Manuscript submitted to AIMS Journal

doi:10.3934/xx.xxxxxxx

STABILIZATION OF A RAYLEIGH BEAM WITH COLLOCATED PIEZOELECTRIC SENSOR/ACTUATOR

Yubo Bai[1], Christophe Prieur[2] and Zhiqiang Wang[*3]

[1]School of Mathematical Sciences, Fudan University, Shanghai, 200433, China

[2]Univ. Grenoble Alpes, CNRS, Grenoble-INP, GIPSA-lab, F-38000, Grenoble, France

[3]School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai, 200433, China

(Communicated by Handling Editor)

Abstract. In this paper, stabilization of a Rayleigh beam with collocated piezoelectric sensor/actuator is studied. With a linear output feedback law, several stability results are achieved. The stability of the closed-loop system depends on the location of the actuator. The sufficient and necessary conditions for strong stability are provided. For almost all choices of the ends of the actuator, the energy of the system decreases to zero in a polynomial way and the explicit polynomial decay rate is obtained. Finally, the system is never exponentially stable for any choice of the ends of the actuator.

1. Introduction and main results.

1.1. **History.** The stabilization problem of flexible structures has attracted much attention in last few decades in the context of beam equation. Boundary control and interior control are two major topics in stabilization problem. Boundary stabilization of Rayleigh beam with only one control feedback was considered in [23] using a compact perturbation method. And [12] investigated the Riesz basis property for the same equation. In [13], boundary stabilization of an Euler–Bernoulli beam with the external disturbance was solved. Dynamic boundary stabilization of an Euler–Bernoulli beam was considered in [15]. Concerning interior stabilization, [2] studied the pointwise feedback control of Euler-Bernoulli beam with two different boundary conditions and obtained the lack of exponential stability when the beam is hinged at both ends. In [18, 19], the output feedback stabilization of Euler-Bernoulli beam with a piezoelectric actuator was considered.

For pointwise feedback control problem of Rayleigh beam, one can refer to [1] which considered the following equations.

$$\begin{aligned} &w_{tt}(x,t) - w_{xxtt}(x,t) + w_{xxxx}(x,t) + w_t(\xi,t)\delta_\xi - w_{xt}(\xi,t)\frac{\mathrm{d}}{\mathrm{d}x}\delta_\xi = 0, \quad 0<x<\pi,\\ &w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0. \end{aligned} \tag{1}$$

2020 *Mathematics Subject Classification.* Primary: 93C20, 93D15; Secondary: 35B40.

Key words and phrases. stabilization, Rayleigh beam, piezoelectric actuator, polynomial decay, frequency domain approach.

*Corresponding author: Zhiqiang Wang.

It is showed that the exponential stability depends on the location of the control point. Indeed, the exponential stability holds if and only if ξ lies in a countable set. After this work, [25] redesigned a collocated output feedback control that leads to the exponential stability and the result does not depend on the position of the control point. The control system in [25] is

$$\begin{aligned}&w_{tt}(x,t)-\alpha w_{xxtt}(x,t)+w_{xxxx}(x,t)=\mathcal{U}(x,t),\quad 0<x<\pi,\\&w(0,t)=w(\pi,t)=w_{xx}(0,t)=w_{xx}(\pi,t)=0,\end{aligned}\tag{2}$$

where the control consists of a pointwise control and a distributed one,

$$\mathcal{U}(x,t)=-u_0(t)\frac{\mathrm{d}}{\mathrm{d}x}\delta_\xi(x)-u_{1,t}(t)[\alpha\delta_\xi(x)+b(x)].$$

Here, u_0 and u_1 are the two (scalar) input signals and

$$b(x)=\begin{cases}\left(\dfrac{1}{\xi}+\dfrac{1}{\pi-\xi}\right)^{-1}\dfrac{x}{\xi}, & \text{for } x\le\xi,\\[2ex] \left(\dfrac{1}{\xi}+\dfrac{1}{\pi-\xi}\right)^{-1}\dfrac{\pi-x}{\pi-\xi}, & \text{for } x>\xi.\end{cases}$$

The feedback control is designed according to [10] which focuses on the exponential stabilization of a linear abstract evolution system.

Concerning stabilization problem for PDEs, Lyapunov method and frequency domain approach are common methods to use. In [5], Lyapunov method was widely used to study boundary stabilization of one dimensional hyperbolic systems. As for frequency domain approach, we express the linear evolution PDEs as an abstract system and then apply some classical results on stability of linear C_0-semigroups. A classical criteria for exponentially stable semigroup was established in [16, 22]. After that, [3] concerned strong stability of one-parameter semigroups. Several years later, [6, 21] were devoted to estimate the polynomial decay rate of solutions to linear evolution equation. A sufficient condition for polynomial stability of linear abstract evolution equation on Hilbert space was proposed in [21] and there were logarithm terms in the decay rate. Then [6] improved the results in [21]. The criteria in [21] was proved to be sufficient and necessary for polynomial stability of linear evolution equation on Hilbert space and the logarithm terms could be removed.

1.2. **Problem under consideration and main results.** In this paper, we consider the control problem modelling the transverse deflection of a Rayleigh beam which is subject to the action of an attached piezoelectric actuator. If we suppose that the beam is hinged at both ends, the equation of Rayleigh beam can be written as (see, for instance, [8, 11])

$$\begin{aligned}&w_{tt}(x,t)-\alpha w_{xxtt}(x,t)+w_{xxxx}(x,t)=u(t)\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x)-\delta_\xi(x)],\quad 0<x<\pi,\\&w(0,t)=w(\pi,t)=w_{xx}(0,t)=w_{xx}(\pi,t)=0,\\&w(x,0)=w^0(x),\ w_t(x,0)=w^1(x).\end{aligned}\tag{3}$$

In the equations above w represents the transverse deflection of the beam, $\alpha>0$ is a physical constant, ξ and η stand for the ends of the actuator ($0<\xi<\eta<\pi$), and δ_y is the Dirac mass at the point y. The control input is given by u.

Here we are interested in the stabilization of system (3). The control u is expressed as a function of the output $w_x(\eta)-w_x(\xi)$. This corresponds to the situation where the output comes from a piezoelectric sensor located on the same interval (ξ,η) as the actuator. In order to choose the feedback control function u, we introduce a suitable Hilbert space and a energy function. Let inner product of $V := H^2(0,\pi)\cap H_0^1(0,\pi)$ be $(w^1,w^2)_V = \int_0^\pi w^1_{xx}\overline{w^2_{xx}}\mathrm{d}x$ and inner product of $H := H_0^1(0,\pi)$ be $(v^1,v^2)_H = \int_0^\pi v^1\overline{v^2} + \alpha v^1_x\overline{v^2_x}\mathrm{d}x$. Then $\mathcal{H} = V\times H$, endowed with the usual product norm, is a complex Hilbert space. Energy function corresponding to the Hilbert space is

$$E(w(t),w_t(t)) = \frac{1}{2}\int_0^\pi (|w_t(x,t)|^2 + \alpha|w_{xt}(x,t)|^2 + |w_{xx}(x,t)|^2)\mathrm{d}x. \tag{4}$$

For every sufficiently smooth solution of (3), we have

$$\begin{aligned}\dot{E}(w(t),w_t(t)) &= \int_0^\pi (w_tw_{tt} + \alpha w_{xt}w_{xtt} + w_{xx}w_{xxt})\mathrm{d}x\\ &= \int_0^\pi (w_tw_{tt} - \alpha w_tw_{xxtt} + w_tw_{xxxx})\mathrm{d}x\\ &= \int_0^\pi w_t(x,t)u(t)\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x) - \delta_\xi(x)]\mathrm{d}x\\ &= -u(t)(w_{xt}(\eta,t) - w_{xt}(\xi,t)).\end{aligned}$$

Inspired by this calculation, we choose feedback control $u(t) := \kappa(w_{xt}(\eta,t) - w_{xt}(\xi,t))$, where $\kappa\ge 0$ is a constant.

Then we focus on the following closed-loop system, for $0<x<\pi$,

$$\begin{aligned}&w_{tt}(x,t) - \alpha w_{xxtt}(x,t) + w_{xxxx}(x,t) = \kappa(w_{xt}(\eta,t) - w_{xt}(\xi,t))\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x)-\delta_\xi(x)],\\ &w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0,\\ &w(x,0) = w^0(x),\ w_t(x,0) = w^1(x).\end{aligned} \tag{5}$$

Then we are interested in the asymptotic stability of closed-loop system (5). In order to state the main results, we give several definitions of stability.

Definition 1.1. We say that (5) is strongly stable in $\mathcal{H}$, if for any (w^0,w^1) in $\mathcal{H}$ we have that

$$E(w(t),w_t(t)) = \frac{1}{2}\|(w(t),w_t(t))\|^2_{\mathcal{H}} \to 0, \quad \text{as } t\to+\infty. \tag{6}$$

In Section 2, we express system (5) as an abstract Cauchy problem for operator $\mathcal{A}$, and the definitions (14) and (15) of the operator $\mathcal{A}$ and the domain $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ are precisely given.

Definition 1.2. We say that (5) is polynomially stable, if there exist two constants $C>0$ and $l>0$ such that for any (w^0,w^1) in $\mathcal{D}(\mathcal{A})$ and any $t\ge 0$,

$$\|(w(t),w_t(t))\|_{\mathcal{H}} \le Ct^{-1/l}\|(w^0,w^1)\|_{\mathcal{D}(\mathcal{A})}. \tag{7}$$

Definition 1.3. We say that (5) is exponentially stable in $\mathcal{H}$, if there exist two constants $C>0$ and $\delta>0$ such that for any (w^0,w^1) in $\mathcal{H}$ and any $t\ge 0$,

$$\|(w(t),w_t(t))\|_{\mathcal{H}} \le C\mathrm{e}^{-\delta t}\|(w^0,w^1)\|_{\mathcal{H}}. \tag{8}$$

Note that the norm on the right-hand of (7) can not be the $\mathcal{H}$-norm. Otherwise, by the semigroup properties, as mentioned in [21], (7) implies (8). Then we state the main results of this paper.

Theorem 1.4. *The system (5) is strongly stable in $\mathcal{H}$ if and only if $\kappa > 0$ and $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ are irrationals.*

Theorem 1.5. *Let $\varepsilon > 0$. There exist a constant $C > 0$, an uncountable set A with Lebesgue measure 0 and a set B_ε with Lebesgue measure 1 (A and B_ε are defined in Section 2), such that for any (w^0, w^1) in $\mathcal{D}(\mathcal{A})$ and for any $t \geq 0$, the solution $(w(t), w_t(t))$ of (5) satisfies*

1. *For any ξ and η with $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to A, (7) holds for $l = 4$.*
2. *For any ξ and η with $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ belong to B_ε, (7) holds for $l = 4 + 4\varepsilon$.*

Theorem 1.6. *For any ξ and η in $(0, \pi)$, the system (5) is not exponentially stable in $\mathcal{H}$, and the polynomial stability (7) does not hold for $0 < l < \frac{1}{2}$.*

These three theorems are the main results of this paper which provide the full analysis for stability of the system (5). Theorem 1.4 gives the necessary and sufficient conditions for strong stability. Theorem 1.5 admits some sufficient conditions of polynomial stability. For the ends of the piezoelectric actuator in an uncountable zero measure set A, we have a decay rate of $t^{-1/4}$. For almost all choices of the ends of the piezoelectric actuator (in a set with Lebesgue measure 1), we have a slower decay rate than the rate of $t^{-1/4}$. Note that the polynomial stability holds for all initial data lying in more regular spaces. Theorem 1.6 shows that the system (5) is never exponentially stable and is not polynomial stable with high decay rate no matter where the piezoelectric actuator is located.

Remark 1.7. These three stability results all highly depend on the location of the piezoelectric actuator. This phenomenon is common in pointwise control and piezoelectric control. As for stabilization problem, one can find similar phenomenon in [1, 2, 18]. As for controllability problem, the results in [4, 9, 24] also depend on the location of the piezoelectric actuator.

Remark 1.8. As we see above, different from the lack of exponential stability in Theorem 1.6, references [1, 25] considered the pointwise or collocated feedback control for Rayleigh beam equation and obtained exponential stability. The reason they obtained the exponential stability is that they apply stronger feedback control. There are two feedback control in (1) but there is only one feedback control in (5). In control system (2), there are also two input signals and the input signal u_1 acts on the whole beam by function b. However the control in (5) only acts on two points of the beam. Another work [10] considered the exponential stabilization of a linear abstract system. However the theory of [10] can not apply to our system, since the exact controllability in the natural energy space of the corresponding open-loop system is necessary but the study in [4] shows that system (3) is not exact controllable in the natural energy space $\mathcal{H}$.

1.3. **Sketch of the proof and outline.** The methods to prove Theorems 1.4 and 1.5 are inspired by the methods used in [18, 19] which studied the output feedback stabilization of Euler-Bernoulli beam with a piezoelectric actuator. We use frequency domain approach to obtain some estimation of the resolvent on the imaginary axis. In the proof of polynomial stability, we use the multiplier method and we provide a careful computation of the first derivative value at $x = \pi$ of the solution to a special linear system (40). Based on the calculation and some results from the theory of Diophantine approximation, we obtain an estimation of polynomial decay rate. As for Theorem 1.6, the lack of exponential stability and the lack of polynomial stability with high decay rate, we prove two results together. In detail, due to well-known results (see [6, 16, 22]), we are necessary to prove the resolvent estimate

$$\sup_{|\beta|\geq 1} |\beta|^{-l}\|(i\beta-\mathcal{A})^{-1}\| = \infty \tag{9}$$

for $0 \leq l < \frac{1}{2}$. Indeed, to prove Theorem 1.6, we succeed to find an increasing sequence $\{\beta_n\}_{n\in\mathbb{N}^*} \subseteq \mathbb{R}$ such that for $0 \leq l < \frac{1}{2}$

$$|\beta_n|^{-l}\|(i\beta_n-\mathcal{A})^{-1}\| \to \infty, \quad \text{as } n \to \infty.$$

This fact implies the lack of exponential stability and the lack of polynomial stability with high decay rate.

To summarize, the complete asymptotic behavior of the system (5) is obtained. The location of the piezoelectric actuator dominates the asymptotic behavior of the system. Especially, under this choice of control function, the exponential stability and the polynomial stability with high decay rate do not hold no matter where the piezoelectric actuator is located.

There is a gap in polynomial stability. The polynomial decay rate (7) with $\frac{1}{2} \leq l < 4$ is still an open problem. Whether exponential stability can be obtained using a different feedback control is an open problem. Stabilization for other types of beam equation with piezoelectric actuator, such as Timoshenko beam equation, is also a challenging open problem since coupled structure in Timoshenko beam equation brings new challenges and difficulties.

The paper is organized as follows. In Section 2 we give some preliminaries on Diophantine approximation and the well-posedness of the closed-loop system. Theorems 1.4, 1.5 and 1.6 are proved in Sections 3, 4 and 5 respectively.

2. Preliminaries.

2.1. Preliminaries on Diophantine approximation.

We need some results from the theory of Diophantine approximation (see [7, 17]). For a real number ρ, we denote by $\|\rho\|_{\mathbb{Z}}$ the difference, taken positively, between ρ and the nearest integer, i.e.,

$$\|\rho\|_{\mathbb{Z}} = \min_{n\in\mathbb{Z}} |\rho - n|.$$

Let us denote by A the set of all irrationals ρ in $(0,1)$ such that if $[0, a_1, \dots, a_n \dots]$ is the expansion of ρ as a continued fraction, then $\{a_n\}_{n=1}^{\infty}$ is bounded. Its Lebesgue measure is equal to zero (see [7]). The following property of this set is essentially useful in our work (see [17]).

Proposition 2.1. *A number ρ is in A if and only if there exists a constant $C > 0$ such that for all strictly positive integer Q,*

$$\|Q\rho\|_{\mathbb{Z}} \geq \frac{C}{Q}. \tag{10}$$

The next proposition, which is proved in [7], shows that an inequality slightly weaker than (10) holds for almost all points in $(0,1)$. This proposition is the definition of set B_ε.

