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Event-based controller synthesis by bounding methods ☆

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ABSTRACT

Two event-triggered algorithms for digital implementation of a continuous-time stabilizing controller are proposed in this work. The first algorithm updates the control value in order to keep the time evolution of a given Lyapunov-like function framed between two auxiliary functions; whereas the second one actualizes the control value so that the state trajectory of the system stays enclosed between two a priori defined templates. In both cases, a natural hybrid formulation of the event-based stabilizing control problem is used to prove the main results of this work. Furthermore, the existence of a minimum inter-event time greater than zero is proved. Numerical simulations are provided to illustrate the digital implementation of the event-sampling algorithms for nonlinear systems.

1. Introduction

Usually, state feedback control laws applied to dynamical systems are implemented digitally; and the core idea of this discrete-time implementation consists in sampling the continuous-time control law periodically with a sufficiently small sampling period. However, this procedure may be constrained in practice. On the one hand, reducing the sampling period to a level that preserves acceptable performance of the controlled system requires a fairly powerful and so expensive hardware [8]. On the other hand, today's systems are complex and compound by several subsystems controlled by a single CPU. Consequently, reducing the communication between the CPU and the subsystems is a challenge of great interest which allows enhancing the ability to control more complex systems and reducing energy consumption.

To reach this goal, numerous control strategies called event-based approaches have been proposed in the literature, see [14] for a recent framework encompassing the most recent existing event-triggered control techniques. They aim to update the control value only when a significant event occurs. Usually, this event is defined as a deviation threshold on the state vector or on the input vector. In this work, new criteria to design event-triggered sampling algorithms for a large class of nonlinear systems are proposed where the control updating decision is based on the dynamical behavior of auxiliary systems.

The first sampling algorithm updates the control value in order to guarantee that the Lyapunov-like function of the event-based system stays framed at each time instant between the Lyapunov functions of the auxiliary systems. The global stability of the event-controlled system is guaranteed without requiring the ISS stability of each subsystem and satisfying a supplementary small gain condition as needed in [3], where scalar interconnected systems are considered. The second sampling algorithm is based on a component by component comparison of the plant state with a priori defined state templates. In fact, in this case, the control updating procedure aims to force the state trajectory of the event-based system to never leave the state enclosure generated by the auxiliary systems. Moreover, the existence problem of a minimal inter-event time bigger than zero is solved. This algorithm is inspired from the design of event-based controllers by using dead-band methods (see, e.g., [9] for an introduction of this method). Consider in particular [13] where only single-input-single-output linear systems are considered. See also recent papers on send-on-delta control techniques dealing with bandlimited signal as in [2].

A preliminary version of this work focused on the case of linear systems has been presented in [11].

The paper is organized as follows. In Section 2 preliminary definitions and notions about hybrid systems, useful to prove our main contributions, are introduced. The problem under consideration is formulated in Section 3 as stability issue of hybrid systems. Sections 4 and 5 state the main contributions of this work regarding the design of event-triggered state feedback controls for nonlinear systems. Numerical simulations are provided in Section 6 when

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focusing on a nonlinear system borrowed from [3]. Section 7 collects concluding remarks.

Notation: In this paper the Euclidean inner product of two vectors x and y will be denoted by $x \cdot y$, the induced norm will be denoted by $|\cdot|$. Given a set $\mathcal{A}$, and a point x, $|x|_{\mathcal{A}}$ is the distance of x relative to $\mathcal{A}$, that is $\inf_{z \in \mathcal{A}} |x-z|$. int $\mathcal{A}$ and $\overline{\mathcal{A}}$ stand respectively for the interior and the closure of $\mathcal{A}$. Given a vector x in $\mathbb{R}^n$, $x^\top$ stands for the transpose of x. The Lie derivative of a function V with respect to the vector f, i.e., $\nabla V \cdot f$ will be denoted by $L_f V$. The inequality operators $\prec, \preceq, \succ$ and $\succcurlyeq$ between vectors must be understood component by component, e.g. $x \prec y$ if and only if $x_i < y_i$ for all i where x_i and y_i are the ith components of x and y respectively. The i-th vector of the canonical basis is denoted by e_i. A function $\alpha : [0,\infty) \to \mathbb{R}$ is of class $\mathcal{K}$ if it is zero at zero, continuous and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A function $\rho : [0,\infty) \to \mathbb{R}$ belongs to $\mathcal{PD}$ (positive definite) if it is continuous, $\rho(s) > 0$ for all $s > 0$ and zero at zero.

2. Basic notions on hybrid systems

This section is devoted to briefly introduce basic definitions and notions on hybrid systems [6] needed to prove the main results of this paper. By definition, hybrid systems are complex dynamical systems that exhibit both continuous and discrete dynamic behavior and viewed as a set of ordinary differential equations (ODE) governed by a finite-state automaton [6]. Mathematically, these dynamical systems can be described as follows:

$$\begin{cases} \dot{x} = f(x) & \text{if } x \in \mathcal{F}, \\ x^+ \in g(x) & \text{if } x \in \mathcal{J}, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ stands for the state of (1) with the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. The set-valued mapping $g : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the reset function of (1). The sets $\mathcal{F}$ and $\mathcal{J}$ are two closed subsets of $\mathbb{R}^n$ respectively called flow and jump sets. Note that, in this work, the design of the two event-triggered sampling algorithms is based on the flow and jump sets. We will define these sets later.

So, the hybrid dynamics involve the notion of *compact hybrid time domain* (see [6, Definition 2.3]). A set E is a compact hybrid time domain if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j),$$

for some finite sequence of times $0 = t_0 \le t_1 \cdots \le t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain. A solution x to (1) consists of a hybrid time domain dom x and a function $x : \text{dom}\, x \to \mathbb{R}^n$ such that $x(t,j)$ is absolutely continuous in t for a fixed j and $(t,j) \in \text{dom}\, x$ satisfying

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t,j) \in \text{dom}\, x$,

$$x(t,j) \in \mathcal{F}, \quad \dot{x}(t,j) = f(x(t,j)),$$

(S2) for all $(t,j) \in \text{dom}\, x$ such that $(t, j+1) \in \text{dom}\, x$,

$$x(t,j) \in \mathcal{J}, \quad x(t,j+1) \in g(x(t,j)).$$

When the state $x(t,j)$ belongs to the intersection of the flow set and of the jump set, then the solution can either flow or jump. Let us emphasize that the state of (1) should be either in $\mathcal{F}$ or in $\mathcal{J}$, and there is no solution issuing from $\mathbb{R}^n \backslash (\mathcal{F} \cup \mathcal{J})$.

A solution x to (1) is said to be *complete* if its domain is unbounded (either in the t-direction or in the j-direction), *Zeno* if it is complete but the projection of dom x onto $\mathbb{R}_{\ge 0}$ is bounded, and *maximal* if there does not exist another solution $\tilde{x}$ to (1) such that x is a truncation of $\tilde{x}$ to some proper subset of its domain. Hereafter, only maximal solutions will be considered.

In the literature (see, e.g., [6, Definition 3.6]), one associates to the hybrid system (1) the following stability definition.

Definition 1. Let $\mathcal{A}$ be a closed subset of $\mathbb{R}^n$ and $\mathcal{H}$ be the hybrid system defined in (1). The set $\mathcal{A}$ is said to be

- stable for $\mathcal{H}$: if for each $\epsilon > 0$ there exists $\delta > 0$ such that each solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \le \delta$ satisfies $|x(t,j)|_{\mathcal{A}} \le \epsilon$ for all $(t,j) \in \text{dom}\, x$;
- pre-attractive for $\mathcal{H}$: if all complete solutions satisfy $\lim_{t+j \to \infty} |x(t,j)|_{\mathcal{A}} = 0$;
- globally pre-asymptotically stable for $\mathcal{H}$: if it is both stable and pre-attractive for $\mathcal{H}$;
- globally asymptotically stable for $\mathcal{H}$: if it is globally pre-asymptotically stable for $\mathcal{H}$ and if each solution to $\mathcal{H}$ is complete.

3. Problem statement

Consider a nonlinear system

$$\dot{x}_p = f_p(x_p, u), \tag{2}$$

where f_p: $\mathbb{R}^{n_p} \times \mathbb{R}^m \to \mathbb{R}^{n_p}$ is continuously differentiable, x_p stands for the state of the plant and u stands for the control.

Assume that there exists a continuous state feedback control law $u = k(x_p)$ for which system (2) in closed loop with k is globally asymptotically stable. Then, the aim of this work is to design event-based sampling algorithms for the stabilizing state feedback control $u = k(x_p)$ by combining reachability analysis with stability analysis of hybrid systems. These sampling algorithms depend on the state of two auxiliary autonomous systems:

$$\dot{x}_a = f_a(x_a), \tag{3a}$$

$$\dot{x}_b = f_b(x_b), \tag{3b}$$

as illustrated in Fig. 1. In (3), $f_a : \mathbb{R}^{n_a} \to \mathbb{R}^{n_a}$ and $f_b : \mathbb{R}^{n_b} \to \mathbb{R}^{n_b}$ are two continuously differentiable functions.

