Robust Stabilization of Chained Systems via Hybrid Control

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***Abstract*—The problem of global robust exponential stabilization of nonholonomic chained systems is addressed and solved by means of a hybrid state feedback control law. It is shown that the control law yields global exponential stability and global robustness against a class of small measurement errors, actuator noises and exogenous disturbances.**

***Index Terms*—Chained systems, exogenous disturbances, hybrid control, measurement errors, robust stabilization.**

I. Introduction

The problem of asymptotic stabilization of nonholonomic systems, and in particular of chained systems, has been widely studied in the last decade and several control laws, yielding diverse asymptotic properties, have been proposed; see [8] and the references therein. On the contrary, the robust stabilization problem for nonholonomic systems is not yet completely solved. Several attempts have been made to study the robustness properties of existing control laws or to robustify given controllers [3], [7]. Most of the robust stabilization results and investigations focus on the problems of parametric uncertainties or model errors, see, e.g., [11] where the problem of local robust stabilization by means of time-varying control laws have been studied; [7], where a similar problem has been addressed using the class of discontinuous control laws introduced in, e.g., [1] and [6], where several types of hybrid control laws have been used to achieve local robustness against unknown parameters or unmodeled dynamics. On the other hand, the fundamental problems of robustness in the presence of sensor noise, external disturbances and actuator disturbances have been only partially addressed; see, e.g., [3] and [17].

These problems are of special interest and relevance whenever discontinuous control laws are employed, as for such control laws classical robustness results and Lyapunov theory are not directly applicable; see, however, [10], where a discontinuous control law, possessing a Lyapunov stability property, has been constructed. Following the line of research started in [1], we make use of a special class of discontinuous control laws, and we show how, adding a proper modification together with a *hybrid variable*, it is possible to obtain a closed-loop system with global stability properties and which is globally robust against measurement noises and exogenous disturbances. The controller construction proposed in this note takes inspiration partly from the results in [14], and partly from the results in [17]. In the former, a hybrid control law achieving robust stabilization of the so-called Artstein circle has been proposed; whereas the latter provides some basic tools to construct two control laws, the so-called *local controller* and *global controller* that, together with the hybrid dynamics, yield a robust closed-loop system. Note, finally, that the main result of the note relies on the general results in [13], which is however not directly applicable because of the special nature of the system considered.

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In the rest of this note, we focus on n-dimensional chained systems with two controls (see [12] for details), i.e., systems described by equations of the form

$$\dot{x}_1 = u_1 \quad \dot{x}_2 = u_2 \quad \dot{x}_3 = x_2 u_1, \ldots, \dot{x}_n = x_{n-1} u_1 \tag{1}$$

and we address the problem of robust exponential stabilization in the presence of measurement errors, external disturbances and actuators errors.

We conclude this section noting that chained systems are asymptotically controllable, thus, by the general results in [15], there exists a hybrid feedback rendering the origin a robustly globally asymptotically stable equilibrium. In the present work such a feedback is *explicitly constructed*, it is shown that only a *finite* number of hybrid variables is needed in the construction and moreover we have *exponential* stabilization. It is worth noting that the problem of stabilization of nonholonomic systems by means of hybrid control has been also studied in [6]. Therein, the authors use a *hysteresis switching logic* rendering the equilibrium attractive with a robustness property. Note, however, that the switching logic therein depends on a *hysteresis constant*, whereas in our context the time between two switches depend only on the measured state.

The note is organized as follows. In Section II, the class of controllers used in the note is introduced and we present the main result of the note, while in Sections III and IV a control law robustly exponentially stabilizing system (1) is described. Section V contains the proof of the main result. Finally, Section VI summarizes the main contributions of this work.

II. Class of Controllers and Main Result

The controllers under consideration admit the following description (see [4] and [16]):

$$u = k(x, s_d) \quad s_d = k_d(x, s_d^-) \tag{2}$$

where s_d evolves in the finite set {1,2}, $k\colon \mathbb{R}^n \times \{1,2\} \to \mathbb{R}^2$ is continuous in x for each fixed s_d, $k_d\colon \mathbb{R}^n \times \{1,2\} \to \{1,2\}$ is a function and s_d^- is defined, at this stage only formally, as $s_d^-(t) = \lim_{s<t} s_d(s)$. For this to make sense, we equip {1,2} with the discrete topology, i.e., every set is an open set. The above controller is hybrid due to the presence of the discrete dynamics of s_d. Denoting with $f\colon \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$ the function defining the right hand-side of the differential equations (1), we can rewrite (1) as

