An LMI condition for the robustness of constant-delay linear predictor feedback with respect to uncertain time-varying input delays

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This paper discusses the robustness of the constant-delay predictor feedback in the case of an uncertain time-varying input delay. Specifically, we study the stability of the closed-loop system when the predictor feedback is designed based on the knowledge of the nominal value of the time-varying delay. By resorting to an adequate Lyapunov–Krasovskiifunctional, we derive an LMI-based sufficient condition ensuring the exponential stability of the closed-loop system for small enough variations of the time-varying delay around its nominal value. These results are extended to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems in the presence of a time-varying delay in the boundary control input.

1. Introduction

Originally motivated by the work of Artstein (1982), linear predictor feedback is an efficient tool for the feedback stabilization of Linear Time-Invariant (LTI) systems with constant input delay. In particular, predictor feedback can be used for controlling plants that are open-loop unstable and in the presence of large input delays. Many extensions have been reported (see, e.g., Krstic, 2009a and the references therein). These include the case of time-varying delay linear predictor feedback (Krstic, 2010b); robustness with respect to disturbance signals (Cai, Bekiaris-Liberis, & Krstic, 2018); truncated predictor (Zhou, Lin, & Duan, 2012); predictor observers in the case of sensor delays (Krstic, 2009a); predictors for nonlinear systems (Bekiaris-Liberis & Krstic, 2013b; Krstic, 2010a); dependence of the delay on the state (Bekiaris-Liberis & Krstic, 2013a); networked control (Selivanov & Fridman, 2016); and the boundary control of partial differential equations (Lhachemi & Prieur, 2019; Prieur & Trélat, 2019).

Most of the predictor feedback strategies reported in the literature assume a perfect knowledge in real-time of the input delay. However, such an assumption might be difficult to fulfill in practice. Consequently, there has been an increased interest in the last decade for the study of the robustness of the predictor feedback with respect to delay mismatches. An example of such a problem was investigated in Krstic (2008) where the exponential stability of the closed-loop system was assessed for unknown constant delays with small enough deviations from the nominal value. The study of the impact of an unknown time-varying delay, but with known nominal value which is used to design the predictor feedback, on the system closed-loop stability was reported in Bekiaris-Liberis and Krstic (2013b). In particular, it was shown that the exponential stability of the closed-loop system is guaranteed for sufficiently small variations of the delay in both amplitude and rate of variation. Such an approach was further investigated in Karafyllis and Krstic (2013) where a small gain condition on the only amplitude of variation of the delay around its nominal value was derived for ensuring the exponential stability of the closed-loop system. However, as underlined in Li, Zhou, and Lin (2014), such a small gain condition might be conservative as it involves norms of matrices which generally grow quickly with their dimensions. In order to reduce such a conservatism, it was proposed in Li et al. (2014) to resort to a Lyapunov–Krasovskii functional approach in the case of constant uncertain delays. By doing so, an LMI-based sufficient condition was derived, for ensuring the asymptotic stability of the closed-loop system with constant uncertain delays.

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The first contribution of this paper deals with the study of the robustness of the constant-delay predictor feedback that has been designed based on the nominal value of an uncertain and time-varying input delay. By taking advantage of classical Lyapunov–Krasovskii functionals (Fridman, 2014), we derive an LMI-based sufficient condition on the amplitude of variation of the input delay around its nominal value that ensures the exponential stability of the closed-loop system. Such an approach was investigated first in Selivanov and Fridman (2016) in the context of networked control. However, the LMI condition derived in this paper differs from the one proposed in Selivanov and Fridman (2016). Three examples are developed showing that, for these case studies, the LMI condition proposed in this paper provides less conservative results than the small gain condition reported in Karafyllis and Krstic (2013) and the LMI condition extracted from Selivanov and Fridman (2016).

The second contribution of this paper deals with the extension of the above result to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems (Curtain & Zwart, 2012) in the presence of a time-varying delay in the boundary control input. The control strategy consists in (1) the use of a predictor feedback to stabilize a finite-dimensional subsystem capturing the unstable modes of the infinite-dimensional system; (2) ensuring that the control law designed on a finite-dimensional truncated model successfully stabilizes the full infinite-dimensional system. Such a control strategy, inspired by Russell (1978) in the case of a delay-free feedback control, was first reported in Prieur and Trélat (2019) for the exponential stabilization of a reaction–diffusion equation with a constant delay in the boundary control. Note that a different approach for tackling the same feedback stabilization problem was reported in Krstic (2009b) via the use of a backstepping boundary controller. Ideas from Prieur and Trélat (2019) were extended to the exponential stabilization of a class of diagonal infinite-dimensional boundary control systems with constant delay in the boundary control in Lhachimi and Prieur (2019). In this present paper, we go beyond (Krstic, 2009b; Lhachimi & Prieur, 2019; Prieur & Trélat, 2019) and assess the robustness of the control strategy reported in Lhachimi and Prieur (2019) in the case of an uncertain and time-varying input delay. Specifically, we show that for time-varying delays presenting (1) a sufficiently small amplitude of variation around its nominal value (with sufficient condition provided by the LMI condition discussed above); (2) a rate of variation that is bounded by an arbitrarily large constant; the infinite-dimensional closed-loop system is exponentially stable.

The remainder of this paper is organized as follows. The robustness of the predictor feedback with respect to uncertain and time-varying delays is investigated in Section 2. The extension of this result to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems is presented in Section 3. The obtained results are applied in Section 4. Finally, concluding remarks are provided in Section 5.

Notation. The sets of non-negative integers, positive integers, real, non-negative real, positive real, and complex numbers are denoted by \( \mathbb{N}, \mathbb{N}^+, \mathbb{R}, \mathbb{R}_+, \mathbb{R}^+_m, \) and \( \mathbb{C} \), respectively. The real and imaginary parts of a complex number \( z \) are denoted by \( \Re z \) and \( \Im z \), respectively. The field \( \mathbb{K} \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \). The set of \( n \)-dimensional vectors over \( \mathbb{K} \) is denoted by \( \mathbb{K}^n \) and is endowed with the Euclidean norm \( \| x \| = \sqrt{x^T x} \). The set of \( n \times m \) matrices over \( \mathbb{K} \) is denoted by \( \mathbb{K}^{n \times m} \) and is endowed with the induced norm denoted by \( \| \cdot \| \). For any symmetric matrix \( P \in \mathbb{R}^{n \times n} \) and \( P > 0 \) (resp. \( P \geq 0 \)) means that \( P \) is positive definite (resp. positive semi-definite). The set of symmetric positive definite matrices of order \( n \) is denoted by \( \mathbb{S}^n_+ \). For any symmetric matrix \( P \in \mathbb{R}^{n \times n} \), \( \lambda_{	ext{min}}(P) \) and \( \lambda_{	ext{max}}(P) \) denote the smallest and largest eigenvalues of \( P \), respectively. For \( M = (m_{ij}) \in \mathbb{C}^{n \times m} \), we introduce

