Automatica 148 (2023) 110790

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Controller design for heterogeneous traffic with bottleneck and disturbances☆

Lina Guan [a,b,c,*], Liguo Zhang [a,b], Christophe Prieur [c]

[a] Faculty of Information Technology, Beijing University of Technology, Beijing, China
[b] Key Laboratory of Computational Intelligence and Intelligent Systems, 100124, Beijing, China
[c] Univ. Grenoble Alpes, CNRS, Grenoble INP, GIPSA-lab, 38000 Grenoble, France

ARTICLE INFO

Article history:
Received 8 December 2021
Received in revised form 27 August 2022
Accepted 28 October 2022
Available online xxxx

Keywords:
Multi-type AR traffic model
Traffic congestion
Disturbances
Backstepping
Optimal tuning
Boundary controller

ABSTRACT

This paper studies an optimal tuning of the boundary controller for a heterogeneous traffic flow model with disturbances in order to alleviate congested traffic. The macroscopic first-order N-class Aw–Rascle traffic model consists of $2N$ hyperbolic partial differential equations. The vehicle size and the driver's behavior characterize the type of vehicle. There are m positive characteristic velocities and $2N - m$ negative characteristic velocities in the congested traffic after linearizing the model equations around the steady state depending on the spatial variable. By using the backstepping method, a controller implemented by a ramp metering at the inlet boundary is designed for rejecting the disturbances to stabilize the $2N \times 2N$ heterogeneous traffic system. The developed controller in terms of proportional integral control is derived from mapping the original system to a target system with a proportional integral boundary control rejecting the disturbances. The integral input-to-state stability of the target system is proved by using the Lyapunov method. Finally, an optimization problem is established and solved for seeking the optimal tuning of the controller.

1. Introduction

Traffic congestion is a pervasive problem that leads to increased fuel consumption and risky driving conditions. It is natural to use the boundary control on the available control signals as the ramp metering or the variable speed limit to stabilize the highway traffic systems. Paper (Tumash, de Wit, & Monache, 2021) contributes to the boundary control design for the multi-directional congested traffic evolving on the large-scale urban networks represented by a continuum two-dimensional plane. In Yu, Park, Bayen, Moura, and Krstic (2021), a reinforcement learning boundary controller is designed to mitigate the stop-and-go congested traffic for the 2×2 quasilinear Aw–Rascle–Zhang (ARZ) partial differential equations (PDEs) model by using the proximal policy optimization which is an algorithm based on the neural networks. In Auriol and Pietri (2022), a delay-robust stabilizing state feedback boundary control law is developed for an underactuated network of two subsystems of a heterodirectional linear first-order $n + m$ hyperbolic PDEs system. In Guan, Zhang, and Prieur (2020), an optimal tuning of PI controller is done for the linearized ARZ traffic model by computing the value of L^2 gain from disturbance to output, that is to measure the disturbance rejection capacity (see the recent survey Mironchenko & Prieur, 2020).

Usually, macroscopic models typically described by PDEs are more suitable to study congested traffic and the disturbances in the traffic flow. In Guan, Zhang, and Prieur (2021), the linearized ARZ traffic flow model with boundary disturbances is mapped into an iISS target system by using a backstepping transformation in order to obtain a full state feedback controller, and we use a backstepping method to derive an observer-based output feedback controller to dissolve traffic congestion resulting from traffic breakdown. The exact boundary controllability of a class of nonlocal conservation laws modeling traffic flow is studied in Bayen, Coron, Nitti, Keimer, and Pflug (2021). In Gupta and Katiyar (2007), the authors propose a new continuum model with an additional anisotropic term that ensures the characteristic velocities can be less than or equal to the macroscopic flow speed. An extension of the speed gradient (SG) model is introduced to study the mixed traffic flow system in Jiang and Wu (2004). Paper (Mohan & Ramadurai, 2017) extends the Aw–Rascle (AR) model for heterogeneous traffic by using the area

☆ This work is supported by the National Natural Science Foundation of China (NSFC, Grant No. 61873007 and No. 62273014), a research grant from project PHC CAI YUANPEI under Grant Number 44029QD, and by MIAI @ Grenoble Alpes (ANR-19-P3IA-0003). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Nikolaos Bekiaris-Liberis under the direction of Editor Miroslav Krstic.

* Corresponding author at: Faculty of Information Technology, Beijing University of Technology, Beijing, China.

E-mail addresses: lina.guan@grenoble-inp.fr (L. Guan), zhangliguo@bjut.edu.cn (L. Zhang), christophe.prieur@gipsa-lab.fr (C. Prieur).

https://doi.org/10.1016/j.automatica.2022.110790
0005-1098/

occupancy and analyzes the properties of the extended model. A new car-following model for the heterogeneous traffic flow is presented in Tang, Huang, Zhao, and Shang (2009). In Mohan and Ramadurai (2017), the macroscopic N-class AR traffic model with the consideration of vehicle size is used because of the validation of simulation. A continuum multi-class traffic model is proposed on the basis of a three-dimensional flow–concentration surface in Mohan and Ramadurai (2021). Paper (Fan & Work, 2015) studies a two-type vehicle heterogeneous traffic model to acquire the overtaking and creeping traffic flows. In Fan, Herty, and Seibold (2014), a generalized ARZ traffic flow system is derived by modifying the pressure relation in the ARZ model and by using the data-fitting method.

The backstepping method is used to derive a boundary controller in some papers. In Yu, Gan, Bayen, and Krstic (2020), a boundary observer for the nonlinear ARZ traffic flow model is designed to estimate the information of the traffic states by using the backstepping method. A controller is designed for the underactuated cascade network of the interconnected PDEs systems by using backstepping in Auriol (2020). Considering the limits of technology and cost, there have been works inspired by Coron, Vazquez, Krstic, and Bastin (2013) to design a control law for the linearized ARZ traffic flow model by using the backstepping transformations (see Yu & Krstic, 2019). Paper (Burkhardt, Yu, & Krstic, 2021) uses the backstepping method to design an output feedback boundary control for the stop-and-go traffic problem of the linearized two-class AR traffic flow system. As an extension of the two-type vehicle traffic flow model in the paper (Burkhardt et al., 2021), this paper generally investigates $N > 2$ vehicle types with the help of some coefficient matrices but in the presence of the unknown and bounded disturbances (high traffic demand) at the inlet and a bottleneck (flow restriction, constant densities) at the outlet of the considered road section. This paper assumes the flow conservation of each vehicle type at the upstream inlet $x = 0$, rather than the constant overall traffic flow entering and leaving the investigated track section in the paper (Burkhardt et al., 2021). In addition to these differences with respect to the paper (Burkhardt et al., 2021), it is natural for the applications to consider the case of a nonuniform steady-state and the transport velocities depending on the spatial variable. Moreover, the objective of this paper is to reject disturbances and alleviate the congested traffic (convergence to the nonuniform steady-state), not to regulate the leaving traffic flow. By means of solving the optimization problem, we obtain the optimal tuning parameters to minimize the likelihood of congested traffic. Paper (Coron et al., 2013) uses a backstepping transformation to design a control law and derives the H^2 exponential stability for a quasilinear 2×2 system of the first-order hyperbolic PDEs. Paper (Bastin, Coron, & Hayat, 2021) studies the sufficient conditions for the local input-to-state stability (ISS) in the sup norm of the general quasilinear hyperbolic systems with the boundary input disturbances. For the one-dimensional parabolic partial differential equations with disturbances at both boundaries, the estimations of the input-to-state stability in the various norms are studied in Karafyllis and Krstic (2017).

Contributions: This paper states a new result on the controller design by using the backstepping method for the linearized multi-type traffic flow hyperbolic system around a nonuniform steady-state to reject disturbances and then to alleviate the congested traffic. Firstly, this work presents the derivation of an extended multi-type AR traffic flow model in the characteristic form. Secondly, we prove the integral input-to-state stability (iISS) of a target system that has a source term of integral form and a proportional–integral (PI) boundary control for rejecting disturbances. Moreover, inspired by Hu, Di Meglio, Vazquez, and Krstic (2016), a controller implemented by ramp metering is designed to robustly stabilize the heterogeneous traffic system by applying the backstepping method to the multi-type vehicle traffic model.

This paper is organized as follows: Section 2 introduces the multi-type AR traffic flow model with the parameters characterizing different vehicle types and the formulation of the control problem to be solved. In Section 3, the iISS of the target system is proved by the Lyapunov method, and a controller is designed by using the backstepping approach. In Section 4, the optimization problem is presented and the numerical results are provided for verifying the existence of the optimal tuning of controller. The paper ends with the concluding remarks in Section 5.

Notation. The set of positive real numbers is represented by $\mathbb{R}_{>0}$. C^0 is the set of continuous functions, and C^1 is the set of continuously differentiable functions. $\max(S)$ is the maximum value of all the elements in S if S is a set. $\partial_t f$ and $\partial_x f$ respectively denote the partial derivatives of a function f with respect to the variables t and x. f' denotes the first derivative of a function f with respect to the variable x, and $\dot{f}$ denotes the first derivative of a function f with respect to the variable t. For a function $\varphi = [\varphi_1, \ldots, \varphi_n]^\top : [0, L] \times [0, +\infty) \to \mathbb{R}^n$, we define the following norms, the L^2-norm

$$\|\varphi\|_{L^2((0,L);\mathbb{R}^n)} = \left(\int_0^L (\varphi_1^2(\xi,t) + \cdots + \varphi_n^2(\xi,t))\,\mathrm{d}\xi\right)^{\frac{1}{2}},$$

the L^∞-norm $\|\varphi\|_{L^\infty((0,L);\mathbb{R}^n)} =$

$$\max\left\{\|\varphi_1\|_{L^\infty((0,L);\mathbb{R})}, \ldots, \|\varphi_n\|_{L^\infty((0,L);\mathbb{R})}\right\},$$

the H^1-norm $\|\varphi\|_{H^1((0,L);\mathbb{R}^n)} =$

$$\left(\int_0^L \left(\|\varphi\|^2_{L^2((0,L);\mathbb{R}^n)} + \|\varphi_x\|^2_{L^2((0,L);\mathbb{R}^n)}\right)\mathrm{d}x\right)^{\frac{1}{2}},$$

and the H^2-norm $\|\varphi\|_{H^2((0,L);\mathbb{R}^n)} = \Big(\int_0^L \Big(\|\varphi\|^2_{L^2((0,L);\mathbb{R}^n)}$

$$+\|\varphi_x\|^2_{L^2((0,L);\mathbb{R}^n)} + \|\varphi_{xx}\|^2_{L^2((0,L);\mathbb{R}^n)} \Big)\,\mathrm{d}x \Big)^{\frac{1}{2}}.$$

$\mathbb{R}^n$ denotes the set of real n-dimensional column vector. $\mathbb{R}^{n\times l}$ denotes the set of real $n \times l$ matrices. $0_{n\times l}$ denotes the $n \times l$ zero matrix. I_n is a n-dimensional identity matrix. $\mathcal{D}_n$ denotes the set of n-dimensional diagonal matrix. $\mathcal{D}_n^+$ denotes the set of n-dimensional diagonal matrix in which the main diagonal entries are positive. The n-dimensional column vector is represented as $M = \begin{bmatrix} M_1 & M_2 & \cdots & M_n \end{bmatrix}^\top$, where the argument M_i ($i = 1, 2, \ldots, n$) is a scalar or a column vector. The diagonal matrix is represented as $M = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ with the diagonal entry d_i ($i = 1, 2, \ldots, n$). The block diagonal matrix is represented as $M = \mathrm{diag}\{M_1, M_2, \ldots, M_n\}$, and the block matrix is represented as $M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{bmatrix}$, where the main diagonal argument M_i ($i = 1, 2, \ldots, n$) and the argument M_{ij} ($i, j = 1, 2, \ldots, n$) are matrices. $[M]_{i,j}$ denotes the entry of matrix M in the ith row and the jth column. $\left\{M_{ij}\right\}_{n_1\le i\le n_2, l_1\le j\le l_2}$ denotes a matrix consisting of the entries of matrix M in the rows from n_1th to n_2th and the columns from l_1th to l_2th. M^{-1} denotes the inverse matrix of a square matrix M. $M^\top$ denotes the transpose of a matrix M. $\lambda(M)$ is the set of all the eigenvalues of a matrix M, and $|\lambda(M)|$ is the set of absolute values of all the eigenvalues if M is a square matrix. The symbol $*$ stands for a symmetric block in a matrix.