Proposition 2.2. *For any $\varepsilon > 0$ there exists a set $B_\varepsilon \subseteq (0,1)$ having Lebesgue measure equal to 1 and a constant $C > 0$, such that for any ρ in B_ε and for all strictly positive integer Q,*

$$\|Q\rho\|_{\mathbb{Z}} \geq \frac{C}{Q^{1+\varepsilon}}. \tag{11}$$

The following proposition on simultaneous approximation (see [7]) used in [24] is quite important to prove Theorem 1.6.

Proposition 2.3. *Let $\rho_1, \ldots, \rho_k$ be k irrationals in $(0,1)$. Then there exists a strictly increasing sequence of natural numbers Q_n such that*

$$Q_n^{\frac{1}{k}} \max_{i=1,\ldots,k} (\|Q_n\rho_1\|_{\mathbb{Z}}, \ldots, \|Q_n\rho_i\|_{\mathbb{Z}}, \ldots, \|Q_n\rho_k\|_{\mathbb{Z}}) \leq \frac{k}{k+1} \quad \forall n \geq 1.$$

2.2. **Well-posedness of (5).** Now we express system (5) as an abstract Cauchy problem. Let us introduce some notions which are also used in [18]. If w is a function in $H^1(0,b) \cap H^1(b,\pi)$, we define $\{w_x\}$ in $L^2(0,\pi)$ by

$$\{w_x\}(x) := \begin{cases} w_x^{\mathcal{D}'(0,b)}(x) & \text{if } x \in (0,b), \\ w_x^{\mathcal{D}'(b,\pi)}(x) & \text{if } x \in (b,\pi), \end{cases}$$

where $w_x^{\mathcal{D}'(0,b)}$ (respectively $w_x^{\mathcal{D}'(b,\pi)}$) denotes the distributional derivative $\frac{\partial w}{\partial x}$ in $\mathcal{D}'(0,b)$ (respectively in $\mathcal{D}'(b,\pi)$). Let $[w]_b := w(b^+) - w(b^-)$. It follows that

$$w_x = \{w_x\} + [w]_b \delta_b \quad \text{in } \mathcal{D}'(0,\pi).$$

Denote $\mathcal{R} := (I - \alpha\partial_{xx})^{-1}$. It is well-known that operator $\mathcal{R}$ is an isomorphism from V' to $L^2(0,\pi)$ and an isomorphism from $H^{-1}(0,\pi)$ to H by Lax-Milgram Theorem. Applying $\mathcal{R}$ to both sides of the first equation of (5) to obtain

$$w_{tt} + \mathcal{R}w_{xxxx} = \kappa(w_{xt}(\eta,t) - w_{xt}(\xi,t))\mathcal{R}\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta(x) - \delta_\xi(x)]. \tag{12}$$

If we introduce $v := w_t$, then we can formally introduce an operator $\mathcal{A}$ on $\mathcal{H}$ with

$$\mathcal{A}(w,v) = \left(v, \mathcal{R}\left(-w_{xxxx} + \kappa(v_x(\eta) - v_x(\xi))\frac{\mathrm{d}}{\mathrm{d}x}[\delta_\eta - \delta_\xi]\right)\right). \tag{13}$$

In order to define the domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$, inspired by [18], let us note

$$\mathcal{A}(w,v) \in \mathcal{H} \Leftrightarrow v \in V \text{ and } -w_{xxx} + \kappa(v_x(\eta) - v_x(\xi))[\delta_\eta - \delta_\xi] \in L^2(0,\pi).$$

Therefore, the restriction of w_{xxx} to each of the intervals $(0,\xi)$, (ξ,η) and (η,π) has to be a $L^2(0,\pi)$ function which means that w belongs to $H^3(0,\xi) \cap H^3(\xi,\eta) \cap H^3(\eta,\pi)$. Using the notion introduced above, we have

$$w_{xxx} = \{w_{xxx}\} + [w_{xx}]_\xi \delta_\xi + [w_{xx}]_\eta \delta_\eta$$

and

$$\begin{aligned} &- w_{xxx} + \kappa(v_x(\eta) - v_x(\xi))[\delta_\eta - \delta_\xi] \\ =& - \{w_{xxx}\} - (\kappa(v_x(\eta) - v_x(\xi)) + [w_{xx}]_\xi)\delta_\xi + (\kappa(v_x(\eta) - v_x(\xi)) - [w_{xx}]_\eta)\delta_\eta. \end{aligned}$$

Then $-w_{xxx} + \kappa(v_x(\eta) - v_x(\xi))[\delta_\eta - \delta_\xi]$ belongs to $L^2(0,\pi)$ provided that all the coefficients in front of the Dirac measures vanish, i.e.,

$$[w_{xx}]_\eta = \kappa(v_x(\eta) - v_x(\xi)) = -[w_{xx}]_\xi.$$

So the definition of $\mathcal{D}(\mathcal{A})$ is

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{(w,v) |\, & w, v \in V, w \in H^3(0,\xi) \cap H^3(\xi,\eta) \cap H^3(\eta,\pi), \\ & w_{xx}(0) = w_{xx}(\pi) = 0, [w_{xx}]_\eta = \kappa(v_x(\eta) - v_x(\xi)) = -[w_{xx}]_\xi\}. \end{aligned} \tag{14}$$

And now we can define the operator $\mathcal{A}$ on $\mathcal{H}$ with

$$\mathcal{A}(w,v) = (v, -\mathcal{R}\{w_{xxx}\}_x), \quad \forall (w,v) \in \mathcal{D}(\mathcal{A}). \tag{15}$$

Then system (5) can be written as the abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = \mathcal{A}z, \quad t > 0, \\ z(0) = (w^0, w^1). \end{cases} \tag{16}$$

The well-posedness of (5) follows an application of the classical semigroup theory.

Theorem 2.4. *If $\kappa > 0$, then $\mathcal{A}$ generates a C^0-semigroup $\mathrm{e}^{t\mathcal{A}}$ of contractions on $\mathcal{H}$.*

According to a classical result of semigroup theory, Theorem 2.4 is proved if we show that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ and that $\mathcal{A}$ is dissipative and that zero belongs to $\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. First notice that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$ obviously. Then we show that $\mathcal{A}$ is dissipative.

Lemma 2.5. *For any $z = (w, v)$ in $\mathcal{D}(\mathcal{A})$ we have that*

$$(\mathcal{A}z, z)_{\mathcal{H}} = 2i\mathrm{Im}\int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x - \kappa|v_x(\eta) - v_x(\xi)|^2. \tag{17}$$

In particular, $\mathrm{Re}(\mathcal{A}z, z)_{\mathcal{H}} = -\kappa|v_x(\eta) - v_x(\xi)|^2 \leq 0$*, i.e., $\mathcal{A}$ is dissipative.*

Proof. For any $z = (w, v)$ in $\mathcal{D}(\mathcal{A})$, we have

$$(\mathcal{A}z, z)_{\mathcal{H}} = \int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x - \int_0^\pi [\mathcal{R}\{w_{xxx}\}_x\overline{v} + \alpha\mathcal{R}\{w_{xxx}\}_{xx}\overline{v_x}]\mathrm{d}x.$$

Notice that $\mathcal{R}f_{xx} = (\mathcal{R}f - f)/\alpha$, then

$$\begin{aligned}(\mathcal{A}z, z)_{\mathcal{H}} &= \int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x - \int_0^\pi [\mathcal{R}\{w_{xxx}\}_x\overline{v} + (\mathcal{R}\{w_{xxx}\} - \{w_{xxx}\})\overline{v_x}]\mathrm{d}x \\ &= \int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x + \int_0^\pi \{w_{xxx}\}\overline{v_x}\mathrm{d}x.\end{aligned}$$

And we have

$$\begin{aligned}\int_0^\pi \{w_{xxx}\}\overline{v_x}\mathrm{d}x &= \int_0^\xi w_{xxx}\overline{v_x}\mathrm{d}x + \int_\xi^\eta w_{xxx}\overline{v_x}\mathrm{d}x + \int_\eta^\pi w_{xxx}\overline{v_x}\mathrm{d}x \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\mathrm{d}x + w_{xx}(\xi^-)\overline{v_x(\xi)} + w_{xx}(\eta^-)\overline{v_x(\eta)} \\ &\quad - w_{xx}(\xi^+)\overline{v_x(\xi)} - w_{xx}(\eta^+)\overline{v_x(\eta)} \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\mathrm{d}x - [w_{xx}]_\xi\overline{v_x(\xi)} - [w_{xx}]_\eta\overline{v_x(\eta)} \\ &= -\int_0^\pi w_{xx}\overline{v_{xx}}\mathrm{d}x - \kappa|v_x(\eta) - v_x(\xi)|^2.\end{aligned}$$

Finally, we have

$$\begin{aligned}(\mathcal{A}z, z)_{\mathcal{H}} &= \int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x - \int_0^\pi w_{xx}\overline{v_{xx}}\mathrm{d}x - \kappa|v_x(\eta) - v_x(\xi)|^2 \\ &= 2i\mathrm{Im}\int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x - \kappa|v_x(\eta) - v_x(\xi)|^2.\end{aligned}$$

□

Next proposition shows that zero belongs to $\rho(\mathcal{A})$ and finishes the proof of Theorem 2.4.

Proposition 2.6. *0 belongs to $\rho(\mathcal{A})$.*

Proof. We have to prove that the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is one-to-one and onto, and that its inverse $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous. Fix any (f, g) in $\mathcal{H}$, and let us investigate the equation $\mathcal{A}(w, v) = (f, g)$, where (w, v) belongs to $\mathcal{D}(\mathcal{A})$. We have to solve

$$\begin{cases} v = f, \\ -\mathcal{R}\{w_{xxx}\}_x = g, \end{cases}$$

which is equipped by boundary conditions. Denote $\lambda = \kappa(f_x(\eta) - f_x(\xi))$, and introduce the solution $\widetilde{w}$ in $H^3(0, \pi)$ of the following equation

$$\begin{cases} -\mathcal{R}\widetilde{w}_{xxxx} = g, \\ \widetilde{w}(0) = \widetilde{w}(\pi) = \widetilde{w}_{xx}(0) = \widetilde{w}_{xx}(\pi) = 0. \end{cases}$$

Since $\mathcal{R}$ is an isomorphism from $H^{-1}(0, \pi)$ to H, $\widetilde{w}$ exists uniquely. Set $w = \widetilde{w} + \widehat{w}$, we need to solve

$$\begin{cases} \mathcal{R}\{\widehat{w}_{xxx}\}_x = 0, \\ \widehat{w}(0) = \widehat{w}(\pi) = \widehat{w}_{xx}(0) = \widehat{w}_{xx}(\pi) = 0, \\ [\widehat{w}_{xx}]_\eta = \lambda = -[\widehat{w}_{xx}]_\xi, \end{cases} \tag{18}$$

where $\widehat{w}$ is to be found in $H^2(0, \pi) \cap H^3(0, \xi) \cap H^3(\xi, \eta) \cap H^3(\eta, \pi)$. Since $\mathcal{R}\{\widehat{w}_{xxx}\}_x = 0$ implies that $\{\widehat{w}_{xxx}\} = 0$ in $L^2(0, \pi)$, from (18) we have

$$\widehat{w}(x) = \begin{cases} a_1 x, & 0 < x < \xi, \\ b_0 + b_1 x + b_2 x^2, & \xi < x < \eta, \\ c_1(x - \pi), & \eta < x < \pi, \end{cases}$$

where a_1, b_0, b_1, b_2, c_1 are constants. Since $H^2(0, \pi)$ is included in $C^1(0, \pi)$ because of the Sobolev Embedding Theorem and $[\widehat{w}_{xx}]_\eta = \lambda = -[\widehat{w}_{xx}]_\xi$, we have

$$\begin{cases} a_1 \xi = b_0 + b_1 \xi + b_2 \xi^2, \\ c_1(\eta - \pi) = b_0 + b_1 \eta + b_2 \eta^2, \\ a_1 = b_1 + 2b_2 \xi, \\ c_1 = b_1 + 2b_2 \eta, \\ -2b_2 = \lambda. \end{cases}$$

Since the determinant of the coefficients matrix of the above 5×5 linear equation is 2π, constants a_1, b_0, b_1, b_2, c_1 are uniquely determined by the linear equation. This proves the existence and uniqueness of $\widehat{w}$ and the existence and uniqueness of (w, v) in $\mathcal{D}(\mathcal{A})$ such that $\mathcal{A}(w, v) = (f, g)$. It remains to prove that the map $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous. Let $(w, v) = \mathcal{A}^{-1}(f, g)$. Then $\|v\|_H = \|f\|_H \leq C\|f\|_V$ because of the Sobolev Embedding Theorem. And we have $\|\widetilde{w}\|_V \leq C\|g\|_H$ because $\mathcal{R}$ is an isomorphism from $H^{-1}(0, \pi)$ to H. And from the linear equation we get $\|\widehat{w}\|_V \leq C|\lambda| = C\kappa|f_x(\eta) - f_x(\xi)| \leq C\|f\|_V$. Therefore, we have $\|(w, v)\|_{\mathcal{H}} \leq C\|(f, g)\|_{\mathcal{H}}$, which shows that the map $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous. Then we obtain that zero belongs to $\rho(\mathcal{A})$ and it concludes the proof of Proposition 2.6 and of Theorem 2.4. □

3. **Strong stability (Proof of Theorem 1.4).** In this section we use Arendt–Batty Theorem (see [3]) to prove strong stability.

First we deal with the only if part. If system (5) is strongly stable in $\mathcal{H}$, it follows from Lemma 2.5 that for any (w^0, w^1) in $\mathcal{D}(\mathcal{A})$ and for any $T \geq 0$,

$$E(w(T), w_t(T)) - E(w^0, w^1) = -\kappa \int_0^T |w_{tx}(\eta, t) - w_{tx}(\xi, t)|^2 \mathrm{d}t.$$

Thus, the condition $\kappa > 0$ is necessary for the energy to decrease. On the other hand, if $\frac{\eta+\xi}{2\pi}$ or $\frac{\eta-\xi}{2\pi}$ is a rational number, there exists a $k \geq 1$ such that

$$\sin\left(\frac{k(\eta+\xi)}{2}\right) \sin\left(\frac{k(\eta-\xi)}{2}\right) = 0.$$

Then the state of the form $(w^0, w^1) = (w_k^0 \sin(kx), w_k^1 \sin(kx))$ gives rise to a solution of (5) whose energy is constant and does not tend to zero. Then $\kappa > 0$ together with $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ are irrationals constitutes a necessary condition for the strong stability of system (5).

Next we check that this condition is also sufficient. Here we need a property of operator $\mathcal{A}$.