\[
\mathcal{R}(M) = \begin{bmatrix} \Re M & -\Im M \\ \Im M & \Re M \end{bmatrix} \in \mathbb{R}^{2n \times 2m}
\]

where \( \Re M \triangleq (\Re m_{ij}) \in \mathbb{R}^{n \times m} \) and \( \Im M \triangleq (\Im m_{ij}) \in \mathbb{R}^{n \times m} \). For any \( i_0 > 0 \), we say that \( \varphi \in \mathcal{C}^0([0; T]; \mathbb{R}) \) is a transition signal over \([0, t_0] \) if \( 0 \leq \varphi \leq 1 \), \( \varphi|_{t_0 = 0} = 0 \), and \( \varphi|_{[0, +\infty]} = 1 \). In Section 3, the notations and terminologies for infinite-dimensional systems are retrieved from Curtain and Zwart (2012).

## 2. Delay-robustness of predictor feedback for LTI systems

### 2.1. Problem setting and existing result

The first part of this paper deals with the feedback stabilization of the following LTI system with delay control:

\[
\dot{x}(t) = Ax(t) + Bu(t - D(t)), \quad t \geq 0,
\]

with \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) such that the pair \( (A, B) \) is stabilizable. Vectors \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) denote the state and the control input, respectively. The command input is subject to an uncertain time-varying delay \( D \in \mathcal{C}^0([0; +\infty]; \mathbb{R}_+) \). We assume that there exist \( D_0 > 0 \) and \( 0 < D < D_0 \) such that \( |D(t) - D_0| \leq D \) for all \( t \geq 0 \). In this context, the following constant-delay linear predictive feedback, which is based on the knowledge of the constant nominal value \( D_0 \), has been proposed in Bekiaris-Liberis and Krstic (2013b):

\[
u(t) = K \left\{ e^{D_0 A} x(t) + \int_{t-D_0}^t e^{(t-s)A} B u(s) ds \right\}
\]

(2)

for \( t > 0 \), where \( K \in \mathbb{R}^{n \times m} \) is a feedback gain such that \( A_0 \triangleq A + BK \) is Hurwitz. The validity of such a control strategy was assessed in Karafyllis and Krstic (2013) via a small gain argument.

**Theorem 1** (Karafyllis & Krstic, 2013). Let \( D_0 > 0 \) be given and let \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( K \in \mathbb{R}^{n \times m} \) be such that \( A_0 = A + BK \) is Hurwitz. Let \( \delta \in (0, D_0) \) be such that

\[
M \| e^{D_0 A} BK \left\{ e^{D_0 x(\delta)} - e^{-i \delta x(\delta)} \right\} \leq \mu,
\]

(3)

where \( M, \mu > 0 \) are constants satisfying \( \| e^{D_0 t} \| \leq Me^{-\mu t} \) for all \( t \geq 0 \). Then, there exist \( N, \sigma > 0 \) such that for all \( x_0 \in \mathbb{R}^n \), \( u_0 \in \mathcal{C}^0([D_0 - \delta, 0]; \mathbb{R}^m) \) with \( u_0(0) = K \left\{ e^{D_0 x_0} + \int_0^{D_0} e^{-sA} B u_0(s) ds \right\} \), and \( D \in \mathcal{C}^0([0; +\infty]; \mathbb{R}_+) \) with \( |D - D_0| \leq \delta \), the solution of (1)-(2) associated with the initial conditions \( x(0) = x_0 \) and \( u(t) = u_0(t) \) for \( D_0 - \delta \leq t \leq 0 \) satisfies for all \( t \geq 0 \) the following estimate:

\[
\| x(t) \| + \max_{-D_0 - \delta \leq s \leq 0} \| u(s) \| \leq Ne^{-\mu t} \left\{ \| x_0 \| + \max_{-D_0 - \delta \leq s \leq 0} \| u_0(s) \| \right\}.
\]

As the left hand-side of (3) is equal to zero when \( \delta = 0 \), a continuity argument shows that there always exists a \( \delta > 0 \) such that (3) holds true. Therefore, Theorem 1 ensures the existence of a sufficiently small amplitude of perturbation \( \delta > 0 \) of the delay \( D(t) \) around its nominal value \( D_0 \) such that the constant-delay linear predictor feedback (2) ensures the exponential stability of the closed-loop system with uncertain time-varying input delays. However, due to the nature of the small gain-condition (3) that involves the norm of matrices (which generally grow quickly as
a function of the matrices dimensions \( n \) and \( m \), the admissible values of \( \delta \) might be conservative (see Li et al., 2014). In particular, from the fact that \( M \geq 1 \) and \( 0 < \mu \leq \mu_M(A_\delta) \equiv \max(\Re(\lambda : \lambda \in \sigma_p(A_\delta))) \), any \( \delta > 0 \) such that the small gain condition (3) holds true satisfies the following estimate:
\[
\delta < \delta_1 \equiv \frac{1}{\|A\|} \left( 1 + \frac{\mu_M(A_\delta)}{\|e^{\theta \delta}AK\|} \right)^b.
\]

To reduce the conservatism, an LMI condition ensuring the exponential stability of the closed-loop system was derived in Selivanov and Fridman (2016) in the context of networked control. The objective of this section it to propose the construction of an alternative LMI for such a problem. Numerical comparisons between the different methods (small gain and LMIs) will be carried out in Sections 2.4 and 4.

### 2.2 Preliminary results

For \( h > 0 \), we denote by \( W \) the space of absolutely continuous functions \( \psi : [-h, 0] \rightarrow \mathbb{R}^n \) with square-integrable derivative endowed with the norm \( \|\psi\|_W \equiv \left( \|\psi(0)\|^2 + \int_0^{h} \|\dot{\psi}(\theta)\|^2d\theta \right)^{1/2} \) (see Kolmanovskii & Myshkis, 2012, Chap. 4, Sec. 1.3).