2. Traffic flow system and control problem

The multi-type AR traffic flow model and the interpretations of the crucial parameters are presented in this section. The preparations for designing a controller are also done including the transformations of the states and the linearization around a nonuniform steady-state. On the basis of the control problem to be solved, the corresponding boundary conditions are derived.

2.1. Multi-type AR traffic flow model

We investigate the multi-type AR traffic flow model in Mohan and Ramadurai (2017) that describes the dynamics of heterogeneous traffic consisting of N vehicle types on a road segment with the length L,

$$\partial_t \rho_i(x,t) + \partial_x\Big(\rho_i(x,t)v_i(x,t)\Big) = 0, \tag{1}$$

$$\partial_t\Big(v_i(x,t) + p_i\left(Ao(\rho)\right)\Big) + v_i(x,t)\partial_x\Big(v_i(x,t) + p_i\left(Ao(\rho)\right)\Big) = \frac{V_{e,i}\left(Ao(\rho)\right) - v_i(x,t)}{\tau_i}, \tag{2}$$

with the independent spatial variable $x \in (0, L)$ and the independent time variable $t \in [0, +\infty)$, where i is the index of vehicle type with $i = 1, 2, \dots, N$, $\rho_i(x,t)$ and $v_i(x,t)$ are respectively the density and the velocity of the vehicle type i. Additionally, the density $\rho_i(x,t)$ is defined as the number of vehicles passing the road section per unit length, and the velocity $v_i(x,t)$ is defined as the average speed of vehicles passing the location x in unit time. The relaxation time τ_i of the vehicle type i is subject to the driving behavior, and the area occupancy $Ao(\rho)$ is formulated as

$$Ao(\rho) = \frac{a^\top \rho}{W}, \tag{3}$$

where $a = (a_1, a_2, \dots, a_N)^\top$ (a_i is the occupied surface per vehicle for type i), $\rho = (\rho_1, \rho_2, \dots, \rho_N)^\top$, and W is the width of the road segment. The area occupancy $Ao(\rho)$ describes the percentage of the road space occupied by all the vehicle classes on the considered road section. In the physical sense, $0 < Ao(\rho) \le 1$.

For the heterogeneous traffic, the traffic pressure function $p_i\left(Ao(\rho)\right)$ of the vehicle type i is an increasing function of the area occupancy $Ao(\rho)$ (see Burkhardt et al., 2021),

$$p_i(Ao(\rho)) = v_i^M\left(\frac{Ao(\rho)}{Ao_i^M}\right)^{\gamma_i}, \quad i = 1, 2, \dots, N, \tag{4}$$

where the free-flow velocity v_i^M and the maximum area occupancy $0 < Ao_i^M \le 1$ of the vehicle type i respectively describe the maximal velocity in the free regime and the maximum percentage of occupied surface in the congested regime, if there is only vehicle class i on the considered road segment. Denoting the maximal free flow speed by v_i^M and the maximal density by ρ_i^M, we assume that the inequalities $0 < v_i \le v_i^M$, $0 < \rho_i \le \rho_i^M$ hold. As described in the paper (Burkhardt et al., 2021), the constant $\gamma_i > 1$ is the pressure exponent of the vehicle type i that can be tuned to get realistic traffic pressure $p_i(Ao(\rho))$.

The steady-state speed–Ao relationship of vehicle class $i (= 1, 2, \dots, N)$ is given by the Greenshield's model in Greenshields, Bibbins, Channing, and Miller (1935) as

$$V_{e,i}(Ao(\rho)) = v_i^M - p_i\left(Ao(\rho)\right) = v_i^M\left(1 - \left(\frac{Ao(\rho)}{Ao_i^M}\right)^{\gamma_i}\right). \tag{5}$$

There is a negative connection from the decreasing function $V_{e,i}\left(Ao(\rho)\right)$ describing the desired velocity of the drivers to the crowded degree.

2.2. Linearization of multi-type AR traffic flow model

Inspired by the case "2 vehicle classes" in Burkhardt et al. (2021), the multi-type AR traffic model (1)–(2) is linearized around a nonuniform steady-state

$$u^* = (\rho_1^*, v_1^*, \rho_2^*, v_2^*, \dots, \rho_N^*, v_N^*)^\top \in C^1([0, L]; \mathbb{R}^{2N}),$$

where ρ_i^*, v_i^* satisfy, for $i = 1, 2, \dots, N$,

$$v_i^*\rho_i^{*\prime} + \rho_i^* v_i^{*\prime} = 0, \tag{6}$$

$$v_i^* v_i^{*\prime} + v_i^* p_i' = \frac{V_{e,i}(Ao(\rho^*)) - v_i^*}{\tau_i}, \tag{7}$$

with $\rho^* = (\rho_1^*, \rho_2^*, \dots, \rho_N^*)^\top$. From (6), note that $\rho_i^* v_i^* = d_i$ with the given constant d_i and the given value for $\rho_i^*(0)$, $i = 1, 2\dots, N$. Assume that there exists a steady state $\rho_i^* > 0$, $v_i^* > 0$, defined on $[0, L]$ satisfying (6)–(7), as done in Bastin and Coron (2017) for a different class of 2×2 hyperbolic systems.

Denoting $(\widetilde{\rho}_1, \widetilde{v}_1, \widetilde{\rho}_2, \widetilde{v}_2, \dots, \widetilde{\rho}_N, \widetilde{v}_N)^\top$ with

$$\widetilde{\rho}_i = \rho_i - \rho_i^* \in H^1([0, L] \times [0, +\infty); \mathbb{R}),$$

$$\widetilde{v}_i = v_i - v_i^* \in H^1([0, L] \times [0, +\infty); \mathbb{R}),$$

$i = 1, 2, \dots, N$, by $\widetilde{u} \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$, the system (1)–(2) is transformed to the following equation, for all $x \in (0, L)$, $t \in [0, +\infty)$,

$$A(\widetilde{u})\widetilde{u}_t(x,t) + B(\widetilde{u})\widetilde{u}_x(x,t) + C(\widetilde{u})\widetilde{u}(x,t) = 0, \tag{8}$$

where, for $i, j = 1, 2, \dots, N$,

$$A(\widetilde{u}) = \begin{bmatrix} A_{11}(\widetilde{u}) & A_{12}(\widetilde{u}) & \cdots & A_{1N}(\widetilde{u}) \\ A_{21}(\widetilde{u}) & A_{22}(\widetilde{u}) & \cdots & A_{2N}(\widetilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1}(\widetilde{u}) & A_{N2}(\widetilde{u}) & \cdots & A_{NN}(\widetilde{u}) \end{bmatrix}, \tag{9}$$

with

$$A_{ij}(\widetilde{u}) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \delta_{ii}(\rho) & 1 \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \delta_{ij}(\rho) & 0 \end{bmatrix}, & \text{if } j \neq i, \end{cases} \tag{10}$$

$$B(\widetilde{u}) = \begin{bmatrix} B_{11}(\widetilde{u}) & B_{12}(\widetilde{u}) & \cdots & B_{1N}(\widetilde{u}) \\ B_{21}(\widetilde{u}) & B_{22}(\widetilde{u}) & \cdots & B_{2N}(\widetilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ B_{N1}(\widetilde{u}) & B_{N2}(\widetilde{u}) & \cdots & B_{NN}(\widetilde{u}) \end{bmatrix}, \tag{11}$$

with

$$B_{ij}(\widetilde{u}) = \begin{cases} \begin{bmatrix} \widetilde{v}_i + v_i^* & \widetilde{\rho}_i + \rho_i^* \\ (\widetilde{v}_i + v_i^*)\delta_{ii}(\rho) & \widetilde{v}_i + v_i^* \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (\widetilde{v}_i + v_i^*)\delta_{ij}(\rho) & 0 \end{bmatrix}, & \text{if } j \neq i, \end{cases} \tag{12}$$

and

$$C(\widetilde{u}) = \begin{bmatrix} C_{11}(\widetilde{u}) & C_{12}(\widetilde{u}) & \cdots & C_{1N}(\widetilde{u}) \\ C_{21}(\widetilde{u}) & C_{22}(\widetilde{u}) & \cdots & C_{2N}(\widetilde{u}) \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1}(\widetilde{u}) & C_{N2}(\widetilde{u}) & \cdots & C_{NN}(\widetilde{u}) \end{bmatrix}, \tag{13}$$

with (14) given in Box I. Therein, for $i, j, k = 1, 2, \dots, N$,

$$\delta_{ij}(\rho) = \partial_{\rho_j} p_i\left(Ao(\rho)\right) = \frac{v_i^M \gamma_i a_j}{Ao_i^M W}\left(\frac{Ao(\rho)}{Ao_i^M}\right)^{\gamma_i - 1},$$

$$\delta_{ij}(\rho^*) = \partial_{\rho_j} p_i\left(Ao(\rho^*)\right) = \frac{v_i^M \gamma_i a_j}{Ao_i^M W}\left(\frac{Ao(\rho^*)}{Ao_i^M}\right)^{\gamma_i - 1},$$

$$C_{ij}(\widetilde{u}) = \begin{cases} \begin{bmatrix} v_i^{*'} & \rho_i^{*'} \\ \frac{1}{\tau_i}\delta_{ii}(\rho^*) + v_i^* \sum_{k=1}^N \sigma_{iki}(\rho^*)\rho_k^{*'} & \frac{1}{\tau_i} + v_i^{*'} + \sum_{k=1}^N \delta_{ik}(\rho)\rho_k^{*'} \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau_i}\delta_{ij}(\rho^*) + v_i^* \sum_{k=1}^N \sigma_{ikj}(\rho^*)\rho_k^{*'} & 0 \end{bmatrix}, & \text{if } j \neq i. \end{cases} \tag{14}$$

Box I.

$$G_{ij}(u^*) = \begin{cases} \begin{bmatrix} v_i^{*'} & \rho_i^{*'} \\ \frac{1}{\tau_i}\delta_{ii}(\rho^*) + v_i^* \sum_{k=1}^N \sigma_{iki}(\rho^*)\rho_k^{*'} - \delta_{ii}(\rho^*)v_i^{*'} & \frac{1}{\tau_i} + v_i^{*'} + \sum_{k=1,k\neq i}^N \delta_{ik}(\rho^*)\rho_k^{*'} \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\tau_i}\delta_{ij}(\rho^*) + v_i^* \sum_{k=1}^N \sigma_{ikj}(\rho^*)\rho_k^{*'} - \delta_{ij}(\rho^*)v_j^{*'} & -\delta_{ij}(\rho^*)\rho_j^{*'} \end{bmatrix}, & \text{if } j \neq i. \end{cases} \tag{19}$$

Box II.

$$\sigma_{ikj}(\rho^*) = \partial_{\rho_k}\delta_{ij}(\rho^*) = \frac{v_i^M \gamma_i(\gamma_i - 1)a_k a_j}{(Ao_i^M W)^2}\left(\frac{Ao(\rho^*)}{Ao_i^M}\right)^{\gamma_i - 2}.$$

Because of the invertibility of $A(\widetilde{u})$, i.e., $|A(\widetilde{u})| \neq 0$, we transform and linearize the system (8) around the nonuniform steady-state u^*, then for all $x \in (0, L)$, $t \in [0, +\infty)$, the linearized system is derived as follows,

$$\widetilde{u}_t(x,t) + F(u^*)\widetilde{u}_x(x,t) = G(u^*)\widetilde{u}(x,t), \tag{15}$$

where, for $i, j = 1, 2, \ldots, N$,

$$F(u^*) = \begin{bmatrix} F_{11}(u^*) & F_{12}(u^*) & \cdots & F_{1N}(u^*) \\ F_{21}(u^*) & F_{22}(u^*) & \cdots & F_{2N}(u^*) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(u^*) & F_{N2}(u^*) & \cdots & F_{NN}(u^*) \end{bmatrix}, \tag{16}$$

with

$$F_{ij}(u^*) = \begin{cases} \begin{bmatrix} v_i^* & \rho_i^* \\ 0 & v_i^* - \rho_i^*\delta_{ii}(\rho^*) \end{bmatrix}, & \text{if } j = i, \\ \begin{bmatrix} 0 & 0 \\ (v_i^* - v_j^*)\delta_{ij}(\rho^*) & -\rho_j^*\delta_{ij}(\rho^*) \end{bmatrix}, & \text{if } j \neq i, \end{cases} \tag{17}$$

and

$$G(u^*) = \begin{bmatrix} G_{11}(u^*) & G_{12}(u^*) & \cdots & G_{1N}(u^*) \\ G_{21}(u^*) & G_{22}(u^*) & \cdots & G_{2N}(u^*) \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}(u^*) & G_{N2}(u^*) & \cdots & G_{NN}(u^*) \end{bmatrix}, \tag{18}$$

with (19) given in Box II.