Lemma 3.1. *The operator $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is compact.*

Proof. Choose any bounded sequence $\{(f_n, g_n)\}_{n\geq 1}$ included in $\mathcal{H}$, our aim is to prove that $\{(w_n, v_n)\}_{n\geq 1} := \{\mathcal{A}^{-1}(f_n, g_n)\}_{n\geq 1}$ is precompact in $\mathcal{H}$. Since

$$\|\mathcal{A}^{-1}(f_n, g_n)\|_{\mathcal{D}(\mathcal{A})} = \|\mathcal{A}^{-1}(f_n, g_n)\|_{\mathcal{H}} + \|(f_n, g_n)\|_{\mathcal{H}}$$

and $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous, we have $\mathcal{A}^{-1} : \mathcal{H} \to \mathcal{D}(\mathcal{A})$ is continuous. Then for any bounded sequence $\{(f_n, g_n)\}_{n\geq 1}$ included in $\mathcal{H}$, $\{\mathcal{A}^{-1}(f_n, g_n)\}_{n\geq 1}$ is bounded in $\mathcal{D}(\mathcal{A})$. Then it is sufficient to prove the embedding $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathcal{H}$ is compact. Let $\{(w_n, v_n)\}_{n\geq 1}$ is any bounded sequence in $\mathcal{D}(\mathcal{A})$, we need to prove there exists a subsequence $\{(w_{n_k}, v_{n_k})\}_{k\geq 1}$ which converges in $\mathcal{H}$. Assume that $\|(w_n, v_n)\|_{\mathcal{D}(\mathcal{A})} \leq C$ for some constant $C > 0$ and then for any $n \geq 1$, we have $\|w_n\|_V \leq C$, $\|v_n\|_V \leq C$ and $\|\mathcal{R}\{w_{n,xxx}\}_x\|_H \leq C$ for some constant $C > 0$. Since embedding $H^2(0, \pi) \hookrightarrow H^1(0, \pi)$ is compact, there exist some v in H and a subsequence $\{v_{n_k}\}_{k\geq 1}$ such that

$$v_{n_k} \to v \quad \text{in } H \quad \text{as } k \to \infty. \tag{19}$$

Notice that $\|\mathcal{R}\{w_{n,xxx}\}_x\|_H \leq C$ implies $\|\{w_{n,xxx}\}\|_{L^2(0,\pi)} \leq C$. Then we have

$$\|w_n\|_V + \|w_n\|_{H^3(0,\xi)} + \|w_n\|_{H^3(\xi,\eta)} + \|w_n\|_{H^3(\eta,\pi)} \leq C$$

for some constant $C > 0$, which implies that there exist some w in V and a subsequence $\{w_{n_k}\}_{k\geq 1}$ such that

$$w_{n_k} \rightharpoonup w \quad \text{in } V \cap H^3(0, \xi) \cap H^3(\xi, \eta) \cap H^3(\eta, \pi) \quad \text{as } k \to \infty,$$

where the notation $\rightharpoonup$ is used to denote weak convergence. So we have $w_{n_k} \to w$ in $H^2(0, \xi) \cap H^2(\xi, \eta) \cap H^2(\eta, \pi)$ and therefore

$$w_{n_k} \to w \quad \text{in } V \quad \text{as } k \to \infty. \tag{20}$$

Equations (19) and (20) show that the embedding $\mathcal{D}(\mathcal{A}) \hookrightarrow \mathcal{H}$ is compact. □

Since $\mathcal{A}^{-1}$ is compact, operator $\mathcal{A}$ is a discrete operator and then $\sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, consists of isolated eigenvalues. Applying Arendt–Batty Theorem, we only need to prove $i\mathbb{R}$ belongs to $\rho(\mathcal{A})$.

Lemma 3.2. *If $\kappa > 0$ and $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ are irrationals, then $i\mathbb{R}$ belongs to $\rho(\mathcal{A})$.*

Proof. Since $\sigma(\mathcal{A})$ consists of isolated eigenvalues, it is sufficient to check that for any β in $\mathbb{R}\setminus\{0\}$ the equation

$$(i\beta-\mathcal{A})(w,v)=(0,0),\quad (w,v)\in\mathcal{D}(\mathcal{A}) \tag{21}$$

admits only the trivial solution $(w,v)=(0,0)$. Using (17) and (21), we obtain that

$$\begin{aligned}0=&((i\beta-\mathcal{A})(w,v),(w,v))_{\mathcal{H}}\\ =&\,i\left(\beta\|(w,v)\|^2_{\mathcal{H}}-2\mathrm{Im}\int_0^\pi v_{xx}\overline{w_{xx}}\mathrm{d}x\right)+\kappa|v_x(\eta)-v_x(\xi)|^2.\end{aligned}$$

Hence $[w_{xx}]_\eta=-[w_{xx}]_\xi=\kappa(v_x(\eta)-v_x(\xi))=0$ and w belongs to $V\cap H^3(0,\pi)$. So (21) yields

$$\begin{cases}-\beta^2 w+\mathcal{R}w_{xxxx}=0,\\ w(0)=w(\pi)=w_{xx}(0)=w_{xx}(\pi)=0.\end{cases}$$

Solving this equation we obtain that when $\beta^2=\lambda_k=\frac{k^4}{1+\alpha k^2}$ for some $k\geq 1$, this equation admits an non-zero solution $w(x)=C\sin(kx)$ where C is a complex constant. Therefore, $v(x)=i\beta w(x)=i\beta C\sin(kx)$. It follows that

$$\begin{aligned}0=&\kappa(v_x(\eta)-v_x(\xi))=i\beta C\kappa k(\cos(k\eta)-\cos(k\xi))\\ =&-2i\beta C\kappa k\sin\left(\frac{k(\eta+\xi)}{2}\right)\sin\left(\frac{k(\eta-\xi)}{2}\right).\end{aligned}$$

Since $\kappa>0$ and $\frac{\eta+\xi}{2\pi}$ and $\frac{\eta-\xi}{2\pi}$ are irrationals, it is necessary that $C=0$. Consequently $w=v=0$. This concludes the proof of Lemma 3.2 and Theorem 1.4. □

4. **Polynomial stability (Proof of Theorem 1.5).** This section is devoted to prove polynomial stability and estimate the decay rate. The methods of proof are inspired by the methods used in [18, 19].

According to Borichev-Tomilov criteria (see [6]), polynomial decay rate (7) is equivalent to the resolvent estimate

$$\sup_{|\beta|\geq 1}|\beta|^{-l}\|(i\beta-\mathcal{A})^{-1}\|<\infty. \tag{22}$$

We prove two different cases ($\frac{\eta\pm\xi}{2\pi}$ belongs to A and $\frac{\eta\pm\xi}{2\pi}$ belongs to B_ε) of (22) simultaneously, so we set

$$l=\begin{cases}4, & \text{if } \dfrac{\eta\pm\xi}{2\pi}\in A,\\ 4+4\varepsilon, & \text{if } \dfrac{\eta\pm\xi}{2\pi}\in B_\varepsilon.\end{cases} \tag{23}$$

We argue by contradiction. If (22) is false, then there exist β_n in $\mathbb{R}$ and (w_n,v_n) in $\mathcal{D}(\mathcal{A})$ for $n\geq 1$ such that

$$\|(w_n,v_n)\|_{\mathcal{H}}=1,\quad |\beta_n|\to\infty \tag{24}$$

and

$$|\beta_n|^l(f_n,g_n)\to(0,0)\quad \text{in } \mathcal{H}, \tag{25}$$

where $(f_n,g_n)=(i\beta_n-\mathcal{A})(w_n,v_n)$, namely,

$$|\beta_n|^l(i\beta_n w_n-v_n)=|\beta_n|^l f_n\to 0\quad\text{in } V, \tag{26}$$

$$|\beta_n|^l(i\beta_n v_n+\mathcal{R}\{w_{n,xxx}\}_x)=|\beta_n|^l g_n\to 0\quad\text{in } H. \tag{27}$$

Note that putting conjugate on (26) and (27) is equivalent to replace $i\beta_n$, w_n and v_n by $-i\beta_n$, $\overline{w_n}$ and $\overline{v_n}$, respectively. So we assume without loss of generality that $\beta_n > 0$.

We obtain a contradiction through two lemmas. The proofs of these two lemmas are given after the proof of Theorem 1.5.

Lemma 4.1. *Let $\beta_n > 0$ and (w_n, v_n) in $\mathcal{D}(\mathcal{A})$ for $n \geq 1$ satisfy (24) and (25) with $l \geq 0$. We have*

$$\lim_{n\to\infty} \|w_n\|_V^2 = \lim_{n\to\infty} \|v_n\|_H^2 = \lim_{n\to\infty} \|\beta_n w_n\|_H^2 = \frac{1}{2}, \tag{28}$$

and

$$\begin{aligned} &3\int_0^\pi |w_{n,xx}|^2 \mathrm{d}x + \beta_n^2 \int_0^\pi [|w_n|^2 + \alpha|w_{n,x}|^2]\mathrm{d}x \\ &= 2\beta_n^2 \int_0^\pi \alpha|w_{n,x}|^2 \mathrm{d}x + O(|\beta_n w_{n,x}(\pi)|^2) + o(1), \end{aligned} \tag{29}$$

where and in what follows, $o(1)$ and $O(1)$ are the infinitesimal and the bounded term respectively as $n \to \infty$.

Remark 4.2. Notice that Lemma 4.1 (and the following Lemma 4.3) holds not only for $l \geq 4$ but also for $0 \leq l < 4$. We only use the expression (23) of l at the end of the proof of Theorem 1.5.

Therefore, we obtain a contradiction to (28) if $\beta_n w_{n,x}(\pi) \to 0$ as $n \to \infty$, because the left-hand side of (29) has a limit 2 but the right-hand side is smaller than 3/2 as $n \to \infty$.

In next lemma, we give the upper bound of $\beta_n w_{n,x}(\pi)$. In what follows, the letters $C, C', C'', \ldots$ denote positive constants which may vary from line to line.

Lemma 4.3. *Let β_n and (w_n, v_n) be the same as in Lemma 4.1. Then there exist a constant $C > 0$ such that*

$$|\beta_n w_{n,x}(\pi)| \leq C \frac{\|g_n\|_H + \|f_n\|_V}{\sqrt{\sin(b_n\pi)^2 + (h(b_n))^2}}, \tag{30}$$

where

$$b_n = \left(\frac{\alpha\beta_n^2 + \beta_n\sqrt{\alpha^2\beta_n^2 + 4}}{2} \right)^{\frac{1}{2}},$$

and the function $h : \mathbb{R} \to \mathbb{R}$ is defined by

$$h(b) = 2\cos b(\pi - \eta + \xi) + 2\cos b(\pi - \eta - \xi) - 2\cos(b\pi) - \cos b(\pi - 2\xi) - \cos b(\pi - 2\eta).$$

Therefore, it is sufficient to estimate the upper bound (30) of $|\beta_n w_{n,x}(\pi)|$. Since $b_n \to \infty$ as $\beta_n \to \infty$, we need to estimate the lower bound of $\sqrt{\sin(b_n\pi)^2 + (h(b_n))^2}$. For b in $\mathbb{N}^*$, we have

$$\begin{aligned} h(b) =& 2\cos(b\pi)(1 - \cos b(\eta + \xi))(\cos b(\eta - \xi) - 1) \\ =& -8\cos(b\pi) \left[\sin\left(b\frac{\eta + \xi}{2}\right) \sin\left(b\frac{\eta - \xi}{2}\right) \right]^2. \end{aligned} \tag{31}$$

Recall the conditions of Theorem 1.5, if $\frac{\eta \pm \xi}{2\pi}$ belong to A, from (10) in Section 2 we see that there exists a small constant $C > 0$ such that for all b in $\mathbb{N}^*$,

$$\left| \sin\left(\frac{b(\eta \pm \xi)}{2} \right) \right| = \left| \sin\left[\pi \left(\frac{b(\eta \pm \xi)}{2\pi} - p \right) \right] \right| \geq \left| \sin\left(\frac{\pi C}{b} \right) \right| \geq \frac{C}{b}. \tag{32}$$

And if $\frac{\eta\pm\xi}{2\pi}$ belong to B_ε, from (11) in Section 2 we have there exists a small constant $C>0$ such that for all b in $\mathbb{N}^*$,

$$\left|\sin\left(\frac{b(\eta\pm\xi)}{2}\right)\right| = \left|\sin\left[\pi\left(\frac{b(\eta\pm\xi)}{2\pi}-p\right)\right]\right| \geq \left|\sin\left(\frac{\pi C}{b}\right)\right| \geq \frac{C}{b^{1+\varepsilon}}. \tag{33}$$

Then by the expressions (23) and (31) for l and $h(b)$, we have

$$|h(b)| \geq \frac{C}{b^l}$$

for b in $\mathbb{N}^*$. Since the function h is uniformly Lipschitz continuous on $\mathbb{R}$, the same inequalities hold (with different constants) for b large enough in

$$\bigcup_{k\in\mathbb{N}^*}\left(k-\frac{C'}{k^l}, k+\frac{C'}{k^l}\right),$$

if the constant $C'>0$ is large enough. On the other hand we may associate with, that constant $C'>0$, a constant $C''>0$ such that

$$|\sin(b\pi)| \geq \frac{C''}{b^l}$$

for b large enough satisfying $|b-k|\geq C'/k^l$ for all k in $\mathbb{N}^*$. It follows that

$$\sqrt{\sin(b\pi)^2+(h(b))^2} \geq \frac{C}{b^l} \tag{34}$$

for b large enough. Gathering together (30) amd (34) we have

$$|\beta_n w_{n,x}(\pi)| \leq C b_n^l(\|g_n\|_H+\|f_n\|_V) \leq C'\beta_n^l(\|g_n\|_H+\|f_n\|_V) \to 0, \quad \text{as } n\to\infty. \tag{35}$$

Therefore, the proof of Theorem 1.5 is achieved.

The remaining thing is to prove Lemmas 4.1 and 4.3.

4.1. **Proof of Lemma 4.1.** We prove this lemma by the multiplier method. From (17), we deduce that

$$\begin{aligned}\kappa|v_{n,x}(\eta)-v_{n,x}(\xi)|^2 &= \mathrm{Re}((i\beta_n-\mathcal{A})(w_n,v_n),(w_n,v_n))_{\mathcal{H}}\\ &= \mathrm{Re}((f_n,g_n),(w_n,v_n))_{\mathcal{H}},\end{aligned}$$

hence using (24) and (25), we have

$$|v_{n,x}(\eta)-v_{n,x}(\xi)|^2 \to 0 \quad \text{as } n\to\infty. \tag{36}$$

On the other hand, (23), (24), (26) and (27) imply that as $n\to\infty$

$$i\beta_n\|w_n\|_V^2-(v_n,w_n)_V \to 0, \tag{37}$$

$$i\beta_n\|v_n\|_H^2+(\mathcal{R}\{w_{n,xxx}\}_x,v_n)_H \to 0. \tag{38}$$

Taking the difference of (37) and (38) we deduce

$$\begin{aligned}i\beta_n(\|w_n\|_V^2-\|v_n\|_H^2) &- \int_0^\pi v_{n,xx}\overline{w_{n,xx}}\mathrm{d}x\\ &- \int_0^\pi [\mathcal{R}\{w_{n,xxx}\}_x\overline{v_n}+\alpha\mathcal{R}\{w_{n,xxx}\}_{xx}\overline{v_{n,x}}]\mathrm{d}x \to 0.\end{aligned}$$

However, by (17), we deduce

$$\begin{aligned}&\mathrm{Im}\left(-\int_0^\pi [\mathcal{R}\{w_{n,xxx}\}_x\overline{v_n}+\alpha\mathcal{R}\{w_{n,xxx}\}_{xx}\overline{v_{n,x}}]\mathrm{d}x+\int_0^\pi v_{n,xx}\overline{w_{n,xx}}\mathrm{d}x\right)\\&=2\mathrm{Im}\int_0^\pi v_{n,xx}\overline{w_{n,xx}}\mathrm{d}x,\end{aligned}$$

which implies that

$$\mathrm{Im}\left(\int_0^\pi [\mathcal{R}\{w_{n,xxx}\}_x\overline{v_n}+\alpha\mathcal{R}\{w_{n,xxx}\}_{xx}\overline{v_{n,x}}]\mathrm{d}x+\int_0^\pi v_{n,xx}\overline{w_{n,xx}}\mathrm{d}x\right)=0.$$

Hence

$$\beta_n(\|w_n\|_V^2-\|v_n\|_H^2)\to 0. \tag{39}$$

Therefore, using (24) and (26) we obtain (28).

Now we prove (29). We eliminate v_n in (27) by using (26), and we have

$$-\beta_n^2 w_n+\mathcal{R}\{w_{n,xxx}\}_x=g_n+i\beta_n f_n \quad \text{in } H, \tag{40}$$

and hence

$$(-\beta_n^2 w_n+\mathcal{R}\{w_{n,xxx}\}_x,\varphi w_{n,x})_H=(g_n+i\beta_n f_n,\varphi w_{n,x})_H \tag{41}$$

for any real function φ in $C^3[0,\pi]$.