**Lemma 1.** Let \( M, N \in \mathbb{R}^{n \times n}, D_0 > 0 \), and \( \delta \in (0, D_0) \) be given. Assume that there exist \( \kappa > 0, P_1, Q \in \mathbb{S}^{n \times n}_+ \), and \( P_2, P_3 \in \mathbb{R}^{n \times n} \) such that \( \Theta(\delta, \kappa) \equiv 0 \) and
\[
\Theta(\delta, \kappa) = \begin{bmatrix}
2\kappa P_1 + P_2^T M + P_2 M & P_2^T - P_3^T & \delta P_2^T N
\end{bmatrix}
\begin{bmatrix}
P_1 - P_2^T & M + \delta P_3 N & -\delta P_2^T N
\end{bmatrix}
\begin{cases}
> 0, & \delta \leq D_0
\end{cases}
\]

Then, there exists \( C_0 > 0 \) such that, for all \( D = D_0 + \delta \leq D_0 \), the trajectory of:
\[
\dot{x}(t) = Mx(t) + N [x(t - D(t)) - x(t - D_0)],
\]
\[
x(t) = x_0(t), \quad t \in [-D_0 - \delta, 0]
\]
with initial condition \( x_0 \in W \) (for \( h = D_0 + \delta \)) satisfies \( \|x(t)\| \leq C_0e^{-\kappa t}\|x_0\|_W \) for all \( t \geq 0 \).

**Proof.** For all \( t \geq 0 \), one has
\[
\dot{x}(t) = Mx(t) + N \int_{t-D_0}^{t-(D-\delta)} \dot{x}(\tau)d\tau.
\]

By classical Lyapunov–Krasovskii functional depending on time derivative for systems with fast varying delays, see Fridman (2014, Sec. 3.2), we introduce \( V(t) = \int_{t-D_0}^{t-D(t)} V_2(t) + V_1(t) \) with \( V_1(t) = x(t)^TP_1x(t) \) and \( V_2(t) = \int_{t-D_0}^{t-D(t)} \dot{\xi}(t)^TP_2\dot{\xi}(t) + \int_{t-D_0}^{t-D(t)} \dot{\xi}(t)^TP_3\dot{\xi}(t)ds \) \( \forall t \geq 0 \), then we have, for all \( t \geq 0 \),
\[
\dot{V}(t) = 2\kappa x(t)^TP_1x(t) + 2\delta \dot{x}(t)^TP_2\dot{x}(t) - 2\kappa \dot{V}_2(t)
\]
\[
- \int_{t-D_0}^{t-D(t)} \delta \dot{\xi}(t)^TP_3\dot{\xi}(t)ds.
\]

The remaining of the proof is now an adaptation of Fridman (2006, Proof of Thm 1). Introducing \( P = \begin{bmatrix} P_1 & 0 & 0 \\ P_2 & P_3 \end{bmatrix} \), where \( P_1, P_2, P_3 \in \mathbb{R}^{n \times n} \) are “slack variables” (Fridman, 2014), we have
\[
x(t)^TP_1\dot{x}(t) = \begin{bmatrix} x(t) \dot{x}(t) \end{bmatrix}^TP_1 \begin{bmatrix} \dot{x}(t) - Mx(t) + N \int_{t-D_0}^{t-(D-\delta)} \dot{x}(\tau)d\tau \\ 0 \end{bmatrix}
\]
\[
\leq \begin{bmatrix} x(t) \dot{x}(t) \end{bmatrix}^TP_1 \begin{bmatrix} I & 0 \\ M - I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \int_{t-D_0}^{t-(D-\delta)} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^TP_2 \begin{bmatrix} 0 & 0 \\ P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \dot{x}(\tau)d\tau.
\]

Now, from the fact that, for any \( a, b \in \mathbb{R}^n, 2a^Tb \leq \|a\|^2 + \|b\|^2 \), we obtain that
\[
\dot{V}(t) + 2\kappa V(t) \leq 2\kappa x(t)^TP_1x(t) + 2\delta x(t)^TP_2\dot{x}(t) + 2\delta \dot{x}(t)^TP_3\dot{x}(t)ds.
\]

With (7)-(8) we deduce that
\[
\dot{V}(t) + 2xV(t) \leq 2\kappa x(t)^TP_1x(t) + 2\delta x(t)^TP_2\dot{x}(t) + 2\delta \dot{x}(t)^TP_3\dot{x}(t)ds.
\]

where it has been used the fact that the sum of the two integral terms is always non-positive, and with
\[
\psi \equiv P_1^{1/2} \begin{bmatrix} 0 & I \\ M - I \end{bmatrix} P_2^{1/2} + P_3 \begin{bmatrix} \kappa P_1 & 0 \\ 0 & \delta Q \end{bmatrix}
\]

From \( \Theta(\delta, \kappa) \leq 0 \), the direct application of the Schur complement yields \( V(t) + \kappa V(t) \leq 0 \). The conclusion follows from the fact that \( \lambda_{\min}(P_1)\|x(t)\|^2 \leq V(t) \leq \max(\lambda_{\min}(P_1), 2\delta \lambda_{\min}(Q))\|x(t)\|_W \) for all \( t \geq 0 \).

By a continuity argument, \( \Theta(\delta, \kappa) < 0 \) implies \( \Theta(\delta, \kappa) \leq 0 \) for all \( \delta > 0 \). We deduce the following result.

**Corollary 1.** Let \( M, N \in \mathbb{R}^{n \times n}, D_0 > 0 \), and \( \delta \in (0, D_0) \) be given. Assume that \( \Theta(\delta, \kappa) < 0 \). Then the conclusions of Lemma 1 hold true for some decay rate \( \kappa > 0 \).

From Lemma 1, the feasibility of the LMI \( \Theta(\delta, \kappa) \leq 0 \) ensures that \( M \) is Hurwitz. The following lemma states a form of converse result.

**Lemma 2.** Let \( D_0 > 0 \) and \( M, N \in \mathbb{R}^{n \times n} \) with \( M \) Hurwitz be given. Let \( P_2 \in \mathbb{S}_+^{n \times n} \) be the unique solution of \( M^T P_2 + P_2 M = -I \) and let \( 0 \leq \kappa < 1/(4\lambda_{\min}(P_2)) \) be given. Introducing \( \delta^* = \delta^*(\kappa) > 0 \) defined by
\[
\delta^* \equiv \min \left( D_0, \frac{\min \left( 1 - 4\kappa \lambda_{\min}(P_2), \lambda_{\min}((M^-1)^T M^-1) \right)}{2\sqrt{2\delta^*e^{D_0}\|N\|_F^2 P_2^- - (M^-1)^T M^-1) P_3} \right)
\]
the LMI \( \Theta(\delta, \kappa) \leq 0 \) is feasible for all \( \delta \in (0, \delta^*) \).