So, the closed-loop system presented in Fig. 1 is more formally written as a hybrid system $\mathcal{H}$:

$$\mathcal{H} : \begin{cases} \begin{cases} \dot{x}_p = f_p(x_p, s) \\ \dot{x}_a = f_a(x_a) \\ \dot{x}_b = f_b(x_b) \\ \dot{s} = 0 \end{cases}, & x \in \mathcal{F}, \\ \begin{cases} x_p^+ \in \{x_p\} \\ x_a^+ \in k_a(x_a, x_p) \\ x_b^+ \in k_b(x_b, x_p) \\ s^+ \in \{k(x_p)\} \end{cases}, & x \in \mathcal{J}, \end{cases} \tag{4}$$

where $x = (x_p^\top, x_a^\top, x_b^\top, s^\top)^\top$ in $\mathbb{R}^n$ stands for the state of this

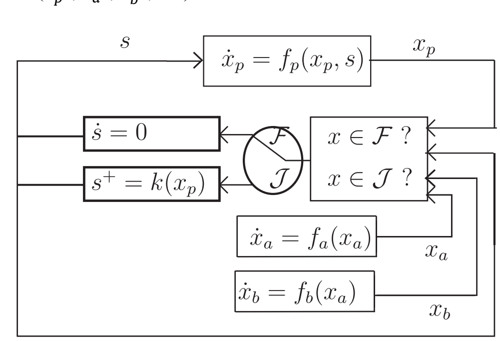

Fig. 1. Event-based sampling algorithm for the state feedback controller $k(x_p)$.

system, $n = n_p + n_a + n_b + m$, k_a and k_b are two set-valued mappings defining the discrete dynamics of x_a and x_b components, and $\mathcal{F}$ and $\mathcal{J}$ are two closed sets. Roughly speaking, this hybrid implementation is the data of a flow and a jump condition, that define respectively when holding and when updating the control input of system (2). To prove our main results, and to derive these piecewise-constant controller design methods, it is assumed that a Lyapunov function for the closed loop system $\dot{x}_p = f_p(x_p, k(x_p))$ is available (without any ISS property), see Assumption 1 for a precise statement of the needed hypothesis.

4. Comparing values of Lyapunov functions (Algorithm 1)

The core idea of the first algorithm is to keep the time evolution of a Lyapunov-like function V_p of (2) with a piecewise constant input framed between two systems called slow system (with state V_s) and fast system (with state V_f). To achieve this goal, first, this algorithm detects online the time instants when the Lyapunov-like function reaches one of the boundary Lyapunov functions. Then, it updates the control values in order to redirect the Lyapunov-like function towards the inside of the region formed by the slow system with state V_s and fast system with state V_f. This procedure is illustrated in Fig. 2.

In this framework, systems (3a) and (3b) will be respectively called slow and fast systems because their Lyapunov functions will bound the Lyapunov function for (2). In the sequel, indices (a, b) are replaced by (f, s) to better fit the context of the present section.

The following assumption will be useful to derive our first main result.

Assumption 1. There exist an open set $\mathcal{O} \subset \mathbb{R}^{n_p + n_p + n_p}$ containing the origin, a continuous function $k : \mathrm{dom}\, k \to \mathbb{R}^m$, a continuously differentiable function $V_p : \mathrm{dom}\, V_p \to \mathbb{R}$, a function $\rho_s \in \mathcal{K}$ such that

- for all $x_p \in \mathrm{dom}\, k$, it holds

$$\alpha_1(|x_p|) \le V_p(x_p) \le \alpha_2(|x_p|); \tag{5}$$

- $\mathrm{dom}\, V_p = \mathrm{dom}\, k$ and $\mathrm{dom}\, V_p \times \mathrm{dom}\, V_p \times \mathrm{dom}\, V_p$ contains a neighborhood of $\mathcal{O}$;
- for all $(x_p, x_f, x_s) \in \mathcal{O}$, it holds

$$(x_p, x_p, x_p) \in \mathcal{O}, \quad k(x_p) \in \mathrm{dom}\, k \tag{6}$$

- for all $(x_p, x_f, x_s) \in \mathcal{O}$, $0 \ne V_p(x_f) \le V_p(x_p) \le V_p(x_s)$, it holds

$$\mathrm{L}_{f_p(\cdot,k(\cdot))} V_p(x_p) < \mathrm{L}_{f_s} V_p(x_s),$$

$$\mathrm{L}_{f_f} V_p(x_f) < \mathrm{L}_{f_p(\cdot,k(\cdot))} V_p(x_p),$$

$$\mathrm{L}_{f_s} V_p(x_s) \le -\rho_s(|x_s|). \tag{7}$$

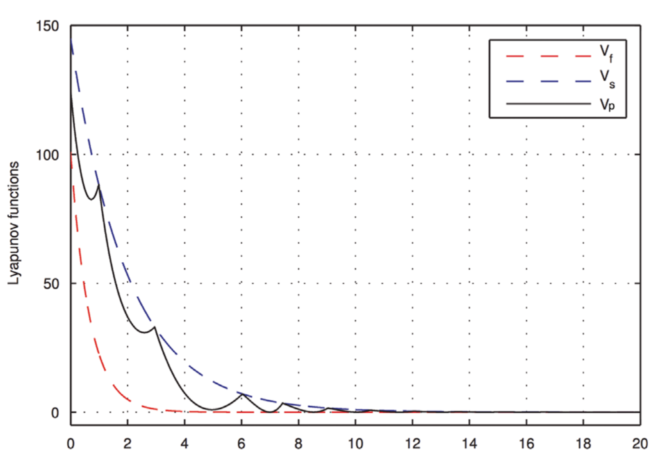

Fig. 2. Graphical illustration of the first event-based sampling algorithm.

Remark 1. For a given closed-loop asymptotically stable nonlinear system $\dot{x}_p = f_p(x_p, k(x_p))$, the slow and fast systems can be respectively defined as follows:

$$\dot{x}_s = -\beta_s f_p(x_s, k(x_s))$$

$$\dot{x}_f = -\beta_f f_p(x_f, k(x_f))$$

where $\beta_s \in (0, 1)$ and $\beta_f > 1$. This approach is employed for the example considered in Section 6.

Remark 2. A stronger assumption than Assumption 1 is when it is assumed moreover that $\mathcal{O} = \mathbb{R}^{n_p + n_p + n_p}$ and when the functions k and V_p are defined on all the space. It yields the following more restrictive assumption: there exist a continuous function $k : \mathbb{R}^{n_p} \to \mathbb{R}^m$, a continuously differentiable function $V_p : \mathbb{R}^{n_p} \to \mathbb{R}$, two functions α_1 and $\alpha_2 \in \mathcal{K}_\infty$ and a function $\rho_s \in \mathcal{K}$ such that, for all $(x_p, x_f, x_s) \in \mathbb{R}^{n_p + n_p + n_p}$, $0 \ne V_p(x_f) \le V_p(x_p) \le V_p(x_s)$, it holds

$$\mathrm{L}_{f_p(\cdot,k(\cdot))} V_p(x_p) < \mathrm{L}_{f_s} V_p(x_s),$$

$$\mathrm{L}_{f_f} V_p(x_f) < \mathrm{L}_{f_p(\cdot,k(\cdot))} V_p(x_p),$$

$$\mathrm{L}_{f_s} V_p(x_s) \le -\rho_s(|x_s|).$$

So, the first main result is to find appropriate sets $\mathcal{F}$ and $\mathcal{J}$ so that the attractor is globally asymptotically stable for the hybrid system $\mathcal{H}$. This problem will be solved in Theorem 1 under Assumption 1, by exploiting Lyapunov methods for hybrid systems (borrowed mainly from [6]).

4.1. First main result

Using Assumption 1, it is possible to exploit the comparison of the values of the Lyapunov function V_p and to derive a piecewise-constant feedback law for the nonlinear control system (2). This is done in our first main result:

Theorem 1. *Under Assumption 1, the set $\{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m$ is globally pre-asymptotically stable for the hybrid system $\mathcal{H}$ defined by*

$$\mathcal{F} = \left\{x \in \mathcal{O} \times \mathrm{dom}\, k, V_p(x_f) \le V_p(x_p) \le V_p(x_s)\right\}$$

$$\mathcal{J} = \mathcal{J}_f \cup \mathcal{J}_s$$

$$\mathcal{J}_f = \left\{ x \in \mathcal{O} \times \mathrm{dom}\, k, \left\{ \begin{array}{c} V_p(x_f) = V_p(x_p) \text{ and } V_p(x_p) \le V_p(x_s) \\ \text{and } \mathrm{L}_{f_f} V_p(x_f) \ge \mathrm{L}_{f_p(\quad \cdot, \quad s)} V_p(x_p) \end{array} \right. \right\}$$

$$\mathcal{J}_s = \left\{ x \in \mathcal{O} \times \mathrm{dom}\, k, \left\{ \begin{array}{c} V_p(x_s) = V_p(x_p) \text{ and } V_p(x_p) \ge V_p(x_f) \\ \text{and } \mathrm{L}_{f_s} V_p(x_s) \le \mathrm{L}_{f_p(\quad \cdot, \quad s)} V_p(x_p) \end{array} \right. \right\}$$

$$k_f(x_f, x_p) = \{x_p\}, \quad k_s(x_s, x_p) = \{x_s\}, \quad \forall x \in \mathcal{J}_f$$

$$k_f(x_f, x_p) = \{x_f\}, \quad k_s(x_s, x_p) = \{x_p\}, \quad \forall x \in \mathcal{J}_s$$

$$k_f(x_f, x_p) = \{x_p\} \cup \{x_f\}, \quad k_s(x_s, x_p) = \{x_p\} \cup \{x_f\}, \quad \forall x \in \mathcal{J}_f \cap \mathcal{J}_s$$

and by (4), with indices (f, s) instead of (a, b).