$$\dot{x} = f(x, u). \tag{3}$$

In this note, we are interested in a notion of robustness to small noise. To this end, consider two functions e and d satisfying the following *regularity assumptions*: e and d are in $\mathcal{L}^{\infty}_{loc}(\mathbb{R}^n \times [0, +\infty); \mathbb{R}^n)$, and are continuous in $x \in \mathbb{R}^n$ for each $t \in [0, +\infty)$. We introduce[1] these functions as a measurement noise e and an external noise d and define the perturbed system with u given by (2), i.e.,

$$\begin{aligned}\dot{x} &= f\left(x(t), k\left(x(t) + e\left(x(t), t\right), s_d(t)\right)\right) + d\left(x(t), t\right)\\ s_d(t) &= k_d\left(x(t) + e\left(x(t), t\right), s_d^-(t)\right).\end{aligned} \tag{4}$$

We have to make precise what we mean by solution of the corresponding differential equation. To this end, we adapt the definition of [4] to the present context of a perturbed hybrid system (see also

[1]We can also consider an actuator noise; see [15].

[5] and [16]). Hence, we rewrite the definition given in [14] and we introduce a nonempty set $\mathcal{RC}$ strictly included in $\mathbb{R}^n \times \{1,2\}$.

Definition 2.1: Given $T > 0$, $(x_0, s_0) \in \mathbb{R}^n \times \{1,2\}$, we say that (X, S_d) is a *solution*, starting from (x_0, s_0), of (4) on $[0,T)$ if the following holds.

1) The map X is absolutely continuous on $[0,T)$.
2) For almost all t in $[0,T)$, we have $\dot{X}(t) = f(X(t), k(X(t) + e(X(t),t), S_d(t)) + d(X(t),t)$.
3) For all $t \in [0,T)$ such that $(X(t), S_d(t))$ is in $\mathcal{RC}$, the map S_d is right-continuous at t.
4) For all $t \in (0,T)$ such that $S_d^-(t)$ exists, one has $S_d(t) = k_d(X(t) + e(X(t),t), S_d^-(t))$.
5) $X(0) = x_0$ and $S_d(0) = k_d(x_0 + e(x_0,0), s_0)$.

In this context, the definition of global exponential stability can be given as follows (we denote the usual Euclidean norm by $|.|$).

Definition 2.2: Let e and d be two functions satisfying our standing regularity assumptions. The origin of the system (4) is said to be a *globally exponentially stable equilibrium* on $\mathbb{R}^n$ if the following two properties hold.

- For every (x_0, s_0) in $\mathbb{R}^n \times \{1,2\}$, there exists a solution of (4) starting from (x_0, s_0). Moreover, all maximal solutions of (4) are defined on $[0, +\infty)$.
- There exists δ of class $\mathcal{K}_\infty$ and $C > 0$ such that, for all $r > 0$ and for all (x_0, s_0) in $\mathbb{R}^n \times \{1,2\}$ with $|x_0| \le \delta(r)$ and for all maximal solutions (X, S_d) of (4) starting from (x_0, s_0), one has

$$|X(t)| \le r e^{-Ct} \qquad \forall t \ge 0. \tag{5}$$

As we are interested in a notion of robustness with respect to small noise, we introduce the following.

Definition 2.3: The controller (k, k_d) is a *robustly globally exponentially stabilizing controller* for (1) if there exists a continuous function $\rho: \mathbb{R}^n \to \mathbb{R}$, such that, for all $x \neq 0$, $\rho(x) > 0$, and such that for any two functions e and d satisfying our standing regularity assumptions and

$$\sup_{\mathbb{R}_{\ge 0}} |e(x,.)| \le \rho(x), \operatorname{esssup}_{\mathbb{R}_{\ge 0}} |d(x,.)| \le \rho(x) \tag{6}$$

for all x in $\mathbb{R}^n$, the origin of (4) is a globally exponentially stable equilibrium on $\mathbb{R}^n$.

The problem of global robust exponential stabilization, in the presence of small measurement and external noises, is solvable.

Theorem 1: There exists a hybrid controller (k, k_d), $k: \{1,2\} \times \mathbb{R}^n \to \mathbb{R}^2$ and $k_d: \mathbb{R}^n \times \{1,2\} \to \{1,2\}$, which is a robustly globally exponentially stabilizing controller for (1).