\(^1\) With the convention \( \delta^* = D_0 \) in the case \( N = 0 \).
\begin{proof}
Let $\delta > 0$ be chosen such that $\Theta(\delta, 0) < 0$ is feasible (see Lemma 2) and, by a continuity argument, let $\kappa > 0$ be such that $\Theta(\delta, \kappa) \leq 0$ is feasible. By the properties of the Aarts transformation (Bresch-Pietti, Prieur, & Trélat, 2018), we have $\kappa \in C^1(\mathbb{R}_+; \mathbb{R}^n)$ and $u \in C^0(-D_0 - \delta, +\infty); \mathbb{R}^m)$. We introduce $z \in C^1([-D_0, +\infty))$ defined for all $t \geq 0$ by (see Aarts, 1982):
\begin{equation}
    z(t) = e^0u(x(t)) + \int_{t-D_0}^t e^{t-s}Bu(s)ds.
\end{equation}
As $u \equiv Kz$, we have for all $t \geq 0$,
\begin{equation}
    \dot{z}(t) = \left( A + \psi(t)BK \right)z(t) + e^{0b}BK\left( \psi \mathbf{z}(t-D(t)) - \psi \mathbf{z}(t-D_0) \right).
\end{equation}
In particular, we have for all $t \geq t_0 \equiv t_0 + D_0 + \delta$ that
\begin{equation}
    \dot{z}(t) = A_3z(t) + e^{0b}BK\left( z(t-D(t)) - z(t-D_0) \right)
\end{equation}
with $A_3 = A + BK$ Hurwitz and the continuously differentiable initial condition $z(t_0)$. Applying Lemma 1, we obtain that $\|z(t)\| \leq C_0e^{-\gamma(t-t_0)}\|z(t_1)\|_{\mathcal{W}}$ for all $t \geq t_1$.

We introduce $V(t) = \|z(t)\|^2/2$ for $t \geq 0$. The use of the Young’s inequality shows that there exist constants $\gamma_1, \gamma_2 > 0$, independent of $x_0$ and $D$, such that for all $t \geq 0$, $V(t) \leq \gamma_1V(t) + \gamma_2\|\psi(t-D(t))\|V(t-D(t)) + \gamma_2\|\psi(t-D_0)\|V(t-D_0)$. We show by induction that, for any $n \in \mathbb{N}$, there exists a constant $c_n > 0$, independent of $x_0$ and $D$, such that $V(t) \leq c_n\|x_0\|^2/2$ for all $t \in [0, n(D_0 - \delta)]$. In the case $n = 1$, we have for all $t \in [0, D_0 - \delta)$, $\|\psi(t-D(t))\| = \|\psi(t-D_0)\|$. Thus $V(t) \leq \gamma_1V(t)$ and we obtain that the property holds true with $c_1 = e^{\gamma(t-D_0-\delta)/2}\|\psi\|_{\mathcal{W}}$. Assume that $V(t) \leq c_{n-1}\|x_0\|^2/2$ for all $t \in [0, n(D_0 - \delta)]$ and $t - D(t) \leq n(D_0 - \delta)$, yielding $V(t) \leq \gamma_1V(t) + \gamma_2c_n\|x_0\|^2$. A straightforward integration shows the existence of the claimed $c_{n+1} > 0$.

Let $t_0 \geq 1$ be such that $t_0(D_0 - \delta) > 1$. This yields $\sup_{t \geq 1} t_0\|z(t)\| \leq c_0\|x_0\|$. From (11), we infer the existence of a constant $c_0 > 0$, independent of $x_0$ and $D$, such that $\sup_{t \geq 1} t_0\|z(t)\| \leq c_0\|x_0\|$. From the definition of $\| \cdot \|_{\mathcal{W}}$, we obtain that $\|z(t)\| \leq \tilde{C}_0e^{-\gamma(t-1)}\|x_0\|$ for all $t \geq 0$ with $\tilde{C}_0 = e^{\gamma(1)}\max(c_0, c_n) > 0$. The conclusion follows from straightforward estimations of $u \equiv Kz$ and (10).
\end{proof}

\section{Applications}

Using the LMI solvers of MATLAB R2017b, we compare the application of the results of: (T1) Theorem 1 from Karafyllis and Krstic (2013); (T2) the LMI condition from Selivanov and Fridman (2016, Thm 2); (T3) Theorem 2. The examples are extracted from Li et al. (2014).

\subsection{Example 1}

With the matrices
\begin{equation*}
    A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & -3 \end{bmatrix},
\end{equation*}
the closed-loop poles are located in $-1 \pm j$. For $D_0 = 1$ s, we obtain (T1) $\delta = 0.0212$ ($\delta t = 0.0400$); (T2) with $\kappa = 0.2$, $\delta = 0.0563$; (T3) with $\kappa = 0.2$, $\delta = 0.0780$. 

\begin{thebibliography}{9}
\bibitem{Aarts1982}
Aarts, J. (1982).
\end{thebibliography}
Example 2. With the matrices

\[
A = \begin{bmatrix}
-2/3 & -1 & 5/3 \\
0 & -1 & 0 \\
1/3 & -1/3 & 2/3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
-2 \\
1
\end{bmatrix},
\]

\[K = \begin{bmatrix}
0.3572 & -0.4853 & 1.1281 \\
0.3925 & 0.5660 & 0.4235
\end{bmatrix}.
\]

the closed-loop poles are located in $-1 \pm j$ and $-2$. For $D_0 = 1$, we obtain (T1) $\delta = 0.0147 (\delta_B = 0.0391)$; (T2) with $\kappa = 0.2$, $\delta = 0.0591$; (T3) with $\kappa = 0.2$, $\delta = 0.0796$.

3. Extension to the feedback stabilization of a class of diagonal infinite-dimensional systems

We extend the results of Theorem 2 to the feedback stabilization of a class of diagonal (infinite-dimensional) boundary control systems exhibiting a finite number of unstable modes by means of a boundary control input that is subject to an uncertain and time-varying delay. In the sequel, $(\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})$ is a separable $K$-Hilbert space.

3.1. Problem setting

Let $D_0 > 0$ and $\delta \in (0, D_0)$ be given. We consider the abstract boundary control system (Curtain & Zwart, 2012):

\[
\begin{align*}
\frac{dX(t)}{dt} &= AX(t), \quad t \geq 0 \\
BX(t) &= \tilde{u}(t) \pm u(t - D(t)), \quad t \geq 0 \\
X(0) &= X_0
\end{align*}
\]  

(13)

with

- $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ a linear (unbounded) operator;
- $B : D(B) \subset \mathcal{H} \to \mathbb{K}^m$ with $D(A) \subset D(B)$ a linear boundary operator;
- $u : [-D_0 - \delta, +\infty) \to \mathbb{K}^m$ with $u|_{[-D_0 - \delta, 0]} = 0$ the boundary control;
- $D : \mathbb{R}_+ \to [D_0 - \delta, D_0 + \delta]$ a time-varying delay.