Inspired by Zhang, Liu, Wong, and Dai (2006), the characteristic polynomial P_{2N} (characteristic variable λ) in this paper is analyzed as follows,

$$\begin{aligned} P_{2N}(\lambda) &= |\lambda I_{2N} - F(u^*)| \\ &= (\lambda - \phi_1)(\lambda - \phi_2)\cdots(\lambda - \phi_{2N-1})(\lambda - \phi_{2N}) \\ &\times \Bigg(1 + \left(\frac{1}{\lambda - \phi_1} - \frac{1}{\lambda - \phi_2}\right)\cdots\left(\frac{1}{\lambda - \phi_{2N-1}} - \frac{1}{\lambda - \phi_{2N}}\right) \\ &(\phi_1 - \phi_3)(\phi_3 - \phi_5)\cdots(\phi_{2N-3} - \phi_{2N-1})(\phi_{2N-1} - \phi_1)\Bigg), \end{aligned} \tag{20}$$

with $\phi_1 = v_1^*$, $\phi_2 = v_1^* - \rho_1^*\delta_{11}(\rho^*)$, $\phi_3 = v_2^*$, $\phi_4 = v_2^* - \rho_2^*\delta_{22}(\rho^*)$, …, $\phi_{2N-1} = v_N^*$, $\phi_{2N} = v_N^* - \rho_N^*\delta_{NN}(\rho^*)$. Assume that $\phi_1 > \phi_2 > \phi_3 > \phi_4 > \cdots > \phi_{2N-1} > \phi_{2N}$, then

$$P_{2N}(\phi_i) < 0, \quad i = 1, 2, \ldots, 2N, \tag{21}$$

$$P_{2N}(\phi_1 + \phi_3 + \cdots + \phi_{2N-1}) > 0, \tag{22}$$

and there is a constant a_i, $i = 1, 2, 3, \ldots, N-1$, on the domain $\phi_{2i} > a_i > \phi_{2i+1}$ such that

$$P_{2N}(a_i) > 0. \tag{23}$$

By using the intermediate value theorem, (21), (22), (23) imply that the polynomial $P_{2N}(\lambda)$ has $2N-1$ distinct positive eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \cdots, \lambda_{2N-1}$ such that

$$\begin{aligned} &\phi_1 + \phi_3 + \cdots + \phi_{2N-1} > \lambda_1 > \phi_1 > \phi_2 > \lambda_2 > a_1 \\ &> \cdots > \lambda_{2N-3} > \phi_{2N-3} > \phi_{2N-2} > \lambda_{2N-2} > a_{N-1} \\ &> \lambda_{2N-1} > \phi_{2N-1} > 0. \end{aligned} \tag{24}$$

From (20), note that if $\lambda < \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \ldots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$, then it holds

$$P_{2N}(\lambda) > 0. \tag{25}$$

Therefore, if $\phi_{2N} < 0$, there is a negative eigenvalue $-\lambda_{2N}$ on the domain $0 > \phi_{2N} > -\lambda_{2N} > \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \ldots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$; if $\phi_{2N} > 0$, there is a negative eigenvalue $-\lambda_{2N}$ on the domain $0 > -\lambda_{2N} > \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \ldots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$ under the following conditions

$$\begin{aligned} P_{2N}(0) = &\phi_1\phi_2\cdots\phi_{2N-1}\phi_{2N} + (\phi_1 - \phi_2)(\phi_3 - \phi_4) \\ &\cdots(\phi_{2N-1} - \phi_{2N})(\phi_1 - \phi_3)(\phi_3 - \phi_5) \\ &\cdots(\phi_{2N-3} - \phi_{2N-1})(\phi_{2N-1} - \phi_1) < 0. \end{aligned} \tag{26}$$

By the analysis of (25), we note that under the condition (26), there is not less than one negative eigenvalue (congested traffic), if $0 > \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \ldots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$. If $0 < \min\{2\phi_3 - \phi_1, 2\phi_5 - \phi_3, \ldots, 2\phi_{2N-1} - \phi_{2N-3}, 2\phi_{2N} - \phi_{2N-1}\}$, all the eigenvalues are positive (free traffic). The analysis of eigenvalues in this paper is actually the generalization of the case $N = 2$ in Burkhardt et al. (2021).

The hyperbolicity of the system (15) is clearly discussed as above, i.e., for all $u^* \in C^1([0, L]; \mathbb{R}^{2N})$, the matrix $F(u^*)$ has $2N$

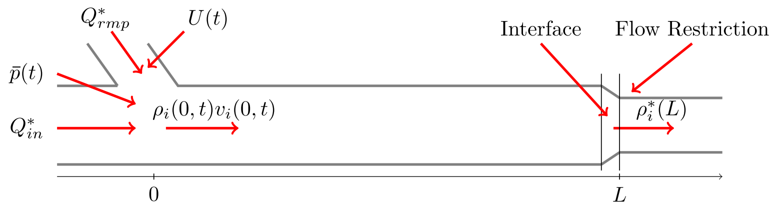

Fig. 1. Multi-type vehicles traffic on a road with disturbances and flow restriction.

real distinct eigenvalues different to zero. Given $2N$ eigenvalues

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0 > -\lambda_{m+1} > \cdots > -\lambda_{2N}, \tag{27}$$

of $F(u^*)$, $\lambda_i \in C^1([0, L]; \mathbb{R}_{>0})$, $i = 1, \ldots, 2N$, that does not depend on t, and assuming that the congestion mode is kept along the trajectory, we denote by m the number of positive eigenvalues. We get that $2N - m$ is the number of waves against the traffic flow (upstream) in the congested traffic due to the reaction of the drivers to their respective leading vehicles, and due to the high value of $Ao(\rho)$. In order to alleviate the traffic congestion, we thus compute the $2N - m$ boundary conditions reducing $\|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})}$. Due to (3), it is done by controlling the sum of the states. Because of $\|Ao(\rho)\|_{L^\infty} \leq C\|Ao(\rho)\|_{H^1}$ with a positive constant C, we will study the scenarios $2N - m \geq 1$ in the H^1 sense in this paper. The two-type vehicle case is investigated in the paper (Burkhardt et al., 2021), where $m = 3$, $N = 2$. With an invertible transformation matrix $T \in C^1([0, L]; \mathbb{R}^{2N\times 2N})$ whose columns are the corresponding right eigenvectors of $2N$ eigenvalues, by using the transformation $\omega = T^{-1}\tilde{u} \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$, the linearized system (15) is rewritten as, for all $x \in (0, L)$, $t \in [0, +\infty)$,

$$\partial_t\omega(x, t) + \Lambda(x)\partial_x\omega(x, t) = M(x)\omega(x, t), \tag{28}$$

where

$$\Lambda = \mathrm{diag}\{\Lambda^+, -\Lambda^-\} \in C^1([0, L]; \mathcal{D}_{2N}),$$
$$\Lambda^+ = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \in C^1([0, L]; \mathcal{D}^+_m),$$
$$\Lambda^- = \mathrm{diag}\{\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{2N}\} \in C^1([0, L]; \mathcal{D}^+_{2N-m}),$$
$$M = T^{-1}G(u^*)T - T^{-1}F(u^*)T' \in C^1([0, L]; \mathbb{R}^{2N\times 2N}).$$

Then, the following definitions are given for the subsequent analysis and investigation,

$$|\Lambda| = \mathrm{diag}\left\{\Lambda^+, \Lambda^-\right\} \in C^1([0, L]; \mathcal{D}^+_{2N}),$$
$$\Lambda' = \mathrm{diag}\left\{\lambda'_1, \ldots, \lambda'_m, -\lambda'_{m+1}, \ldots, -\lambda'_{2N}\right\} \in C^0([0, L]; \mathcal{D}_{2N}),$$
$$(\Lambda^+)' = \mathrm{diag}\{\lambda'_1, \lambda'_2, \ldots, \lambda'_m\} \in C^0([0, L]; \mathcal{D}^+_m),$$
$$(\Lambda^-)' = \mathrm{diag}\{\lambda'_{m+1}, \lambda'_{m+2}, \ldots, \lambda'_{2N}\} \in C^0([0, L]; \mathcal{D}^+_{2N-m}),$$

where λ'_i ($i = 1, 2, \ldots, 2N$) is the derivative of λ_i with respect to the spatial variable x.

2.3. Problem statement

The control problem is motivated by alleviating the congestion on a road segment with the disturbances at the inlet boundary and the flow restriction at the downstream boundary. For example, the occurrence of traffic congestion is attributed to the excess capacity of a bottleneck at the downstream outlet and the high traffic demand (modeled as the disturbances) at the upstream inlet of the considered road section.

In order to alleviate the traffic congestion, we design a boundary control law to reject disturbances for an investigated road segment, on which a ramp metering is installed at the inlet $x = 0$ and a constant density is kept at the outlet $x = L$,

$$\rho_i(L, t) = \rho^*_i(L), \quad \forall t \in [0, +\infty), \tag{29}$$

for $i = 1, 2, \ldots, N$. As described in Piccoli and Garavello (2006), the interface at the bottleneck plays a key role in the analysis of the boundary condition at the inlet of a bottleneck. It is a buffer zone for velocity drop (the velocity in the interface is continuously decreasing from the left boundary of the interface to $x = L$). The value of the velocity limit $v_i(L, \cdot)$ is derived from the constant density $\rho^*_i(L)$ and the measurement of the flux $q_i(L, \cdot)$ at the inlet of a bottleneck, for $i = 1, 2, \ldots, N$. The diagram of the control model is illustrated in Fig. 1.

