generated by noninductive current sources, μ_0 is the permeability of free space, F is the diamagnetic function, C_2 and C_3 are geometric coefficients, V_ρ is the spatial derivative of the plasma volume, and B_{ϕ_0} is the toroidal magnetic field at the geometric center of the plasma. Some important variable definitions are given in Table I.

Neglecting the diamagnetic effect caused by poloidal currents and using a cylindrical approximation of the plasma geometry ($\rho \ll R_0$, where R_0 is the major plasma radius) the coefficients in (27) simplify as follows:

$$F \approx R_0 B_{\phi_0}, \quad C_2 = C_3 = 4\pi^2 \frac{\rho}{R_0}, \quad V_\rho = 4\pi^2 \rho R_0.$$

TABLE I
Variable Definition

Variables	*Description*	*Units*
ψ	Poloidal magnetic flux profile	Tm^2
ϕ	Toroidal magnetic flux profile	Tm^2
q	Safety factor profile $q \doteq d\phi/d\psi$	
R_0	Location of the magnetic center	m
B_{ϕ_0}	Toroidal magnetic field at the center	T
ρ	Equivalent radius of the magnetic surfaces	m
a	Location of the last closed magnetic surface	m
r	Normalized spatial variable $r \doteq \rho/a$	
t	Time	s
V	Plasma Volume	m^3
F	Diamagnetic Function	Tm
C_2, C_3	Geometric coefficients	
$\eta_{\|}$	Parallel resistivity	Ωm
η	Normalized diffusivity coefficient $\eta_{\|}/(\mu_0 a^2)$	
μ_0	Permeability of free space: $4\pi \times 10^{-7}$	Hm^{-1}
j_{ni}	Non-inductive effective current density	Am^{-2}
j	Normalized non-inductive effective current density $\mu_0 a^2 R_0 j_{ni}$	
j_ϕ	Effective current density	Am^{-2}
j_ω	Inductive current density	Am^{-2}
j_{eccd}	ECCD current density	Am^{-2}
j_{lh}	LHCD current density	Am^{-2}
j_{bs}	Bootstrap current density	Am^{-2}
I_p	Total plasma current	A
P_{lh}	Lower Hybrid antenna power	W
$N_{\|}$	Hybrid wave parallel refractive index	

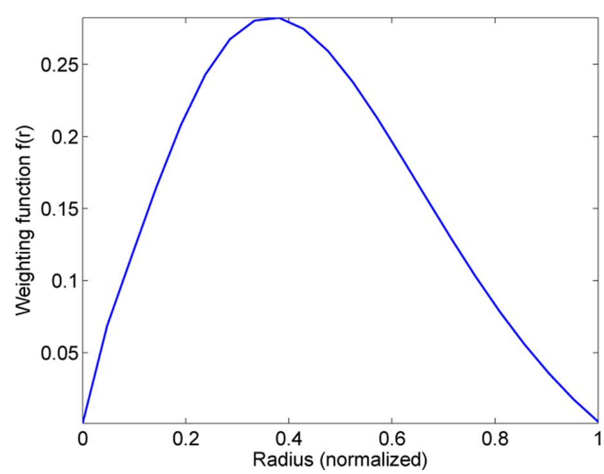

Fig. 3. Function f verifying the conditions of Theorem 3.2 for an exponential η with time-varying parameters. $f_{\min} = 0.001$, $f_{\max} = 0.2823$.

Defining a normalized spatial variable $r = \rho/a$, where a (assumed constant) is the equivalent (minor) radius of the last closed magnetic surface, the simplified model is obtained as in [2] and [37]

$$\psi_t(r,t) = \frac{\eta_{\|}(r,t)}{\mu_0 a^2} \left(\psi_{rr} + \frac{1}{r}\psi_r \right) + \eta_{\|}(r,t) R_0 j_{ni}(r,t) \quad (28)$$