It is assumed that $(A, B)$ is a boundary control system:

1. The disturbance-free operator $A_0$, defined on the domain $D(A_0) \supset D(A) \cap \ker(B)$ by $A_0 \Delta A|_{D(A_0)}$ is the generator of a $C_0$-semigroup $S$ on $\mathcal{H}$;
2. There exists a bounded operator $B \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, called a lifting operator, such that $R(B) \subset D(A)$, $AB \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, and $BB = I_{\mathbb{K}^m}$.

where $\ker(B)$ stands for the kernel of $B$ and $R(B)$ denotes the range of $B$.

In the following developments, we assume that the boundary control system exhibits a diagonal structure:

Assumption 1. The disturbance-free operator $A_0$ is a Riesz spectral operator (Curtain & Zwart, 2012), i.e., is a linear and closed operator with simple eigenvalues $\lambda_n$ and corresponding eigenvectors $\phi_n \in D(A_0)$, $n \in \mathbb{N}^+$, that satisfy:

1. $\{\phi_n, n \in \mathbb{N}^+\}$ is a Riesz basis (Christensen, 2016);
   - $\text{span}_{n \in \mathbb{N}^+} \phi_n = \mathcal{H};$
   - there exist constants $m_0, M_0 \in \mathbb{R}_+$ such that for all $N \in \mathbb{N}^+$ and all $\alpha_1, \ldots, \alpha_N \in \mathbb{K}$,
     \[
     m_0 \sum_{n=1}^N |\alpha_n|^2 \leq \left\| \sum_{n=1}^N \alpha_n \phi_n \right\|_{\mathcal{H}}^2 \leq M_0 \sum_{n=1}^N |\alpha_n|^2.
     \]  

(14)

2. The closure of $\{\lambda_n, n \in \mathbb{N}^+\}$ is totally disconnected, i.e. for any distinct $a, b \in [\lambda_n, n \in \mathbb{N}^+]$, $[a, b] \not\subset [\lambda_n, n \in \mathbb{N}^+]$.

We also assume that the system presents a finite number of unstable modes and that the set composed of the real part of the stable modes does not accumulate at 0:

Assumption 2. There exist $N_0 \in \mathbb{N}^+$ and $\alpha \in \mathbb{R}_+$ such that $\text{Re} \lambda_n \leq -\alpha$ for all $n \geq N_0 + 1$.

As $\{\phi_n, n \in \mathbb{N}^+\}$ is a Riesz basis, we can introduce its biorthogonal sequence $\{\psi_n, n \in \mathbb{N}^+\}$, i.e., $\langle \phi_k, \psi_n \rangle_{\mathcal{H}} = \delta_{k,l}$ if and only if $k = l$. Then, we have for all $x \in \mathcal{H}$ the following series expansion: $x = \sum_{n \geq 1} \langle x, \psi_n \rangle_{\mathcal{H}} \phi_n$. As $A_0$ is a Riesz-spectral operator, then $\psi_n$ is an eigenvector of the adjoint operator $A_0^*$ associated with the eigenvalue $\lambda_n$.

3.2. Spectral decomposition and finite dimensional truncated model

Under the assumption that $\tilde{u} \in C^2([0, +\infty); \mathbb{K}^m)$ and $X_0 \in D(A)$ such that $\tilde{X}(0) = \tilde{u}(0) = u(0) - D(0) = 0$ (i.e., $X_0 \in D(A_0)$), there exists a unique classical solution $X \in C^0([R_+; D(A))] \cap C^1(R_+; \mathcal{H})$ of (13); see, e.g., Curtain and Zwart (2012, Th. 3.3.3).

Then,

\[
X(t) = \sum_{n \in \mathbb{N}^+} \langle X(t), \psi_n \rangle_{\mathcal{H}} \phi_n = \sum_{n \in \mathbb{N}^+} c_n(t) \phi_n,
\]

where $c_n(t) \equiv \langle X(t), \psi_n \rangle_{\mathcal{H}}$. We infer that $c_n \in C^1([R_+; \mathbb{K})$ and, from (13), we have for all $t \geq 0$ the following spectral decomposition (Lhachemi & Shorten, 2019):

\[
\begin{align*}
\dot{c}_n(t) &= \langle AX(t), \psi_n \rangle_{\mathcal{H}} \\
&= \langle A_0 (X(t) - \tilde{B} u(t)), \psi_n \rangle_{\mathcal{H}} + \langle A \tilde{B} u(t), \psi_n \rangle_{\mathcal{H}} \\
&= \langle X(t) - \tilde{B} u(t), A_0^* \psi_n \rangle_{\mathcal{H}} + \langle A \tilde{B} u(t), \psi_n \rangle_{\mathcal{H}} \\
&= \lambda_n c_n(t) - \lambda_n \langle \tilde{B} u(t), \psi_n \rangle_{\mathcal{H}} + \langle A \tilde{B} u(t), \psi_n \rangle_{\mathcal{H}},
\end{align*}
\]

(15)

where it has been used that $\mathcal{E} \langle X(t) - \tilde{B} u(t) \rangle = \tilde{u}(t) - \tilde{u}(t) = 0$, showing that $X(t) - \tilde{B} u(t) \in D(A) \cap \ker(\mathcal{E}) = D(A_0)$.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ be the canonical basis of $\mathbb{K}^m$. Introducing $B_{i,n,k} \equiv -\log(\rho(B_{i,k}) \psi_n \rangle_{\mathcal{H}} + \langle A \tilde{B} u(t), \psi_n \rangle_{\mathcal{H}}$, we obtain from (15) that the following linear ODE holds true for all $t \geq 0$

\[
\dot{Y}(t) = A_{n0} Y(t) + B_{n0} \tilde{u}(t - D(t)),
\]

(16)

where $A_{n0} = \text{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{K}^{N_0 \times N_0}$, $B_{n0} = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq m}$, and $Y(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_{N_0}(t) \end{bmatrix}$, $\begin{bmatrix} \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix} \in \mathbb{K}^{N_0}$.

(17)

Under the following assumption, we obtain the existence of a feedback gain $\mathbb{K}^{N_0 \times N_0}$ such that $A_{n0} + B_{n0} \tilde{K}$ is Hurwitz.

Assumption 3. $(A_{n0}, B_{n0})$ is stabilizable.