First we deal with the right-hand side of (41). We obtain by using (27) and (28)

$$(g_n,\varphi w_{n,x})_H\le C\|g_n\|_H\|w_{n,x}\|_H\le C\|g_n\|_H\|w_n\|_V\to 0. \tag{42}$$

Notice that

$$\begin{aligned}(f_n,\varphi w_{n,x})_H=&\int_0^\pi [f_n\varphi\overline{w_{n,x}}+\alpha f_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\=&-\int_0^\pi [(f_n\varphi)_x\overline{w_n}+\alpha f_{n,xx}\varphi\overline{w_{n,x}}]\mathrm{d}x\\&+\alpha(\varphi(\pi)f_{n,x}(\pi)\overline{w_{n,x}(\pi)}-\varphi(0)f_{n,x}(0)\overline{w_{n,x}(0)}).\end{aligned} \tag{43}$$

By (26) and (28) and $H^2(0,\pi)$ included in $C^1(0,\pi)$, we deduce that

$$\beta_n\left|\int_0^\pi [(f_n\varphi)_x\overline{w_n}+\alpha f_{n,xx}\varphi\overline{w_{n,x}}]\mathrm{d}x\right|\le C\|f_n\|_V\|\beta_n w_n\|_H\to 0 \tag{44}$$

and that

$$\begin{aligned}\beta_n\varphi(\pi)f_{n,x}(\pi)\overline{w_{n,x}(\pi)}&=\varphi(\pi)o(\beta_n w_{n,x}(\pi)),\\ \beta_n\varphi(0)f_{n,x}(0)\overline{w_{n,x}(0)}&=\varphi(0)o(\beta_n w_{n,x}(0)).\end{aligned} \tag{45}$$

It follows from (43), (44) and (45) that

$$(i\beta_n f_n,\varphi w_{n,x})_H=\varphi(\pi)o(\beta_n w_{n,x}(\pi))+\varphi(0)o(\beta_n w_{n,x}(0))+o(1). \tag{46}$$

Now we turn to the left-hand side of (41). Using integrating by parts we have

$$\begin{aligned}(-\beta_n^2 w_n,\varphi w_{n,x})_H=&-\beta_n^2\int_0^\pi [w_n\varphi\overline{w_{n,x}}+\alpha w_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\=&\beta_n^2\int_0^\pi [\varphi_x|w_n|^2+\varphi w_{n,x}\overline{w_n}+\alpha w_{n,xx}\varphi\overline{w_{n,x}}]\mathrm{d}x\\&-\beta_n^2\alpha(\varphi(\pi)|w_{n,x}(\pi)|^2-\varphi(0)|w_{n,x}(0)|^2).\end{aligned}$$

Hence we have

$$
\begin{aligned}
2\mathrm{Re}(-\beta_n^2 w_n, \varphi w_{n,x})_H =& \beta_n^2 \int_0^\pi \varphi_x[|w_n|^2 - \alpha|w_{n,x}|^2]\mathrm{d}x \\
&- \beta_n^2 \alpha(\varphi(\pi)|w_{n,x}(\pi)|^2 - \varphi(0)|w_{n,x}(0)|^2).
\end{aligned} \tag{47}
$$

Denote $I := (\mathcal{R}\{w_{n,xxx}\}_x, \varphi w_{n,x})_H$. Since $\mathcal{R}f_{xx} = (\mathcal{R}f - f)/\alpha$ and $\mathcal{R}$ is an isomorphism from $L^2(0,\pi)$ to V, then we obtain by integrating by parts

$$
\begin{aligned}
I =& -\int_0^\pi \{w_{n,xxx}\}(\varphi\overline{w_{n,x}})_x \mathrm{d}x \\
=& -\left(\int_0^\xi + \int_\xi^\eta + \int_\eta^\pi\right)[w_{n,xxx}\varphi_x\overline{w_{n,x}} + \varphi w_{n,xxx}\overline{w_{n,xx}}]\mathrm{d}x \\
=& \int_0^\pi [-\varphi_x|w_{n,xx}|^2 + 3\varphi_x|w_{n,xx}|^2 + w_{n,xx}\varphi_{xx}\overline{w_{n,x}}]\mathrm{d}x \\
&+ \left(\int_0^\xi + \int_\xi^\eta + \int_\eta^\pi\right)\varphi w_{n,xx}\overline{w_{n,xxx}}\mathrm{d}x + \varphi_x(\eta)\overline{w_{n,x}(\eta)}[w_{n,xx}]_\eta \\
&+ \varphi_x(\xi)\overline{w_{n,x}(\xi)}[w_{n,xx}]_\xi + \varphi(\eta)[|w_{n,xx}|^2]_\eta + \varphi(\xi)[|w_{n,xx}|^2]_\xi \\
=& \int_0^\pi 3\varphi_x|w_{n,xx}|^2\mathrm{d}x + \left(\int_0^\xi + \int_\xi^\eta + \int_\eta^\pi\right)\varphi w_{n,xx}\overline{w_{n,xxx}}\mathrm{d}x \\
&+ \varphi_x(\eta)\overline{w_{n,x}(\eta)}[w_{n,xx}]_\eta + \varphi_x(\xi)\overline{w_{n,x}(\xi)}[w_{n,xx}]_\xi + \varphi(\eta)[|w_{n,xx}|^2]_\eta \\
&+ \varphi(\xi)[|w_{n,xx}|^2]_\xi + \int_0^\pi [w_{n,x}\varphi_{xx}\overline{w_{n,xx}} - \varphi_{xxx}|w_{n,x}|^2 - w_{n,x}\varphi_{xx}\overline{w_{n,xx}}]\mathrm{d}x \\
&+ \left(\int_0^\xi + \int_\xi^\eta + \int_\eta^\pi\right) w_{n,x}\varphi_x\overline{w_{n,xxx}}\mathrm{d}x + \varphi_x(\eta)w_{n,x}(\eta)[\overline{w_{n,xx}}]_\eta \\
&+ \varphi_x(\xi)w_{n,x}(\xi)[\overline{w_{n,xx}}]_\xi + \varphi_{xx}(\pi)|w_{n,x}(\pi)|^2 - \varphi_{xx}(0)|w_{n,x}(0)|^2 \\
=& \left(\int_0^\xi + \int_\xi^\eta + \int_\eta^\pi\right)[w_{n,x}\varphi_x\overline{w_{n,xxx}} + \varphi w_{n,xx}\overline{w_{n,xxx}}]\mathrm{d}x \\
&+ \int_0^\pi [3\varphi_x|w_{n,xx}|^2 - \varphi_{xxx}|w_{n,x}|^2]\mathrm{d}x + \varphi(\eta)[|w_{n,xx}|^2]_\eta + \varphi(\xi)[|w_{n,xx}|^2]_\xi \\
&+ 2\mathrm{Re}(\varphi_x(\eta)\overline{w_{n,x}(\eta)}[w_{n,xx}]_\eta + \varphi_x(\xi)\overline{w_{n,x}(\xi)}[w_{n,xx}]_\xi) \\
&+ \varphi_{xx}(\pi)|w_{n,x}(\pi)|^2 - \varphi_{xx}(0)|w_{n,x}(0)|^2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
2\mathrm{Re} I =& \int_0^\pi [3\varphi_x|w_{n,xx}|^2 - \varphi_{xxx}|w_{n,x}|^2]\mathrm{d}x + \varphi(\eta)[|w_{n,xx}|^2]_\eta + \varphi(\xi)[|w_{n,xx}|^2]_\xi \\
&+ 2\mathrm{Re}(\varphi_x(\eta)\overline{w_{n,x}(\eta)}[w_{n,xx}]_\eta + \varphi_x(\xi)\overline{w_{n,x}(\xi)}[w_{n,xx}]_\xi) \\
&+ \varphi_{xx}(\pi)|w_{n,x}(\pi)|^2 - \varphi_{xx}(0)|w_{n,x}(0)|^2.
\end{aligned}
$$

Using a classical interpolation inequality (see [20]), for all s in $[0,2)$,

$$
\|u\|^2_{H^s(0,\pi)} \le C\|u\|_{H^2(0,\pi)}\|u\|_{L^2(0,\pi)},
$$

and (28) and $\beta_n \to \infty$, we obtain that $w_n \to 0$ in $H^s(0,\pi)$ for all s in $[0,2)$. Hence $w_n \to 0$ in $C^1(0,\pi)$ since embedding $H^2(0,\pi) \hookrightarrow C^1(0,\pi)$ is continuous. And we have $[w_{n,xx}]_\eta \to 0$ and $[w_{n,xx}]_\eta \to 0$ because of (14) and (36). Therefore,

$$2\mathrm{Re}I = \int_0^\pi 3\varphi_x|w_{n,xx}|^2\mathrm{d}x + \varphi(\eta)[|w_{n,xx}|^2]_\eta + \varphi(\xi)[|w_{n,xx}|^2]_\xi + o(1).$$

Notice that

$$\begin{aligned}[|w_{n,xx}|^2]_\eta =& w_{n,xx}(\eta^+)\overline{w_{n,xx}(\eta^+)} - w_{n,xx}(\eta^-)\overline{w_{n,xx}(\eta^-)}\\ =&[w_{n,xx}]_\eta\overline{w_{n,xx}(\eta^+)} + w_{n,xx}(\eta^-)[\overline{w_{n,xx}}]_\eta\\ =&2\mathrm{Re}([w_{n,xx}]_\eta\overline{w_{n,xx}(\eta^+)}) - |[w_{n,xx}]_\eta|^2\\ =&2\kappa\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))\overline{w_{n,xx}(\eta^+)}) + o(1).\end{aligned}$$

Similarly,

$$\begin{aligned}[|w_{n,xx}|^2]_\xi =& -2\kappa\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))\overline{w_{n,xx}(\xi^+)}) + o(1)\\ =& -2\kappa\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))([\overline{w_{n,xx}}]_\xi + \overline{w_{n,xx}(\xi^-)})) + o(1)\\ =& -2\kappa\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))\overline{w_{n,xx}(\xi^-)}) + o(1).\end{aligned}$$

Consequently, we have

$$\begin{aligned}2\mathrm{Re}I =& \int_0^\pi 3\varphi_x|w_{n,xx}|^2\mathrm{d}x + 2\kappa\varphi(\eta)\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))\overline{w_{n,xx}(\eta^+)})\\ &- 2\kappa\varphi(\xi)\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))\overline{w_{n,xx}(\xi^-)}) + o(1).\end{aligned} \tag{48}$$

Gathering together (41), (42), (46), (47) and (48) with $\varphi(x) = x$, we obtain

$$\begin{aligned}&3\int_0^\pi |w_{n,xx}|^2\mathrm{d}x + \beta_n^2\int_0^\pi [|w_n|^2 - \alpha|w_{n,x}|^2]\mathrm{d}x = (\alpha\pi\beta_n\overline{w_{n,x}(\pi)} + o(1))\beta_n w_{n,x}(\pi)\\ &- 2\kappa\mathrm{Re}((v_{n,x}(\eta) - v_{n,x}(\xi))(\eta\overline{w_{n,xx}(\eta^+)} - \xi\overline{w_{n,xx}(\xi^-)})) + o(1).\end{aligned} \tag{49}$$

Now we estimate $w_{n,xx}(\eta^+)$ and $w_{n,xx}(\xi^-)$ as $n \to \infty$. We have the following result.

Proposition 4.4. *Let β_n and (w_n, v_n) be the same as in Lemma 4.1, then we have*

$$\begin{aligned}|w_{n,xx}(\eta^+)|^2 &= O(|\beta_n w_{n,x}(\pi)|^2 + 1),\\ |w_{n,xx}(\xi^-)|^2 &= O(|\beta_n w_{n,x}(\pi)|^2 + 1).\end{aligned}$$

Since we do not know the asymptotic behavior of $|\beta_n w_{n,x}(\pi)|^2$ right now, we add 1 into O to specify that $|w_{n,xx}(\eta^+)|^2$ and $|w_{n,xx}(\xi^-)|^2$ are only bounded terms if $\beta_n w_{n,x}(\pi) \to 0$.

Proof of Proposition 4.4. Since (40) holds in H, we obtain by differentiating both sides of (40)

$$-\beta_n^2 w_{n,x} + \mathcal{R}\{w_{n,xxx}\}_{xx} = g_{n,x} + i\beta_n f_{n,x} \quad \text{in } L^2(0,\pi). \tag{50}$$

Multiplying each term in (40) by $\varphi\overline{w_{n,x}}$ where φ in $C^3[0,\pi]$ is a real function, and integrating over $(0,\xi)$ and (η,π), we obtain

$$-\beta_n^2\int_0^\xi w_n\varphi\overline{w_{n,x}}\mathrm{d}x + \int_0^\xi \mathcal{R}\{w_{n,xxx}\}_x\varphi\overline{w_{n,x}}\mathrm{d}x = \int_0^\xi (g_n + i\beta_n f_n)\varphi\overline{w_{n,x}}\mathrm{d}x, \tag{51}$$

$$-\beta_n^2\int_\eta^\pi w_n\varphi\overline{w_{n,x}}\mathrm{d}x + \int_\eta^\pi \mathcal{R}\{w_{n,xxx}\}_x\varphi\overline{w_{n,x}}\mathrm{d}x = \int_\eta^\pi (g_n + i\beta_n f_n)\varphi\overline{w_{n,x}}\mathrm{d}x. \tag{52}$$

Multiplying each term in (50) by $(\varphi\overline{w_{n,x}})_x$ and integrating over $(0,\xi)$ and (η,π), we obtain

$$\begin{aligned}&-\beta_n^2\int_0^\xi w_{n,x}(\varphi\overline{w_{n,x}})_x\mathrm{d}x+\int_0^\xi \mathcal{R}\{w_{n,xxx}\}_{xx}(\varphi\overline{w_{n,x}})_x\mathrm{d}x\\&=\int_0^\xi (g_n+i\beta_n f_n)_x(\varphi\overline{w_{n,x}})_x\mathrm{d}x,\end{aligned}\tag{53}$$

and

$$\begin{aligned}&-\beta_n^2\int_\eta^\pi w_{n,x}(\varphi\overline{w_{n,x}})_x\mathrm{d}x+\int_\eta^\pi \mathcal{R}\{w_{n,xxx}\}_{xx}(\varphi\overline{w_{n,x}})_x\mathrm{d}x\\&=\int_\eta^\pi (g_n+i\beta_n f_n)_x(\varphi\overline{w_{n,x}})_x\mathrm{d}x.\end{aligned}\tag{54}$$

Adding (51) by $\alpha\cdot$(53) and adding (52) by $\alpha\cdot$(54), we obtain

$$\begin{aligned}&-\beta_n^2\int_0^\xi [w_n\varphi\overline{w_{n,x}}+\alpha w_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\&+\int_0^\xi [\mathcal{R}\{w_{n,xxx}\}_x\varphi\overline{w_{n,x}}+\alpha\mathcal{R}\{w_{n,xxx}\}_{xx}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\=&\int_0^\xi [g_n\varphi\overline{w_{n,x}}+\alpha g_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x+i\beta_n\int_0^\xi [f_n\varphi\overline{w_{n,x}}+\alpha f_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x,\end{aligned}\tag{55}$$

and

$$\begin{aligned}&-\beta_n^2\int_\eta^\pi [w_n\varphi\overline{w_{n,x}}+\alpha w_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\&+\int_\eta^\pi [\mathcal{R}\{w_{n,xxx}\}_x\varphi\overline{w_{n,x}}+\alpha\mathcal{R}\{w_{n,xxx}\}_{xx}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\=&\int_\eta^\pi [g_n\varphi\overline{w_{n,x}}+\alpha g_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x+i\beta_n\int_\eta^\pi [f_n\varphi\overline{w_{n,x}}+\alpha f_{n,x}(\varphi\overline{w_{n,x}})_x]\mathrm{d}x.\end{aligned}\tag{56}$$