with the boundary conditions

$$\psi_r(0,t) = 0$$

and

$$\psi_r(1,t) = -\frac{R_0 \mu_0 I_p(t)}{2\pi} \quad (29)$$

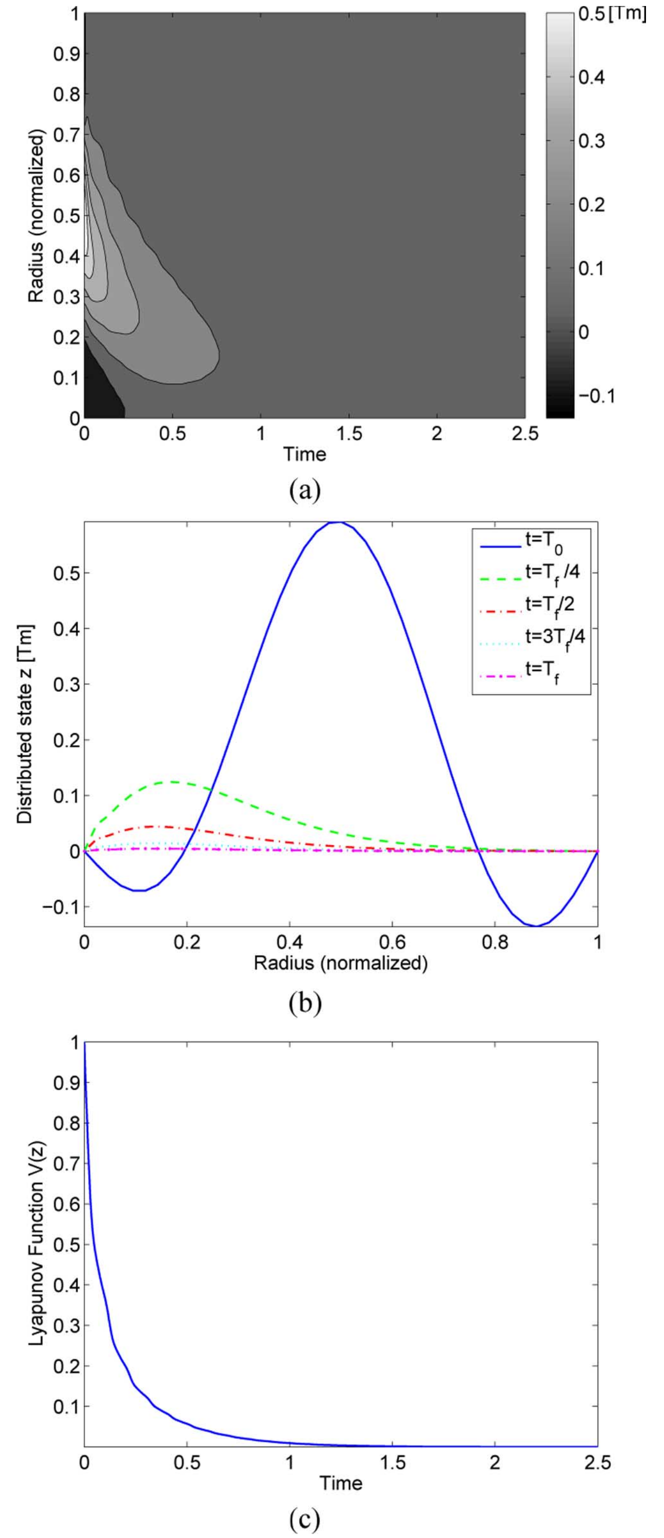

Fig. 4. Response of the nominal system without control action. (a) Contour plot of the solution to the PDE. (b) Time slices of the solution to the PDE. (c) Normalized evolution of the Lyapunov function.