Remark 2.4: Some observations are in order.

1) In [17, Prop. 3], for any sufficiently small disturbances, the controller renders the origin of the perturbed closed-loop system attractive and locally stable. In Theorem 1, imposing more structure on the perturbations we obtain a global result.
2) The hybrid controller (k, k_d) can be explicited (see Section IV). Note also, that, to prove Theorem 1, we need to define two controllers: the "local controller" and the "global controller". The local controller is continuous on $\mathbb{R}^n \setminus \{x_1 = 0\}$, therefore it is necessary to *use* another controller (the global controller) on a neigborhood of $\{x_1 = 0\}$ (more precisely on a cone) and to *define* an adequate switching strategy.
3) The idea of *switching* between a *local* and a *global* controller to achieve stabilization in the large has been advocated in several papers, and typically in the context of stabilization of unstable equilibria of mechanical systems. Note, however, that in the present context the distance between the sets in which the local and the global controllers are defined is always zero, and moreover we aim at achieving robust asymptotic stability rather than asymptotic stability.

We define the local controller in Section III-A and the global one in Section III-B. We join the domain of definition of this feedbacks by means of a hysteresis, as detailed in Section IV, and finally we prove Theorem 1 in Section V.

III. Components of the Hysteresis

A. Local Controller

Consider (1) and the control law $u_l: \mathbb{R}^n \to \mathbb{R}^2$ defined by

$$\begin{aligned} u_{1l}(x) &= -x_1 \\ u_{2l}(x) &= p_2 x_2 + p_3 \frac{x_3}{x_1} + p_4 \frac{x_4}{x_1^2} + \cdots + p_n \frac{x_n}{x_1^{n-2}} \end{aligned} \tag{7}$$

with the p_i such that the matrix

$$A = \begin{bmatrix} p_2+1 & p_3 & p_4 & \cdots & p_{n-1} & p_n \\ -1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & n-1 \end{bmatrix}$$

is Hurwitz. (A simple analysis shows that the eigenvalues of the matrix A can be arbitrarily assigned by a proper selection of the coefficients p_i). Let $P = P^T > 0$ be such that $A^T P + PA < 0$, and let z be a variable in $\mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} z = z(x) &= Y^T P Y, \quad \text{if } x_1 \neq 0 \\ &= +\infty, \qquad \text{if } x_1 = 0 \end{aligned} \tag{8}$$

for all x in $\mathbb{R}^n$, with $Y \in \mathbb{R}^{n-1}$ defined by $Y = Y(x) = [(x_2/x_1)\ (x_3/x_1^2)\ \cdots\ (x_n/x_1^{n-1})]^T$, $\forall x \in \mathbb{R}^n$, $x_1 \neq 0$. Let e and $d: \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ be two perturbations satisfying our standing regularity assumptions. The closed-loop system in consideration in this section is the system

$$\dot{x} = f(x, u_l(x+e)) + d \tag{9}$$

and for such a system the following statement holds.

Lemma 3.1: There exists a continuous function $\rho_l: \mathbb{R} \to \mathbb{R}$ satisfying $\rho_l(\xi) > 0$, $\forall \xi \neq 0$, such that for all e, $d: \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ satisfying our standing regularity assumptions and

$$\sup_{\mathbb{R}_{\ge 0}} |e(x,.)| \le \rho_l(x_1), \operatorname{esssup}_{\mathbb{R}_{\ge 0}} |d(x,.)| \le \rho_l(x_1) \tag{10}$$

for all x in $\mathbb{R}^n$, and for all x_0 satisfying $z(x_0) \le M$, there exists a Carathéodory solution X of (9) starting from x_0 and all such Carathéodory solutions are maximally defined on $[0, +\infty)$.

Moreover, there exists a function δ_l of class $\mathcal{K}_\infty$ and $C > 0$ such that, for all r and M, and for all x_0 satisfying $|x_0| \le \delta_l(r)$ and $z(x_0) \le M$, we have $|X(t)| \le r\sqrt{M} e^{-Ct}$ and $z(t) \le M e^{-Ct}$, for all $t \ge 0$.