Denote (55) and (56) by $I_1^\xi+I_2^\xi=I_3^\xi+I_4^\xi$ and $I_1^\eta+I_2^\eta=I_3^\eta+I_4^\eta$ respectively. Now we estimate each term of (55) and (56). Obviously

$$|I_3^\nu|\le C\|g_n\|_H\|w_n\|_V=o(1),\quad \nu=\xi,\eta.\tag{57}$$

Using integrating by parts, we have

$$\begin{aligned}I_4^\xi=&i\beta_n\int_0^\xi [f_n\varphi\overline{w_{n,x}}-\alpha f_{n,xx}\varphi\overline{w_{n,x}}]\mathrm{d}x\\&+i\beta_n[\varphi(\xi)f_{n,x}(\xi)\overline{w_{n,x}}(\xi)-\varphi(0)f_{n,x}(0)\overline{w_{n,x}}(0)].\end{aligned}$$

Notice that

$$\left|i\beta_n\int_0^\xi [f_n\varphi\overline{w_{n,x}}-\alpha f_{n,xx}\varphi\overline{w_{n,x}}]\mathrm{d}x\right|\le C\|f_n\|_V\|\beta_n w_n\|_H=o(1),$$

then we obtain

$$I_4^\xi=i\beta_n[\varphi(\xi)f_{n,x}(\xi)\overline{w_{n,x}}(\xi)-\varphi(0)f_{n,x}(0)\overline{w_{n,x}}(0)]+o(1),\tag{58}$$

and similarly obtain

$$I_4^\eta=i\beta_n[\varphi(\pi)f_{n,x}(\pi)\overline{w_{n,x}}(\pi)-\varphi(\eta)f_{n,x}(\eta)\overline{w_{n,x}}(\eta)]+o(1).\tag{59}$$

Again using integrating by parts, we deduce

$$\begin{aligned} I_1^\xi =&\beta_n^2\int_0^\xi[\varphi_x|w_n|^2+w_{n,x}\varphi\overline{w_n}+\alpha w_{n,xx}\varphi\overline{w_{n,x}}]\mathrm{d}x\\ &-\beta_n^2\varphi(\xi)|w_n(\xi)|^2-\beta_n^2\alpha(\varphi(\xi)|w_{n,x}(\xi)|^2-\varphi(0)|w_{n,x}(0)|^2), \end{aligned}$$

and therefore,

$$\begin{aligned} 2\mathrm{Re}I_1^\xi =&\beta_n^2\int_0^\xi\varphi_x[|w_n|^2-\alpha|w_{n,x}|^2]\mathrm{d}x\\ &-\beta_n^2\varphi(\xi)|w_n(\xi)|^2-\beta_n^2\alpha(\varphi(\xi)|w_{n,x}(\xi)|^2-\varphi(0)|w_{n,x}(0)|^2). \end{aligned} \tag{60}$$

Similarly we have

$$\begin{aligned} 2\mathrm{Re}I_1^\eta =&\beta_n^2\int_\eta^\pi\varphi_x[|w_n|^2-\alpha|w_{n,x}|^2]\mathrm{d}x\\ &+\beta_n^2\varphi(\eta)|w_n(\eta)|^2-\beta_n^2\alpha(\varphi(\pi)|w_{n,x}(\pi)|^2-\varphi(\eta)|w_{n,x}(\eta)|^2). \end{aligned} \tag{61}$$

Finally we deal with I_2^ξ and I_2^η. We have

$$\begin{aligned} I_2^\xi =&\int_0^\xi[-\mathcal{R}\{w_{n,xxx}\}(\varphi\overline{w_{n,x}})_x+(\mathcal{R}\{w_{n,xxx}\}-\{w_{n,xxx}\})(\varphi\overline{w_{n,x}})_x]\mathrm{d}x\\ &+\mathcal{R}\{w_{n,xxx}\}(\xi)\varphi(\xi)\overline{w_{n,x}(\xi)}\\ =&-\int_0^\xi w_{n,xxx}(\varphi\overline{w_{n,x}})_x\mathrm{d}x+\mathcal{R}\{w_{n,xxx}\}(\xi)\varphi(\xi)\overline{w_{n,x}(\xi)}\\ =:&\ I_{21}^\xi+I_{22}^\xi. \end{aligned} \tag{62}$$

First notice that

$$\begin{aligned} \|\mathcal{R}\{w_{n,xxx}\}\|_{C^0(0,\pi)} \le& C\|\mathcal{R}\{w_{n,xxx}\}\|_H\le C\|\{w_{n,xxx}\}\|_{H^{-1}(0,\pi)}\\ =&C\|w_{n,xxx}-[w_{n,xx}]_\xi\delta_\xi-[w_{n,xx}]_\eta\delta_\eta\|_{H^{-1}(0,\pi)}\\ \le&C(\|w_{n,xxx}\|_{H^{-1}(0,\pi)}+[w_{n,xx}]_\xi+[w_{n,xx}]_\eta)\\ \le&C\|w_n\|_V+o(1)=O(1), \end{aligned}$$

and therefore,

$$I_{22}^\xi\le\|\mathcal{R}\{w_{n,xxx}\}\varphi\overline{w_{n,x}}\|_{C^0(0,\pi)}=o(1). \tag{63}$$

Then we estimate I_{21}^ξ. We have

$$\begin{aligned} I_{21}^\xi =&-\int_0^\xi[w_{n,xxx}\varphi_x\overline{w_{n,x}}+w_{n,xxx}\varphi\overline{w_{n,xx}}]\mathrm{d}x\\ =&\int_0^\xi[-\varphi_x|w_{n,xx}|^2+3\varphi_x|w_{n,xx}|^2+w_{n,xx}\varphi_{xx}\overline{w_{n,x}}+w_{n,xx}\varphi\overline{w_{n,xxx}}]\mathrm{d}x\\ &-\varphi_x(\xi)\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-)-\varphi(\xi)|w_{n,xx}(\xi^-)|^2\\ =&\int_0^\xi[3\varphi_x|w_{n,xx}|^2+w_{n,xx}\varphi\overline{w_{n,xxx}}+w_{n,x}\varphi_x\overline{w_{n,xxx}}-\varphi_{xxx}|w_{n,x}|^2]\mathrm{d}x\\ &-\varphi_x(\xi)\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-)-\varphi(\xi)|w_{n,xx}(\xi^-)|^2-\varphi_x(\xi)w_{n,x}(\xi)\overline{w_{n,xx}(\xi^-)}\\ &+\varphi_{xx}(\xi)|w_{n,x}(\xi)|^2-\varphi_{xx}(0)|w_{n,x}(0)|^2. \end{aligned}$$

So

$$\begin{aligned}2\mathrm{Re}I_{21}^{\xi}=&\int_0^{\xi}[3\varphi_x|w_{n,xx}|^2-\varphi_{xxx}|w_{n,x}|^2]\mathrm{d}x-2\varphi_x(\xi)\mathrm{Re}(\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-))\\&-\varphi(\xi)|w_{n,xx}(\xi^-)|^2+\varphi_{xx}(\xi)|w_{n,x}(\xi)|^2-\varphi_{xx}(0)|w_{n,x}(0)|^2\\=&3\int_0^{\xi}\varphi_x|w_{n,xx}|^2\mathrm{d}x-\varphi(\xi)|w_{n,xx}(\xi^-)|^2\\&-2\varphi_x(\xi)\mathrm{Re}(\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-))+o(1).\end{aligned}\tag{64}$$

Consequently, gathering together (62), (63) and (64), we obtain

$$\begin{aligned}2\mathrm{Re}I_2^{\xi}=&3\int_0^{\xi}\varphi_x|w_{n,xx}|^2\mathrm{d}x-\varphi(\xi)|w_{n,xx}(\xi^-)|^2\\&-2\varphi_x(\xi)\mathrm{Re}(\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-))+o(1),\end{aligned}\tag{65}$$

and similarly we have

$$\begin{aligned}2\mathrm{Re}I_2^{\eta}=&3\int_{\eta}^{\pi}\varphi_x|w_{n,xx}|^2\mathrm{d}x+\varphi(\eta)|w_{n,xx}(\eta^+)|^2\\&+2\varphi_x(\eta)\mathrm{Re}(\overline{w_{n,x}(\eta)}w_{n,xx}(\eta^+))+o(1).\end{aligned}\tag{66}$$

Taking $\varphi(x)=x$ and gathering together (55), (57), (58), (60) and (65), we obtain

$$\begin{aligned}&3\int_0^{\xi}|w_{n,xx}|^2\mathrm{d}x+\beta_n^2\int_0^{\xi}[|w_n|^2-\alpha|w_{n,x}|^2]\mathrm{d}x-2\mathrm{Re}(\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-))\\&-\xi(|w_{n,xx}(\xi^-)|^2+|\beta_nw_n(\xi)|^2+\alpha|\beta_nw_{n,x}(\xi)|^2)\\=&\beta_n\overline{w_{n,x}(\xi)}o(1)+o(1),\end{aligned}\tag{67}$$

and similarly, gathering together (56), (57), (59), (61) and (66), we obtain

$$\begin{aligned}&3\int_{\eta}^{\pi}|w_{n,xx}|^2\mathrm{d}x+\beta_n^2\int_{\eta}^{\pi}[|w_n|^2-\alpha|w_{n,x}|^2]\mathrm{d}x+2\mathrm{Re}(\overline{w_{n,x}(\eta)}w_{n,xx}(\eta^+))\\&+\eta(|w_{n,xx}(\eta^+)|^2+|\beta_nw_n(\eta)|^2+\alpha|\beta_nw_{n,x}(\eta)|^2)\\=&\pi\alpha|\beta_nw_{n,x}(\pi)|^2+\beta_n(\overline{w_{n,x}(\pi)}+\overline{w_{n,x}(\eta)})o(1)+o(1),\end{aligned}\tag{68}$$

Focus on (68), since $w_{n,x}(\eta)=o(1)$, we have

$$2\mathrm{Re}(\overline{w_{n,x}(\eta)}w_{n,xx}(\eta^+))=w_{n,xx}(\eta^+)o(1).$$

Using Cauchy inequality, we deduce

$$|w_{n,xx}(\eta^+)|^2=O(|\beta_nw_{n,x}(\pi)|^2+1).$$

Taking difference of (68) and (67), we obtain

$$\begin{aligned}&3\left(\int_{\eta}^{\pi}-\int_0^{\xi}\right)|w_{n,xx}|^2\mathrm{d}x+\beta_n^2\left(\int_{\eta}^{\pi}-\int_0^{\xi}\right)[|w_n|^2-\alpha|w_{n,x}|^2]\mathrm{d}x\\&+\eta(|w_{n,xx}(\eta^+)|^2+|\beta_nw_n(\eta)|^2+\alpha|\beta_nw_{n,x}(\eta)|^2)\\&+\xi(|w_{n,xx}(\xi^-)|^2+|\beta_nw_n(\xi)|^2+\alpha|\beta_nw_{n,x}(\xi)|^2)\\=&\pi\alpha|\beta_nw_{n,x}(\pi)|^2-2\mathrm{Re}[\overline{w_{n,x}(\eta)}w_{n,xx}(\eta^+)+\overline{w_{n,x}(\xi)}w_{n,xx}(\xi^-)]\\&+\beta_n(\overline{w_{n,x}(\pi)}+\overline{w_{n,x}(\eta)})o(1)+\beta_n\overline{w_{n,x}(\xi)}o(1)+o(1).\end{aligned}$$

Using $w_{n,x}(\xi) = o(1)$, $w_{n,x}(\eta) = o(1)$ and Cauchy inequality, we obtain

$$|w_{n,xx}(\xi^-)|^2 = O(|\beta_n w_{n,x}(\pi)|^2 + 1).$$

This concludes the proof of Proposition 4.4. □

Thanks to Proposition 4.4 and (49), finally we arrive to

$$3\int_0^\pi |w_{n,xx}|^2 \mathrm{d}x + \beta_n^2 \int_0^\pi [|w_n|^2 - \alpha |w_{n,x}|^2] \mathrm{d}x = O(|\beta_n w_{n,x}(\pi)|^2) + o(1),$$

which is equivalent to (29).

This concludes the proof of Lemma 4.1.

4.2. **Proof of Lemma 4.3.** To simplify the notations we drop the subscript n in the proof of Lemma 4.3. In order to help the reading of the proof of (30) we split it in three steps as follows.

Step 1. We compute the expression (80) of $w_x(\pi)$ by solving a 4×4 linear system and denote $w_x(\pi) := H_1/H_2$;

Step 2. We estimate the upper bound of $|H_1|$;

Step 3. We estimate the lower bound of $|H_2|$.

Let us now proceed with the formal proof of (30).

Step 1. (Computation of the expression (80) of $w_x(\pi)$): We compute $w_x(\pi)$ by solving system (40). Setting $\widetilde{F} = g + i\beta f$, we solve the system

$$\begin{cases} -\beta^2 w + \mathcal{R}\{w_{xxx}\}_x = \widetilde{F} \quad \text{in } (0,\pi), \\ w(0) = w(\pi) = w_{xx}(0) = w_{xx}(\pi) = 0, \\ [w]_\xi = [w]_\eta = [w_x]_\xi = [w_x]_\eta = 0, \\ [w_{xx}]_\eta = -[w_{xx}]_\xi = \kappa(i\beta(w_x(\eta) - w_x(\xi)) - (f_x(\eta) - f_x(\xi))). \end{cases}$$

We denote $F = (I - \alpha\partial_{xx})\widetilde{F}$. Here we firstly assume (w,v) belongs to $\mathcal{D}(\mathcal{A}) \cap \{(w,v) | w \in H^4(0,\xi) \cap H^4(\xi,\eta) \cap H^4(\eta,\pi)\}$ and then obtain the upper bound (30) of $w_x(\pi)$. Then the density argument shows that the same bound holds for (w,v) in $\mathcal{D}(\mathcal{A})$. Since w belongs to $V \cap H^4(0,\xi) \cap H^4(\xi,\eta) \cap H^4(\eta,\pi)$, we have $[w_{xxx}]_\xi = [w_{xxx}]_\eta = 0$ (see [1]). Then we need to solve

$$\begin{cases} \{w_{xxx}\}_x + \alpha\beta^2 w_{xx} - \beta^2 w = F \quad \text{in } (0,\pi), \\ w(0) = w(\pi) = w_{xx}(0) = w_{xx}(\pi) = 0, \\ [w]_\xi = [w]_\eta = [w_x]_\xi = [w_x]_\eta = [w_{xxx}]_\xi = [w_{xxx}]_\eta = 0, \\ [w_{xx}]_\eta = -[w_{xx}]_\xi = \kappa(i\beta(w_x(\eta) - w_x(\xi)) - (f_x(\eta) - f_x(\xi))). \end{cases}$$

Let λ, μ, σ and γ be given numbers. We first solve the following Cauchy problem in $(0,\xi)$

$$\begin{cases} w_{xxxx} + \alpha\beta^2 w_{xx} - \beta^2 w = F \quad \text{in } (0,\xi), \\ w(0) = 0,\ w_x(0) = \lambda,\ w_{xx}(0) = 0,\ w_{xxx}(0) = \mu, \end{cases} \tag{69}$$

and then solve the following (backwards) Cauchy problem in (η,π)

$$\begin{cases} w_{xxxx} + \alpha\beta^2 w_{xx} - \beta^2 w = F \quad \text{in } (\eta,\pi), \\ w(\pi) = 0,\ w_x(\pi) = \sigma,\ w_{xx}(\pi) = 0,\ w_{xxx}(\pi) = \gamma, \end{cases} \tag{70}$$

and finally solve the following Cauchy problem in (ξ, η)

$$\begin{cases} w_{xxxx} + \alpha\beta^2 w_{xx} - \beta^2 w = F \quad \text{in } (\xi, \eta), \\ w(\xi^+) = w(\xi^-),\ w_x(\xi^+) = w_x(\xi^-),\ w_{xxx}(\xi^+) = w_{xxx}(\xi^-), \\ w_{xx}(\xi^+) = w(\xi^-) - \kappa(i\beta(w_x(\eta^+) - w_x(\xi^-)) - (f_x(\eta) - f_x(\xi))). \end{cases} \tag{71}$$