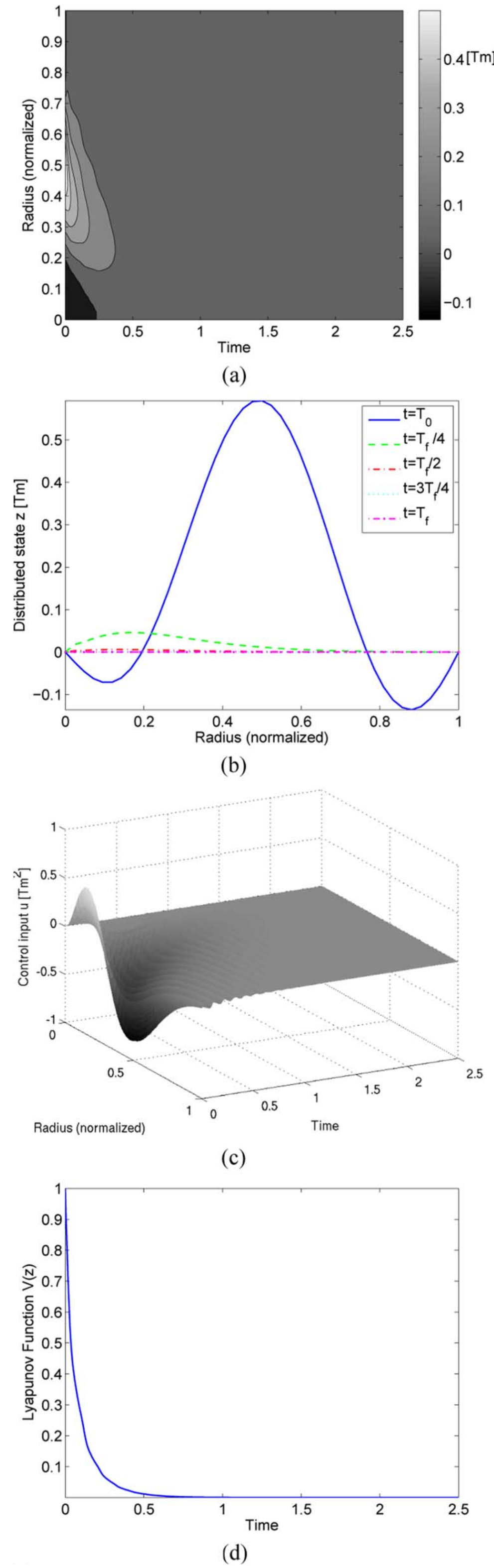

Fig. 5. Response of the nominal system with unconstrained control action ($\gamma = 1.6$). (a) Contour plot of the solution to the PDE. (b) Time slices of the solution to the PDE. (c) Evolution of the control u. (d) Normalized evolution of the Lyapunov function.

where I_p is the total plasma current, and with the initial condition

$$\psi(r, t_0) = \psi_0(r). \tag{30}$$

For the purposes of this paper, j_{ni} is considered as having one main component, which is the lower hybrid current drive