Remark 3.2: Lemma 3.1 states that, for any $M > 0$, the region $z(x) \le M$ is robustly forward invariant, i.e., it is positively invariant in the presence of a class of measurement and external noise. Moreover, any trajectory in such a region converges exponentially to the origin. Note that a somewhat simpler and local version of Lemma 3.1 has been given in [17]. ◇

Proof of Lemma 3.1: To begin with note that the Carathéodory conditions are met for system (9). Therefore, we have the existence of a unique forward Carathéodory solution of (9) for any initial condition.

Let x in $\mathbb{R}^n$ be such that $x_1 \neq 0$. Let us impose that $\rho_l(\xi) \le |\xi|/2$, $\forall \xi \in \mathbb{R}$. The function $x \mapsto Y(x)^T A^T P Y(x)$ is continuous on $\mathbb{R}^n \setminus \{x, x_1 = 0\}$. As a result, there exists ρ_l and $C_1 > 0$ such that for all perturbations satisfying (10), we have $\dot{z} \leq C_1(Y^T(x)A_T PY(x) + Y^T(x)PAY(x))$. This implies with $A^T P + PA < 0$ and the positivity of P, the existence of two real numbers $C_2 > 0$ and $C_3 > 0$, such that $\dot{z} \leq -C_2|Y|^2 \leq -C_3 z$. Therefore, there exists a function ρ_l such that, for all e and d satisfying our standing regularity assumptions and (10), along all Carathéodory solutions of (9) the variable z, hence, the variable Y, converges (exponentially) to zero.

Furthermore, due to (8) and the positivity of P, there exists $C_4 > 0$ such that, for all z, $|x_2|/|x_1| \leq C_4\sqrt{z}, \cdots$, and $|x_n|/|x_1|^{n-1} \leq C_4\sqrt{z}$. Thus, by (1) and (7), along all Carathéodory solutions, x_1 tends exponentially to zero.

We conclude that choosing the function $\rho_l\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ sufficiently small and satisfying $\rho_l(\xi) > 0, \forall \xi \neq 0$, the properties of stability and attractivity, for all perturbations satisfying (10), are established, and this completes the proof of Lemma 3.1. □

B. Global Controller

Let $\mu > 0$ and consider the control law u_g defined on $\mathbb{R}^n$ by

$$u_{1g} = 1 \quad u_{2g} = -\mu x_2. \tag{11}$$

The closed-loop system in consideration in this section is

$$\dot{x} = f\left(x, u_g(x+e)\right) + d \tag{12}$$

and for such a system the following statement holds.

Lemma 3.3: There exists a continuous function $\rho_g\colon \mathbb{R}^n \to \mathbb{R}$ satisfying $\rho_g(x) > 0, \forall x \neq 0$, such that, for any initial condition, all perturbed systems (12), where e and d are two functions satisfying our standing regularity assumptions and (6) with $\rho = \rho_g$, admit a unique Carathéodory solution, defined for all $t \geq 0$.

Moreover, there exists a function δ_g of class $\mathcal{K}_\infty$ such that, for any $r > 0$ and for any $M > 0$, there exists a time $T_g = T_g(M, \delta_g(r))$ such that, for all Carathéodory solutions X of (12) with initial condition x_0 with $|x_0| \leq \delta_g(r)$, one has $z(X(t)) \leq M$ for all $t \geq T_g$, and $|X(t)| \leq r$, for all $t \leq T_g$.

Remark 3.4: Lemma 3.3 states that, for any $M > 0$, the trajectories of (12) enter the region $z(x) \leq M$ in finite time, while remaining bounded up to this time. ⋄

Proof of lemma 3.3: To begin with note that, by the regularity assumptions on e and d, (12) satisfies the Carathéodory conditions. Therefore, there exists a unique forward Carathéodory solution of (12) for any initial condition. Moreover, a direct integration of the closed-loop system (3)–(11) yields

$$\begin{aligned}
x_1(t) &= x_{10} + t \quad x_2(t) = x_{20}e^{-\mu t}\\
x_3(t) &= x_{30} + \frac{x_{20}}{\mu}(1 - e^{-\mu t}), \quad \ldots\\
x_n(t) &= \sum_{j=3}^{n} \frac{x_{j0}t^{n-j}}{(n-j)!}\\
&\quad + \sum_{j=4}^{n} \frac{(-1)^{j-4}x_{20}t^{n-j+1}}{\mu^{j-3}(n-j+1)!}\\
&\quad + \frac{(-1)^{n-3}x_{20}}{\mu^{n-2}}(1 - e^{-\mu t}).
\end{aligned} \tag{13}$$