Then λ, μ, σ and γ have to be chosen such that

$$\begin{cases} w(\eta^+) = w(\eta^-),\ w_x(\eta^+) = w_x(\eta^-),\ w_{xxx}(\eta^+) = w_{xxx}(\eta^-), \\ w_{xx}(\eta^+) = w(\eta^-) + \kappa(i\beta(w_x(\eta^+) - w_x(\xi^-)) - (f_x(\eta) - f_x(\xi))). \end{cases} \tag{72}$$

The ODE $w_{xxxx} + \alpha\beta^2 w_{xx} - \beta^2 w = F$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} w \\ w_x \\ w_{xx} \\ w_{xxx} \end{pmatrix} = M \begin{pmatrix} w \\ w_x \\ w_{xx} \\ w_{xxx} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ F \end{pmatrix}, \quad \text{with} \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta^2 & 0 & -\alpha\beta^2 & 0 \end{pmatrix},$$

where $\beta > 0$ is large enough. Solving the eigenfunction $y^4 + \alpha\beta^2 y^2 - \beta^2 = 0$, we have $y_{1,2} = \pm a$ and $y_{3,4} = \pm ib$, where

$$a^2 := \frac{-\alpha\beta^2 + \beta\sqrt{\alpha^2\beta^2 + 4}}{2}, \quad -b^2 := -\frac{\alpha\beta^2 + \beta\sqrt{\alpha^2\beta^2 + 4}}{2}, \tag{73}$$

with $a, b > 0$. With the notation of a and b, we obtain that the solution matrix e^{xM} corresponding to M has the form

$$\mathrm{e}^{xM} = \frac{1}{a^2 + b^2}\begin{pmatrix} u_1'(x) & u_1(x) & u_2'(x) & u_2(x) \\ u_1''(x) & u_1'(x) & u_2''(x) & u_2'(x) \\ u_1'''(x) & u_1''(x) & u_2'''(x) & u_2''(x) \\ u_1''''(x) & u_1'''(x) & u_2''''(x) & u_2'''(x) \end{pmatrix}, \tag{74}$$

where

$$u_1(x) = \frac{b^2}{a}\sinh(ax) + \frac{a^2}{b}\sin(bx), \quad u_2(x) = \frac{1}{a}\sinh(ax) - \frac{1}{b}\sin(bx).$$

Therefore, solution of (69) is

$$\begin{pmatrix} w \\ w_x \\ w_{xx} \\ w_{xxx} \end{pmatrix}(x) = \mathrm{e}^{xM}\begin{pmatrix} 0 \\ \lambda \\ 0 \\ \mu \end{pmatrix} + \int_0^x \mathrm{e}^{(x-y)M} F(y)\mathrm{d}y \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad x \in (0, \xi),$$

and we obtain

$$\begin{pmatrix} w \\ w_x \\ w_{xx} \\ w_{xxx} \end{pmatrix}(\xi^-) = \mathrm{e}^{\xi M}\begin{pmatrix} 0 \\ \lambda \\ 0 \\ \mu \end{pmatrix} + \int_0^\xi \mathrm{e}^{(\xi-y)M} F(y)\mathrm{d}y \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{75}$$

Similarly, solving (70) backward, we have

$$\begin{pmatrix} w \\ w_x \\ w_{xx} \\ w_{xxx} \end{pmatrix}(\eta^+) = \mathrm{e}^{(\eta-\pi)M}\begin{pmatrix} 0 \\ \sigma \\ 0 \\ \gamma \end{pmatrix} + \int_\pi^\eta \mathrm{e}^{(\eta-y)M} F(y)\mathrm{d}y \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{76}$$

Denoting $G := \kappa(i\beta(w_x(\eta^+) - w_x(\xi^-)) - (f_x(\eta) - f_x(\xi)))$ and solving (71) forward, we have

$$\begin{aligned}\begin{pmatrix} w \\ w_x \\ w_{xx} \\ w_{xxx}\end{pmatrix}(\eta^-) =& \mathrm{e}^{(\eta-\xi)M}\begin{pmatrix} w(\xi^-) \\ w_x(\xi^-) \\ w_{xx}(\xi^-) - G \\ w(\xi^-)\end{pmatrix} + \int_\xi^\eta \mathrm{e}^{(\eta-y)M}F(y)\mathrm{d}y\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \\ =& \mathrm{e}^{\eta M}\begin{pmatrix}0\\ \lambda\\0\\ \mu\end{pmatrix} - G\mathrm{e}^{(\eta-\xi)M}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \int_0^\eta \mathrm{e}^{(\eta-y)M}F(y)\mathrm{d}y\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.\end{aligned} \tag{77}$$

Then thanks to (75), (76) and (77), (72) is equivalent to

$$\begin{aligned}&\mathrm{e}^{(\eta-\pi)M}\begin{pmatrix}0\\ \sigma\\0\\ \gamma\end{pmatrix} + \int_\pi^\eta \mathrm{e}^{(\eta-y)M}F(y)\mathrm{d}y\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \\ =& \mathrm{e}^{\eta M}\begin{pmatrix}0\\ \lambda\\0\\ \mu\end{pmatrix} + G(I - \mathrm{e}^{(\eta-\xi)M})\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \int_0^\eta \mathrm{e}^{(\eta-y)M}F(y)\mathrm{d}y\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\end{aligned}$$

namely

$$\begin{pmatrix}0\\ \sigma\\0\\ \gamma\end{pmatrix} = \mathrm{e}^{\pi M}\begin{pmatrix}0\\ \lambda\\0\\ \mu\end{pmatrix} + G(\mathrm{e}^{(\pi-\eta)M} - \mathrm{e}^{(\pi-\xi)M})\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \int_0^\pi \mathrm{e}^{(\pi-y)M}F(y)\mathrm{d}y\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{78}$$

Next we reorganize the linear system (78). First we introduce some notions

$$\begin{aligned}&\boldsymbol{a} := -\begin{pmatrix}u_1(\pi)\\ u_1'(\pi)\\ u_1''(\pi)\\ u_1'''(\pi)\end{pmatrix}, \quad \boldsymbol{b} := \begin{pmatrix}u_2(\pi)\\ u_2'(\pi)\\ u_2''(\pi)\\ u_2'''(\pi)\end{pmatrix}, \quad \boldsymbol{c} := \begin{pmatrix}0\\ a^2+b^2\\ 0\\ 0\end{pmatrix}, \quad \boldsymbol{d} := \begin{pmatrix}0\\0\\0\\ a^2+b^2\end{pmatrix} \\ &\boldsymbol{e} := \begin{pmatrix}u_2'(\pi-\eta) - u_2'(\pi-\xi)\\ u_2''(\pi-\eta) - u_2''(\pi-\xi)\\ u_2'''(\pi-\eta) - u_2'''(\pi-\xi)\\ u_2''''(\pi-\eta) - u_2''''(\pi-\xi)\end{pmatrix}, \quad \boldsymbol{j} := \int_0^\pi \begin{pmatrix}u_2(\pi-y)\\ u_2'(\pi-y)\\ u_2''(\pi-y)\\ u_2'''(\pi-y)\end{pmatrix}F(y)\mathrm{d}y, \\ &\Delta = \frac{i\beta}{a^2+b^2}\left(\int_0^\xi u_2'(\xi-y)F(y)\mathrm{d}y + \int_\eta^\pi u_2'(\eta-y)F(y)\mathrm{d}y\right) + f_x(\eta) - f_x(\xi),\end{aligned}$$

and constant $S = i\kappa\beta(a^2+b^2)^{-1}$. Then substituting solution matrix (74) and G into (78), we have

$$\begin{aligned}&\begin{pmatrix}\boldsymbol{a} + Su_1'(\xi)\boldsymbol{e} & \boldsymbol{b} + Su_2'(\xi)\boldsymbol{e} & \boldsymbol{c} - Su_1'(\pi-\eta)\boldsymbol{e} & \boldsymbol{d} - Su_2'(\pi-\eta)\boldsymbol{e}\end{pmatrix}\begin{pmatrix}\lambda\\ \mu\\ \sigma\\ \gamma\end{pmatrix} \\ &= \boldsymbol{j} - \kappa\Delta\boldsymbol{e}.\end{aligned} \tag{79}$$

According to the solution of linear system and recalling $w_x(\pi)=\sigma$, we have

$$w_x(\pi)=\frac{\det\left|\boldsymbol{a}+Su_1'(\xi)\boldsymbol{e}\quad \boldsymbol{b}+Su_2'(\xi)\boldsymbol{e}\quad \boldsymbol{j}-\kappa\Delta\boldsymbol{e}\quad \boldsymbol{d}-Su_2'(\pi-\eta)\boldsymbol{e}\right|}{\det\left|\boldsymbol{a}+Su_1'(\xi)\boldsymbol{e}\quad \boldsymbol{b}+Su_2'(\xi)\boldsymbol{e}\quad \boldsymbol{c}-Su_1'(\pi-\eta)\boldsymbol{e}\quad \boldsymbol{d}-Su_2'(\pi-\eta)\boldsymbol{e}\right|},\tag{80}$$

and denote by H_1 the numerator of $w_x(\pi)$ and by H_2 the denominator of $w_x(\pi)$, i.e., $w_x(\pi)=H_1/H_2$.

Our aim is to estimate the upper bound of $|\beta w_x(\pi)|$ as $\beta\to\infty$. Therefore, we need to estimate the upper bound of $|H_1|$ and the lower bound of $|H_2|$. Before we estimate $|H_1|$ and $|H_2|$, we claim some property of a and b. Recalling (73), as $\beta\to\infty$, we have

$$\begin{aligned}&a^2+b^2=\beta\sqrt{\alpha^2\beta^2+4}\sim\alpha\beta^2,\quad b^2\sim\alpha\beta^2,\\&a^2=\frac{\alpha\beta^2}{2}\left(\sqrt{1+\frac{4}{\alpha^2\beta^2}}-1\right)\sim\frac{\alpha\beta^2}{2}\frac{1}{2}\frac{4}{\alpha^2\beta^2}=\frac{1}{\alpha}.\end{aligned}\tag{81}$$

Step 2. (Estimation of the upper bound of $|H_1|$): We calculate the expression of H_1,

$$\begin{aligned}H_1=&\det\left|\boldsymbol{a}\quad \boldsymbol{b}\quad \boldsymbol{j}\quad \boldsymbol{d}\right|+\det\left|Su_1'(\xi)\boldsymbol{e}\quad \boldsymbol{b}\quad \boldsymbol{j}\quad \boldsymbol{d}\right|+\det\left|\boldsymbol{a}\quad Su_2'(\xi)\boldsymbol{e}\quad \boldsymbol{j}\quad \boldsymbol{d}\right|\\&+\det\left|\boldsymbol{a}\quad \boldsymbol{b}\quad -\kappa\Delta\boldsymbol{e}\quad \boldsymbol{d}\right|+\det\left|\boldsymbol{a}\quad \boldsymbol{b}\quad \boldsymbol{j}\quad -Su_2'(\pi-\eta)\boldsymbol{e}\right|\\=&(a^2+b^2)^2\left\{\int_0^\pi[P_1(y)+i\kappa\beta P_2(y)]F(y)\mathrm{d}y-i\kappa\beta\int_0^\xi P_3(y)F(y)\mathrm{d}y\right.\\&\left.-i\kappa\beta\int_\eta^\pi P_4(y)F(y)\mathrm{d}y+H_{15}\right\}\\=:&\,(a^2+b^2)^2(H_{11}+H_{12}+H_{13}+H_{14}+H_{15}),\end{aligned}\tag{82}$$

where

$$P_1(y)=\frac{(a^2+b^2)}{ab}(\sinh(a\pi)\sin(by)-\sin(b\pi)\sinh(ay)),$$

$$\begin{aligned}P_2(y)=&\frac{\sin(by)}{2b}[-2\cosh a(\pi-\eta+\xi)-2\cosh a(\pi-\eta-\xi)+2\cosh(a\pi)\\&+\cosh a(\pi-2\xi)+\cosh a(\pi-2\eta)]\\&+\frac{\cosh(a\eta)-\cosh(a\xi)}{2b}[\sin b(\pi-\eta-y)-\sin b(\pi-\eta+y)\\&-\sin b(\pi-\xi-y)-\sin b(\pi+\xi-y)]\\&+\frac{\sin(b\pi)}{2b}[\cosh a(\eta+\xi-y)+\cosh a(\eta-\xi-y)\\&-\cosh a(2\xi-y)-\cosh(ay)]\\&+\frac{\sinh(ay)}{2a}[-2\cos b(\pi-\eta+\xi)-2\cos b(\pi-\eta-\xi)+2\cos(b\pi)\\&+\cos b(\pi-2\xi)+\cos b(\pi-2\eta)]\\&+\frac{\cos(b\eta)-\cos(b\xi)}{2a}[\sinh a(\pi-\eta-y)-\sinh a(\pi-\eta+y)\\&-\sinh a(\pi-\xi-y)-\sinh a(\pi+\xi-y)]\\&+\frac{\sinh(a\pi)}{2a}[\cos b(\eta+\xi-y)+\cos b(\eta-\xi-y)-\cos b(2\xi-y)-\cos(by)],\end{aligned}$$

$$
\begin{aligned}
P_3(y) = & -\frac{\cosh(a\eta) - \cosh(a\xi)}{2b}[\sin b(\pi + \xi - y) + \sin b(\pi - \xi + y)] \\
& + \frac{\sin(b\pi)}{2b}[\cosh a(\eta + \xi - y) + \cosh a(\eta - \xi + y) \\
& - \cosh a(2\xi - y) - \cosh(ay)] \\
& - \frac{\cos(b\eta) - \cos(b\xi)}{2a}[\sinh a(\pi + \xi - y) + \sinh a(\pi - \xi + y)] \\
& + \frac{\sinh(a\pi)}{2a}[\cos b(\eta + \xi - y) + \cos b(\eta - \xi + y) - \cos b(2\xi - y) - \cos(by)], \\
P_4(y) = & -\frac{\cosh(a\eta) - \cosh(a\xi)}{2b}[\sin b(\pi + \eta - y) + \sin b(\pi - \eta + y)] \\
& + \frac{\sin(b\pi)}{2b}[\cosh a(2\eta - y) + \cosh(ay) \\
& - \cosh a(\eta + \xi - y) - \cosh a(\eta - \xi - y)] \\
& - \frac{\cos(b\eta) - \cos(b\xi)}{2a}[\sinh a(\pi + \eta - y) + \sinh a(\pi - \eta + y)] \\
& + \frac{\sinh(a\pi)}{2a}[\cos b(2\eta - y) + \cos(by) - \cos b(\eta + \xi - y) - \cos b(\eta - \xi - y)],
\end{aligned}
$$

and

$$
\begin{aligned}
H_{15} = & \kappa(a^2 + b^2)\left[\frac{1}{a}(\cos(b\eta) - \cos(b\xi))\sinh(a\pi)\right. \\
& \left. - \frac{1}{b}(\cosh(a\eta) - \cosh(a\xi))\sin(b\pi)\right](f_x(\eta) - f_x(\xi)).
\end{aligned}
$$

Then we estimate H_1, namely H_{11}, H_{12}, H_{13}, H_{14} and H_{15}, as $\beta \to \infty$. Using (81), we have

$$
|H_{15}| \le C(a^2 + b^2)|f_x(\eta) - f_x(\xi)| \le C(a^2 + b^2)\|f\|_V \le C\beta^2\|f\|_V. \tag{83}
$$