Before proving this result, let us note that it is quite different from that obtained in [19]. In fact, in [19], it is assumed that the time derivative of the Lyapunov function is known at each time instant. In Theorem 1, we relax this assumption by comparing the Lyapunov-like function V_p with the Lyapunov functions V_s and V_f linked to the slow and fast systems respectively. The proof of the previous result is based on the computation of a weak Lyapunov function and the LaSalle invariance principle is applied. The computation of a strict Lyapunov function is still an open question. It would yield to a simpler proof by applying [6, Theorem 3.18].

Proof. Let us introduce the following function $V: \mathrm{dom}\, V \to \mathbb{R}$, defined by, for all $x \in \mathrm{dom}\, V := \mathbb{R}^{n_p} \times \mathbb{R}^{n_s} \times \mathrm{dom}\, V_p \times \mathbb{R}^m$,

$$V(x) = V_p(x_s),$$

and denote

$$\mathcal{A} = \{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m. \tag{8}$$

Following [6, Definition 3.16], V is a Lyapunov function candidate. Note that, due to (5) and the expression of $\mathcal{F}$ in Theorem 1, we have, for all $x \in \mathcal{F}$,

$$|x_f| \le \alpha_1^{-1}(V_p(x_f)) \le \alpha_1^{-1}(V_p(x_p)) \le \alpha_1^{-1}(V_p(x_s)) \le \alpha_1^{-1}\alpha_2(|x_s|)$$

and similarly

$$|x_p| \le \alpha_1^{-1}\alpha_2(|x_s|).$$

Therefore, for all $x \in \mathcal{F}$,

$$|x|_{\mathcal{A}} = |x_p| + |x_f| + |x_s| \le (I + 2\alpha_1^{-1}\alpha_2)(|x_s|) \tag{9}$$

where I stands for the identity map. Therefore, noting $|x_s| \le |x|_{\mathcal{A}}$ and using again (5), the definition of V implies that, for all x in $\mathcal{F}$,

$$V(x) \le \alpha_2(|x_s|) \le \alpha_2(|x|_{\mathcal{A}}) \tag{10a}$$

and

$$V(x) \ge \alpha_1(|x_s|) \ge \alpha_1(I + 2\alpha_1^{-1}\alpha_2)^{-1}(|x|_{\mathcal{A}}). \tag{10b}$$

Let us now introduce the map

$$G : \begin{pmatrix} x_p \\ x_f \\ x_s \\ s \end{pmatrix} \rightrightarrows \begin{pmatrix} \{x_p\} \\ k_f(x_f, x_p) \\ k_s(x_s, x_p) \\ \{k(x_p)\} \end{pmatrix} \tag{11}$$

which is the right-hand side of the discrete dynamics of (4). Note that, due to (6), $G(\mathcal{J}) \subset \mathcal{F}$ and observe that $\mathcal{F} \cup \mathcal{J} = \mathcal{F}$. Therefore, defining $\tilde{\alpha}_1 \in \mathcal{K}_\infty$ by $\tilde{\alpha}_1 = \alpha_1(I + 2\alpha_1^{-1}\alpha_2)^{-1}$, it is deduced from (10):

$$\tilde{\alpha}_1(|x|_{\mathcal{A}}) \le V(x) \le \alpha_2(|x|_{\mathcal{A}}), \quad \forall x \in \mathcal{F} \cup \mathcal{J} \cup G(\mathcal{J}). \tag{12}$$

Now denote by F the map given by the right-hand side of the continuous dynamics of (4), that is the map

$$F : \begin{pmatrix} x_p \\ s \\ x_f \\ x_s \end{pmatrix} \mapsto \begin{pmatrix} f_p(x_p, s) \\ 0 \\ f_f(x_f) \\ f_s(x_s) \end{pmatrix}.$$

Due to (7), (9), $\rho_s \in \mathcal{K}$, and the expression of V, it holds, $\forall x \in \mathcal{F}$:

$$L_F V(x) \le -\rho_s(|x_s|) \le -\rho_s(I + 2\alpha_1^{-1}\alpha_2)^{-1}(|x|_{\mathcal{A}}). \tag{13}$$

Moreover it clearly follows from the expression of V and G that, for all $x \in \mathcal{J}$, $V(G(x)) = V(x)$. Therefore with [6, Proof of the uniform stability in Theorem 3.18] (see also the comment at the beginning of the proof of [6, Proposition 3.24]), the closed set $\mathcal{A}$ is stable for the hybrid system $\mathcal{H}$.

To prove that $\mathcal{A}$ is pre-attractive for $\mathcal{H}$ as defined in Definition 1, let us first prove that each solution to $\mathcal{H}$ is bounded. Due to $L_F V(x) \le 0$, for all x in $\mathcal{F}$, and $V(G(x)) = x$, for all x in $\mathcal{J}$, it follows that V is bounded for any solution to $\mathcal{H}$. With (24), it follows that the distance to $\mathcal{A}$ is bounded. Due to the expression of $\mathcal{A}$ in (8), it follows that all components of the solution are bounded, except maybe the last one, that is the s-component. Due to the flow and jump dynamics of this component in (4), and since the function k is continuous, the boundedness of the x_p component implies the boundedness of the s-component. Therefore all solutions to $\mathcal{H}$ are bounded.

To complete the proof of the pre-attractivity, consider now a complete solution to $\mathcal{H}$. The proof of the convergence to $\mathcal{A}$ relies on the invariance principle, see, e.g., [18, Theorem 4.3]. To do that, let us first note that the *Standing Assumption* of [18, Definition 2.3] is satisfied since the flow and jump sets are closed, and since the dynamics in the hybrid system $\mathcal{H}$ is continuous. Consider a solution $\bar{x}$ to $\mathcal{H}$ which is included in a level set of the function V. Let us show that this solution $\bar{x}$ should be in $\mathcal{A}$, i.e., that the x_p, x_f and x_s components of $\bar{x}$ should be 0.

Note that for each solution to $\mathcal{H}$ such that it jumps at a given time (t, j) then, either the state is in $\mathcal{A}$ (and then the solution has to stay in $\mathcal{A}$ after time (t,j)), or, due to (7) and the expressions of $\mathcal{F}$ and $\mathcal{J}$, the solution has to flow after (t, j).

Due to (13) and since $\rho_s(I + 2\alpha_1^{-1}\alpha_2)^{-1} \in \mathcal{PD}$, then the solution $\bar{x}$ cannot flow, except if it is in $\mathcal{A}$. Moreover, it cannot jump, except if it is in $\mathcal{A}$, since, as it has been proven just before, after each jump, either the solution is in $\mathcal{A}$ or it has to flow. Therefore $\bar{x}$ should be in $\mathcal{A}$.

Therefore with [18, Theorem 4.3], any precompact solution to $\mathcal{H}$ approaches $\mathcal{A}$. Moreover, as we have already proven, each complete solution is bounded and thus precompact. Therefore each complete solution to $\mathcal{H}$ approaches $\mathcal{A}$, and $\mathcal{A}$ is pre-attractive for $\mathcal{H}$.

This concludes the proof of Theorem 1.□

Remark 3. It is worth pointing out that, from the stability point of view, the event-triggered sampling algorithm can be designed only with the slow Lyapunov function. Here, we have preferred to present the general case where the convergence rate of the event-based system must belong into a desired interval.

4.2. Existence of a minimum inter-event time

Let us prove the existence of a (strictly) positive duration between two jump instants for solutions to the hybrid system $\mathcal{H}$ considered in Theorem 1. This property is crucial when implementing this control strategy, since it prevents infinite number of events and infinite number of control updates for digital implementations. This minimal inter-event time is proven in the next result.

We are now in position to state the existence of a minimal inter-event time, in the context of Theorem 1, as long as the solution stays outside a neighborhood of the attractor $\mathcal{A}$. To be more precise, we have

Proposition 1. *Assume Assumption 1 and $\mathcal{O} = \mathbb{R}^{n_p + n_p + n_p}$ hold. For all $0 < r < R$, there exists $t_{\min} > 0$ such that for all solutions to $\mathcal{H}$ jumping at two hybrid times $(t_1, j_1) < (t_2, j_2)$, and satisfying*

$$r \le |x(t,j)|_{\mathcal{A}} \le R, \quad \forall (t_1, j_1) \le (t,j) \le (t_2, j_2), \tag{14}$$

it holds $t_2 - t_1 \ge t_{\min}$.