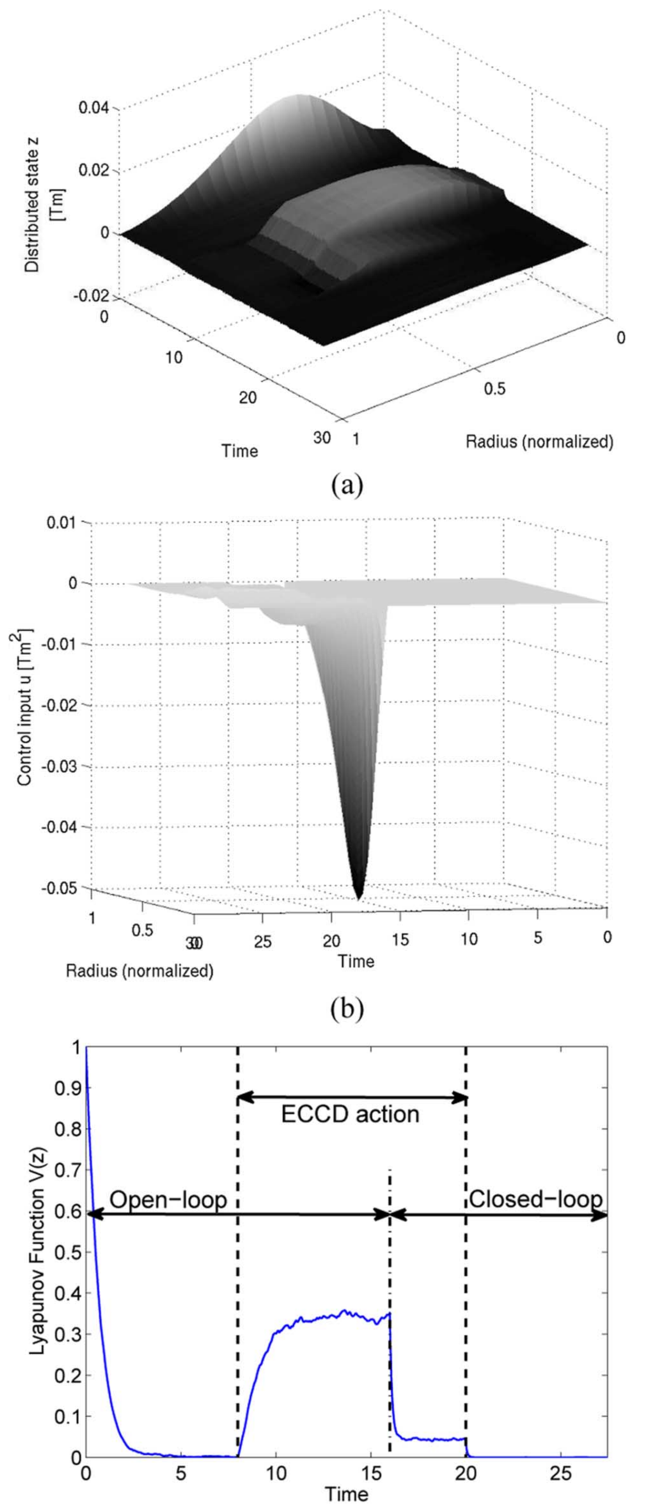

Fig. 6. Response of the disturbed system, disturbance applied at $t = 8$ s and removed at $t = 20$ s with unconstrained control action beginning at $t = 16$ s($\gamma = 0.75$). (a) Evolution of the z profile in time. (b) Evolution of the control u. (c) Normalized evolution of the Lyapunov function.

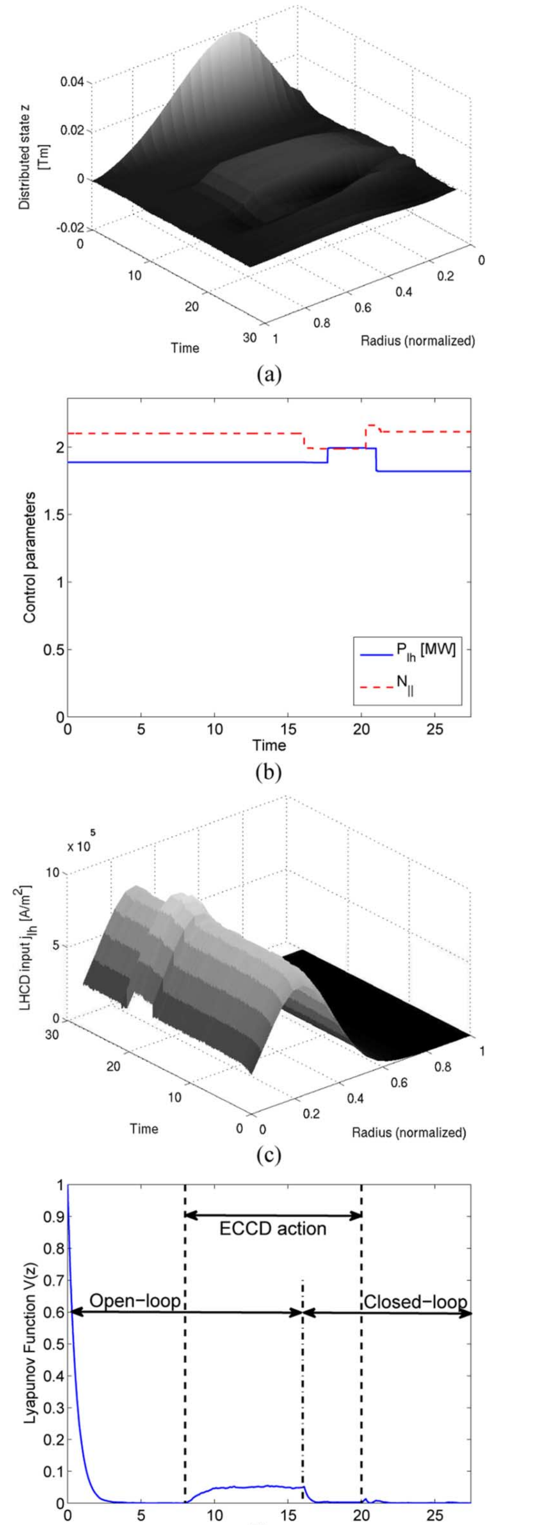

Fig. 7. Response of the disturbed system, disturbance applied at $t = 8$ s and removed at $t = 20$ s with constrained control action beginning at $t = 16$ s($\gamma = 0.6$). (a) Evolution of the z profile in time. (b) Antenna parameters used to calculate the control input. (c) Evolution of the actual j_{lh} applied to the system. (d) Normalized evolution of the Lyapunov function.

(LHCD) current deposit j_{lh}. The extension to other noninductive actuators is possible with minor modifications. Considering the evolution of the system around an equilibrium $(\overline{\psi}, \overline{j})$ and assuming an ideal tracking of the total plasma current, the evolution of ψ is given by (1)–(3). Defining $z \doteq \nabla\tilde{\psi} \cdot \overrightarrow{\rho}$, $\eta \doteq$

$\eta_{\|}/(\mu_0 a^2)$, and $u \doteq \tilde{j}$, where $\tilde{\psi} \doteq \psi - \overline{\psi}$ and $\tilde{j} \doteq j - \overline{j}$, properties P_1, P_2, and P_3 hold and, thus, the results of Sections III and IV apply. Furthermore, the implementation of a state feedback is possible due to the online availability of the magnetic flux profiles using the Equinox code. See [5].