Since $F = (I - \alpha\partial_{xx})(g + i\beta f)$, we deduce by integrating by parts that

$$
\begin{aligned}
H_{11} = & \int_0^\pi P_1(y)F(y)\mathrm{d}y = \frac{a^2 + b^2}{ab}\int_0^\pi (\sinh(a\pi)\sin(by) - \sin(b\pi)\sinh(ay))F(y)\mathrm{d}y \\
= & \frac{a^2 + b^2}{ab}\left\{\int_0^\pi [(\sinh(a\pi)\sin(by) - \sin(b\pi)\sinh(ay))g(y)\right. \\
& + \alpha(b\sinh(a\pi)\cos(by) - a\sin(b\pi)\cosh(ay))g'(y)]\mathrm{d}y \\
& \left. + i\beta\int_0^\pi (\sinh(a\pi)\sin(by) - \sin(b\pi)\sinh(ay))(f(y) - \alpha f''(y))\mathrm{d}y\right\},
\end{aligned}
$$

hence

$$
\begin{aligned}
|H_{11}| \le & C\beta\|\sinh(a\pi)\sin(b\cdot) - \sin(b\pi)\sinh(a\cdot)\|_H\|g\|_H \\
& + C\beta^2\|\sinh(a\pi)\sin(b\cdot) - \sin(b\pi)\sinh(a\cdot)\|_{L^2(0,\pi)}\|f\|_V.
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
\|\sinh(a\pi)\sin(b\cdot) - \sin(b\pi)\sinh(a\cdot)\|_{L^2(0,\pi)} \le & C, \\
\|\sinh(a\pi)\sin(b\cdot) - \sin(b\pi)\sinh(a\cdot)\|_H \le & Cb \le C\beta,
\end{aligned}
$$

we have

$$
|H_{11}| \le C\beta^2(\|g\|_H + \|f\|_V). \tag{84}
$$

Next, we estimate $H_{12}+H_{13}+H_{14}$.

$$\begin{aligned}H_{12}=&i\kappa\beta\int_0^\pi P_2(y)F(y)\mathrm{d}y\\=&i\kappa\beta\int_0^\pi[P_2(y)g(y)+\alpha P_2'(y)g'(y)]\mathrm{d}y-\kappa\beta^2\int_0^\pi P_2(y)(f(y)-\alpha f''(y))\mathrm{d}y\\&-i\kappa\beta\alpha(P_2(\pi)g'(\pi)-P_2(0)g'(0)),\\H_{13}=&-i\kappa\beta\int_0^\xi P_3(y)F(y)\mathrm{d}y\\=&-i\kappa\beta\int_0^\xi[P_3(y)g(y)+\alpha P_3'(y)g'(y)]\mathrm{d}y+\kappa\beta^2\int_0^\xi P_3(y)(f(y)-\alpha f''(y))\mathrm{d}y\\&+i\kappa\beta\alpha(P_3(\xi)g'(\xi)-P_3(0)g'(0)),\\H_{14}=&-i\kappa\beta\int_\eta^\pi P_4(y)F(y)\mathrm{d}y\\=&-i\kappa\beta\int_\eta^\pi[P_4(y)g(y)+\alpha P_4'(y)g'(y)]\mathrm{d}y+\kappa\beta^2\int_\eta^\pi P_4(y)(f(y)-\alpha f''(y))\mathrm{d}y\\&+i\kappa\beta\alpha(P_4(\pi)g'(\pi)-P_4(\eta)g'(\eta)).\end{aligned}$$

We first deal with trace terms. Expressions of P_2, P_3 and P_4 show that

$$P_4(\pi)-P_2(\pi)=0,\quad P_2(0)-P_3(0)=0,\quad P_3(\xi)=0,\quad P_4(\eta)=0,$$

therefore, trace terms in $H_{12}+H_{13}+H_{14}$ vanish. Then we obtain

$$\begin{aligned}|H_{12}+H_{13}+H_{14}|\le&C\beta(\|P_2\|_H+\|P_3\|_H+\|P_4\|_H)\|g\|_H\\&+C\beta^2(\|P_2\|_{L^2(0,\pi)}+\|P_3\|_{L^2(0,\pi)}+\|P_4\|_{L^2(0,\pi)})\|f\|_V.\end{aligned}$$

Noticing that

$$\|P_j\|_{L^2(0,\pi)}\le C,\quad \|P_j\|_H\le Cb\le C\beta,\quad j=2,3,4,$$

we have

$$|H_{12}+H_{13}+H_{14}|\le C\beta^2(\|g\|_H+\|f\|_V).\tag{85}$$

Gathering together (83), (84) and (85), we obtain

$$|H_{11}+H_{12}+H_{13}+H_{14}+H_{15}|\le C\beta^2(\|g\|_H+\|f\|_V).\tag{86}$$

In the estimate of $H_{11}+H_{12}+H_{13}+H_{14}+H_{15}$, we involve the value of g' at a point. In fact, we assume g is smooth enough at first and then use the density argument to claim the estimate (86) also holds for g in H.

Step 3. (Estimation of the lower bound of $|H_2|$): We calculate the expression of H_2,

$$\begin{aligned}H_2=&\det\begin{vmatrix}\boldsymbol{a}&\boldsymbol{b}&\boldsymbol{c}&\boldsymbol{d}\end{vmatrix}+\det\begin{vmatrix}Su_1'(\xi)\boldsymbol{e}&\boldsymbol{b}&\boldsymbol{c}&\boldsymbol{d}\end{vmatrix}+\det\begin{vmatrix}\boldsymbol{a}&Su_2'(\xi)\boldsymbol{e}&\boldsymbol{c}&\boldsymbol{d}\end{vmatrix}\\&+\det\begin{vmatrix}\boldsymbol{a}&\boldsymbol{b}&-Su_1'(\pi-\eta)\boldsymbol{e}&\boldsymbol{d}\end{vmatrix}+\det\begin{vmatrix}\boldsymbol{a}&\boldsymbol{b}&\boldsymbol{c}&-Su_2'(\pi-\eta)\boldsymbol{e}\end{vmatrix}\\=&-(a^2+b^2)^2\left\{\frac{(a^2+b^2)^2}{ab}\sinh(a\pi)\sin(b\pi)+i\kappa\beta(a^2+b^2)\left[\frac{\sin(b\pi)}{2b}(2\cosh(a\pi)\right.\right.\\&+\cosh a(\pi-2\xi)+\cosh a(\pi-2\eta)-2\cosh a(\pi-\eta+\xi)-2\cosh a(\pi-\eta-\xi))\\&+\frac{\sinh(a\pi)}{2a}(2\cos b(\pi-\eta+\xi)+2\cos b(\pi-\eta-\xi)-2\cos(b\pi)\\&\left.\left.-\cos b(\pi-2\xi)-\cos b(\pi-2\eta))\right]\right\}\\=:&-(a^2+b^2)^2H_2'.\end{aligned}\tag{87}$$

Then we estimate the lower bound of $|H_2'|$. Denote

$$h(b) = 2\cos b(\pi-\eta+\xi) + 2\cos b(\pi-\eta-\xi) \\ - 2\cos(b\pi) - \cos b(\pi-2\xi) - \cos b(\pi-2\eta)$$

and

$$l(a) = 2\cosh a(\pi-\eta+\xi) + 2\cosh a(\pi-\eta-\xi) \\ - 2\cosh(a\pi) - \cosh a(\pi-2\xi) - \cosh a(\pi-2\eta).$$

With the asymptotic property of a and b, namely (81), simple calculation shows that there exist $C_1, C_2 > 0$ such that for any fixed $0 < \xi < \eta < \pi$, we have $C_1 \le -l(a) \le C_2$, for β large enough. Then we obtain

$$\begin{aligned} |H_2'|^2 =& (a^2+b^2)^2 \left[\left(\frac{a^2+b^2}{ab}\right)^2 \sinh(a\pi)^2 \sin(b\pi)^2 \right. \\ & \left. + \kappa^2\beta^2 \left(\frac{\sinh(a\pi)}{2a} h(b) - \frac{\sin(b\pi)}{2b} l(a) \right)^2 \right] \\ \ge & C'\beta^4(\beta^2\sin(b\pi)^2 + \beta^2(h(b))^2 - \sin(b\pi)^2) \\ \ge & C\beta^6(\sin(b\pi)^2 + (h(b))^2) \end{aligned}$$

for β large enough. It follows that

$$|H_2'| \ge C\beta^3 \sqrt{\sin(b\pi)^2 + (h(b))^2}. \tag{88}$$

Therefore, gathering together (80), (82), (86), (87) and (88), we finally obtain that

$$|\beta w_x(\pi)| \le C \frac{\|g\|_H + \|f\|_V}{\sqrt{\sin(b\pi)^2 + (h(b))^2}}. \tag{89}$$

This concludes the proof of Lemma 4.3.

5. **The lack of exponential stability and the lack of polynomial stability with high decay rate (Proof of Theorem 1.6).** In this section, we prove two results in Theorem 1.6 together by providing a counter example. The exponential stability of (5) is equivalent to $i\mathbb{R}$ is included in $\rho(\mathcal{A})$ and the resolvent estimate

$$\sup_{\beta\ge 1} \|(i\beta - \mathcal{A})^{-1}\| < \infty \tag{90}$$

holds due to a well-known result (see [16, 22]). Combined with the discussion in the previous section, our aim is to prove

$$\sup_{\beta\ge 1} |\beta|^{-l} \|(i\beta - \mathcal{A})^{-1}\| = \infty \tag{91}$$

for $0 \le l < \frac{1}{2}$.

First we give the inspiration of Theorem 1.6. Recalling Remark 4.2, we notice that actually we can deduce (28) and (29) from (24) and (25) with $l = 0$. Therefore, if we try to prove the exponential stability in the same way as in the proof of Theorem 1.5, we are necessary to claim

$$\inf_{b\ge 1} \sqrt{\sin(b\pi)^2 + (h(b))^2} \ge C > 0.$$

However, this claim is false. In fact

$$\inf_{b\geq 1}\sqrt{\sin(b\pi)^2+(h(b))^2}\leq \inf_{b\in\mathbb{N}^*}\sqrt{\sin(b\pi)^2+(h(b))^2}=\inf_{b\in\mathbb{N}^*}|h(b)|$$
$$=8\inf_{b\in\mathbb{N}^*}\left(\sin\left(b\frac{\eta+\xi}{2}\right)\sin\left(b\frac{\eta-\xi}{2}\right)\right)^2.$$

Since $\frac{\eta\pm\xi}{2\pi}$ are irrational numbers,

$$\inf_{b\geq 1}\sqrt{\sin(b\pi)^2+(h(b))^2}\leq 8\inf_{b\in\mathbb{N}^*}\left(\sin\left(b\frac{\eta+\xi}{2}\right)\sin\left(b\frac{\eta-\xi}{2}\right)\right)^2=0.$$

These facts inspire us to prove the exponential stability never holds no matter where ξ and η are located, as precisely stated in Theorem 1.6.

Now let us turn to prove Theorem 1.6. Our aim is to find β_n in $\mathbb{R}$, (w_n, v_n) in $\mathcal{D}(\mathcal{A})$ for $n = 1, 2, \ldots$ such that

$$\|(w_n, v_n)\|_{\mathcal{H}} = 1, \quad \beta_n \to \infty \tag{92}$$

and

$$|\beta|^l (f_n, g_n) \to (0, 0) \quad \text{in } \mathcal{H}, \tag{93}$$

for $0 \leq l < \frac{1}{2}$ where $(f_n, g_n) := (i\beta_n - \mathcal{A})(w_n, v_n)$. If such β_n and (w_n, v_n) can be found, then (91) holds which completes the proof of Theorem 1.6.

To find such β_n and (w_n, v_n), we need to use Proposition 2.3 on simultaneous approximation. If $\frac{\eta\pm\xi}{2\pi}$ are rational numbers, then the system is not strongly stable. Now assume $\frac{\eta\pm\xi}{2\pi}$ are irrational numbers. Applying Proposition 2.3 we obtain the existence of a strictly increasing sequence of positive integers $\{q_n\}_{n\geq 1}$ such that

$$\left|\sin\left(q_n\frac{\eta+\xi}{2}\right)\right|\leq\frac{\pi}{\sqrt{q_n}},\quad \left|\sin\left(q_n\frac{\eta-\xi}{2}\right)\right|\leq\frac{\pi}{\sqrt{q_n}}\quad \forall n\geq 1. \tag{94}$$

As we mentioned in Remark 4.2, β_n, w_n and v_n should satisfy some necessary conditions of (92) and (93),

$$|v_{n,x}(\eta) - v_{n,x}(\xi)|^2 \to 0 \quad \text{as } n \to \infty,$$

and

$$\lim_{n\to\infty}\|w_n\|_V^2 = \lim_{n\to\infty}\|v_n\|_H^2 = \lim_{n\to\infty}\|\beta_n w_n\|_H^2 = \frac{1}{2},$$

i.e., (36) and (28). We set

$$\beta_n = \frac{q_n^2}{\sqrt{1+\alpha q_n^2}} \to \infty \quad \text{as } n \to \infty, \tag{95}$$

and

$$v_n(x) = \frac{1}{\sqrt{\pi} d_n}\sin(q_n x), \tag{96}$$

where $\{d_n\}_{n\geq 1}$ is a sequence that will be chosen later. Since q_n is integer, we have v_n belongs to V and

$$\|v_n\|_H^2 = \frac{1+\alpha q_n^2}{2d_n^2}.$$

Therefore, it is necessary that d_n must satisfy

$$\lim_{n\to\infty} d_n = +\infty, \quad \lim_{n\to\infty}\frac{1+\alpha q_n^2}{d_n^2} = 1. \tag{97}$$

Using (94), (96) and (97), we obtain that

$$|v_{n,x}(\eta) - v_{n,x}(\xi)| = \frac{q_n}{\sqrt{\pi} d_n} \left| \sin\left(q_n \frac{\eta+\xi}{2}\right) \sin\left(q_n \frac{\eta-\xi}{2}\right) \right| \to 0. \tag{98}$$

Then we need to define w_n. Notice that v_n belongs to $C^\infty[0,\pi]$ and

$$\begin{aligned} [w_{n,xx}]_\eta = -[w_{n,xx}]_\xi =& \kappa(v_{n,x}(\eta) - v_{n,x}(\xi)) \\ =& -\frac{\kappa q_n}{\sqrt{\pi} d_n} \sin\left(q_n \frac{\eta+\xi}{2}\right) \sin\left(q_n \frac{\eta-\xi}{2}\right), \end{aligned}$$

therefore, from the definition of f_n we have

$$[f_{n,xx}]_\eta = -[f_{n,xx}]_\xi = iL_n,$$

where

$$L_n = \kappa\beta_n(v_{n,x}(\eta) - v_{n,x}(\xi)) = -\frac{\kappa q_n \beta_n}{\sqrt{\pi} d_n} \sin\left(q_n \frac{\eta+\xi}{2}\right) \sin\left(q_n \frac{\eta-\xi}{2}\right). \tag{99}$$

Using (94), (95) and (97), we have

$$|L_n| \le \kappa\pi^{\frac{3}{2}} \left|\frac{\beta_n}{d_n}\right| \le L \quad \forall n \ge 1, \tag{100}$$

where $L > 0$ is a constant. Assume $f_n = iF_n$, F_n is a real function, therefore, we have $[F_{n,xx}]_\eta = -[F_{n,xx}]_\xi = L_n$. As required, w_n belongs to $V \cap H^3(0,\xi) \cap H^3(\xi,\eta) \cap H^3(\eta,\pi)$ and $w_{n,xx}(0) = w_{n,xx}(\pi) = 0$, therefore, F_n belongs to $V \cap H^3(0,\xi) \cap H^3(\xi,\eta) \cap H^3(\eta,\pi)$ and $F_{n,xx}(0) = F_{n,xx}(\pi) = 0$. Then we have

$$F_{n,xxx} = \{F_{n,xxx}\} + [F_{n,xx}]_\eta \delta_\eta + [F_{n,xx}]_\xi \delta_\xi = \{F_{n,xxx}\} + L_n(\delta_\eta - \delta_\xi). \tag{101}$$