Then, we can employ the strategy presented in Section 2 to ensure the exponential feedback stabilization of the finite-dimensional truncated dynamics (16). The objective is now to assess that such a strategy ensures the stabilization of the full infinite-dimensional system.

Remark 2. In general, even for problems originally defined over the real field $\mathbb{K} = \mathbb{R}$, the spectral decomposition (16) might be complex-valued due to the incursion into the complex plan to define the eigenstructures $(\lambda_n, \phi_n)$ of the system; typical examples of such systems are strings and beams. Consequently, we need in the sequel the following complex-version of Theorem 2.
Corollary 2. In the context of Theorem 2 but with complex-valued A, B, K, and \( x_0 \), i.e., \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{m \times n}, K \in \mathbb{C}^{m \times n}, \) and \( x_0 \in \mathbb{C}^n \), the conclusions of Theorem 2 hold true with \( M = \mathcal{R}(A_1) \) and \( N = \mathcal{R}(e^{D_0}B) \). In this case, the matrices of (5) are such that \( P_1, Q \in \mathbb{S}_{++}^m \) and \( P_2, P_3 \in \mathbb{R}^{n \times n} \).

Proof. For \( z(t) \in \mathbb{C}^n \), we infer that (12) is equivalent to

\[
\dot{z}(t) = \mathcal{R}(A)z(t) + \mathcal{R}(e^{D_0}B)Kz(t) + \mathcal{R}(e^{D_0}B)Kz(t) - (D(t) - D_0)z(t)
\]

for the nominal case and is of class \( \mathcal{H} \). Therefore, the existence of a classical trajectory \( X(\cdot) \) and the control input \( u(\cdot) \) of the closed-loop dynamics (18a)-(18e) satisfy the robustness assessment of the control strategy with respect to uncertain and time-varying delays \( D(t) \).

3.4. Exponential stability of the closed-loop system

The exponential stability of the closed-loop system (18a)-(18e) in the nominal case \( D(t) = D_0 \) has been assessed in Lhachemi and Prieur (2019). The contribution of this paper relies on the following robustness assessment of the control strategy with respect to uncertainty and time-varying delays \( D(t) \).

Theorem 3. Let \((A, B)\) be an abstract boundary control system such that Assumptions 1, 2, and 3 hold true. There exist \( \delta \in (0, D_0) \) and \( \eta > 0 \) such that, for any given \( \delta_0 > 0 \), we have the existence of a constant \( C_1 > 0 \) such that, for any \( X_0 \in \mathcal{D}(A_0) \) and \( D \in C^1[0, \infty); \mathbb{R}^m \) with \( |D - D_0| \leq \delta \) and \( \sup_{t \in \mathbb{R}_+} |D(t)| \leq \delta \), the trajectory \( X(\cdot) \) and the control input \( u(\cdot) \) of the closed-loop dynamics (18a)-(18e) satisfy

\[
\|X(t)\|_{\mathcal{H}} + \|u(t)\| \leq C_1 e^{-\eta t} \delta_0 \text{ for all } t \geq 0.
\]

Furthermore, if \( \eta > 0 \) is such that \( \Theta(\delta, \kappa) \leq 0 \) is feasible, then the decay rate \( \eta \) can be taken as any element of \( [0, \kappa] \) if \( \alpha > \kappa \) or \( \eta \in (0, \alpha) \) if \( \alpha \leq \kappa \).

Proof. Since \( \delta \in (0, D_0) \) and \( \kappa > 0 \) such that \( \Theta(\delta, \kappa) \leq 0 \) is feasible (see Lemma 2). We introduce \( \eta \in [0, \kappa] \) if \( \alpha > \kappa \) or \( \eta \in (0, \alpha) \) if \( \alpha \leq \kappa \). Thus, we can select \( \alpha \in (0, 1) \) such that \( \alpha \delta \leq \alpha(1 - \epsilon) > \eta \). Let \( \delta_0 > 0 \) be arbitrarily given. Let \( X_0 \in \mathcal{D}(A_0) \) and \( D \in C^1[0, \infty); \mathbb{R}^m \) such that \( |D - D_0| \leq \delta \) and \( \sup_{t \in \mathbb{R}_+} |D(t)| \leq \delta \). Further, if \( \kappa > 0 \) is such that \( \Theta(\delta, \kappa) \leq 0 \) is feasible, then the decay rate \( \eta \) can be taken as any element of \( [0, \kappa] \) if \( \alpha > \kappa \) or \( \eta \in (0, \alpha) \) if \( \alpha \leq \kappa \).

From Lemma 3, we denote by \( X \in C^0(0, \infty); \mathcal{D}(A) \) of the unique classical solution of the closed-loop system (18a)-(18e) and \( u \in C^1[0, \infty); \mathbb{R}^m \) the associated control input. Thus, (16) holds true for all \( t \geq 0 \). Furthermore, as \( u = \psi K Z \) with \( Z \) given by (19) and \( u \in C^1[0, \infty); \mathbb{R}^m \) of the closed-loop system (18a)-(18e) admits a unique classical solution \( X \in C^0(0, \infty); \mathcal{D}(A) \) and \( \kappa > 0 \) independent of \( X_0 \) and \( D \). From (14) and (17), we have that \( \|X(0)\| \leq \|X_0\|_{\mathcal{H}} \sqrt{\mathcal{M}} \). This yields, along with \( \kappa > 0 \), \( \|Y(t)\| = \sup_{t \geq 0} |u(\cdot)| \) for all \( t \geq 0 \). In order to assess the exponential stability of the full infinite-dimensional system, we introduce for all \( t \geq 0 \)

\[
V(t) = \frac{1}{2} \sum_{k=1}^{N_0} \|X(t) - \bar{B}(t), \psi_k\|^2 \geq 0,
\]

which is such that \( V(t) \leq \|X(t) - \bar{B}(t)\|^2/2m_\kappa < +\infty \) and \( V \in C^1(0, \infty); \mathbb{R} \). The quantity \( V(t) \) is used to derive an upper bound of \( \|X(t)\|_{\mathcal{H}} \) as follows. Noting that

\[
\frac{1}{2} \sum_{k=1}^{N_0} \|X(t) - \bar{B}(t), \psi_k\|^2 \\
\leq \|X(t)\|^2 + \|\bar{B}(t)\|^2 \\
\leq \|Y(t)\|^2 + \frac{1}{m_\kappa} \|\bar{B}(t)\|^2
\]
we obtain that
\[
V(t) = \frac{1}{2} \sum_{k=1}^{N_0} |(X(t) - B\tilde{u}(t), \psi_k)|^2 - \frac{1}{2} \sum_{k=1}^{N_0} |(X(t) - B\tilde{u}(t), \psi_k)|^2 \\
\geq \frac{1}{2M_R} \|X(t) - B\tilde{u}(t)\|_H^2 - \|Y(t)\|^2 - \frac{1}{m_R} \|B\tilde{u}(t)\|_H^2.
\]