B. Illustration of Stability: Numerical Computation of the Lyapunov Function

In order to test the proposed control law in Corollary 3.6 for the nominal system, we consider an identified estimate of the normalized plasma resistivity $\eta(r,t) = A(t)e^{\lambda(t)r}$, with $A(t) \doteq 0.0107 - 0.0014\cos 40\pi t$ and $\lambda(t) \doteq 6.1 + 0.8\sin 20\pi t$ for all $t \in [0,T)$. In particular, $0.0093 \leq A(t) \leq 0.0121$ and $4.3 \leq \lambda(t) \leq 6.9$ for all $t \in [0,T)$. The limits for the variations were chosen from data extracted from Tore Supra shot 35109, described in [37]. A function f, satisfying the conditions of Theorem 3.2 for these values of η, has been numerically computed using Mathematica. It is depicted in Fig. 3. It should be noted that, in practice, the knowledge of these coefficients does not need to be exact. It is enough to find a common weighting function f valid on a rich enough set of profiles (and, thus, on convex combinations of those profiles). Moreover, since α in (9) is positive, it provides a robustness margin with respect to small numerical errors.

Using this f, the time-evolution of (4) with boundary conditions (5), initial condition (6) and of the associated Lyapunov function V without control action ($u_{ctrl} = 0$), for an arbitrary numerical value of the initial condition, is shown in Fig. 4. The guaranteed convergence rate α is indeed respected but is conservative. This is not unexpected, since inequality (9) holds for all values of r and the central and edge diffusivities vary considerably (almost by a factor 1000).

Finally, the response of the system using the control defined in Corollary 3.6, with $\gamma = 1.6$ is shown in Fig. 5. Comparing Figs. 4(c) and 5(d), we can verify that the exponential decrease of V with the control defined in Corollary 3.6 is indeed increased by at least $e^{-\gamma t}$, in agreement with the theoretical results.

C. Illustration of ISS Property: Tokamak Simulation With an Unconstrained Controller

In order to test the controller defined in Corollary 3.6 in a more realistic setting, not only considering the evolution of the diffusion equation but also the dynamics of the diffusivity coefficients and other system parameters, the simulator presented in [37] was used to test the behavior of the system under the effect of disturbances and neglected inputs. In particular, the effect of the variation of the so-called bootstrap current (a plasma self-generated current source proportional to the inverse of the magnetic flux gradient that introduces a nonlinearity in the system dynamics) around the equilibrium and the electron cyclotron current drive (ECCD) input, turned on for $8s \leq t \leq 20s$, acts as unknown exogeneous current sources in the evolution equation. For a rigorous treatment, they can be considered as disturbances both in the state and input (as in Theorem 4.2 and Corollary 4.3). The variation of the resistivity coefficients is caused mainly by variations in the temperature profile, which is affected by the LH antenna.