As a result $\lim_{t\to\infty} z(t) = 0$, so there exists a finite time $t^*_{x(0)} \geq 0$ (minimally defined) such that $z(t^*_{x(0)}) < M$. Moreover, for any $C > 0$, and by (8) and (13), $T = \sup\{t^*_{x(0)}, |x(0)| \leq C\}$ is finite, and this implies the stability property as stated in Lemma 3.3. Finally, insensitivity with respect to small perturbations e and d satisfying (6) results from the right-continuity of (12). □

IV. Definition of the Hybrid Controller

In this section, we define the hybrid controller robustly stabilizing system (1). To this end, for any strictly positive real number M, we define the subset Γ_M of $\mathbb{R}^n$ as

$$\Gamma_M = \{x, x_1 \neq 0,\ z < M\} \tag{14}$$

where z is defined by (8). In the three-dimensional case, these sets are cones with axis x_1 and are symmetric with respect to $\{x_1 = 0\}$. Let $M_6 > \cdots > M_1 > 0$. For simplicity, in what follows, for all $i \in \{1, \ldots, 6\}$, we define $\Gamma_i := \Gamma_{M_i}$. The hybrid controller (k, k_d) is defined making a hysteresis between u_l and u_g on Γ_5 and Γ_2, i.e.,

$$\begin{aligned}
k : \{1,2\} \times \mathbb{R}^n &\to \mathbb{R}^2\\
(s_d, x) &\mapsto u_l(x), \quad \text{if } s_d = 1,\ x_1 \neq 0\\
&\qquad 0, \qquad\ \ \text{if } s_d = 1, x_1 = 0\\
&\qquad u_g(x), \quad \text{if } s_d = 2
\end{aligned} \tag{15}$$

$$\begin{aligned}
k_d : \mathbb{R}^n \times \{1,2\} &\to \{1,2\}\\
(x, s_d) &\mapsto 1, \qquad \text{if } x \in \Gamma_2 \cup \{0\}\\
&\qquad s_d, \qquad \text{if } x \in \Gamma_5 \setminus \Gamma_2\\
&\qquad 2, \qquad \text{if } x \notin \Gamma_5 \cup \{0\}.
\end{aligned} \tag{16}$$

V. Properties of the Solutions and Proof of Theorem 1

A. Properties of the Solutions

In this section, we study some properties of the solutions of all perturbed systems (4) in closed-loop with the hybrid controller (15) and (16). To this end, consider the sets[2] $\Sigma_{1\to 2} = \mathrm{clos}(\Gamma_3) \setminus \Gamma_1, \Sigma_{2\to 1} = \mathrm{clos}(\Gamma_6) \setminus \Gamma_4$, the set (see [14] for a similar, yet simpler, situation)

$$\mathcal{RC} = \mathbb{R}^n \times \{1,2\} \setminus \left(\Sigma_{1\to 2} \times \{2\} \bigcup \Sigma_{2\to 1} \times \{1\}\right). \tag{17}$$

Let $\rho\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that

- for all $x \neq 0$, $\rho(x) > 0$;
- we have

$$\rho(x) \leq \rho_l(x_1),\ \forall x \in \Gamma_6 \tag{18}$$

$$\rho(x) \leq \rho_g(x),\ \forall x \in \mathbb{R}^n \setminus \Gamma_1 \tag{19}$$

  where ρ_l and ρ_g are defined in Lemma 3.1 and Lemma 3.3, respectively;
- the following implication holds, $\forall i \in \{1, \cdots, 5\}$, $\forall e \in \mathbb{R}^n$, $|e| \leq \rho(x)$

$$z(x+e) \leq M_i \Rightarrow z(x) \leq M_{i+1}; \tag{20}$$

- the following implication holds, $\forall i \in \{2, \cdots, 6\}$, $\forall e \in \mathbb{R}^n$, $|e| \leq \rho(x)$:

$$z(x+e) \geq M_i \Rightarrow z(x) \geq M_{i-1}. \tag{21}$$

Let e and d be two functions satisfying our standing regularity assumptions and (6). To begin with, we show that (4) has a solution for any initial condition (the proof is similar to the proof of [14, Prop. 6].

Lemma 5.1: For all (x_0, s_0) in $\mathbb{R}^n \times \{1,2\}$, there exists a solution of (4) starting from (x_0, s_0).