We can derive the following equation on the basis of the flow conservation at the upstream inlet $x = 0$, for all $t \in [0, +\infty)$,

$$Q^*_{in} + \bar{p}(t) + Q^*_{rmp} + \Theta U(t) = \begin{bmatrix} \rho_1(0,t)v_1(0,t) \\ \rho_2(0,t)v_2(0,t) \\ \vdots \\ \rho_N(0,t)v_N(0,t) \end{bmatrix}, \tag{30}$$

where $Q^*_{in} \in \mathbb{R}^N$ is a vector whose entries are the constant inflow of each vehicle class, and $\bar{p} \in C^1([0, +\infty); \mathbb{R}^N)$ is a vector whose entries are the unknown disturbances of the flow rate of each vehicle class and serves as an exogenous variable depending on the time variable t. The actuation signal vector $U \in C^0([0, +\infty); \mathbb{R}^{2N-m})$ with a coefficient matrix $\Theta \in \mathbb{R}^{N\times 2N-m}$ is implemented by an on-ramp metering at the upstream boundary of the considered road segment. The matrix Θ is the control matrix describing the impact of the control input on the flow of each vehicle class. To alleviate the congestion, we want to minimize the area occupancy when the total inflow at the inlet consisting of the total inflow at the ramp $0 \leq Q^*_{rmp} + \Theta U \leq Q^{max}_{rmp}$ (Q^{max}_{rmp} is the flux limit on the on-ramp), and the total inflow at the inlet $0 \leq Q^*_{in} + \bar{p} \leq Q^{max}_{in}$ (Q^{max}_{in} is the flux limit of the incoming road) are limited by the maximum flow $Q_{max} \geq \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}(Q^*_{in} + \bar{p} + Q^*_{rmp} + \Theta U) \geq 0$, and $0 < v_i(0, \cdot) \leq v^M_i$, $0 < Ao(\rho(0, \cdot)) \leq \max\{Ao^M_1, Ao^M_2, \ldots, Ao^M_N\}$. From (6), the nominal on-ramp flux rate $Q^*_{rmp} \in \mathbb{R}^N$ satisfies the relation

$$Q^*_{in} + Q^*_{rmp} = \begin{bmatrix} \rho^*_1(0)v^*_1(0) \\ \rho^*_2(0)v^*_2(0) \\ \vdots \\ \rho^*_N(0)v^*_N(0) \end{bmatrix} = \begin{bmatrix} \rho^*_1(L)v^*_1(L) \\ \rho^*_2(L)v^*_2(L) \\ \vdots \\ \rho^*_N(L)v^*_N(L) \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix}. \tag{31}$$

Eq. (31) represents the sum of the inflow Q^*_{in} and the referenced input on-ramp flux rate Q^*_{rmp}, as the referenced input, is equivalent to the steady-state flow at the inlet and outlet boundaries of the considered road segment. Then, (30) shows that the control input is implemented to reject the disturbances $\bar{p}$.

From the boundary condition at $x = L$, by combining (30) with (31) and linearizing, the boundary conditions are derived, for all $t \in [0, +\infty)$,

$$A_1\tilde{u}(0, t) = \bar{p}(t) + \Theta U(t), \tag{32}$$
$$B_1\tilde{u}(L, t) = 0, \tag{33}$$

with

$$A_1 = \mathrm{diag}\left\{[v^*_1(0), \rho^*_1(0)], \ldots, [v^*_N(0), \rho^*_N(0)]\right\} \in \mathbb{R}^{N\times 2N},$$

$$B_1 = \mathrm{diag}\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \ldots, \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\} \in \mathbb{R}^{2N\times 2N}.$$

For the sake of alleviating the congestion and preventing the capacity drop, a controller is designed by using the backstepping approach in this paper. In the next subsection, a Riemann coordinate transformation of the state ω is dealt with in order to make the development and analysis of the controller easier.

2.4. Riemann coordinates transformation

By the transformation

$$R = \begin{bmatrix}R^+\\ R^-\end{bmatrix} = \Psi\omega, \tag{34}$$

with $\Psi = \mathrm{diag}\left\{\Psi^+, \Psi^-\right\} \in C^\infty([0, L]; \mathcal{D}^+_{2N})$,

$$\Psi^+ = \mathrm{diag}\left\{e^{-\int_0^x \frac{[M(s)]_{1,1}}{\lambda_1(s)}\,\mathrm{d}s}, e^{-\int_0^x \frac{[M(s)]_{2,2}}{\lambda_2(s)}\,\mathrm{d}s}, \ldots, e^{-\int_0^x \frac{[M(s)]_{m,m}}{\lambda_m(s)}\,\mathrm{d}s}\right\} \in C^\infty([0, L]; \mathcal{D}^+_m),$$

$$\Psi^- = \mathrm{diag}\left\{e^{\int_0^x \frac{[M(s)]_{m+1,m+1}}{\lambda_{m+1}(s)}\,\mathrm{d}s}, e^{\int_0^x \frac{[M(s)]_{m+2,m+2}}{\lambda_{m+2}(s)}\,\mathrm{d}s}, \ldots, e^{\int_0^x \frac{[M(s)]_{2N,2N}}{\lambda_{2N}(s)}\,\mathrm{d}s}\right\} \in C^\infty([0, L]; \mathcal{D}^+_{2N-m}),$$

from $\omega \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$ to the new variable $R \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N})$ with $R^+ : [0, L] \times [0, +\infty) \to \mathbb{R}^m$, $R^- : [0, L] \times [0, +\infty) \to \mathbb{R}^{2N-m}$, we derive the following system with a simpler source term in which all the diagonal entries of the coefficient matrix are zero, for all $x \in (0, L)$, $t \in [0, +\infty)$,

$$R_t(x, t) + \Lambda(x)R_x(x, t) = \Sigma(x)R(x, t), \tag{35}$$

$$R_{in}(t) = K_P R_{out}(t) + \Gamma_0(\bar{p}(t) + \Theta U(t)), \tag{36}$$

where

$$\Sigma = \begin{bmatrix}\Sigma^{++} & \Sigma^{+-}\\ \Sigma^{-+} & \Sigma^{--}\end{bmatrix} \in C^1([0, L]; \mathbb{R}^{2N\times 2N}),$$

$$R_{in} = \begin{bmatrix}R^+(0, \cdot)\\ R^-(L, \cdot)\end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}),$$

$$R_{out} = \begin{bmatrix}R^+(L, \cdot)\\ R^-(0, \cdot)\end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}),$$

$$K_P = \begin{bmatrix}0_{m\times m} & \Gamma_1\\ \Gamma_3 & 0_{2N-m\times 2N-m}\end{bmatrix} \in \mathbb{R}^{2N\times 2N},$$

$$\Gamma_0 = \begin{bmatrix}\Gamma_2\\ 0_{2N-m\times N}\end{bmatrix} \in \mathbb{R}^{2N\times N},$$

with

$$\Sigma^{++} = \left\{\epsilon_{ij}\right\}_{1\le i,j\le m} \in C^1([0, L]; \mathbb{R}^{m\times m}),$$
$$\Sigma^{+-} = \left\{\epsilon_{ij}\right\}_{1\le i\le m, m+1\le j\le 2N} \in C^1([0, L]; \mathbb{R}^{m\times 2N-m}),$$
$$\Sigma^{-+} = \left\{\epsilon_{ij}\right\}_{m+1\le i\le 2N, 1\le j\le m} \in C^1([0, L]; \mathbb{R}^{2N-m\times m}),$$
$$\Sigma^{--} = \left\{\epsilon_{ij}\right\}_{m+1\le i\le 2N, m+1\le j\le 2N} \in C^1([0, L]; \mathbb{R}^{2N-m\times 2N-m}),$$

and $\epsilon_{ij} \in C^1([0, L])$,

$$\epsilon_{ij} = \begin{cases}0, & \text{if } j = i,\\ [\Psi]_{i,i}\cdot[M]_{i,j}\cdot[\Psi]^{-1}_{j,j}, & \text{if } j \ne i.\end{cases}$$

There are matrices $\Upsilon_1 \in \mathbb{R}^{m\times N}$ and $\Upsilon_2 \in \mathbb{R}^{2N-m\times 2N}$ such that $\Upsilon_1 A_1 T^+(0) \in \mathbb{R}^{m\times m}$ and $\Upsilon_2 B_1 T^-(L) \in \mathbb{R}^{2N-m\times 2N-m}$ are invertible, and we obtain

$$\Gamma_1 = -(\Upsilon_1 A_1 T^+(0))^{-1}\Upsilon_1 A_1 T^-(0),$$
$$\Gamma_2 = (\Upsilon_1 A_1 T^+(0))^{-1}\Upsilon_1,$$
$$\Gamma_3 = -\Psi^-(L)(\Upsilon_2 B_1 T^-(L))^{-1}\Upsilon_2 B_1 T^+(L)(\Psi^+(L))^{-1},$$

$$T^+(0) = \left\{T^0_{ij}\right\}_{1\le i\le 2N, 1\le j\le m} \in \mathbb{R}^{2N\times m},$$
$$T^-(0) = \left\{T^0_{ij}\right\}_{1\le i\le 2N, m+1\le j\le 2N} \in \mathbb{R}^{2N\times 2N-m},$$
$$T^+(L) = \left\{T^L_{ij}\right\}_{1\le i\le 2N, 1\le j\le m} \in \mathbb{R}^{2N\times m},$$
$$T^-(L) = \left\{T^L_{ij}\right\}_{1\le i\le 2N, m+1\le j\le 2N} \in \mathbb{R}^{2N\times 2N-m},$$

and

$$T^0_{ij} = [T(0)]_{i,j}, \quad T^L_{ij} = [T(L)]_{i,j}.$$

Since the transformation (34) is invertible, the linearized system in terms of density and velocity has the same stability property as the system (35)–(36). Inspired by Coron et al. (2013), we are now in a position to design the controller.

3. Controller design

3.1. Target system

Consider the backstepping transformations, for all $x \in (0, L)$, $t \in [0, +\infty)$,

$$Z^+(x, t) = R^+(x, t), \tag{37}$$

$$Z^-(x, t) = R^-(x, t) - \int_x^L G^1(x, \xi)R^+(\xi, t)\,\mathrm{d}\xi - \int_x^L G^2(x, \xi)R^-(\xi, t)\,\mathrm{d}\xi, \tag{38}$$

where G^1, G^2 are piecewise differentiable kernels defined on the triangular domain $\mathbb{T} = \left\{(x, \xi) \in \mathbb{R}^2 \,|\, 0 \le x \le \xi \le L\right\}$ as described in Hu et al. (2016).

A system can be precisely controlled by only tuning the proportional gain, but the stability is relatively weakened, and even an unstable state occurs. In practical control engineering, the PI controller is mainly used to improve the stable property of the controlled system. Inspired by Hu et al. (2016), the following target system is introduced, for all $x \in (0, L)$, $t \in [0, +\infty)$,

$$Z_t(x, t) + \Lambda(x)Z_x(x, t) = \Sigma_1(x)Z(x, t) + \int_x^L C_1(x, \xi)Z(\xi, t)\,\mathrm{d}\xi + k_1(x)Z_{out}(t), \tag{39}$$

$$Z_{in}(t) = K_P Z_{out}(t) + X(t), \tag{40}$$

$$X(t) = K_I\int_0^t Z_{out}(\sigma)\,\mathrm{d}\sigma + \Gamma_0\bar{p}(t), \tag{41}$$

where

$$Z = \begin{bmatrix}Z^+\\ Z^-\end{bmatrix} \in H^1([0, L] \times [0, +\infty); \mathbb{R}^{2N}),$$

$$\Sigma_1 = \begin{bmatrix}\Sigma^{++} & \Sigma^{+-}\\ 0_{2N-m\times m} & 0_{2N-m\times 2N-m}\end{bmatrix} \in C^1([0, L]; \mathbb{R}^{2N\times 2N}),$$

$$C_1 = \begin{bmatrix} C^+ & C^- \\ 0_{2N-m\times m} & 0_{2N-m\times 2N-m} \end{bmatrix},$$
$$k_1 = \begin{bmatrix} 0_{m\times m} & 0_{m\times 2N-m} \\ K_1 & 0_{2N-m\times 2N-m} \end{bmatrix} \in C^1([0, L]; \mathbb{R}^{2N\times 2N}),$$
$$Z_{in}(\cdot) = \begin{bmatrix} Z^+(0, \cdot) \\ Z^-(L, \cdot) \end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}),$$
$$Z_{out}(\cdot) = \begin{bmatrix} Z^+(L, \cdot) \\ Z^-(0, \cdot) \end{bmatrix} \in L^\infty([0, +\infty); \mathbb{R}^{2N}),$$
$$K_I = \begin{bmatrix} K_I^{11} & K_I^{12} \\ 0_{2N-m\times m} & 0_{2N-m\times 2N-m} \end{bmatrix} \in \mathbb{R}^{2N\times N},$$

with $K_I^{11} \in \mathbb{R}^{m\times m}$, $K_I^{12} \in \mathbb{R}^{m\times 2N-m}$. Here k_1 is a strictly upper triangular matrix, and C^+, C^- are given as the piecewise differentiable solutions to the Volterra integral equations, for all (x, ξ) in $\mathbb{T}$,

$$C^+(x, \xi) = \Sigma^{+-}(x)G^1(x, \xi) + \int_x^\xi C^-(x, s)G^1(s, \xi)\,ds, \tag{42}$$
$$C^-(x, \xi) = \Sigma^{+-}(x)G^2(x, \xi) + \int_x^\xi C^-(x, s)G^2(s, \xi)\,ds. \tag{43}$$

The system (39)–(41) is considered under the initial conditions,

$$Z(\cdot, 0) = Z_0(\cdot) \in L^\infty([0, L]; \mathbb{R}^{2N}), \tag{44}$$
$$X(0) = X_0 = \Gamma_0\bar{p}(0) \in \mathbb{R}^{2N}. \tag{45}$$

The exponential stability for the H^1-norm of the target system (39)–(41) is as follows. It is based on a sufficient condition that would be checked numerically in Section 4.