Proof. Pick $0 < r < R$. Due to (10), there exists $\underline{M}$ and $\overline{M}$ such that for all x satisfying $r \le |x|_{\mathcal{A}} \le R$, it holds $\underline{M} \le V(x) \le \overline{M}$. Let the set $\mathcal{S} = \{x, \underline{M} \le V(x) \le \overline{M}\}$. Since V satisfies (12) in the proof of Theorem 1 and since $\tilde{\alpha}_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{K}_\infty$, the projection of $\mathcal{S}$ in the first $3n_p$ components is compact. Moreover, due to the continuity of the function k, due to the fact that the image of a compact set by a continuous function is compact, and due to the expression of the dynamics of the s-variable in (4), it follows, on the one hand, that the solutions to $\mathcal{H}$, that are included in $\mathcal{S}$, evolve in a compact set, denoted K, whose projection in the $3n_p$ components does not contain the origin.

On the other hand, due to (7) in Assumption 1 and the expressions of the flow and jump sets in Theorem 1, after each jump, each solution is either in $\mathcal{A}$, or it is in the following subset of the flow set described by (7):

$$\mathcal{F}_{sub} = \left\{ x, \left\{ \begin{array}{l} 0 \neq V_p(x_f) \le V_p(x_p) \le V_p(x_s), \\ \mathrm{L}_{f_p(\cdot,k(\cdot))} V_p(x_p) < \mathrm{L}_{f_s} V_p(x_s), \\ \mathrm{L}_{f_f} V_p(x_f) < \mathrm{L}_{f_p(\cdot,k(\cdot))} V_p(x_p). \end{array} \right. \right\}.$$

The set $\mathcal{F}_{sub}$ has a positive distance to the jump set $\mathcal{J}$ defined in Theorem 1. Moreover, since the flow dynamics defining $\mathcal{H}$ is defined by a continuous map, and since the set K is compact, we get that all solutions evolving in K have a finite maximal speed M. Thus, flowing in K from $\mathcal{F}_{sub}$ to $\mathcal{J}$ needs a uniform positive time (with a positive lower bound $t_{\min}$ given by the distance between $\mathcal{F}_{sub}$ to $\mathcal{J}$ over M). Therefore any solution, evolving in K and having a jump, has to flow for at least a uniform nonzero positive time $t_{\min} > 0$. This minimal time between jumps is uniform for all solutions evolving in K and *a priori* valid only for solutions whose state is in K and thus for all solutions to $\mathcal{H}$ satisfying (14).

This concludes the proof of Proposition 1.□

Note that the proof of the previous result has some connections with the proof of [5, Proposition 6], where intermediate (continuous and discrete) time and uniform boundedness of solutions in compact sets are considered.

Since a consequence of Proposition 1 is that all solutions to $\mathcal{H}$ are complete, combining Theorem 1 and Proposition 1, we get the following:

Corollary 1. *Under Assumption 1, with $\mathcal{O} = \mathbb{R}^{n_p+n_p+n_p}$, the set $\{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m$ is globally asymptotically stable for the hybrid system $\mathcal{H}$ defined in Theorem 1.*

Proof. To prove this corollary, due to Theorem 1, it remains to prove that all solutions are complete.

Let us prove by contradiction that all (maximal) solutions are complete. To do that, let us consider a maximal and incomplete solution $\overline{x}$ to $\mathcal{H}$. Then its hybrid time domain is bounded in the t-direction. Therefore due to Proposition 1, for any $0 < r < R$, the solution cannot stay in the set $\{x,\ r < |x|_{\mathcal{A}} < R\}$. Since each solution to $\mathcal{H}$ is bounded (as proven in the proof of Theorem 1), there exists $\overline{R} > 0$ such that the solution lies in $\{x, |x|_{\mathcal{A}} < \overline{R}\}$. Therefore, we get that for any $r > 0$, the solution $\overline{x}$ cannot stay in the set: $\{x, r < |x|_{\mathcal{A}}\}$. In other words, there exists a sequence of time (t_n, j_n) in dom $\overline{x}$ such that, denoting $\overline{x}_n = \overline{x}(t_n, j_n)$, it holds $|\overline{x}_n|_{\mathcal{A}} \to 0$, as $n \to \infty$. Since the solution $\overline{x}$ is incomplete, the hybrid time domain dom $\overline{x}$ is bounded, and there exists a subsequence (also denoted (t_n, j_n)) and (T, J) in $[0, \infty) \times \mathbb{N}$ such that $(t_n, j_n) \to (T, J)$, as $n \to \infty$.

Now, using again the boundedness of solutions to $\mathcal{H}$, there exists a subsequence such that $\overline{x}_n \to \overline{x}_\infty$ with $\overline{x}_\infty \in \mathcal{A}$.

Moreover, if $\overline{x}$ reaches the attractor, then (by the stability property proven in Theorem 1), it remains at the attractor and the solution is also defined for all hybrid times. Therefore, T should be the maximal value of dom $\overline{x}$ in the t-direction and J should be the maximal value of dom $\overline{x}$ in the j-direction. Now, defining $\tilde{x}$ as the function equal to $\overline{x}$ for all (t, j) in dom $\overline{x}$ and equal to $\overline{x}_\infty$ hereafter, we get a solution to $\mathcal{H}$, and thus a contradiction with the maximality of $\overline{x}$.

Therefore all solutions to $\mathcal{H}$ are complete (maybe there are some Zeno solutions to $\mathcal{H}$). This completes the proof of Corollary 1.□

5. Comparing the state components (Algorithm 2)

The second event-based sampling algorithm compares the state of the nonlinear control system (2) and the states of two asymptotically stable auxiliary systems. Here these systems are called lower and upper systems and their dynamics are described respectively by

$$\dot{x}_l = f_l(x_l), \tag{15a}$$

$$\dot{x}_u = f_u(x_u). \tag{15b}$$

In (15), $f_l : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ and $f_u : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$ are two continuously differentiable functions, where n_c is a positive integer which stands for the dimension of an additional state vector which can be used to design the upper and lower systems. Note that, here the indices (a, b) of the hybrid system introduced in (4) are replaced by (l, u). So, at each time instant where the state trajectory of (2) with a piecewise constant input intersects the state trajectories generated by the bounding stable systems (15), the sampling algorithm updates the control value in order to redirect the state trajectory of (2) towards the inside of the region defined by the upper x_u and lower x_l solutions to (15). This procedure is illustrated in Fig. 3. Before introducing the main assumption needed to prove our results in this context, let us start by defining a projection operator.

Definition 2. Let $\pi : \mathbb{R}^{n_p} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_p}$ be the projection operator defined by, for all $(x_p, x_c) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_c}$, $\pi(x_p, x_c) = x_p$.

So, the following assumption will be useful:

Assumption 2. There exist an open set $\mathcal{O} \subset \mathbb{R}^{n_p} \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}) \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c})$ containing the origin, a continuous function k : dom $k \to \mathbb{R}^m$, two continuously differentiable functions V_l : dom $V_l \to \mathbb{R}$ and V_u : dom $V_u \to \mathbb{R}$, two functions α_1 and $\alpha_2 \in \mathcal{K}_\infty$ and a function $\rho \in \mathcal{K}$ such that

- for all $(x_p, x_c) \in$ dom V_l, it holds

$$\alpha_1(|(x_p, x_c)|) \le V_l(x_p, x_c) \le \alpha_2(|(x_p, x_c)|), \tag{16a}$$

and for $(x_p, x_c) \in$ dom V_u, it holds

$$\alpha_1(|(x_p, x_c)|) \le V_u(x_p, x_c) \le \alpha_2(|(x_p, x_c)|); \tag{16b}$$

- dom $k \times$ dom $V_l \times$ dom V_u contains a neighborhood of $\mathcal{O}$;
- for all $(x_p, x_l, x_u) \in \mathcal{O}$, it holds

$$x_p, x_l, x_u \in \mathcal{O}, \quad k(x_p) \in \text{dom } k; \tag{17}$$

- for all $(x_p, x_l) \in$ dom $k \times$ dom V_l such that $\pi(x_l) \le x_p \neq 0$, $\exists i = 1, \ldots, n_p$, $\pi(x_l)_i = x_{pi}$ and $\pi(x_l) \neq x_p$, it holds

$$(\pi(f_l(x_l)) - f_p(x_p, k(x_p))) \cdot e_i < 0 \tag{18}$$

where e_i denotes the i-th vector of the basis of $\mathbb{R}^n_p$;

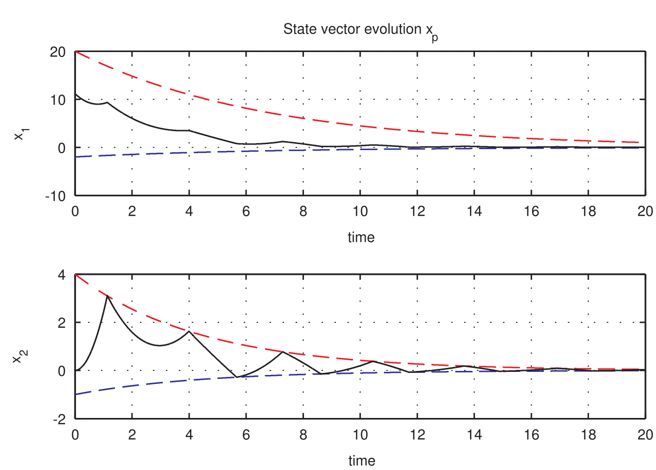

Fig. 3. Graphical illustration of the second event-based sampling algorithm.