[2]In the forthcoming discussion, for a given subset Ω of $\mathbb{R}^n$, the closure, the interior and the boundary of Ω are denoted as $\mathrm{clos}(\Omega)$, $\mathrm{int}\Omega$ and $\partial\Omega$, respectively.

To describe further properties of the trajectories of the considered system, we need to recall the definition of *switch* time.

Definition 5.2: A map S_d: $[0, T) \to \{1, 2\}$ is said to have a *switch* at time t if S_d is not continuous at t. Let (X, S_d) be a solution of (4) and consider the problem of locating the points where S_d may have a switch. To address this problem, for all t in (0,T), consider the sets

$$S_d^p(t) = \left\{ s : \exists t_n \in [t, T), t_n \underset{n\to\infty}{\longrightarrow} t, S_d(t_n) \underset{n\to\infty}{\longrightarrow} s \right\}$$
$$S_d^m(t) = \left\{ s : \exists t_n \in [t_0, t], t_n \underset{n\to\infty}{\longrightarrow} t, S_d(t_n) \underset{n\to\infty}{\longrightarrow} s \right\}.$$

With the aid of such sets it is easy to provide some properties of trajectories with a *switch*, as detailed in the following statement.

Lemma 5.3: Let (X, S_d) be a solution of (4) such that S_d has a switch at time $t \in (0, T)$.

- If the switch is such that $2 \in S_d^m(t)$ and $1 \in S_d^p(t)$, then $X(t)$ is in $\mathrm{clos}(\Gamma_3) \setminus \Gamma_1$.
- If the switch is such that $1 \in S_d^m(t)$ and $2 \in S_d^p(t)$, then $X(t)$ is in $\mathrm{clos}(\Gamma_6) \setminus \Gamma_4$.

As a third result, we can prove that, as usual, maximal solutions of (4) must blow up if their domains of definition are bounded.

Lemma 5.4: Let (X, S_d) be a maximal solution of (4) defined on $[0, T)$. Suppose $T < +\infty$, then $\limsup_{t\to T} |X(t)| = +\infty$.

We conclude this series of preliminary results, by discussing the behavior of the solutions between switches.

Lemma 5.5: Let (X, S_d) be a maximal solution of (4) defined on $[0, T)$. Then, $T = +\infty$ and only three cases are possible.

1) There exists no switch and X is a Carathéodory solution of (9) on $[0, +\infty)$ and remains in Γ_6.
2) There exists a time σ in $(0, +\infty)$ such that
   - the map X is a Carathéodory solution of (12) on $[0, \sigma)$ and is not in Γ_1;
   - the map X is a Carathéodory solution of (9) on $[\sigma, +\infty)$ and remains in Γ_6.
3) There exists two switches σ_1 and σ_2 in $(0, +\infty)$, such that
   - the map X is a Carathéodory solution of (9) on $[0, \sigma_1)$ and remains in Γ_6;
   - the map X is a Carathéodory solution of (12) on $[\sigma_1, \sigma_2)$ and is not in Γ_1;
   - the map X is a solution of (9) on $[\sigma_2, +\infty)$ and remains in Γ_6.

Proof of lemma 5.5: To begin with, we establish the following fact.

Claim 5.6: There cannot exist two consecutive switches σ_1,σ_2 such that $2 \in S_d^m(\sigma_1)$, $1 \in S_d^p(\sigma_1)$ and $2 \in S_d^p(\sigma_2)$.

Proof of Claim 5.6: Suppose the conclusion of Claim 5.6 does not hold. Then, by Lemma 5.3, $X(\sigma_1) \in \mathrm{clos}(\Gamma_3) \setminus \Gamma_1$, $X(\sigma_2) \in \mathrm{clos}(\Gamma_6) \setminus \Gamma_4$ and, by (14)

$$M_1 < z\,(X(\sigma_1)) \le M_3 \text{ or } X_1(\sigma_1) = 0 \tag{22}$$
$$M_4 < z\,(X(\sigma_2)) \le M_6 \text{ or } X_1(\sigma_2) = 0. \tag{23}$$

Moreover, by Definition 2.1, $S_d(t) = 1$ for all t in (σ_1, σ_2). Therefore, by (15), X is a solution of (9) on (σ_1, σ_2) and, by (16), $X(t) + e(X(t), t)$ is in Γ_5 and, thus, by (20), $X(t)$ is in Γ_6, for all t in (σ_1, σ_2).