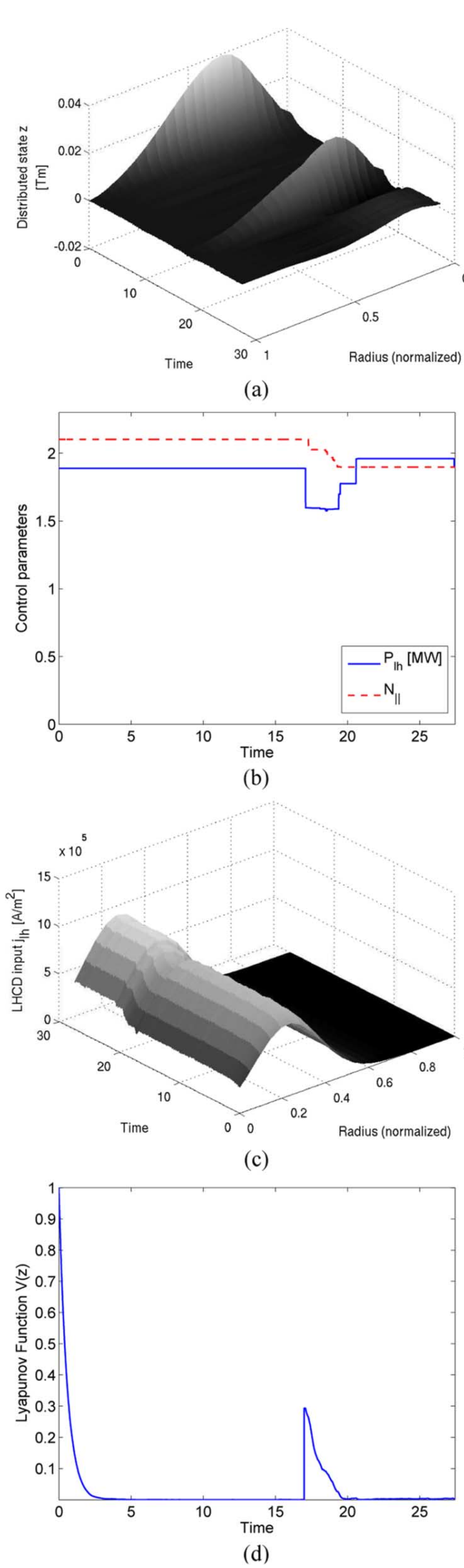

Fig. 8. Response of the system, change of reference applied at $t = 17$ s with constrained control action beginning at $t = 4$ s($\gamma = 0.6$). (a) Evolution of the z profile in time. (b) Antenna parameters used to calculate the control input. (c) Evolution of the actual j_{lh} applied to the system. (d) Normalized evolution of the Lyapunov function.

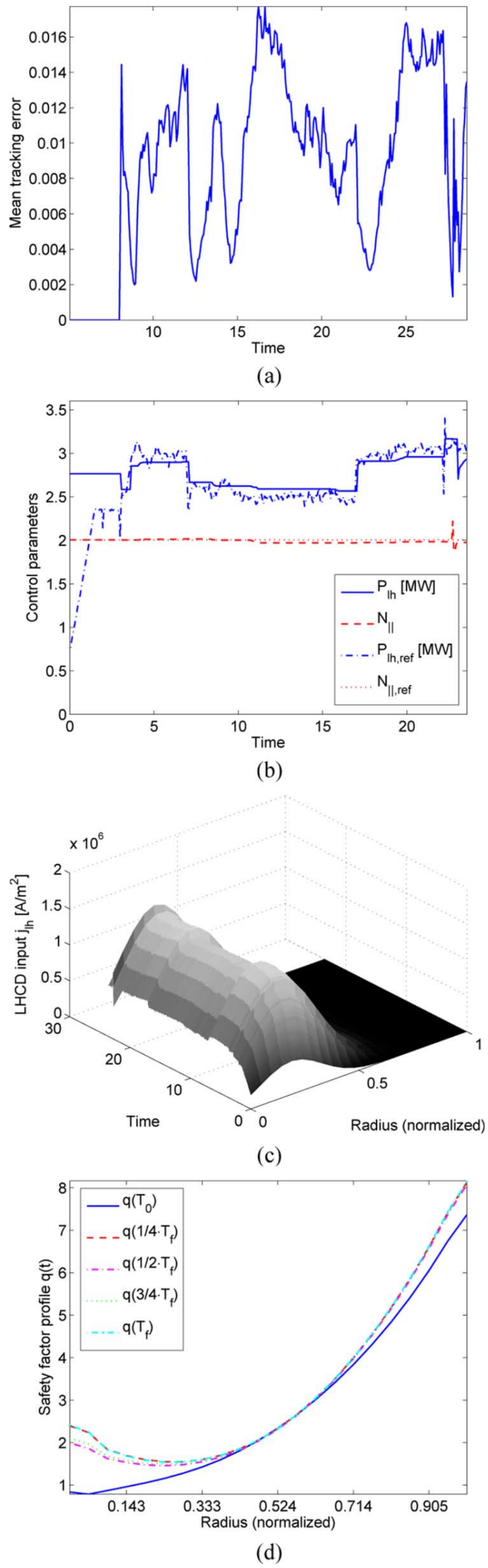

Fig. 9. Response of the system, with constrained control action beginning at $t = 3.1$ s($\gamma = 2.5$). (a) Evolution of the normalized mean error in time. (b) Antenna parameters used to calculate the control input. (c) Evolution of the actual j_{lh} applied to the system. (d) Evolution of the safety factor profile.