- for all $(x_p, x_u) \in \text{dom}\, k \times \text{dom}\, V_u$ such that $0 \neq x_p \preceq \pi(x_u)$, $\exists i = 1, \dots, n_p$, $x_{pi} = \pi(x_{ui})$, and $x_p \neq \pi(x_u)$, it holds

$$(\pi(f_u(x_u)) - f_p(x_p, k(x_p))) \cdot e_i > 0; \tag{19}$$

- for all $(x_l, x_u) \in \text{dom}\, V_l \times \text{dom}\, V_u$, it holds

$$\begin{aligned} &\text{L}_{f_l} V_l(x_l) \le -\rho(|x_l|), \\ &\text{L}_{f_u} V_u(x_u) \le -\rho(|x_u|). \end{aligned} \tag{20}$$

This assumption is slightly different from the ones considered in [11], where it is necessary to consider additional functions (compare in particular with [11, Assumption 2]).

Remark 4. This assumption holds for any bounded monotone system (see [20]), where x_c is not needed for the upper and lower systems. Some methods to design the lower and upper systems are proposed in [16,17]. In addition, for controllable linear systems one can compute a state transformation matrix such that the transformed linear system is monotone with respect to the new state vector (see [11,15]). Furthermore, a large class of biological and biotechnological systems satisfies this assumption (see [21,1,7,4]).

The second main result of this paper is to derive a hybrid implementation of a piecewise constant controller for (2) under Assumption 2. This is done formally in Theorem 2 by using techniques on hybrid systems.

5.1. Second main result

Our second main result exploits the order preserving properties of dynamical systems [20] so that the state trajectory of the event-based controlled system, described in appropriate coordinates, stays framed between the state trajectories generated by the upper and lower systems.

We need a mild technical assumption on the comparison functions considered in Assumption 2:

Assumption 3. The functions α_1, α_2, and ρ in Assumption 2 are such that there exist $\alpha_3 \in \mathcal{K}_\infty$, and three positive values M_1, M_2 and M_3 satisfying

$$\alpha_1(s+s') \le M_1(\alpha_1(s) + \alpha_1(s')) \quad \forall s \ge 0,\ s' \ge 0, \tag{21a}$$

$$\alpha_2(s) + \alpha_2(s') \le M_2 \alpha_3(s+s'), \quad \forall s \ge 0,\ s' \ge 0. \tag{21b}$$

$$\rho(s+s') \le M_3(\rho(s) + \rho(s')), \quad \forall s \ge 0,\ s' \ge 0. \tag{21c}$$

It can be checked that Assumption 3 holds for any linear controllable system selecting k as a stabilizing controller.

Using Assumptions 2 and 3, it is possible to exploit the comparison of the components of the state to derive a piecewise constant controller for the nonlinear control system (2). It yields a hybrid system of the form

$$\tilde{\mathcal{H}}: \begin{cases} \begin{cases} \dot{x}_p = f_p(x_p, s) \\ \dot{x}_l = f_l(x_l) \\ \dot{x}_u = f_u(x_u) \\ \dot{s} = 0 \end{cases}, & \tilde{x} \in \tilde{\mathcal{F}} \\ \begin{cases} x_p^+ = x_p \\ x_l^+ = x_l \\ x_u^+ = x_u \\ s^+ = k(x_p) \end{cases}, & \tilde{x} \in \tilde{\mathcal{J}} \end{cases} \tag{22}$$

where $\tilde{x} = (x_p^\top, x_l^\top, x_u^\top, s^\top)^\top$ in $\mathbb{R}^{\tilde{n}}$, $\tilde{n} = n_p + (n_p + n_c) + (n_p + n_c) + m$. This hybrid system is considered in our second main result:

Theorem 2. *Under Assumptions 2 and 3, the set $\{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m$ is globally pre-asymptotically stable for the hybrid system $\tilde{\mathcal{H}}$ defined by*

$$\tilde{\mathcal{F}} = \{\tilde{x} \in \mathcal{O} \times \text{dom}\, k, \pi(x_l) \preceq x_p \preceq \pi(x_u)\}$$

$$\tilde{\mathcal{J}} = \pi(x_l) \preceq x_p \preceq \pi(x_u) \left\{ \begin{aligned} &\pi(x_l) \preceq x_p \preceq \pi(x_u) \\ &\textit{and } \exists i = 1, \dots, n_p, \textit{ such that} \\ &\tilde{x} \in \mathcal{O} \times \text{dom}\, k, \left\{ \begin{matrix} \pi(x_l)_i = x_{pi} \textit{ and } (\pi(f_l(x_l)) - f_p(x_p, k(x_p))) \cdot e_i \ge 0 \\ \textit{or} \\ \pi(x_u)_i = x_{pi} \textit{ and } (\pi(f_u(x_u)) - f_p(x_p, k(x_p))) \cdot e_i \le 0 \end{matrix} \right\} \end{aligned} \right\}$$

and by (22).

Proof. Let us introduce the following function $\tilde{V} : \text{dom}\, \tilde{V} \to \mathbb{R}$, defined by, for all $\tilde{x} \in \text{dom}\, \tilde{V} := \mathbb{R}^{n_p} \times \text{dom}\, V_l \times \text{dom}\, V_u \times \mathbb{R}^m$,

$$\tilde{V}(\tilde{x}) = V_l(x_l) + V_u(x_u),$$

and denote $\tilde{\mathcal{A}} = \{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m$. Following [6, Definition 3.16], $\tilde{V}$ is a Lyapunov function candidate. Note that, due to the expression of $\tilde{\mathcal{F}}$ in Theorem 2, we have, for all $\tilde{x} \in \tilde{\mathcal{F}}$, and for all $i = 1, \dots, n_p$,

$$|x_{pi}| \le \max\{|x_{li}|, |x_{ui}|\} \le |x_{li}| + |x_{ui}|,$$

and thus

$$|x_p| \le |x_l| + |x_u|.$$

Therefore, for all $\tilde{x} \in \tilde{\mathcal{F}}$,

$$|\tilde{x}|_{\tilde{\mathcal{A}}} = |x_p| + |x_l| + |x_u| \le 2(|x_l| + |x_u|) \tag{23}$$

and, using (16) and (21b), the definition of $\tilde{V}$ implies that, for all $\tilde{x}$ in $\tilde{\mathcal{F}}$,

$$\tilde{V}(\tilde{x}) \le \alpha_2(|x_l|) + \alpha_2(|x_u|) \le M_2 \alpha_3(|x_l| + |x_u|) \le M_2 \alpha_3(|\tilde{x}|_{\tilde{\mathcal{A}}}), \tag{24a}$$

and

$$\tilde{V}(\tilde{x}) \ge \alpha_1(|x_l|) + \alpha_1(|x_u|) \ge M_1^{-1} \alpha_1(|x_l| + |x_u|) \ge M_1^{-1} \alpha_1(\tfrac{1}{2} |\tilde{x}|_{\tilde{\mathcal{A}}}). \tag{24b}$$

where (21a) has been used in the second inequality.

Moreover, as in the proof of Theorem 1, using (17), denoting by G the map given by the right-hand side of the discrete dynamics of (22), then it holds

$$\tilde{\mathcal{F}} \cup \tilde{\mathcal{J}} \cup G(\tilde{\mathcal{J}}) = \tilde{\mathcal{F}}$$

Therefore, defining $\tilde{\alpha}_1$ and $\tilde{\alpha}_2 \in \mathcal{K}_\infty$ by $\tilde{\alpha}_1(s) = M_1^{-1} \alpha_1(\frac{1}{2} s)$ and $\tilde{\alpha}_2(s) = M_2 \alpha_3(s)$, for all $s \ge 0$, it is deduced from (24):

$$\tilde{\alpha}_1(|\tilde{x}|_{\tilde{\mathcal{A}}}) \le \tilde{V}(\tilde{x}) \le \tilde{\alpha}_2(|\tilde{x}|_{\tilde{\mathcal{A}}}), \quad \forall \tilde{x} \in \tilde{\mathcal{F}} \cup \tilde{\mathcal{J}} \cup G(\tilde{\mathcal{J}}).$$

Now denote by F the map given by the right-hand side of the continuous dynamics of (22). Due to (20), (21c), (23), $\rho \in \mathcal{K}$, and the expression of $\tilde{V}$, it holds, for all $\tilde{x} \in \tilde{\mathcal{F}}$,

$$\text{L}_F \tilde{V}(\tilde{x}) \le -\rho(|x_l|) - \rho(|x_u|) \le -M_3^{-1} \rho(|x_l| + |x_u|) \le -M_3^{-1} \rho(\tfrac{1}{2} |\tilde{x}|_{\tilde{\mathcal{A}}}). \tag{25}$$

Therefore, with [6, Proof of the uniform stability in Theorem 3.18], the closed set $\tilde{\mathcal{A}}$ is stable for the hybrid system $\tilde{\mathcal{H}}$.