Note now that condition (18) implies that the conclusions of Lemma 3.1 hold, hence, by (22), Γ_3 is forward invariant. Therefore $z(X(\sigma_2)) \le M_3$, and this contradicts (23). □

We are now ready to prove that $T = +\infty$. For, suppose $T < +\infty$, then by Claim 5.6 and (15) there exists $\sigma < T$ such that X is a solution of (9) (if $S_d(\sigma) = 1$) or of (12) (if $S_d(\sigma) = 2$) on $[\sigma, T)$. However, by Lemma 5.4, $\limsup_{t\to T} |X(t)| = +\infty$, and this contradicts the conclusions of Lemma 3.1 (if $S_d(\sigma) = 1$) or of Lemma 3.3 (if $S_d(\sigma) = 2$). As a consequence, $T = +\infty$.

To complete the proof of Lemma 5.5, note that, for all t in $[0, +\infty)$, $S_d(t) = 1$ (resp. $S_d(t) = 2$) implies, by (16), that $X(t) + e(X(t), t)$ is in Γ_5 (respectively, not in Γ_2) and thus, by (6), (20) and (21), $X(t)$ is in Γ_6 (respectively, not in Γ_1). □

Remark 5.7: An interesting consequence of Claim 5.6 is that, along any trajectory of the perturbed system, there are strictly less than four switches. ◇

B. Proof of Theorem 1

Existence of solutions follows from Lemma 5.1 and *maximality* from Lemma 5.5.

Global Stability: Let $r > 0$ and consider perturbations e, d: $\mathbb{R}^n \times \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ satisfying our standing regularity assumptions and (6), for all x in $\mathbb{R}^n$. Let (X, S_d) be a solution of (4) with $|X(0)| < \min(\delta_l(r), \delta_g(r))$, and defined on $[0, +\infty)$. To prove the global stability property, we now show that, for all $t \ge 0$

$$|X(t)| \le \max\left(r\sqrt{M_6}, r, \delta_l^{-1}(r)\sqrt{M_6}, \delta_g^{-1}(\sqrt{M_6}r), \delta_l^{-1}\left(\delta_g^{-1}(\sqrt{M_6}r)\right)\sqrt{M_6}\right). \tag{24}$$

To this end, note that, by Lemma 5.5, only three cases may occur.

1) Suppose Case 1 of Lemma 5.5 holds. Then, by Lemma 3.1 and (14), (24) holds.
2) Suppose Case 2 of Lemma 5.5 holds. Then, by Lemma 3.3, we have $\sigma \le T_g(M_1, \delta_g(r))$ and thus (24) holds for all t in $[0, \sigma)$. Moreover, for all t in $[\sigma, +\infty)$, X is a Carathéodory solution of (9) such that $|X(\sigma)| \le \delta_l(\delta_l^{-1}(r))$. Thus, we conclude (24) by Lemma 3.1.
3) Suppose Case 3 of Lemma 5.5 holds. Then, Lemma 3.1 implies (24), for all t in $[0, \sigma_1)$. Moreover, for all t in $[\sigma_1, \sigma_2)$, X is a Carathéodory solution of (12) such that $|X(\sigma)| \le \delta_g(\delta_g^{-1}(\sqrt{M_6}r))$. Thus, by Lemma 3.1, (24) holds for all t in $[\sigma_1, \sigma_2)$. Finally, for all t in $[\sigma_2, +\infty)$, X is a Carathéodory solution of (9) such that $|X(\sigma)| \le \delta_l(\delta_l^{-1}(\delta_g^{-1}(\sqrt{M_6 r})))$. Hence, one has (24) by Lemma 3.1.

Global Exponential Attractivity: Let us prove that there exists a function R: $\mathbb{R} \to \mathbb{R}$ of class $\mathcal{K}_\infty$ such that, for all $r > 0$, for all perturbations e, d: $\mathbb{R}^n \times \mathbb{R}_{\ge 0} \to \mathbb{R}^n$ satisfying our standing regularity assumptions and (6) and for all (X, S_d) solutions of (4) with $|X(0)| \le \min(\delta_l^{-1}(r), \delta_g^{-1}(r))$, we have $|X(t)| \le R(r)e^{-Ct}$, $\forall t \ge 0$. To prove this, by Lemma 5.5, only three cases may occur.