Theorem 1. *The steady-state $Z(x, t) \equiv 0$ of the system (39)–(41) is integral input-to-state stable for the H^1-norm if there exist positive constants $\alpha, q_1, q_2, q_3, q_4$, diagonal positive-definite matrices $P_1, P_4 \in \mathbb{R}^{2N\times 2N}$, a symmetric positive-definite matrix $P_2 \in \mathbb{R}^{2N\times 2N}$ and a matrix $P_3 \in \mathbb{R}^{2N\times 2N}$ such that the following matrix inequalities hold, for all $x \in [0, L]$,*

(i)

$$\Omega(x) = \begin{bmatrix} \Omega_{11}(x) & \Omega_{12}(x) & \Omega_{13}(x) & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23}(x) & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44}(x) \end{bmatrix} \geq 0, \tag{46}$$

where

$$\Omega_{11}(x) = -\Lambda'(x)P_1 - \alpha P_1 - \Bigg(\Sigma_1^\top(x)P_1 + P_1\Sigma_1(x) + q_1Lv_1^2I_{2N} + \left(\frac{L}{q_1} + \frac{L}{q_2}\right)C_1^\top(0, x)C_1(0, x)\Bigg),$$

$$\Omega_{12}(x) = -P_3K_I - P_1k_1(x),$$

$$\Omega_{13}(x) = -\Lambda'(x)P_3 - \alpha P_3 - \Sigma_1^\top(x)P_3,$$

$$\Omega_{14} = 0_{2N\times 2N},$$

$$\Omega_{22} = \frac{1}{L}E_2P_1 - \frac{1}{L}K_P^\top E_1P_1K_P - \frac{1}{L}K_I^\top E_1P_4K_I,$$

$$\Omega_{23}(x) = -\frac{1}{L}K_P^\top E_1P_1 - \frac{1}{L}\left(K_P^\top M_1 + M_2\right) - K_I^\top P_2 - k_1^\top(x)P_3,$$

$$\Omega_{24} = -\frac{1}{L}K_I^\top E_1P_4K_P,$$

$$\Omega_{33} = -\frac{1}{L}E_1P_1 - \frac{1}{L}\left(M_1 + M_1^\top\right) - \alpha P_2 - q_2Lv_2^2I_{2N},$$

$$\Omega_{34} = 0_{2N\times 2N},$$

$$\Omega_{44}(x) = \frac{1}{L}E_2P_4 - \frac{1}{L}K_P^\top E_1P_4K_P - \frac{1}{q_4}k_1^\top(x)k_1(x),$$

with

$$M_1 = \begin{bmatrix} \Lambda^+(0)P_3^{++} & \Lambda^+(0)P_3^{+-} \\ -\Lambda^-(L)P_3^{-+} & -\Lambda^-(L)P_3^{--} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -\Lambda^+(L)P_3^{++} & -\Lambda^+(L)P_3^{+-} \\ \Lambda^-(0)P_3^{-+} & \Lambda^-(0)P_3^{--} \end{bmatrix},$$

$$P_3^{++} = \{P_3\}_{1\leq i,j\leq m} \in \mathbb{R}^{m\times m},$$
$$P_3^{+-} = \{P_3\}_{1\leq i\leq m, m+1\leq j\leq 2N} \in \mathbb{R}^{m\times 2N-m},$$
$$P_3^{-+} = \{P_3\}_{m+1\leq i\leq 2N, 1\leq j\leq m} \in \mathbb{R}^{2N-m\times m},$$
$$P_3^{--} = \{P_3\}_{m+1\leq i\leq 2N, m+1\leq j\leq 2N} \in \mathbb{R}^{2N-m\times 2N-m},$$

$E_1 = diag\left\{\Lambda^+(0), \Lambda^-(L)\right\}$, $E_2 = diag\left\{\Lambda^+(L), \Lambda^-(0)\right\}$, $v_1 = \max(\lambda(P_1))$, $v_2 = \max(|\lambda(P_3)|)$,

(ii)

$$M(x) = \left(-\Lambda'(x) - \alpha I_{2N}\right)P_4 - \Bigg(\Sigma_1^\top(x)P_4 + P_4\Sigma_1(x) + (q_3L + q_4)v_3^2I_{2N} + \frac{L}{q_3}C_1^\top(0, x)C_1(0, x)\Bigg) \geq 0, \tag{47}$$

with $v_3 = \max(\lambda(P_4))$.

In other words, there exist positive constants b_1, c_1 such that, for every $Z_0 \in H^1\left((0, L); \mathbb{R}^{2N}\right)$, $X_0 \in \mathbb{R}^{2N}$, and for any $\bar{p}$ such that $\dot{\bar{p}} \in C^0[0, +\infty)$, the solution $Z \in C^0\left([0, +\infty); H^1\left((0, L); \mathbb{R}^{2N}\right)\right)$, $X \in C^0\left([0, +\infty); \mathbb{R}^{2N}\right)$ to the Cauchy problem (39)–(41), (44)–(45) is defined on $[0, +\infty) \times [0, L]$ and satisfies

$$\|Z(\cdot, t)\|^2_{H^1((0,L);\mathbb{R}^{2N})} + |X(t)|^2 \leq c_1e^{-\alpha t}\left(\|Z_0\|^2_{H^1((0,L);\mathbb{R}^{2N})} + |X_0|^2\right) + b_1\int_0^t \dot{\bar{p}}^\top(s)\dot{\bar{p}}(s)\,ds, \quad \forall t \in [0, +\infty). \tag{48}$$

Remark. This theorem is in fact very general and could be applied to other control problems modeled by the hyperbolic systems. □

Proof. The following H^1 Lyapunov function candidate is introduced for the stability analysis of the system (39)–(41), for all $t \in [0, +\infty)$,

$$V\left(Z(x, \cdot), X(\cdot), Z_t(x, \cdot)\right) = V_1 + V_2 + V_3 + V_4, \tag{49}$$

where

$$V_1 = \int_0^L Z^\top(x, \cdot)\mathcal{P}_1(x)Z(x, \cdot)\,dx, \tag{50}$$
$$V_2 = \int_0^L \left(Z^\top(x, \cdot)\mathcal{P}_3(x)X(\cdot) + X^\top(\cdot)\mathcal{P}_3^\top(x)Z(x, \cdot)\right)dx, \tag{51}$$
$$V_3 = LX^\top(\cdot)P_2X(\cdot), \tag{52}$$
$$V_4 = \int_0^L Z_t^\top(x, \cdot)\mathcal{P}_4(x)Z_t(x, \cdot)\,dx, \tag{53}$$

and for all $x \in [0, L]$,

$$\mathcal{P}_1(x) \triangleq P_1\text{diag}\left\{e^{-\mu x}I_m, e^{\mu x}I_{2N-m}\right\},$$

$$\mathcal{P}_3(x) \triangleq P_3\text{diag}\left\{e^{-\frac{\mu}{2}x}I_m, e^{\frac{\mu}{2}x}I_{2N-m}\right\},$$

$$\mathcal{P}_4(x) \triangleq P_4 \mathrm{diag}\left\{e^{-\mu x} I_m, e^{\mu x} I_{2N-m}\right\}.$$

By definition, the notation Z_t must be understood as, for all $x \in [0, L]$,

$$Z_t(x,\cdot) \triangleq -\Lambda(x)Z_x(x,\cdot) + \Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot).$$

Under the definition of V and straightforward estimations, there exists a positive real constant β such that, for every Z, we can obtain the following inequality,

$$\frac{1}{\beta}\int_0^L \left(\|Z(x,\cdot)\|_{L^2}^2 + |X(\cdot)|^2 + \|Z_x(x,\cdot)\|_{L^2}^2\right)\mathrm{d}x \le V \le \beta\int_0^L \left(\|Z(x,\cdot)\|_{L^2}^2 + |X(\cdot)|^2 + \|Z_x(x,\cdot)\|_{L^2}^2\right)\mathrm{d}x. \quad (54)$$

By time differentiation of (39) and (40), Z_t can be shown to satisfy the following equations, for all $x \in [0, L]$,

$$Z_{tt}(x,\cdot) = -\Lambda(x)Z_{tx}(x,\cdot) + \Sigma_1(x)Z_t(x,\cdot) + \int_x^L C_1(x,\xi)Z_t(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)\dot{Z}_{out}(\cdot), \quad (55)$$

$$\dot{Z}_{in}(\cdot) = K_P\dot{Z}_{out}(\cdot) + \dot{X}(\cdot). \quad (56)$$

Taking time derivative of V_1 along the solutions to (39)–(41) and using integration by parts, the following result is achieved,

$$\begin{aligned}\dot{V}_1 =\ & Z_{out}^\top(\cdot)\left(K_P^\top \bar{E}_1 P_1 K_P - e^{-\mu L}\bar{E}_2 P_1\right)Z_{out}(\cdot) \\ & + Z_{out}^\top(\cdot)K_P^\top \bar{E}_1 P_1 X(\cdot) + X^\top(\cdot)P_1\bar{E}_1 K_P Z_{out}(\cdot) \\ & + X^\top(\cdot)\bar{E}_1 P_1 X(\cdot) \\ & + \int_0^L Z^\top(x,\cdot)\left(\Lambda'(x)\mathcal{P}_1(x) - \mu|\Lambda(x)|\mathcal{P}_1(x)\right)Z(x,\cdot)\,\mathrm{d}x \\ & + \int_0^L \Bigg(\left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot)\right)^\top \\ & \mathcal{P}_1(x)Z(x,\cdot) + Z^\top(x,\cdot)\mathcal{P}_1(x) \\ & \left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot)\right)\Bigg)\mathrm{d}x, \quad (57)\end{aligned}$$

with

$$\bar{E}_1 = \mathrm{diag}\left\{\Lambda^+(0), e^{\mu L}\Lambda^-(L)\right\},$$

$$\bar{E}_2 = \mathrm{diag}\left\{\Lambda^+(L), e^{\mu L}\Lambda^-(0)\right\}.$$

By taking time derivative of V_2 along the solutions to (39)–(41) and using integration by parts, we get