The original equilibrium was chosen from experimental data drawn from Tore Supra shot 35109. The effect of the ECCD antennas was overemphasized in order to better illustrate its action on the state and the Lyapunov function (the power in the simulation was chosen as three times the actual capacity of these actuators). A controller parameter $\gamma = 0.75$ was found to yield acceptable results (both in terms of the amplitude of the control and the effect of the noisy measurements in the system). The results are shown in Fig. 6, with control action starting at $t = 16$ s. While a steady-state error remains when the ECCD is turned on, it is significantly reduced by the feedback action. The convergence speed is also noticeably improved.

D. Exploiting the Lyapunov Approach: Tokamak Simulation with a Constrained Controller

In view of a possible implementation of the control law in a real tokamak experiment, strict constraints have to be imposed on the control action. For this application, the actuator considered is the current density generated by the lower hybrid waves. This current deposit profile $j_{lh}(r,t)$ depends on two main physical parameters: the power delivered by the antennas $P_{lh}(t)$ and the parallel refractive index $N_{\|}(t)$. In Tore Supra, two LH antennas exist and their parameters may vary in the following manner: $P_{lh,1} \leq 1.5$ MW, $P_{lh,2} \leq 3$ MW, $N_{\|,1} \in$[1.43,2.37] and $N_{\|,2} \in$[1.67,2.33]. However, in this paper, only one set of parameters $(P_{lh}, N_{\|})$ is used to derive a controller that illustrates the usefulness of the control Lyapunov function, as defined in Proposition 4.1, from a practical standpoint.

Based on Proposition 4.1, we propose to choose, at each time step, a couple $(P_{lh}^*, N_{\|}^*)$ as follows:

$$(P_{lh}^*, N_{\|}^*) = \arg \min_{(P_{lh}, N_{\|}) \in \mathcal{U}} \int_0^1 f(r) \left[\eta u(P_{lh}, N_{\|})\right]_r zdr \tag{31}$$

subject to the constraints

$$0 \geq \int_0^1 f(r) \left[\eta u(P_{lh}^*, N_{\|}^*)\right]_r zdr \geq -\gamma V(z) \tag{32}$$

where $\mathcal{U} \doteq [P_{lh,min}, P_{lh,\max}] \times [N_{\|,min}, N_{\|,\max}]$ and $u : \mathcal{U} \to C^\infty([0,1])$ is a nonlinear function representing the relation between the engineering parameters and the variations in the j_{lh} profile as presented in [37].

Remark 5.1: The inequality in the left-hand side of (32) guarantees that the worst case of the optimization scheme is $\int_0^1 f(r) \left[\eta u(P_{lh}^*, N_{\|}^*)\right]_r zdr = 0$. In other words, the closed-loop system verifies the ISS inequalities of Theorem 4.2 and Corollary 4.3 for a value of $\beta \geq \alpha$. The inequality in the right-hand side of (32) is not necessary for the stability of system (4)–(6), but aims to limit the contribution of the controller on the rate of convergence of the closed-loop system. If, for all time, $(P_{lh}, N_{\|}) \in \mathcal{U}$ exists so that the control proposed in Corollary 3.6 is exactly $u(P_{lh}, N_{\|})$, then it is a solution to the constrained optimization problem.

Since solving this optimization problem analytically is quite difficult, a numerical method using a gradient-descent algorithm on the discretized parameter space was implemented in practice. As the state dynamics describe the system deviation from an equilibrium, choosing $u = 0$ (i.e., $(P_{lh}, N_{\|}) = (\overline{P}_{lh}, \overline{N}_{\|})$) always gives a feasible starting point. In general, we might not find a solution of the proposed problem (31), and we could have problems facing local minima, but under simulation with data taken from Tore Supra shots 35109 and 31463 (the first generated by modulating the LH power, the second including also ECCD action), the results are satisfying.

The values of u and u_r for the different vertices of the parameter grid were calculated offline to allow real-time control. In this case, the mean time taken by the algorithm to determine the control values was 432 μs using a Matlab function running on a processor at 2.54 GHz.

For the first simulation, using an equilibrium point taken from Tore Supra shot 35109, we introduce a disturbance as in the previous section, corresponding to three times the maximum ECCD power for $8s \leq t \leq 20s$ and then activate the control at $t = 16$ s to attenuate its effect. The results are shown in Fig. 7. It can be seen that despite the constraints, the attenuation of the disturbance is very effective, with the value of the Lyapunov function rapidly reduced once the feedback control is activated. The control value was updated every 0.1 s, which is much greater than the required computing time.