To conclude the proof of Theorem 2 it remains to prove the pre-attractivity. First let us note that all the solutions to $\tilde{\mathcal{H}}$ are bounded. To do that, from $L_F \tilde{V} \le 0$, for all x in $\mathcal{F}$, and $\tilde{V}(G(x)) \le V(x)$, for all x in $\mathcal{J}$, and from the expression of $\tilde{\mathcal{A}}$ and (24b), we first deduce (as in the proof of Theorem 1) the boundedness of all components of any solution to $\tilde{\mathcal{H}}$, except the s-component. Then the boundedness of the s-component follows from the continuity of the function k and from jump dynamics in (22). The remaining proof of the pre-attractivity relies on the invariance principle.

To apply this invariance property, note that, since the flow and jump sets of $\tilde{\mathcal{H}}$ are closed and since the dynamics is continuous, it follows that the *Standing Assumption* of [18, Definition 2.3] is satisfied. Consider a solution $\bar{x}$ to $\tilde{\mathcal{H}}$ which is included in a level set of the function $\tilde{V}$. Let us show that this solution $\bar{x}$ should be in $\tilde{\mathcal{A}}$, i.e., that the x_p, x_l and x_u components of $\bar{x}$ should be 0.

Due to (25) and since $\rho \in \mathcal{PD}$, the solution cannot flow, except if it is in $\tilde{\mathcal{A}}$. Moreover, it cannot jump, except if it is in $\tilde{\mathcal{A}}$, since, as it has been proven just before, after each jump, either the solution is in $\tilde{\mathcal{A}}$ or it has to flow. Therefore $\bar{x}$ should be in $\tilde{\mathcal{A}}$.

Therefore with [18, Theorem 4.3], any precompact solution to $\tilde{\mathcal{H}}$ approaches $\tilde{\mathcal{A}}$. Moreover each complete solution that is not in $\mathcal{A}$ is bounded and thus precompact. Therefore each complete solution to $\tilde{\mathcal{H}}$ approaches $\tilde{\mathcal{A}}$, and $\tilde{\mathcal{A}}$ is pre-attractive for $\tilde{\mathcal{H}}$.

This concludes the proof of Theorem 2.□

Remark 5. Theorems 1 and 2 state two event-triggered algorithms to design, by emulation, event-based controllers. The main difference between the two theorems are the event-triggering conditions. Indeed, Theorem 1 compares the norms of the state of three systems (or more precisely the values of the Lyapunov function), whereas Theorem 2 exploits the comparison of the states componentwise. It gives different piecewise-constant controllers for (2).

Under the additional assumption $\mathcal{O} = \mathbb{R}^{n_p} \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}) \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c})$, it is possible to prove the existence of a positive minimal inter-event time in the context of Theorem 2. This could be done in a similar way as in Proposition 1. This yields the following result which is similar to Corollary 2.

Corollary 2. *Under Assumption 2, with $\mathcal{O} = \mathbb{R}^{n_p} \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}) \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c})$, the set $\{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m$ is globally asymptotically stable for the hybrid system $\tilde{\mathcal{H}}$ defined in Theorem 2.*

Proof. To prove this result, assuming $\mathcal{O} = \mathbb{R}^{n_p} \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c}) \times (\mathbb{R}^{n_p} \times \mathbb{R}^{n_c})$, we follow the steps of proof of Proposition 1 and we prove that the solutions enjoy a minimal inter-event time property as long as the state evolves in a given compact set. This yields a completeness property of solutions to $\tilde{\mathcal{H}}$, and implies with Theorem 2 that the set $\{0\} \times \{0\} \times \{0\} \times \mathbb{R}^m$ is globally asymptotically stable for $\tilde{\mathcal{H}}$.□

6. Illustrative example

Consider the nonlinear system borrowed from [3]

$$\begin{cases} \dot{x}_{p1} &= x_{p1}x_{p2} + x_{p1}^2 u_1 \\ \dot{x}_{p2} &= x_{p1}^2 + u_2 \end{cases} \tag{26}$$

with state variables $(x_{p1}, x_{p2}) \in \mathbb{R}^2$ and the input vector $u = (u_1, u_2) \in \mathbb{R}^2$. This system is stable under the following state-feedback control law [3]:

$$u_1(x_p) = -k_1 x_{p1} \quad \text{and} \quad u_2(x_p) = -k_2 x_{p2} \tag{27}$$

where $k_1 = 16$ and $k_2 = 6$ and admits the following quadratic function as a Lyapunov function:

$$V_p(x_p) = \tfrac{1}{2} x_p^\top x_p \ . \tag{28}$$

By direct computation one can show that

$$\dot{V}_p(x_p) = -\mu(|x_p|) = -(x_{p1}^2, x_{p2})Q(x_{p1}^2, x_{p2})^\top$$

where Q is the positive definite matrix $Q = \begin{bmatrix} k_1 & -1 \\ -1 & k_2 \end{bmatrix}$. Then, it is clear that there exist functions α_1 and $\alpha_2 \in \mathcal{K}_\infty$, satisfying

$$\alpha_1(|x_p|) \le V_p(x_p) \le \alpha_2(|x_p|) , \quad \forall x_p \in \mathbb{R}^2. \tag{29}$$

6.1. *Applying method from Theorem 1*

First, following Remark 1, given two positive values such that $\beta_s \in (0,1)$ and $\beta_f > 1$, let us design the "slow" and "fast" systems:

$$\begin{cases} \dot{x}_{s1} &= \beta_s(x_{s1}x_{s2} - k_1 x_{s1}^3) \\ \dot{x}_{s2} &= \beta_s(x_{s1}^2 - k_2 x_{s2}) \end{cases} \tag{30}$$

$$\begin{cases} \dot{x}_{f1} &= \beta_f(x_{f1}x_{f2} - k_1 x_{f1}^3) \\ \dot{x}_{f2} &= \beta_f(x_{f1}^2 - k_2 x_{f2}) \end{cases} \tag{31}$$

Letting $\mathcal{O} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, due to (28), we get that Assumption 1 holds and thus, with Corollary 1, the hybrid system $\mathcal{H}$ is globally asymptotically stable.

Let us compare the solutions of (26) in closed loop with (27) with solutions to the hybrid system $\mathcal{H}$ given by Theorem 1. More precisely, consider the initial conditions $x_s(0) = (6, -5)^\top$ for (30), $x_f(0) = (2, -1)^\top$ for (31), and $x(0) = (4, -3)^\top$ for (26) in closed loop with (27) and for $\mathcal{H}$ considered in Theorem 1. The time evolution are given in Fig. 4. As shown in this figure, with only 8 control updating time instants, one obtains a similar result to the case of continuous-time control. Moreover, the same example has been considered in [3] where a similar result is obtained with 38 events.

Fig. 5 contains a zoom on the time evolution of $V_p(x_s)$, $V_p(x_f)$ and $V_p(x_p)$. It can be noted that, when closing the loop with the piecewise constant control u, the Lyapunov-like function of the plant stays always framed between $V_p(x_f)$ and $V_p(x_p)$.

6.2. *Applying method from Theorem 2*

Before designing the upper and lower systems, note that the sign of the state trajectory of the first state variable of the system (26), in closed loop with (27), is constant and depends on the sign of the initial value. Thus, system (26) under the control law (27) is monotone with respect to the positive orthant $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ if $x_{p1}(0) \ge 0$ and monotone with respect to the orthant $\mathbb{R}_{\le 0} \times \mathbb{R}_{\ge 0}$ if $x_{p1}(0) \le 0$. So, for $x_{p1}(0)$ positive, one can use the following upper and lower systems:

$$\begin{cases} \dot{x}_{u1} = & x_{u1}x_{u2} - k_1 x_{u1}^3 \\ \dot{x}_{u2} = & x_{u1}^2 - k_2 x_{u2} \end{cases} \tag{32}$$

$$\begin{cases} \dot{x}_{l1} &= x_{l1}x_{l2} - k_1 x_{l1}^3 \\ \dot{x}_{l2} &= x_{l1}^2 - k_2 x_{l2} \end{cases} \tag{33}$$

and, for negative $x_{p1}(0)$, one can use the following upper and lower systems:

$$\begin{cases} \dot{y}_{u1} &= y_{u1}x_{u2} - k_1 y_{u1}^3 \\ \dot{x}_{u2} &= y_{u1}^2 - k_2 x_{u2} \end{cases} \tag{34}$$

$$\begin{cases} \dot{y}_{l1} &= y_{l1}x_{l2} - k_1 y_{l1}^3 \\ \dot{x}_{l2} &= y_{l1}^2 - k_2 x_{l2} \end{cases} \tag{35}$$

where $y_{l1}(0) \in [0, -x_{p1}(0)]$ and $y_{u1}(0) \ge -x_{p1}(0)$.