$$\begin{aligned}\dot{V}_2 \le\ & Z_{out}^\top(\cdot)\left(K_P^\top \bar{M}_1 + \bar{M}_2\right)X(\cdot) + X^\top(\cdot)\bar{M}_1 X(\cdot) \\ & + X^\top(\cdot)\left(\bar{M}_1^\top K_P + \bar{M}_2^\top\right)Z_{out}(\cdot) + X^\top(\cdot)\bar{M}_1^\top X(\cdot) \\ & + \int_0^L Z^\top(x,\cdot)\left(\Lambda'(x)\mathcal{P}_3(x) - \frac{\mu}{2}|\Lambda(x)|\mathcal{P}_3(x)\right)X(\cdot)\,\mathrm{d}x \\ & + \int_0^L X^\top(\cdot)\left(-\frac{\mu}{2}\mathcal{P}_3^\top(x)|\Lambda(x)| + \mathcal{P}_3^\top(x)\Lambda'(x)\right)Z(x,\cdot)\,\mathrm{d}x \\ & + \int_0^L \left(Z^\top(x,\cdot)\mathcal{P}_3(x)K_I Z_{out}(\cdot)\right. \\ & \left. + Z_{out}^\top(\cdot)K_I^\top \mathcal{P}_3^\top(x)Z(x,\cdot)\right)\mathrm{d}x \\ & + \kappa_1\int_0^L Z^\top(x,\cdot)\mathcal{P}_3(x)\Gamma_0\left(Z^\top(x,\cdot)\mathcal{P}_3(x)\Gamma_0\right)^\top\mathrm{d}x \\ & + \frac{L}{\kappa_1}\dot{\bar{p}}^\top(\cdot)\dot{\bar{p}}(\cdot) \\ & + \int_0^L \Bigg(\left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot)\right)^\top \\ & \mathcal{P}_3(x)X(\cdot) + X^\top(\cdot)\mathcal{P}_3^\top(x) \\ & \left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot)\right)\Bigg)\mathrm{d}x, \quad (58)\end{aligned}$$

with a positive constant κ_1 and

$$\bar{M}_1 = \begin{bmatrix} \Lambda^+(0)P_3^{++} & \Lambda^+(0)P_3^{+-} \\ -e^{-\frac{\mu}{2}L}\Lambda^-(L)P_3^{-+} & -e^{\frac{\mu}{2}L}\Lambda^-(L)P_3^{--}\end{bmatrix},$$

$$\bar{M}_2 = \begin{bmatrix} -e^{-\frac{\mu}{2}L}\Lambda^+(L)P_3^{++} & -e^{\frac{\mu}{2}L}\Lambda^+(L)P_3^{+-} \\ \Lambda^-(0)P_3^{-+} & \Lambda^-(0)P_3^{--}\end{bmatrix}.$$

By taking time derivative of V_3 along the solutions to (39)–(41), we can derive the following result with a positive constant κ_2,

$$\dot{V}_3 \le LZ_{out}^\top(\cdot)K_I^\top P_2 X(\cdot) + LX^\top(\cdot)P_2 K_I Z_{out}(\cdot) + L\kappa_2 X^\top(\cdot)P_2\Gamma_0\left(X^\top(\cdot)P_2\Gamma_0\right)^\top + \frac{L}{\kappa_2}\dot{\bar{p}}^\top(\cdot)\dot{\bar{p}}(\cdot). \quad (59)$$

Taking time derivative of V_4 along the solutions to (39)–(41), (55) and using integration by parts, we get

$$\begin{aligned}\dot{V}_4 \le\ & \dot{Z}_{out}^\top(\cdot)\left(K_P^\top \bar{E}_1 P_4 K_P - e^{-\mu L}\bar{E}_2 P_4\right)\dot{Z}_{out}(\cdot) \\ & + \dot{Z}_{out}^\top(\cdot)K_P^\top P_4\bar{E}_1 K_I Z_{out}(\cdot) \\ & + Z_{out}^\top(\cdot)K_I^\top \bar{E}_1 P_4 K_P \dot{Z}_{out}(\cdot) \\ & + Z_{out}^\top(\cdot)K_I^\top \bar{E}_1 P_4 K_I Z_{out}(\cdot) + \frac{1}{\kappa_3}\dot{\bar{p}}^\top(\cdot)\dot{\bar{p}}(\cdot) \\ & + \kappa_3\dot{Z}_{out}^\top(\cdot)K_P^\top \bar{E}_1 P_4\Gamma_0\left(K_P^\top \bar{E}_1 P_4\Gamma_0\right)^\top\dot{Z}_{out}(\cdot) \\ & + \kappa_4 Z_{out}^\top(\cdot)K_I^\top \bar{E}_1 P_4\Gamma_0\left(K_I^\top \bar{E}_1 P_4\Gamma_0\right)^\top Z_{out}(\cdot) \\ & + \frac{1}{\kappa_4}\dot{\bar{p}}^\top(\cdot)\dot{\bar{p}}(t) + \dot{\bar{p}}(\cdot)^\top\Gamma_0^\top \bar{E}_1 P_4\Gamma_0\dot{\bar{p}}(\cdot) \\ & + \int_0^L Z_t^\top(x,\cdot)\left(\Lambda'(x)\mathcal{P}_4(x) - \mu|\Lambda(x)|\mathcal{P}_4(x)\right)Z_t(x,\cdot)\,\mathrm{d}x \\ & + \int_0^L \Bigg(\left(\Sigma_1(x)Z_t(x,\cdot) + \int_x^L C_1(x,\xi)Z_t(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)\dot{Z}_{out}(\cdot)\right)^\top \\ & \mathcal{P}_4(x)Z_t(x,\cdot) + Z_t^\top(x,\cdot)\mathcal{P}_4(x) \\ & \left(\Sigma_1(x)Z_t(x,\cdot) + \int_x^L C_1(x,\xi)Z_t(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)\dot{Z}_{out}(\cdot)\right)\Bigg)\mathrm{d}x, \quad (60)\end{aligned}$$

with positive constants κ_3 and κ_4.

The three rightmost integrals in (57), (58) and (60) are considered individually,

$$\begin{aligned}& \int_0^L \Bigg(\left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot)\right)^\top \\ & \mathcal{P}_1(x)Z(x,\cdot) + Z^\top(x,\cdot)\mathcal{P}_1(x) \\ & \left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,\mathrm{d}\xi + k_1(x)Z_{out}(\cdot)\right)\Bigg)\mathrm{d}x \\ \le\ & \int_0^L \Big(\left(\Sigma_1(x)Z(x,\cdot) + k_1(x)Z_{out}(\cdot)\right)^\top\mathcal{P}_1(x)Z(x,\cdot) \\ & + Z^\top(x,\cdot)\mathcal{P}_1(x)\left(\Sigma_1(x)Z(x,\cdot) + k_1(x)Z_{out}(\cdot)\right)\Big)\mathrm{d}x \\ & + q_1 L e^{2\mu L}\nu_1^2\int_0^L Z^\top(x,\cdot)Z(x,\cdot)\,\mathrm{d}x \\ & + \frac{L}{q_1}\int_0^L \left(C_1(0,x)Z(x,\cdot)\right)^\top\left(C_1(0,x)Z(x,\cdot)\right)\mathrm{d}x. \quad (61)\end{aligned}$$

Similarly, we derive the inequalities for the other two integrals,

$$\int_0^L \left(\left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,d\xi + k_1(x)Z_{out}(\cdot)\right)^\top \mathcal{P}_3(x)X(\cdot) + X^\top(\cdot)\mathcal{P}_3^\top(x) \left(\Sigma_1(x)Z(x,\cdot) + \int_x^L C_1(x,\xi)Z(\xi,\cdot)\,d\xi + k_1(x)Z_{out}(\cdot)\right) \right) dx$$
$$\le \int_0^L \left((\Sigma_1(x)Z(x,\cdot) + k_1(x)Z_{out}(\cdot))^\top \mathcal{P}_3(x)X(\cdot) + X^\top(\cdot)\mathcal{P}_3^\top(x)(\Sigma_1(x)Z(x,\cdot) + k_1(x)Z_{out}(\cdot)) \right) dx$$
$$+ q_2 L e^{\mu L}\nu_2^2 \int_0^L X^\top(\cdot)X(\cdot)\,dx + \frac{L}{q_2}\int_0^L (C_1(0,x)Z(x,\cdot))^\top (C_1(0,x)Z(x,\cdot))\,dx, \quad (62)$$

$$\int_0^L \left(\left(\Sigma_1(x)Z_t(x,\cdot) + \int_x^L C_1(x,\xi)Z_t(\xi,\cdot)\,d\xi + k_1(x)\dot{Z}_{out}(\cdot)\right)^\top \mathcal{P}_4(x)Z_t(x,\cdot) + Z_t^\top(x,\cdot)\mathcal{P}_4(x) \left(\Sigma_1(x)Z_t(x,\cdot) + \int_x^L C_1(x,\xi)Z_t(\xi,\cdot)\,d\xi + k_1(x)\dot{Z}_{out}(\cdot)\right) \right) dx$$
$$\le \int_0^L \left((\Sigma_1(x)Z_t(x,\cdot))^\top \mathcal{P}_4(x)Z_t(x,\cdot) + Z_t^\top(x,\cdot)\mathcal{P}_4(x)\Sigma_1(x)Z_t(x,\cdot) \right) dx$$
$$+ (q_3 L + q_4)e^{2\mu L}\nu_3^2 \int_0^L Z_t^\top(x,\cdot)Z_t(x,\cdot)\,dx + \frac{L}{q_3}\int_0^L (C_1(0,x)Z_t(x,\cdot))^\top (C_1(0,x)Z_t(x,\cdot))\,dx$$
$$+ \frac{1}{q_4}\int_0^L (k_1(x)\dot{Z}_{out}(\cdot))^\top (k_1(x)\dot{Z}_{out}(\cdot))\,dx. \quad (63)$$

Using (57)–(63), there exists a constant $\alpha > 0$ such that, for all $t \ge 0$,

$$\dot V = \dot V_1 + \dot V_2 + \dot V_3 + \dot V_4 \le -\alpha V - \int_0^L \begin{bmatrix} Z(x,\cdot) \\ Z_{out}(\cdot) \\ X(\cdot) \\ \dot{Z}_{out}(\cdot) \end{bmatrix}^\top \bar{\Omega}(x) \begin{bmatrix} Z(x,\cdot) \\ Z_{out}(\cdot) \\ X(\cdot) \\ \dot{Z}_{out}(\cdot) \end{bmatrix} dx - \int_0^L Z_t^\top(x,\cdot)\bar{M}(x)Z_t(x,\cdot)\,dx$$
$$+ \dot{\bar{p}}^\top(\cdot)\left(\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4}\right)I_{2N} + \Gamma_0^\top \bar{E}_1 P_4 \Gamma_0\right)\dot{\bar{p}}(\cdot), \quad (64)$$

where, for all $x \in [0, L]$,

$$\bar{\Omega}(x) = \begin{bmatrix} \bar{\Omega}_{11}(x) & \bar{\Omega}_{12}(x) & \bar{\Omega}_{13}(x) & \bar{\Omega}_{14} \\ * & \bar{\Omega}_{22} & \bar{\Omega}_{23}(x) & \bar{\Omega}_{24} \\ * & * & \bar{\Omega}_{33} & \bar{\Omega}_{34} \\ * & * & * & \bar{\Omega}_{44}(x) \end{bmatrix} \quad (65)$$