(14)

Using the triangular inequality, this yields for all \( t \geq 0 \),
\[
\|X(t)\|_H \leq \|B\tilde{u}(t)\|_H + \sqrt{2M_R \left( V(t) + \|Y(t)\|^2 + \frac{1}{m_R} \|B\tilde{u}(t)\|_H^2 \right)}.
\]

Noting that \( t - D(t) \geq t - D_0 - \delta \), we have \( \|\tilde{u}(t)\| = \|u(t - D(t))\| \leq C_1 e^{\|D(t)\|} \|X_0\|_H / \sqrt{m_R} \). As \( B \) is bounded and \( \|Y(t)\| \leq C_1 e^{\|D(t)\|} \|X_0\|_H / \sqrt{m_R} \), the proof will be complete if we can show the existence of \( \tilde{C}_1 > 0 \), independent of \( X_0 \) and \( D \), such that \( V(t) \leq \tilde{C}_1 e^{2\delta t} \|X_0\|_H^2 \). To do so, we compute for \( t \geq 0 \) the time derivative of \( V \) as follows:
\[
\dot{V}(t) = \sum_{k=1}^{N_0} \text{Re} \left\{ \left( \frac{dX}{dt}(t) - B\tilde{u}(t), \psi_k \right)_H \right\} \langle X(t) - B\tilde{u}(t), \psi_k \rangle_H \\
\geq \sum_{k=1}^{N_0} \text{Re} \left\{ \left| (A\tilde{u}(t), \psi_k)_H - (\tilde{B}(t), \psi_k)_H \right| \right\} \langle X(t) - B\tilde{u}(t), \psi_k \rangle_H \\
\leq -2\alpha V(t) \\
+ \sum_{k=1}^{N_0} \left( \left| (\tilde{A}\tilde{u}(t), \psi_k)_H \right|^2 + \left| (\tilde{B}(t), \psi_k)_H \right|^2 \right) \langle X(t) - B\tilde{u}(t), \psi_k \rangle_H \\
\leq \frac{1}{2\alpha} \sum_{k=1}^{N_0} \left( \left| (\tilde{A}\tilde{u}(t), \psi_k)_H \right|^2 + \left| (\tilde{B}(t), \psi_k)_H \right|^2 \right)
\]

(15)

For all \( t \geq D_0 + \delta + t_0 \), as \( t - D(t) \geq t_0 \) and thus \( \psi(t) - D(t) = 1 \), we have
\[
\dot{u}(t) = u(t - D(t)) = KZ(t - D(t)) = \sum_{i=1}^{m} \{K_i Z(t - D(t))\} e_i
\]


where \( K_i \) stands for ith line of \( K \). We deduce that
\[
\sum_{k=1}^{N_0} \left| (A\tilde{u}(t), \psi_k)_H \right|^2 \leq m \sum_{i=1}^{m} \left| (A\tilde{u}_i, \psi_k)_H \right|^2 |K_i Z(t - D(t))|^2 \\
\leq \frac{m}{m_R} \left( \sum_{i=1}^{m} \left| (A\tilde{u}_i, \psi_k)_H \right|^2 \right) \|K_i Z(t - D(t))\|^2.
\]

(14)

Similarly, for all \( t \geq t_1 \triangleq 2(D_0 + \delta) + t_0 \),
\[
\dot{u}(t - D(t)) = K\tilde{Z}(t - D(t)) \equiv \sum_{i=1}^{m} \{K_i Z(t - D(t))\} e_i \\
+ K\tilde{B}_0 K [Z(t - 2D(t)) - Z(t - D(t) - D_0)]
\]

(20)

with \( \tilde{B}_0 \triangleq e^{D_0 \delta} B_0 \). We deduce that
\[
\sum_{k=1}^{N_0} \left( \left| (\tilde{B}(t), \psi_k)_H \right|^2 \right) \leq \frac{2\beta m}{m_R} \left( \sum_{i=1}^{m} \left| (B\tilde{u}_i, \psi_k)_H \right|^2 \right) \|Z(t - D(t))\|^2 \\
+ \frac{2\beta m}{m_R} \left( \sum_{i=1}^{m} \left| (B\tilde{u}_i, \psi_k)_H \right|^2 \right) \|K_i A_i\|^2,
\]

where \( \beta \triangleq (1 + \delta)^2 \). Thus, introducing the constants \( k_1, k_2 \geq 0 \) defined by:
\[
k_1 = \frac{m}{2\epsilon a \alpha m_R} \left( \sum_{i=1}^{m} \left| (A\tilde{u}_i, \psi_k)_H \right|^2 + 2\beta \left| (B\tilde{u}_i, \psi_k)_H \right|^2 \right),
\]

\[
k_2 = \frac{\beta m}{\epsilon a \alpha} \sum_{i=1}^{m} \left| (B\tilde{u}_i, \psi_k)_H \right|^2,
\]

we obtain that, for all \( t \geq t_1 \),
\[
\dot{V}(t) \leq -2\alpha V(t) + \alpha(t) + \alpha(t) \geq 0 \text{ defined by:}
\]
\[
\alpha(t) = k_1 \|Z(t - D(t))\|^2 + k_2 \|Z(t - 2D(t)) - Z(t - D(t) - D_0)\|^2 \\
\leq k_1 e^{2\delta t} \|X_0\|_H^2,
\]

where \( k_1 = \tilde{C}_1 e^{2\delta t} (D(t) + \delta) \{k_1 + 2k_2 e^{2\delta t} (e^{2\delta t} - 1)\} / m_R \). Now, as \( V \) is of class \( e^{t} \) over \( \mathbb{R}_+ \), we obtain after integration that, for all \( t \geq t_1 \),
\[
V(t) \leq e^{-2\alpha(t-t_1)} V(t_1) + e^{-2\alpha(t-t_1)} \int_{t_1}^{t} e^{2\alpha(t-t')} \alpha(t) dt \\
\leq e^{-2\alpha(t-t_1)} V(t_1) + \frac{k_3}{2(\alpha - \eta)} e^{-2\alpha(t-t_1)} \|X_0\|_H^2
\]

where it has been used that \( \alpha > \eta \).