The second proposed scenario is a change of operating point, where both equilibria were drawn from Tore Supra shot 35109. The control action starts at $t = 4$ s and the change of reference is applied at $t = 17$ s. The results can be seen in Fig. 8. It is interesting to see the behavior of the Lyapunov function under the constrained control: even though an exponentially decreasing upper bound exists, the actual shape is more irregular than in the unconstrained case (similar to a time-varying gain guaranteeing at all times a negative derivative for the Lyapunov function).

Finally, a more complicated tracking scenario is proposed, where a time-varying reference is generated from Tore Supra shot 31463 (which involves both LH and ECCD action). Furthermore, only one equilibrium point is calculated, corresponding to the mean value of the parameters applied during the shot instead of one for each point of the trajectory. Fig. 9 represents: (a) the mean tracking error, (b) the values for the engineering parameters of the LH antenna, (c) the LH current deposit profile, and (d) the safety factor profile. This result illustrates the robustness of the controller with respect to deviations from the calculated equilibrium (used in the computation of the feedback).

VI. Conclusion

In this paper, a strict Lyapunov function was found for a diffusion equation with time-varying distributed coefficients. This function guarantees some ISS properties for the system and allows for the construction of simple control laws that maintain these properties and improve the performance of the system. A particularly important contribution was a robustness study of the system with respect to disturbances and errors in the model and measurements, since for most physical applications, the exact values and behavior of the diffusivity coefficients are not well known. Another contribution is the consideration of the distributed and time-varying nature of these coefficients in the nominal scenario without constraining their rate of variation. Finally, the proposed Lyapunov function design was applied to the control of the gradient of the poloidal magnetic flux profile in the Tore Supra tokamak, with the objective of safety factor regulation.

Future work will be devoted to the implementation and testing of the proposed constrained control law with a more complex simulation code, METIS and/or CRONOS (see [1] and [2], respectively). These codes include energy and momentum conservation laws as well as refined plasma/wave interaction descriptions for the antennas. Some effort will also be devoted to the estimation of the diffusivity coefficients in view of an experimental implementation on Tore Supra.

Appendix

Proof of Theorem 2.1: This proof is organized as follows.

a) First, an auxiliary problem in 2-D Cartesian coordinates under symmetry conditions is formulated.

b) Next, the existence and uniqueness of solutions to the auxiliary problem are shown using [18, Theor. 5.1.21, Cor. 5.1.22, pp. 206–208], which, in turn, imply the existence and uniqueness of solutions to the problem (4)–(6).

i) Consider the following 2-D Cartesian auxiliary system:

$$\begin{aligned}\zeta_t(x_1,x_2,t) = &\eta(x_1,x_2,t)\Delta\zeta(x_1,x_2,t)\\ &+ F(x_1,x_2,t), \forall\,(x_1,x_2,t)\\ &\in \Omega\times[0,T]\end{aligned} \tag{33}$$

with symmetric boundary condition

$$\zeta_\nu(x_1,x_2,t) = 0,\ \ \forall\,(x_1,x_2,t)\in\partial\Omega\times[0,T] \tag{34}$$

where ζ_ν is the derivative of ζ in the outward normal direction to $\partial\Omega$, and with the symmetric initial condition $\zeta_0 \in \mathcal{C}^{3+\alpha_c}(\overline{\Omega})$, $0<\alpha_c<1$

$$\zeta(x_1,x_2,0) = \zeta_0(x_1,x_2),\ \ \forall\,(x_1,x_2)\in\Omega \tag{35}$$

where Δ is the Laplacian $F(x_1,x_2,t) \doteq \eta(x_1,x_2,t)u(x_1,x_2,t)$. This system is equivalent, when imposing a central symmetry condition and sufficient regularity of the initial condition, to (1)–(3).

ii) To apply [18, Theor. 5.1.21, Cor. 5.1.22, pp. 206–208] it must be shown first that the diffusive operators verify a uniform ellipticity condition in $\overline{\Omega}$. This is trivially verified as a direct consequence of P_1 and, therefore, Theorem 5.1.21 gives the existence and uniqueness of solutions, and Corollary 5.1.22 establishes the desired regularity (so that the gradient is in $C^{2+\alpha_c,1+\alpha_c/2}(\overline{\Omega}\times[0,T])$). This degree of regularity is sufficient to ensure that all of the integrals used for the definition of the Lyapunov function and its time derivative are well defined. This concludes the proof of Theorem 2.1.

■

Existence, uniqueness, and regularity results are also valid when the control input is of the form proposed in Corollary 3.6 (which amounts to a feedback in the variable ζ). and can extend to certain forms of nonhomogeneous boundary conditions thanks to the structure of the operators considered in [18].

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