1) Suppose Case 1 of Lemma 5.5 holds. Then, by Lemma 3.1 and (14), we have, for all $t \ge 0$, $|X(t)| \le r\sqrt{M_6 e^{-Ct}}$.
2) Suppose Case 2 of Lemma 5.5 holds. Then, by Lemma 3.3, $\sigma \le T_g(M_1, \delta_g(r))$. Moreover, for all t in $[0, \sigma]$, $|X(t)| \le r$, and for all $t \ge \sigma$, $|X(t)| \le \delta_l^{-1}(r)\sqrt{M_6}e^{-C(t-\sigma)}$.
3) Suppose Case 3 of Lemma 5.5 holds. The inequality $|X(\sigma_1)| \le r\sqrt{M_6}$ and Lemma 3.3 yield $\sigma_2 - \sigma_1 \le T_g(M_1, r\sqrt{M_6})$. Moreover due to Lemma 3.1, $z(0) \le M_6$ and $z(\sigma_1) \ge M_1$, we have $\sigma_1 \le (1/C)\ln(M_6/M_1)$, and, for all $t \ge \sigma_2$, $|X(t)| \le \delta_l^{-1}(\delta_g^{-1}(r\sqrt{M_6}))\sqrt{M_6}e^{-C(t-\sigma_2)}$.

It is immediate to infer the robust global exponential stability property from the global stability and the exponential attractivity.

VI. Conclusion

The problem of global robust exponential stabilization of nonholonomic chained systems in the presence of sensor noise and external disturbances has been addressed. It has been shown that the problem is solvable by means of a simple hybrid control law, i.e., it is possible to achieve global exponential stability of the zero equilibrium in the presence of (small) perturbations vanishing at the origin. The control law retains the basic properties of the discontinuous control laws proposed in [1], namely exponential convergence rate and lack of oscillatory behavior. The results presented in this note are based on the general theory developed in [15]. In this respect, the main contribution of this work is to show that, for a large class of nonholonomic systems, a robustly stabilizing control law can be explicitly designed, and it is possible to obtain explicit bounds on the admissible perturbations.

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Nonregular Feedback Linearization: A Nonsmooth Approach

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***Abstract*—In this note, we address the problem of exact linearization via nonsmooth nonregular feedback. A criterion of nonregular static state feedback linearizability is presented for a class of nonlinear affine systems with two control inputs, and its application to nonholonomic systems is briefly discussed.**

***Index Terms*—Nonlinear systems, nonregular feedback linearization, nonsmooth analysis.**

I. INTRODUCTION

Feedback linearization is a standard technique for control of many nonlinear systems. Since the pioneering work of Krener [20], which addressed linearization of nonlinear systems via state diffeomorphisms, the problem of linearization has been studied using increasingly more general transformations. The problem of regular static state feedback linearization was elegantly solved in [2] and [18]. The problem of regular dynamic state feedback linearization was first initiated in [7] and then extensively addressed in many references; see, for example, [6], [15], and the references therein. Dynamic feedback linearizability is closely related to the differential flatness of nonlinear systems [12], [13]. The problem of nonregular state feedback linearization was studied in [14] and [27].

Nonregular state feedback linearization is a rigorous design mechanism. In comparison with regular dynamic feedback linearization, this approach does not introduce any additional dynamics, while it is applicable to a broad class of practical engineering systems, such as robots with flexible joints [14]. By combining nonregular feedback linearization with backstepping design, the nonregular backstepping design approach provides a Lyapunov-function-based recursive design mechanism for a class of nonlinear systems [28]. This approach can avoid undesired cancellation of the beneficial nonlinearities and enhance robustness and softness through appropriate backstepping design of Lyapunov functions.

On the other hand, many practical systems do not admit any smooth static or dynamic state stabilizer due to the violation of the well-known necessary condition [3]. To cope with this difficulty, many innovative nonsmooth control approaches have been proposed in recent years. Among these, the problem of state equivalence for the singular case, i.e., the nested sequence of involutive distributions of the systems containing singular distributions was extensively investigated [4], [5]; a non-Lipschitz continuous feedback approach combining the theory of homogeneous systems and the idea of adding a power integrator was developed for global stabilization of several classes of nonlinear systems with uncontrollable unstable linearization [22], [25], [26]; and a generalized p-normal form was proposed which includes several

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