Due to (32)–(35), the functions $V_u : \mathbb{R}^2 \to \mathbb{R}$ and $V_l : \mathbb{R}^2 \to \mathbb{R}$ can be defined by

$$V_u(x_u) = V_p(x_u) \quad \text{and} \quad V_l(x_l) = V_p(x_l). \tag{36}$$

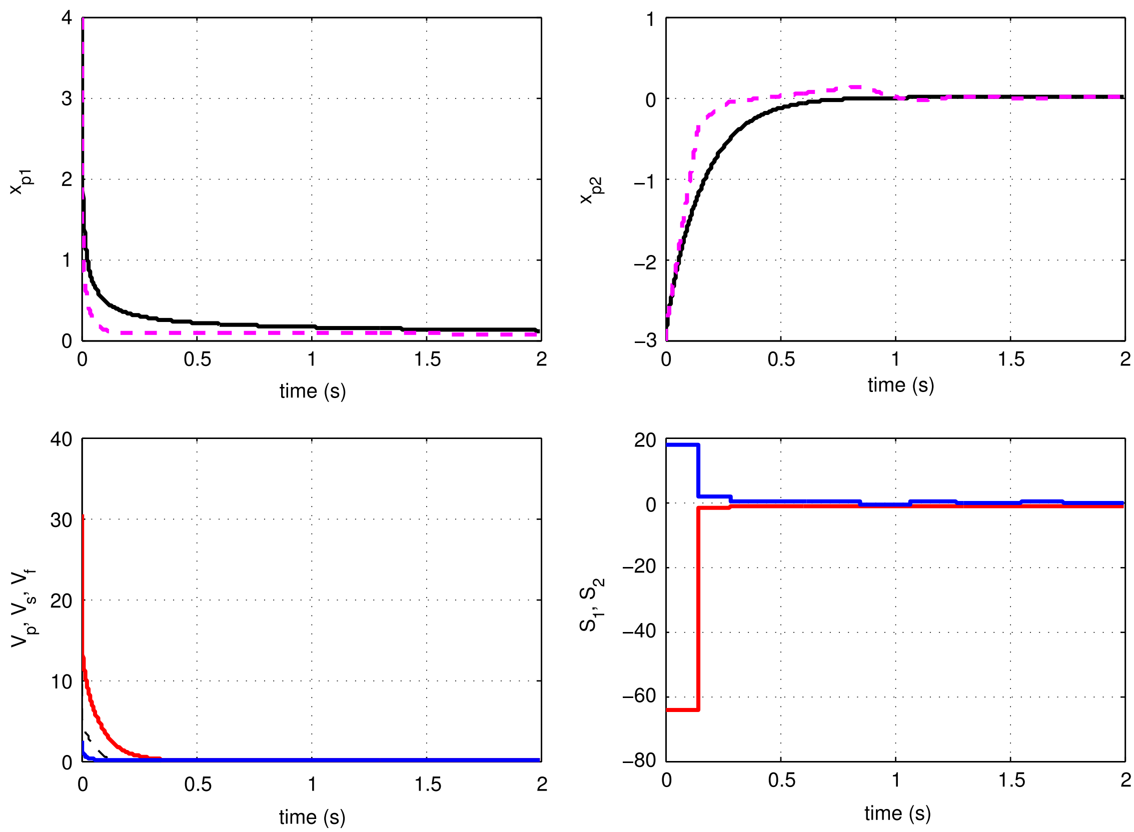

Fig. 4. Top pictures show the time evolution of the state variables (dashed lines stand for the event based controlled system and continuous lines stand for the continuous-time controlled system). Bottom left picture shows the time evolution of $V_p(x_s)$, $V_p(x_f)$ (continuous lines) and $V_p(x_p)$ (dashed line). Bottom right picture shows the time evolution of the piecewise constant control law.

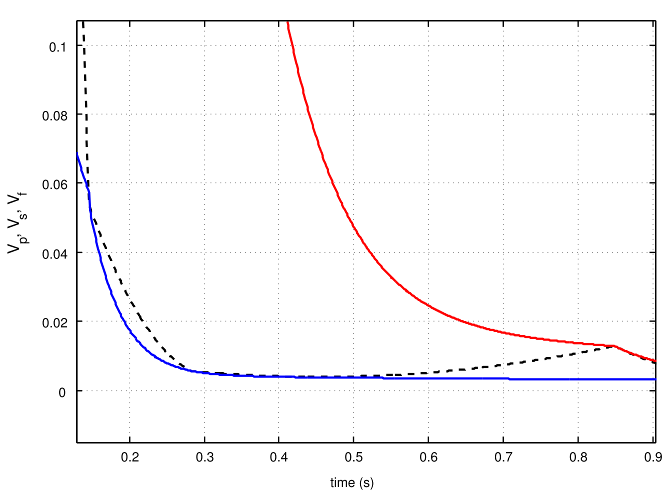

Fig. 5. Time evolution of $V_p(x_p)$ (dashed black line), $V_p(x_s)$ (red line), and $V_p(x_f)$ (blue line). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

Letting $\mathcal{O} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, due to (28) and (36), we get that Assumptions 2 and 3 hold and thus, with Corollary 2, the hybrid system $\mathcal{H}$ is globally asymptotically stable.

Now, let us compare the solutions of (26) in closed loop with (27) with solutions to the hybrid system $\mathcal{H}$ given by Theorem 2. Consider the initial conditions $x_u(0) = (8, 0)^\top$ for (32), $x_u(0) = (1, -6)^\top$ for (33), and $x_p(0) = (4, -3)^\top$ for (26) in closed loop with (27) and for $\mathcal{H}$ considered in Theorem 2. The time evolutions are given in Fig. 6. As shown in this figure, with only 11 control updating time instants, one obtains a similar result to the case of continuous-time control. Note that, here the components (u_1, u_2) of the piecewise control law are updated separately. In fact, we preferred to do that to show the possibility to use our method to deal with the case of distributed control systems [3] where the complex system is composed of scalar subsystems.

Fig. 7 shows a zoom on the time evolution of state variables: upper and lower systems (in red and blue bold lines), event-based controlled system (in dashed lines) and continuous-time controlled system (in tiny black lines). So, it can be noted that, when closing the loop with the piecewise constant control u, the state trajectories of this system stay always framed between the state trajectories of the upper and lower systems.

7. Conclusion

Based on bounding methods for nonlinear systems and stability analysis of hybrid systems, two event-triggered stabilizing controllers have been proposed for nonlinear system for which a static stabilizing controller is given. The control updating strategy is based either on the value of a Lyapunov-like function, or on the comparison of the plant state with given state templates. The obtained stability properties are illustrated by numerical simulations on a nonlinear system. With the first event-based controller, one guarantees that the speed of convergence of the system is included in a given interval formed by the convergence rate of the auxiliary systems (fast and slow); whereas with the second event-based controller, one ensures that the behavior of the system does not violate some specifications defined by a state enclosure.

This paper lets some issues open. In particular, it could be interesting to combine these event-sampling algorithms with interval observer design methods in order to replace, in the control updating criteria, the state vector by its estimate. A