It remains to evaluate \( V(t) \) for \( 0 \leq t \leq t_1 \). From the exponential estimate of \( Z(t) \), we have that \( \|Z(t)\| \leq C_0 \|X_0\|_H / \sqrt{m_R} \) for all \( t \geq 0 \). From (20), we deduce the existence of a constant \( \tilde{C}_1 > 0 \), independent of \( X_0 \) and \( D \), such that \( \|Z(t)\| \leq \tilde{C}_1 \|X_0\|_H \) for all \( t \geq 0 \). Then, from (21), the facts that \( \sup_{t \in \mathbb{R}_+} |\dot{D}(t)| \leq \delta \) and \( \sup_{t \in \mathbb{R}_+} |\dot{\psi}(t)| < +\infty \), and \( V(0) \leq \|X_0\|_H^2 / (2m_R) \), we obtain the existence of a constant \( \tilde{C}_1 > 0 \), independent of \( X_0 \) and \( D \), such that \( V(t) \leq \tilde{C}_1 \|X_0\|_H^2 \) for all \( t \in [0, t_1] \). Consequently, we obtain that
\[
V(t) \leq \tilde{C}_1 e^{2\delta t} \|X_0\|_H^2 \text{ for all } t \geq 0 \text{ with } \tilde{C}_1 = \tilde{C}_1 e^{2\delta t} + \frac{k_3}{2(\alpha - \eta)} \|X_0\|_H^2.
\]

This completes the proof. \( \square \)

4. Illustrative example

We consider the following one-dimensional reaction–diffusion equation on \( (0, L) \) with a delayed Dirichlet boundary control:
\[
\begin{align*}
Y_t(x, t) &= aY(x, t) + cY(x, t) \quad (x, t) \in \mathbb{R}_+ \times (0, L) \\
\tilde{Y}(y, 0) &= u(t - D(t)), \quad t > 0
\end{align*}
\]

with \( a, c > 0 \), \( y(x, t) \in \mathbb{R} \), and \( u(t) \in \mathbb{R}^2 \). Introducing the \( \mathbb{R} \)-Hilbert space \( H = L^2(0, L) \) with \( \langle f, g \rangle_H = \int_0^L f g dx \), it is well-known that the above reaction–diffusion equation can be
written under the form of the abstract boundary control system (13) with \( X(t) = y(t, \cdot) \in H, A^f = a^f + cf \) on the domain \( D(A) = A^f(0, L) \), and the boundary operator \( B_0f = (f(0), f(L)) \) on the domain \( D(B) = H^1(0, L) \). An example of lifting operator \( B \) associated with \((A, B)\) is given for any \((u_1, u_2) \in \mathbb{R}^2 \) by \([B(u_1, u_2)](x) = u_1 + (u_2 - u_1)x/L \) with \( x \in (0, L) \). It is well-known that the disturbance-free operator \( A_0 \) is a Riesz-spectral operator that generates a \( C_0 \)-semigroup with \( \lambda_n = c - an \pi^2/L^2 \) and \( \phi_n(x) = \sqrt{2} \pi n \sin(n \pi x/L), n \geq 1 \). Then, the boundary control system \((A, B)\) satisfies Assumptions 1 and 2. Furthermore, straightforward computations show that \( b_{n,1} = an \pi \sqrt{2}/L^2 \) and \( b_{n,2} = (-1)^{n-1} \sqrt{2}/L^2 \). As the eigenvalues are simple and \( b_{n,k} \neq 0 \) for all \( n \geq 1 \) and \( k \in \{1, 2\} \), we obtain from the Kalman condition that \((A_{N_0}, B_{N_0})\) is controllable, fulfilling Assumption 3. Thus, one can compute a feedback gain \( K \in \mathbb{R}^{m \times N_0} \) such that \( A_{N_0} + B_{N_0}K \) is Hurwitz and then apply the result of Theorem 3 for ensuring the exponential stability of the closed-loop system.

For numerical computations, we set \( a = c = 0.5 \) and \( L = 2 \pi \). In this configuration, we have two unstable modes \( \lambda_1 = 0.375 \) and \( \lambda_2 = 0 \) while the two first stable modes are such that \( \lambda_3 = -0.625 \) and \( \lambda_4 = -1.5 \). Setting \( N_0 = 3 \), the feedback gain \( K \in \mathbb{R}^{3 \times 3} \) is computed to place the poles of the closed-loop truncated model \( A_N = A_{N_0} + B_{N_0}K \) at the points \(-0.75, -1, -1.25\). Over the range \( D_0 \in (0, 5) \), Fig. 1 depicts: (1) the estimate \( \hat{\delta} \) (4) on the admissible values of \( \delta > 0 \) given by Theorem 1 taken from Karafyllis and Krstic (2013); (2) with decay rate \( \kappa = 0.2 \), the admissible values of \( \delta > 0 \) based on Selivanov and Fridman (2016, Thm 2) and Theorem 2. For the studied example, the values of \( \delta \) provided by Theorem 2 are significantly less conservative.

For numerical simulations, we set the nominal value of the delay to \( D_0 = 1 s \). In this case, Theorem 3 ensures the exponential stability of the closed-loop system with decay rate \( \kappa = 0.2 \) for values of \( \delta \) up to \( \delta = 0.260 \). We set the initial condition \( x(0) = -x(2L/3 - x)L - x \) and the time-varying delay \( D(t) = 1 + 0.25 \sin(3 \pi t + \pi/4) \) is such that \( c \leq 2 c^2 \) and is such that \( |D(t) - D_0| \leq 0.25 \leq 0.260 \) and \( \hat{D}(t) \leq 0.75 \pi < + \infty \) for all \( t \geq 0 \). The transition time \( t_0 \) is taken as \( t_0 = 0.5 s \) while the transition signal \( \psi[\hat{D}(t)] \) is selected as the restriction over \( [0, t_0] \) of the unique quintic polynomial function \( f \) satisfying \( f(0) = f'(0) = f''(0) = f''(t_0) = f'''(t_0) = 0 \) and \( f''(t_0) = 1 \). The employed numerical scheme relies on the discretization of the reaction–diffusion equation using its first 10 modes. The time domain evolution of the closed-loop system is depicted in Figs. 2–3. As expected from Theorem 3, both the system state and the control input converge to zero.

5. Conclusion

This paper discussed first the use of predictor feedback for the stabilization of finite-dimensional LTI systems in the presence of an uncertain time-varying delay in the control input. By means of a Lyapunov–Krasovskii functional, it has been derived an LMI-based sufficient condition ensuring the exponential stability of the closed-loop system for small enough variations of the time-varying delay around its nominal value. Then, this result has been extended to the feedback stabilization of a class of diagonal infinite-dimensional boundary control systems.

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