Therefore, we denote

$$F_{n,xx} := \phi_n - L_n I_{(\xi,\eta)}, \tag{102}$$

where ϕ_n belongs to H, $\phi_{n,x} = \{F_{n,xxx}\}$ and $I_{(\xi,\eta)}$ is the characteristic function of (ξ,η). Since (93), we require $|\beta_n|^l \|F_n\|_V \to 0$, i.e., $|\beta_n|^l \|F_{n,xx}\|_{L^2(0,\pi)} \to 0$, namely $|\beta_n|^l \|\phi_n - L_n I_{(\xi,\eta)}\|_{L^2(0,\pi)} \to 0$. Here we use piecewise linear function to approximate $L_n I_{(\xi,\eta)}$. Let

$$c_n = \frac{C}{q_n^m}, \tag{103}$$

where $2l < m < 2 - 2l$ and $C > 0$ is a constant such that $c_1 < \frac{\eta-\xi}{2}$. Note that $0 \le l < \frac{1}{2}$ implies the existence of m. And define

$$\phi_n = \begin{cases} 0, & x \in (0,\xi), \\ \dfrac{L_n}{c_n}(x-\xi), & x \in (\xi, \xi+c_n), \\ L_n, & x \in (\xi+c_n, \eta-c_n), \\ -\dfrac{L_n}{c_n}(x-\eta), & x \in (\eta-c_n, \eta), \\ 0, & x \in (\eta,\pi), \end{cases} \tag{104}$$

Then we have

$$\|\phi_n - L_n I_{(\xi,\eta)}\|^2_{L^2(0,\pi)} = 2\int_\xi^{\xi+c_n} \left(\frac{L_n}{c_n}(x-\xi) - L_n\right)^2 \mathrm{d}x = \frac{2}{3} L_n^2 c_n. \tag{105}$$

Since $|L_n| \leq L$, $q_n \to \infty$, $m > 2l$ and (103), we have $|\beta_n|^l \|\phi_n - L_n I_{(\xi,\eta)}\|_{L^2(0,\pi)} \to 0$. Next we calculate $\|w_n\|_V$. Since $(f_n, g_n) := (i\beta_n - \mathcal{A})(w_n, v_n)$ and the definition of $\mathcal{A}$, i.e. (15), we obtain that

$$w_n = \frac{1}{i\beta_n}(f_n + v_n) = \frac{1}{\beta_n}(F_n - iv_n),$$

which implies that

$$w_{n,xx} = \frac{1}{\beta_n}(F_{n,xx} - iv_{n,xx}) = \frac{1}{\beta_n}(\phi_n - L_n I_{(\xi,\eta)} - iv_{n,xx}).$$

Then we have

$$\begin{aligned} \|w_n\|_V^2 = \|w_{n,xx}\|^2_{L^2(0,\pi)} =& \frac{1}{\beta_n^2}(\|\phi_n - L_n I_{(\xi,\eta)}\|^2_{L^2(0,\pi)} + \|v_{n,xx}\|^2_{L^2(0,\pi)}) \\ =& \frac{1}{\beta_n^2}\left(\frac{2}{3}L_n^2 c_n + \frac{q_n^4}{2d_n^2}\right) = \frac{1+\alpha q_n^2}{2d_n^2} + \frac{2L_n^2 c_n}{3\beta_n^2} \to \frac{1}{2}. \end{aligned} \tag{106}$$

According to (92), we also need $\|w_n\|_V^2 + \|v_n\|_H^2 = 1$, which implies that

$$d_n^2 = \frac{1+\alpha q_n^2}{1 - (2L_n^2 c_n)/(3\beta_n^2)}. \tag{107}$$

The last condition we need to check is $|\beta_n|^l (i\beta_n v_n + \mathcal{R}\{w_{n,xxx}\}_x) \to 0$ in H. We denote $\{w_{n,xxx}\}$ as

$$\{w_{n,xxx}\} = \frac{1}{\beta_n}(\{F_{n,xxx}\} - iv_{n,xxx}) = \frac{1}{\beta_n}(\phi_{n,x} - iv_{n,xxx}) =: s_n + it_n. \tag{108}$$

Denote $G_n := \|i\beta_n v_n + \mathcal{R}\{w_{n,xxx}\}_x\|_H^2$, then by integrating by parts, we have

$$\begin{aligned} G_n =& \int_0^\pi [((\mathcal{R}t_n)_x + \beta_n v_n)^2 + (\mathcal{R}s_{n,x})^2 + \alpha((\mathcal{R}t_n)_{xx} + \beta_n v_{n,x})^2 + \alpha(\mathcal{R}s_{n,xx})^2]\mathrm{d}x \\ =& \int_0^\pi [(\mathcal{R}t_{n,x})^2 + 2\beta_n v_n (\mathcal{R}t_n)_x + (\beta_n v_n)^2 + (\mathcal{R}s_{n,x})^2 + (\mathcal{R}t_n - t_n)\mathcal{R}t_{n,xx} \\ &+ 2(\mathcal{R}t_n - t_n)\beta_n v_{n,x} + \alpha(\beta_n v_{n,x})^2 + (\mathcal{R}s_n - s_n)\mathcal{R}s_{n,xx}]\mathrm{d}x \\ =& \int_0^\pi [(\beta_n v_n)^2 - t_n \mathcal{R}t_{n,xx} - 2\beta_n t_n v_{n,x} + \alpha(\beta_n v_{n,x})^2 - s_n \mathcal{R}s_{n,xx}]\mathrm{d}x \\ =& \int_0^\pi \left[\beta_n^2(v_n^2 + \alpha v_{n,x}^2) + \frac{1}{\alpha}(t_n^2 + s_n^2 - t_n\mathcal{R}t_n - s_n\mathcal{R}s_n) - 2\beta_n t_n v_{n,x}\right]\mathrm{d}x. \end{aligned}$$

We know from [14] that for any ψ in $L^2(0,\pi)$,

$$\begin{aligned} \mathcal{R}\psi =& c\sinh\frac{x}{\sqrt{\alpha}} - \frac{1}{\sqrt{\alpha}}\int_0^x \sinh\left(\frac{x-s}{\sqrt{\alpha}}\right)\psi(s)\mathrm{d}s, \\ c =& \left(\sqrt{\alpha}\sinh\frac{\pi}{\sqrt{\alpha}}\right)^{-1}\int_0^\pi \sinh\left(\frac{\pi - s}{\sqrt{\alpha}}\right)\psi(s)\mathrm{d}s. \end{aligned}$$

Then we can calculate $\mathcal{R}t_n$ and $\mathcal{R}s_n$ as

$$t_n = -\frac{1}{\beta_n}v_{n,xxx} = \frac{q_n^3}{\sqrt{\pi}\beta_n d_n}\cos(q_n x),$$

$$\begin{aligned}\mathcal{R}t_n =&\frac{q_n}{d_n\sqrt{\pi(1+\alpha q_n^2)}}\left[\cos(q_n x)-\cosh\frac{x}{\sqrt{\alpha}}\right.\\&\left.+\left(\sinh\frac{\pi}{\sqrt{\alpha}}\right)^{-1}\left(\cosh\frac{\pi}{\sqrt{\alpha}}-\cos(q_n\pi)\right)\sinh\frac{x}{\sqrt{\alpha}}\right],\\ s_n =& \frac{1}{\beta_n}\phi_{n,x}=\frac{L_n}{\beta_n c_n}(I_{(\xi,\xi+c_n)}-I_{(\eta-c_n,\eta)}),\end{aligned}$$

$$\mathcal{R}s_n=\begin{cases}0, & x\in(0,\xi),\\ \frac{L_n}{\beta_n c_n}\left(1-\cosh\frac{x-\xi}{\sqrt{\alpha}}\right), & x\in(\xi,\xi+c_n),\\ \frac{L_n}{\beta_n c_n}\left(1-\cosh\frac{c_n}{\sqrt{\alpha}}\right), & x\in(\xi+c_n,\eta-c_n),\\ \frac{L_n}{\beta_n c_n}\left(\cosh\frac{x-\eta+c_n}{\sqrt{\alpha}}-\cosh\frac{c_n}{\sqrt{\alpha}}\right), & x\in(\eta-c_n,\eta),\\ 0, & x\in(\eta,\pi).\end{cases}$$

Then we calculate the single term in G_n. We have

$$\int_0^\pi \beta_n^2(v_n^2+\alpha v_{n,x}^2)\mathrm{d}x=\beta_n^2\|v_n\|_H^2=\beta_n^2\frac{1+\alpha q_n^2}{2d_n^2},$$

$$\int_0^\pi 2\beta t_n v_{n,x}\mathrm{d}x=2\beta\int_0^\pi \frac{q_n^4}{\pi\beta_n d_n^2}\cos(q_n x)^2\mathrm{d}x=\frac{q_n^4}{d_n^2},$$

$$\int_0^\pi \frac{t_n^2}{\alpha}\mathrm{d}x=\frac{q_n^6}{2\alpha\beta_n^2 d_n^2},$$

$$\int_0^\pi \frac{s_n^2}{\alpha}\mathrm{d}x=\frac{2L_n^2}{\alpha\beta_n^2 c_n},$$

$$\int_0^\pi \frac{t_n\mathcal{R}t_n}{\alpha}\mathrm{d}x=\frac{q_n^2}{2\alpha d_n^2}+\frac{2q_n^2}{\pi\sqrt{\alpha}d_n^2(1+\alpha q_n^2)}\left(\sinh\frac{\pi}{\sqrt{\alpha}}\right)^{-1}\left(\cos(q_n\pi)-\cosh\frac{\pi}{\sqrt{\alpha}}\right),$$

$$\int_0^\pi \frac{s_n\mathcal{R}s_n}{\alpha}\mathrm{d}x=\frac{L_n^2}{\alpha\beta_n^2 c_n^2}\left(c_n\left(1+\cosh\frac{c_n}{\sqrt{\alpha}}\right)-2\sqrt{\alpha}\sinh\frac{c_n}{\sqrt{\alpha}}\right).$$

Consequently, we obtain

$$\begin{aligned}G_n =&\frac{L_n^2}{\alpha\beta_n^2 c_n}\left(1-\cosh\frac{c_n}{\sqrt{\alpha}}\right)+\frac{2L_n^2}{\sqrt{\alpha}\beta_n^2 c_n^2}\sinh\frac{c_n}{\sqrt{\alpha}}\\&+\frac{2q_n^2}{\pi\sqrt{\alpha}d_n^2(1+\alpha q_n^2)}\left(\sinh\frac{\pi}{\sqrt{\alpha}}\right)^{-1}\left(\cos(q_n\pi)-\cosh\frac{\pi}{\sqrt{\alpha}}\right).\end{aligned}\tag{109}$$

On the one hand, $l<\frac{1}{2}$ and (97) imply that

$$\left|\frac{2q_n^2\beta_n^{2l}}{\pi\sqrt{\alpha}d_n^2(1+\alpha q_n^2)}\left(\sinh\frac{\pi}{\sqrt{\alpha}}\right)^{-1}\left(\cos q_n\pi-\cosh\frac{\pi}{\sqrt{\alpha}}\right)\right|\to 0\quad\text{as } n\to\infty.$$

On the other hand, since $c_n\to 0$ as $n\to\infty$, using Taylor expansion, we obtain

$$\begin{aligned}1-\cosh\frac{c_n}{\sqrt{\alpha}}&=-\frac{c_n^2}{2\alpha}+O(c_n^4),\\ \sinh\frac{c_n}{\sqrt{\alpha}}&=\frac{c_n}{\sqrt{\alpha}}+O(c_n^3).\end{aligned}$$

Then we have

$$\beta_n^{2l} G_n = \frac{2L_n^2}{\alpha \beta_n^{2-2l} c_n} - \frac{L_n^2 c_n}{2\alpha^2 \beta_n^{2-2l}} + o(1). \tag{110}$$

Using $|L_n| \leq L$, $\beta_n \sim q_n/\sqrt{\alpha}$, $2l < m < 2-2l$ and (103), we finally have $\beta_n^{2l} G_n \to 0$ as $n \to \infty$.

Now we have already proven $\beta_n \to \infty$ and (w_n, v_n) in $\mathcal{D}(\mathcal{A})$ such that (92) and (93) hold. Then we have proved that for any $0 < \xi < \eta < \pi$

$$\sup_{|\beta| \geq 1} |\beta|^{-l} \|(i\beta - \mathcal{A})^{-1}\| = \infty, \tag{111}$$

for $0 \leq l < \frac{1}{2}$. This concludes the proof of Theorem 1.6.

Acknowledgments. Yubo Bai would like to acknowledge the funding from the China Scholarship Council (No. 202206100101). Zhiqiang Wang is partially supported by the Natural Sciences Foundation of China, under grant 11971119.

REFERENCES

[1] K. Ammari, Z. Liu and M. Tucsnak, Decay rates for a beam with pointwise force and moment feedback, *Math. Control Signals Systems*, **15** (2002), 229–255.

[2] K. Ammari and M. Tucsnak, Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force, *SIAM J. Control Optim.*, **39** (2000), 1160–1181.

[3] W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.*, **306** (1988), 837–852.

[4] Y. Bai, C. Prieur and Z. Wang, Exact controllability for a Rayleigh beam with piezoelectric actuator, 2023, Preprint on webpage at https://hal.science/hal-03964846.

[5] G. Bastin and J.-M. Coron, *Stability and boundary stabilization of 1-D hyperbolic systems*, vol. 88 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser/Springer, 2016.

[6] A. Borichev and Y. Tomilov, Optimal polynomial decay of functions and operator semigroups, *Math. Ann.*, **347** (2010), 455–478.

[7] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45, Cambridge University Press, New York, 1957.

[8] E. Crawley and E. Anderson, Detailed models of piezoceramic actuation of beams, in *30th Structures, Structural Dynamics and Materials Conference*, 1989, 2000–2010.

[9] E. Crépeau and C. Prieur, Control of a clamped-free beam by a piezoelectric actuator, *ESAIM Control Optim. Calc. Var.*, **12** (2006), 545–563.

[10] R. F. Curtain and G. Weiss, Exponential stabilization of well-posed systems by colocated feedback, *SIAM J. Control Optim.*, **45** (2006), 273–297.

[11] Ph. Destuynder, I. Legrain, L. Castel and N. Richard, Theoretical, numerical and experimental discussion on the use of piezoelectric devices for control-structure interaction, *European J. Mech. A Solids*, **11** (1992), 181–213.

[12] B.-Z. Guo, Basis property of a Rayleigh beam with boundary stabilization, *J. Optim. Theory Appl.*, **112** (2002), 529–547.

[13] B.-Z. Guo and F.-F. Jin, The active disturbance rejection and sliding mode control approach to the stabilization of the Euler–Bernoulli beam equation with boundary input disturbance, *Automatica*, **49** (2013), 2911–2918.

[14] B.-Z. Guo, J.-M. Wang and C.-L. Zhou, On the dynamic behavior and stability of controlled connected Rayleigh beams under pointwise output feedback, *ESAIM Control Optim. Calc. Var.*, **14** (2008), 632–656.

[15] B.-Z. Guo and K.-Y. Yang, Dynamic stabilization of an Euler–Bernoulli beam equation with time delay in boundary observation, *Automatica*, **45** (2009), 1468–1475.

[16] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. Differential Equations*, **1** (1985), 43–56.

[17] S. Lang, *Introduction to Diophantine approximations*, 2nd edition, Springer-Verlag, New York, 1995.

[18] P. Le Gall, C. Prieur and L. Rosier, Output feedback stabilization of a clamped-free beam, *Internat. J. Control*, **80** (2007), 1201–1216.

[19] P. Le Gall, C. Prieur and L. Rosier, Stabilization of a clamped-free beam with collocated piezoelectric sensor/actuator, *Int. J. Tomogr. Stat.*, **6** (2007), 104–109.

[20] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York-Heidelberg, 1972, Translated from the French by P. Kenneth.

[21] Z. Liu and B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, *Z. Angew. Math. Phys.*, **56** (2005), 630–644.

[22] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.*, **284** (1984), 847–857.

[23] B. Rao, A compact perturbation method for the boundary stabilization of the Rayleigh beam equation, *Appl. Math. Optim.*, **33** (1996), 253–264.

[24] M. Tucsnak, Regularity and exact controllability for a beam with piezoelectric actuator, *SIAM J. Control Optim.*, **34** (1996), 922–930.

[25] G. Weiss and R. F. Curtain, Exponential stabilization of a Rayleigh beam using collocated control, *IEEE Trans. Automat. Control*, **53** (2008), 643–654.

Received xxxx 20xx; revised xxxx 20xx; early access xxxx 20xx.