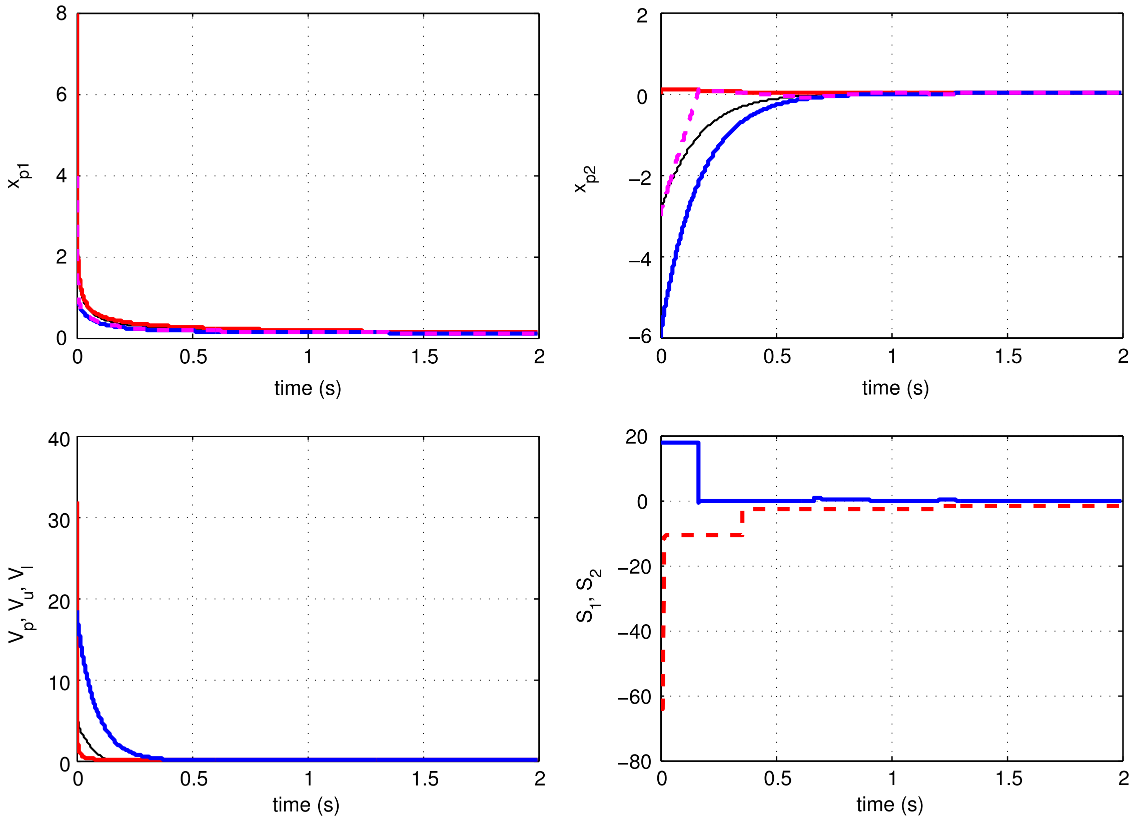

Fig. 6. Top pictures show the time evolution of the state variables (dashed lines stand for the event based controlled system and continuous lines stand for the continuous-time controlled system). Bottom left picture shows the time evolution of the Lyapunov functions of the three systems (bold lines stand for the upper and lower systems and the tiny line stands for the event based controlled system). Bottom right picture shows the time evolution of the piecewise constant control law.

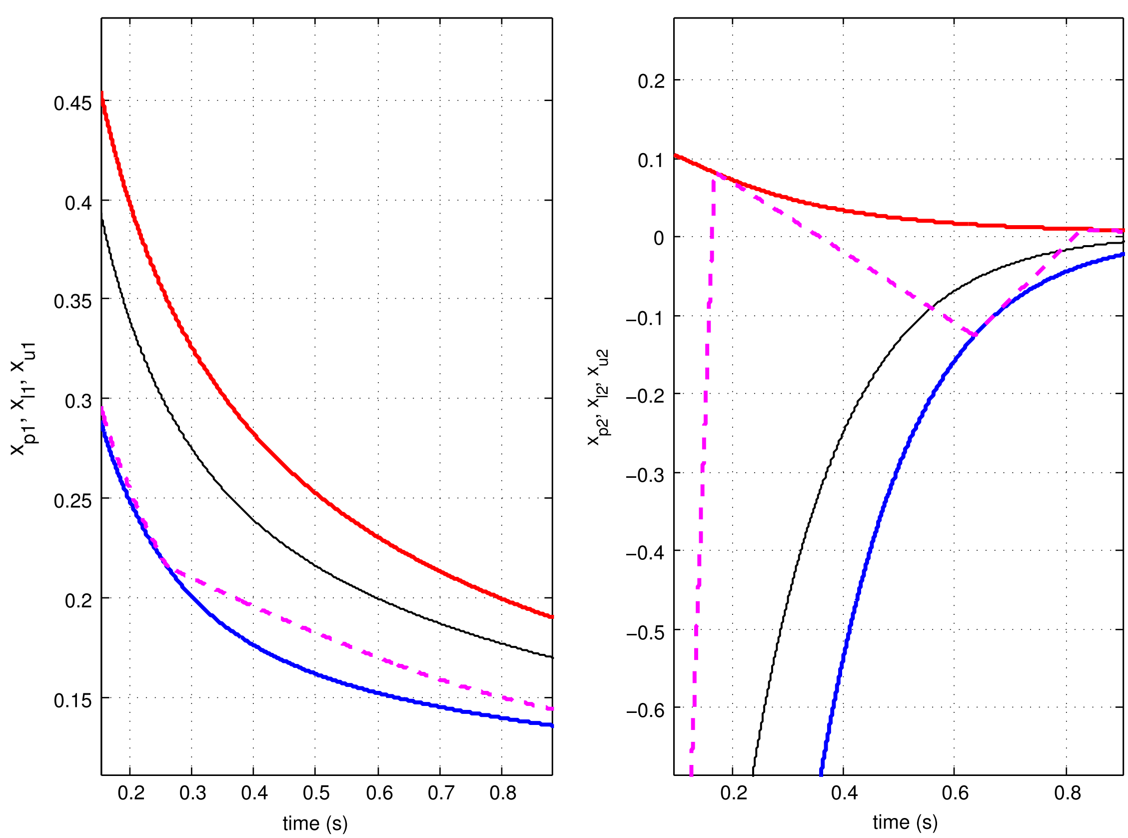

Fig. 7. A zoom on time evolution of the state variables. Bold curves stand for the state variables of the upper and lower systems. Tiny curves stand for the state variables of the continuous-time controlled system and the dashed curves show the state variables of the event-based controlled system. (For interpretation of the references to color in this figure, the reader is referred to the web version of this paper.)

preliminary result in that direction has been presented in [12]. It could be also interesting to apply the present work with particular class of nonlinear systems, as the ones with saturating inputs (see, e.g., [10] where reset controllers have been computed for such systems).

References

[1] D. Angeli, E.D. Sontag, Oscillations in I/O monotone systems, IEEE Trans. Circuits Syst. Spl. Issue Syst. Biol. 55 (2008) 66–176.
[2] A. Beschi, S. Dormido, J. Sanchez, A. Visioli, Stability analysis of symmetric send-on-delta event-based control systems, in: American Control Conference, Washington, DC, USA, 2013, pp. 1771–1776.
[3] C. De Persis, R. Sailer, F. Wirth, Parsimonious event-triggered distributed control: a Zeno free approach, Automatica 49 (2013) 2116–2124.
[4] D. Dochain, State and parameter estimation in chemical and biochemical processes: a tutorial, J. Process Control 13 (8) (2003) 801–818.
[5] R. Goebel, R. Sanfelice, A. Teel, Hybrid dynamical systems, IEEE Control Syst. Mag. 29 (2) (2009) 28–93.
[6] R. Goebel, R.G. Sanfelice, A.R. Teel, Hybrid Dynamical Systems: Modeling, Stability, and Robustness, Princeton University Press, Princeton and Oxford, 2012.
[7] M. Hadj-Sadok, J.-L. Gouzé, Estimation of uncertain models of activated sludge processes with interval observers, J. Process Control 11 (2001) 299–310.
[8] P. Hsu, S. Sastry, The effect of discretized feedback in a closed loop system, in: The 26th IEEE Conference on Decision and Control. Los Angeles, California (CA), USA, 1987, pp. 1518–1523.
[9] D. Lehmann, Event-Based State-Feedback Control, Logos Verlag, Berlin, 2011.
[10] T. Loquen, S. Tarbouriech, C. Prieur, Stability analysis for reset systems with input saturation, in: The 46th IEEE Conference on Decision and Control (CDC). New Orleans, LA, USA, 2007, pp. 3272–3277.
[11] N. Meslem, C. Prieur, Event-triggered algorithm for continuous-time systems based on reachability analysis, in: The 52nd IEEE Conference on Decision and Control. Florence, Italy, 2013, pp. 2048–2053.
[12] N. Meslem, C. Prieur, Event-based stabilizing controller using a state observer, in: The First IEEE International Conference on Event-based Control, Communication, and Signal Processing, Krakow, Poland, 2015.
[13] A. Pawlowski, A. Cervin, J. Guzman, M. Berenguel, Generalized predictive control with actuator deadband for event-based approaches, IEEE Trans. Ind. Informatics 10 (1) (2014) 523–537.
[14] R. Postoyan, P. Tabuada, D. Nesic, A. Anta, A framework for the event-triggered stabilization of nonlinear systems, IEEE Trans. Autom. Control 60 (4) (2015) 982–996.
[15] T. Raissi, D. Efimov, A. Zolghadri, Interval state estimation for a class of nonlinear systems, IEEE Trans. Autom. Control 57 (1) (2012) 260–265.
[16] N. Ramdani, N. Meslem, Y. Candau, A hybrid bounding method for computing an over-approximation for the reachable set of uncertain nonlinear systems, IEEE Trans. Autom. Control 54 (10) (2009) 2352–2364.
[17] N. Ramdani, N. Meslem, Y. Candau, Computing reachable sets for uncertain nonlinear monotone systems, Nonlinear Anal.: Hybrid Syst. 4 (2) (2010) 263–278.
[18] R. Sanfelice, R. Goebel, A.R. Teel, Invariance principles for hybrid systems with connections to detectability and asymptotic stability, IEEE Trans. Autom. Control 52 (12) (2007) 2282–2297.
[19] A. Seuret, C. Prieur, N. Marchand, Stability of non-linear systems by means of event-triggered sampling algorithms, IMA J. Math. Control Inf. 31 (3) (2014) 415–433.
[20] H. Smith, Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems, Mathematical Surveys and Monographs, American Mathematical Society, Providence, 2008.
[21] E. Sontag, Molecular systems biology and control, Eur. J. Control 11 (4–5) (2005) 396–435.