with

$$\bar{\Omega}_{11}(x) = \mu|\Lambda(x)|\mathcal{P}_1(x) - \Lambda'(x)\mathcal{P}_1(x) - \alpha\mathcal{P}_1(x) - \kappa_1\mathcal{P}_3(x)\Gamma_0(\mathcal{P}_3(x)\Gamma_0)^\top - \left(\Sigma_1^\top(x)\mathcal{P}_1(x) + \mathcal{P}_1(x)\Sigma_1(x) + q_1 L e^{2\mu L}\nu_1^2 I_{2N} + \left(\frac{L}{q_1} + \frac{L}{q_2}\right)C_1^\top(0,x)C_1(0,x)\right),$$
$$\bar{\Omega}_{12}(x) = -\mathcal{P}_3(x)K_I - \mathcal{P}_1(x)k_1(x),$$
$$\bar{\Omega}_{13}(x) = \frac{\mu}{2}|\Lambda(x)|\mathcal{P}_3(x) - \Lambda'(x)\mathcal{P}_3(x) - \alpha\mathcal{P}_3(x) - \Sigma_1^\top(x)\mathcal{P}_3(x),$$
$$\bar{\Omega}_{14} = 0_{2N\times 2N},$$
$$\bar{\Omega}_{22} = \frac{e^{-\mu L}}{L}\bar{E}_2 P_1 - \frac{1}{L}K_P^\top \bar{E}_1 P_1 K_P - \frac{1}{L}K_I^\top \bar{E}_1 P_4 K_I - \frac{\kappa_4}{L}K_I^\top \bar{E}_1 P_4 \Gamma_0 (K_I^\top \bar{E}_1 P_4 \Gamma_0)^\top,$$
$$\bar{\Omega}_{23}(x) = -\frac{1}{L}K_P^\top \bar{E}_1 P_1 - \frac{1}{L}\left(K_P^\top \bar{M}_1 + \bar{M}_2\right) - K_I^\top P_2 - k_1^\top(x)\mathcal{P}_3(x),$$
$$\bar{\Omega}_{24} = -\frac{1}{L}K_I^\top \bar{E}_1 P_4 K_P,$$
$$\bar{\Omega}_{33} = -\frac{1}{L}\bar{E}_1 P_1 - \frac{1}{L}\left(\bar{M}_1 + \bar{M}_1^\top\right) - \kappa_2 P_2 \Gamma_0 (P_2\Gamma_0)^\top - \alpha P_2 - q_2 L e^{\mu L}\nu_2^2 I_{2N},$$
$$\bar{\Omega}_{34} = 0_{2N\times 2N},$$
$$\bar{\Omega}_{44}(x) = \frac{e^{-\mu L}}{L}\bar{E}_2 P_4 - \frac{1}{L}K_P^\top \bar{E}_1 P_4 K_P - \frac{\kappa_3}{L}K_P^\top \bar{E}_1 P_4 \Gamma_0 \left(K_P^\top \bar{E}_1 P_4 \Gamma_0\right)^\top - \frac{1}{q_4}k_1^\top(x)k_1(x),$$

and

$$\bar{M}(x) = \left(-\Lambda'(x) + \mu|\Lambda(x)| - \alpha I_{2N}\right)\mathcal{P}_4(x) - \left(\Sigma_1^\top(x)\mathcal{P}_4(x) + \mathcal{P}_4(x)\Sigma_1(x) + (q_3 L + q_4)e^{2\mu L}\nu_3^2 I_{2N} + \frac{L}{q_3}C_1^\top(0,x)C_1(0,x) \right). \quad (66)$$

Under the conditions (46), (47), $\exists\, \mu, \kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0$ small enough, such that $\bar{\Omega} \ge 0$ and $\bar{M} \ge 0$, thus

$$\dot V \le -\alpha V + \alpha_1 \dot{\bar{p}}^\top(\cdot)\dot{\bar{p}}(\cdot), \quad (67)$$

with $\alpha_1 = \max\left(\lambda\left(\left(\frac{L}{\kappa_1} + \frac{L}{\kappa_2} + \frac{1}{\kappa_3} + \frac{1}{\kappa_4}\right)I_{2N} + \Gamma_0^\top P_4 \bar{E}_1 \Gamma_0\right)\right)$. Thus along the solutions to the system (39)–(41), for all $t \in [0, +\infty)$,

$$V \le V(0)e^{-\alpha t} + \alpha_1 \int_0^t \dot{\bar{p}}^\top(s)\dot{\bar{p}}(s)\,ds. \quad (68)$$

Combining this relation with (54), there exist positive constants $c_1 = \beta^2$, $b_1 = \beta\alpha_1$ such that, for all $t \in [0, +\infty)$,

$$\int_0^L \left(\|Z(x,t)\|_{L^2}^2 + |X(t)|^2 + \|Z_x(x,t)\|_{L^2}^2\right) dx \le c_1 e^{-\alpha t}\left(\int_0^L \left(\|Z_0(x)\|_{L^2}^2 + |X_0|^2 + \|Z_x(x,0)\|_{L^2}^2\right) dx\right) + b_1 \int_0^t \dot{\bar{p}}^\top(s)\dot{\bar{p}}(s)\,ds, \quad (69)$$

completing the proof of Theorem 1. □

Based on the invertibility of backstepping transformation, it can be shown that the H^1 norm of the system (39)–(41) is equivalent to the H^1 norm of the system (35)–(36). Thus, the exponential stability of the H^1 norm of the system (39)–(41) implies the corresponding one for the H^1 norm of the system (35)–(36).

3.2. Control law

Take time derivative and spatial derivative on (37)–(38), and substitute them into (39)–(41) to get the following equations of the kernels G^1 and G^2, for all $(x,\xi)\in\mathbb{T}$,

$$\Lambda^-(x)G^1_x(x,\xi)-G^1_\xi(x,\xi)\Lambda^+(\xi)$$
$$=G^1(x,\xi)\left((\Lambda^+)'(\xi)+\Sigma^{++}(\xi)\right)+G^2(x,\xi)\Sigma^{-+}(\xi), \tag{70}$$
$$\Lambda^-(x)G^2_x(x,\xi)+G^2_\xi(x,\xi)\Lambda^-(\xi)$$
$$=G^2(x,\xi)\left(-(\Lambda^-)'(\xi)+\Sigma^{--}(\xi)\right)+G^1(x,\xi)\Sigma^{+-}(\xi), \tag{71}$$

with the boundary conditions

$$G^1(x,x)\Lambda^+(x)+\Lambda^-(x)G^1(x,x)=\Sigma^{-+}(x), \tag{72}$$
$$G^2(x,x)\Lambda^-(x)-\Lambda^-(x)G^2(x,x)=-\Sigma^{--}(x), \tag{73}$$
$$G^1(x,L)\Lambda^+(L)-G^2(x,L)\Lambda^-(L)\Gamma_3=K_1(x). \tag{74}$$

These equations are under-determined, and to ensure well-posedness, additional boundary conditions are added,

$$G^2_{ij}(0,\xi)=g^2_{ij}(\xi),\quad 1\le j<i\le 2N-m, \tag{75}$$

for some arbitrary functions g^2_{ij}, $1\le j<i\le 2N-m$.

There is a matrix $\overline{\Theta}\in\mathbb{R}^{2N-m\times m}$ such that $\overline{\Theta}\Gamma_2\Theta\in\mathbb{R}^{2N-m\times 2N-m}$ is invertible, then we deduce, from (36), (37), (38), (40), (41), the following controller defined as, $\forall t\in[0,+\infty)$,

$$\begin{aligned}U(t)=&\left(\overline{\Theta}\Gamma_2\Theta\right)^{-1}\overline{\Theta}\int_0^t\left(K_I^{11}R^+(L,\sigma)+K_I^{12}R^-(0,\sigma)\right)\,d\sigma\\ &-\left(\overline{\Theta}\Gamma_2\Theta\right)^{-1}\overline{\Theta}K_I^{12}\int_0^t\int_0^L\Big(G^1(0,\xi)R^+(\xi,\sigma)\\ &\qquad+G^2(0,\xi)R^-(\xi,\sigma)\Big)\,d\xi\,d\sigma\\ &-\left(\overline{\Theta}\Gamma_2\Theta\right)^{-1}\overline{\Theta}\Gamma_1\int_0^L\Big(G^1(0,\xi)R^+(\xi,t)\\ &\qquad+G^2(0,\xi)R^-(\xi,t)\Big)\,d\xi.\end{aligned} \tag{76}$$

Due to the dependence of U on the parameter Γ_2 and the inclusion of the parameter Γ_2 in the coefficient matrix Γ_0, Γ_0 has an effect on U and thus has an impact on the iISS of the system (35)–(36). Under the conditions of Theorem 1, the target system (39)–(41) is integral input-to-state stable. Thus, using the invertibility of backstepping transformation, the original system (35)–(36) is integral input-to-state stable in the H^1-norm with the control law (76).

4. Optimal tuning of the controller and numerical studies

In Theorem 1, we theoretically assume that there are P_1, P_2, P_3 and P_4 such that Ω and M satisfy the inequalities (46) and (47). In this section, an optimization problem is presented and solved for verifying the existences of P_1, P_2, P_3, and P_4 and obtaining the optimal values of parameters of the designed controller. The experiment is set and the results of computation are presented and discussed.

4.1. Optimal tuning of the controller

From (3), we note that $Ao(\rho)$ depends on the density vector ρ. The higher value of $\|\rho\|_{L^\infty((0,L);\mathbb{R}^N)}$ is, the higher value of $\|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})}$ is, then the traffic congestion is more possible to happen. Even though the traffic system has been stabilized, traffic congestion easily happens again due to the high road occupancy. In order to minimize the probability of the re-occurrence of the congested traffic after stabilization, we set the following optimization problem to derive the optimal tuning of the control law U,

$$\min_{\rho_i^*(0),v_i^*(0),K_I,\alpha,q_1,q_2,q_3,P_1,P_2,P_3,P_4}\|Ao(\rho)\|_{L^\infty((0,L);\mathbb{R})}$$
$$\text{subject to (46) and (47).} \tag{77}$$

From (76), the value of U depends on the parameters Γ_1, Γ_2, K_I^{11}, K_I^{12} and the kernels G^1, G^2 at $x=0$. From the definitions of K_P, K_I and Γ_2, we notice that the controller U actually depends on the parameters K_I, $\rho_i^*(0)$, $v_i^*(0)$, $\rho_i^*(L)$, and $v_i^*(L)$ $(i=1,2,\ldots,N)$, while the values of them for the optimal tuning can be obtained by solving the optimization problem (77). Due to $\rho_i^*v_i^*=d_i$, $i=1,2,\ldots,N$, K_I, $\rho_i^*(0)$ are the key parameters of the controller for the given $v_i^*(0)$.

4.2. Numerical studies

For numerical computation, the traffic parameters of two vehicle classes on a considered road section in the congested regime are chosen as in Burkhardt et al. (2021), see Table 1. The spatial variable x is discretized on the domain $[0,L]$. Given $v_1^*(0)=50$ km/h and $v_2^*(0)=25$ km/h, the values of ρ_1^*, ρ_2^* on the domain $[0,L]$ are derived by solving the ordinary differential equations (6)–(7) with the initial values $\rho_1^*(0)$, $\rho_2^*(0)$. By using a linear search method, we compute $\rho_1^*(0)$, $\rho_2^*(0)$, and solve the optimization problem (77). These parameters are crucial for the control gains. We obtain the optimal values of $\rho_1^*(0)$, $\rho_2^*(0)$ in Table 1 and see Fig. 2 for the plot of the nonuniform steady-state. The relationships $a_1<a_2$, $\tau_1<\tau_2$ and $v_1^*(0)>v_2^*(0)$ in Table 1 illustrate that, class 1 represents small and fast vehicles, and class 2 describes big and slow vehicles. When $\alpha\to 0$, $q_1=10^{-6}$, $q_2=1$, $q_3=10^{-5}$, $q_4=10^{-6}$,

$$P_1=\text{diag}\{2.0624, 2.4130, 7.1381, 2.5177\}\times 10^3,$$

$$P_2=1.8797\times 10^4\times I_4,$$

$$P_3=\begin{bmatrix}-14.3656 & -0.1018 & -0.0520 & -0.0969\\ 0.1162 & -16.6918 & 0.0284 & -0.1367\\ 0.1040 & -0.0498 & -43.7981 & 1.2582\\ -0.1063 & -0.1313 & 0.6899 & 17.0344\end{bmatrix},$$

and

$$P_4=\text{diag}\{2.2409, 2.5573, 3.8620, 2.4711\}\times 10^3.$$

The solution to the optimization problem (77) gives the following control gains

$$K_I^{11}=\begin{bmatrix}-20 & 30 & 30\\ -24 & -7 & 26\\ -10 & 20 & -30\end{bmatrix}\times 10^{-5},\quad K_I^{12}=\begin{bmatrix}60\\30\\20\end{bmatrix}\times 10^{-5},$$
$$\Gamma_1=\begin{bmatrix}-0.785\\ 1.047\\ -4.204\end{bmatrix}\times 10^{-4},\ \Gamma_2=\begin{bmatrix}0 & 0.0469\\ 0 & -0.0625\\ 0.0332 & 0.2051\end{bmatrix},$$

for which the conditions of Theorem 1 are satisfied. As a final remark, let us explain how to simulate the system (35)–(36) in a closed loop with the control (76) and these control gains. It asks in particular to discretize the piecewise continuously differentiable kernel functions G^1 and G^2 by following the approach of Hu et al. (2016). See also Chen, Vazquez, and Krstic (2022) where discontinuous kernel functions are numerically computed for a different control problem.

Table 1
Selected values of parameters.

Name	Symbol	Value	Unit
Number of vehicle class	N	2	1
Relaxation time	τ_1	30	s
	τ_2	60	s
Pressure exponent	γ_1	2.5	1
	γ_2	2	1
Free-flow velocity	v_1^M	80	$\frac{km}{h}$
	v_2^M	60	$\frac{km}{h}$
Maximum $Ao(\rho)$	Ao_1^M	0.9	1
	Ao_2^M	0.85	1
Occupied surface per vehicle	a_1	10	m^2
	a_2	42	m^2
Steady-state density at the inlet	$\rho_1^*(0)$	110	$\frac{veh}{km}$
	$\rho_2^*(0)$	70	$\frac{veh}{km}$
Steady-state velocity at the inlet	$v_1^*(0)$	50	$\frac{km}{h}$
	$v_2^*(0)$	25	$\frac{km}{h}$
Road width	W	6.5	m
Road length	L	1	km
Number of grid points	N_x	40	1

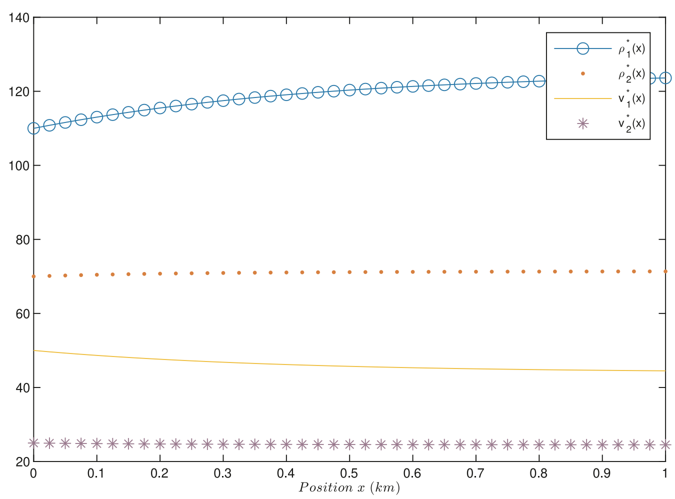

Fig. 2. Relation between the spatial variable x and the nonuniform steady-state $u^* = (\rho_1^*, v_1^*, \rho_2^*, v_2^*)^\top$.

5. Conclusion

The robust control problem was studied to stabilize the multi-type linearized AR traffic flow system. A controller was designed by using backstepping and the optimal tuning was done numerically.

Inspired by Coron et al. (2013), the H^2 locally iISS and state estimation problem will be studied for the quasilinear system in future research. It would be of interest to solve this analogous problem by using more complicated backstepping transformations to simplify the target system.

References

Auriol, Jean (2020). Output feedback stabilization of an underactuated cascade network of interconnected linear PDE systems using a backstepping approach. *Automatica*, *117*, Article 108964.

Auriol, Jean, & Pietri, Delphine Bresch (2022). Robust state-feedback stabilization of an underactuated network of interconnected n+m hyperbolic PDE systems. *Automatica*, *136*(110040).

Bastin, Georges, & Coron, Jean-Michel (2017). A quadratic Lyapunov function for hyperbolic density–velocity systems with nonuniform steady states. *Systems & Control Letters*, *104*, 66–71.

Bastin, Georges, Coron, Jean-Michel, & Hayat, Amaury (2021). Input-to-state stability in sup norms for hyperbolic systems with boundary disturbances. *Nonlinear Analysis*, *208*.

Bayen, Alexandre, Coron, Jean-Michel, Nitti, Nicola De, Keimer, Alexander, & Pflug, Lukas (2021). Boundary controllability and asymptotic stabilization of a nonlocal traffic flow model. *Vietnam Journal of Mathematics*, *49*(3), 957–985.

Burkhardt, Mark, Yu, Huan, & Krstic, Miroslav (2021). Stop-and-go suppression in two-class congested traffic. *Automatica*, *125*, Article 109381.

Chen, Guangwei, Vazquez, Rafael, & Krstic, Miroslav (2022). Rapid stabilization of Timoshenko Beam by PDE backstepping. arXiv preprint arXiv:2207.04746.

Coron, Jean-Michel, Vazquez, Rafael, Krstic, Miroslav, & Bastin, Georges (2013). Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *SIAM Journal on Control and Optimization*, *51*(3), 2005–2035.

Fan, Shimao, Herty, Michael, & Seibold, Benjamin (2014). Comparative model accuracy of a data-fitted generalized Aw-Rascle-Zhang model. *Networks and Heterogeneous Media*, *9*(2), 239–268. http://dx.doi.org/10.3934/nhm.2014.9.239.

Fan, Shimao, & Work, Daniel B. (2015). A heterogeneous multiclass traffic flow model with creeping. *SIAM Journal of Applied Mathematics*, *75*(2), 813–835. http://dx.doi.org/10.1137/140977977.

Greenshields, B. D., Bibbins, J. R., Channing, W. S., & Miller, H. H. (1935). A study of traffic capacity. *Highway Research Board Proceedings*.

Guan, Lina, Zhang, Liguo, & Prieur, Christophe (2020). Optimal PI controller rejecting disturbance for ARZ traffic model. In *59th conference on decision and control (CDC)* (pp. 5665–5670). Jeju Island, Republic of Korea: http://dx.doi.org/10.1109/CDC42340.2020.9304249.

Guan, Lina, Zhang, Liguo, & Prieur, Christophe (2021). Optimal observer-based output feedback controller for traffic congestion with bottleneck. *International Journal of Robust and Nonlinear Control*, *31*, 7087–7106.

Gupta, A. K., & Katiyar, V. K. (2007). A new multi-class continuum model for traffic flow. *Transportmetrica*, *3*(1), 73–85.

Hu, Long, Di Meglio, Florent, Vazquez, Rafael, & Krstic, Miroslav (2016). Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Transactions on Automatic Control*, *61*(11), 3301–3314.

Jiang, Rui, & Wu, QingSong (2004). Extended speed gradient model for mixed traffic. *Transportation Research Record*, *1883*(1), 78–84.

Karafyllis, Iasson, & Krstic, Miroslav (2017). ISS in different norms for 1-D parabolic PDEs with boundary disturbances. *SIAM Journal on Control and Optimization*, *55*(3), 1716–1751. http://dx.doi.org/10.1137/16M1073753.

Mironchenko, Andrii, & Prieur, Christophe (2020). Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Review*, *62*(3), 529–614.

Mohan, Ranju, & Ramadurai, Gitakrishnan (2017). Heterogeneous traffic flow modelling using second-order macroscopic continuum model. *Physics Letters. A*, *381*(3), 115–123.

Mohan, Ranju, & Ramadurai, Gitakrishnan (2021). Multi-class traffic flow model based on three dimensional flow–concentration surface. *Physica A: Statistical Mechanics and its Applications*, *577*(126060).

Piccoli, Benedetto, & Garavello, Mauro (2006). *Traffic flow on networks, vol. 1*. American institute of mathematical sciences.

Tang, Tieqiao, Huang, Haijun, Zhao, Shougen, & Shang, Huayan (2009). A new dynamic model for heterogeneous traffic flow. *Physics Letters. A*, *373*(29), 2461–2466.

Tumash, Liudmila, de Wit, Carlos Canudas, & Monache, Maria Laura Delle (2021). Boundary control for multi-directional traffic on urban networks. In *60th conference on decision and control (CDC)* (pp. 2671–2676). Austin, Texas, USA.

Yu, Huan, Gan, Qijian, Bayen, Alexandre, & Krstic, Miroslav (2020). PDE traffic observer validated on freeway data. *IEEE Transactions on Control Systems Technology*, *29*, 1048–1060.

Yu, Huan, & Krstic, Miroslav (2019). Traffic congestion control for Aw-Rascle-Zhang model. *Automatica*, *100*, 38–51.

Yu, Huan, Park, Saehong, Bayen, Alexandre, Moura, Scott, & Krstic, Miroslav (2021). Reinforcement learning versus PDE backstepping and PI control for congested freeway traffic. *IEEE Transactions on Control Systems Technology*, 1–17.

Zhang, Peng, Liu, Ruxun, Wong, S. C., & Dai, Shiqiang (2006). Hyperbolicity and kinematic waves of a class of multi-population partial differential equations. *European Journal of Applied Mathematics*, *17*, 171–200.

Lina Guan received her B.S. degree in Mathematics and Applied Mathematics and M.S. degree in Electric Machines and Electric Apparatus from Shenyang University of Technology, China in 2008 and 2011, and Ph.D. degree in Automatique-Productique from Université Grenoble Alpes, France in 2022. She is working as a Ph.D. student at Faculty of Information Technology at Beijing University of Technology, China (since 2018). She was a visiting Ph.D. student at Gipsa-lab, Grenoble, France (2019–2021). Her research focuses on the control of partial differential equations with the application of intelligent transportation system.

Liguo Zhang received the Ph.D. degree in Control Theory and Applications from Beijing University of Technology (BJUT), China, in 2006. He was a visiting professor (2011–2012) at the Department of Civil and Environmental Engineering, University of California, Berkeley. From 2014, he has been a full professor of the School of Electronic Information and Control Engineering, BJUT. He is currently the Deputy Director of Faculty of Information Technology, and the Dean of the Department of Artificial Intelligence and Automation, BJUT. His research interests include hybrid systems, intelligent transportation systems, and control of distributed parameter systems. He has been an Associate Editor of the IMA Journal Mathematical Control and Information, the Guest Editor of the International Journal of Distributed Sensor Networks. He was the member of the International Program Committee for the 58th, 59th, and 61st IEEE Conference on Decision and Control.

Christophe Prieur graduated in Mathematics from the Ecole Normale Supérieure de Cachan, France in 2000. He received the Ph.D. degree in 2001 in Applied Mathematics from the Université Paris-Sud, France. From 2002 he was an associate researcher CNRS at the laboratory SATIE, Cachan, France, and at the LAAS, Toulouse, France (2004–2010). In 2010 he joined the Gipsa-lab, Grenoble, France where he is currently a senior researcher of the CNRS (since 2011). He was the Program Chair of the 9th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2013), the 14th European Control Conference (ECC 2015) and the 61st IEEE Conference on Decision and Control (CDC 2022). He has been a member of the IEEE-CSS Conference Editorial Board and the EUCA Conference Editorial Board, and an associate editor of the IMA J. Mathematical Control and Information, the IEEE Trans. on Automatic Control, the European J. of Control, the IEEE Trans. on Control Systems Technology, and the Optimal Control Applications and Methods. He is currently an associate editor of the AIMS Evolution Equations and Control Theory, the SIAM Journal of Control and Optimization and the Mathematics of Control, Signals, and Systems. He is a senior editor of the IEEE Control Systems Letters, and an editor of the IMA Journal of Mathematical Control and Information. His current research interests include nonlinear control theory, hybrid systems, and control of partial differential equations, with applications including navigation and object tracking, fluid dynamics, and fusion control. He is an IMA Fellow, and an IEEE Fellow.