Automatica 86 (2017) 29–37

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Stochastic stability of Markov jump hyperbolic systems with application to traffic flow control[☆]

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ARTICLE INFO

Article history:
Received 8 June 2016
Received in revised form 18 May 2017
Accepted 29 July 2017

Keywords:
Stochastic hyperbolic conservation laws
Markov process
Stochastic stability
Lyapunov function
Traffic flow control

ABSTRACT

In this paper, we investigate the stochastic stability of linear hyperbolic conservation laws governed by a finite-state Markov chain. Both system matrices and boundary conditions are subject to the Markov switching. The existence and uniqueness of weak solutions are developed for the stochastic hyperbolic initial–boundary value problem. By means of Lyapunov techniques some sufficient conditions are obtained by seeking the balance between the boundary condition and the transition probability of the Markov process. Particularly, boundary feedback control of a stochastic traffic flow model is developed for the freeway transportation system by integrating the on-ramp metering with the speed limit control.

1. Introduction

Many physical or engineering processes may be represented by the hyperbolic partial differential equations (PDEs) of conservation laws in one space dimension, such as Saint-Venant equation for open channels (de Halleux, Prieur, Coron, d'Andréa Novel, & Bastin, 2003), Euler equation for gas pipes (Gugat, Dick, & Leugering, 2011), and Aw–Rascle equation for road traffic (Aw & Rascle, 2000). In such systems, the system matrices and the boundary conditions can both be subject to abrupt changes in their structures and parameters induced by the external causes or the internal mechanism. For example, in the freeway transportation systems, it can be the phase transition of traffic modes (Colombo, 2003), or the random flux at the boundaries (Haut, Bastin, Coron, & d'Andréa Novel, 2007). In such situations, it is more realistic to model the dynamic behaviors of these processes with switched hyperbolic systems.

Many results have been made for boundary stability of switched hyperbolic systems. In Amin, Hante, and Bayen (2012), the exponential stability is given under arbitrary switching using the propagation of solutions along the characteristics. In Prieur, Girard, and Witrant (2014), using Lyapunov techniques some sufficient conditions are obtained for the exponential stability uniformly. Switching boundary control for semilinear hyperbolic balance equations is considered in Hante, Leugering, and Seidman (2009). In Lamare, Girard, and Prieur (2015), stabilizing switching controllers are developed based on the steepest decent selection of the Lyapunov function.

In this paper, we consider a class of switched hyperbolic systems, named the Markov jump linear hyperbolic (MJLH) systems, in which mode switching is governed by a Markov chain and all modes are linear hyperbolic conservation laws. The boundary stabilization for hyperbolic systems (Coron, d'Andréa Novel, & Bastin, 2007; Li, 1994) and the stochastic stability for the Markov jump linear (MJL) systems of continuous-time (Costa, Fragoso, & Todorov, 2013) or discrete-time cases (Boukas, 2005) have been studied for many years independently. The main contribution of this work is that the boundary stochastic stability for the MJLH systems is firstly obtained by means of Lyapunov techniques. The matrix inequality condition is based on the balance between the boundary condition of the hyperbolic conservation laws and the transition probability of the Markov process.

A second contribution of our work is the application to the boundary control of freeway traffic. Due to the existence of a large number of uncertainties, such as demand variability and capacity decrease, the local freeway traffic may randomly lie in the free-flow mode or in the congestion mode (Boel & Mihaylova, 2006; Sumalee, Zhong, Pan, & Szeto, 2011). Then we develop a two-mode

☆ This work is supported by the National Natural Science Foundation of China (NSFC, Grant Nos. 61374076and 61533002) and the International Cooperation and Exchange Program of NSFC (Grant No. 61111130119). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Xiaobo Tan under the direction of Editor Miroslav Krstic.

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http://dx.doi.org/10.1016/j.automatica.2017.08.007
0005-1098/

MJLH model to represent the quasilinear Aw–Rascle equation and design boundary feedback strategies by integrating the on-ramp metering with the speed limiting control. Theoretical contributions guarantee the stochastic exponential convergence of the MJLH traffic flow model, even with different transition probabilities of the Markov chain.

This paper is organized as follows. The class of MJLH systems and the wellposedness of weak solutions are given in Section 2. In Section 3, the main result on the sufficient conditions of the exponentially mean-square stable are derived for MJLH systems. Numerical computation of the conditions is discussed in Section 4. Finally, in Section 5, as a matter of illustration, an application of boundary feedback control of freeway traffic based on the MJLH traffic flow model is presented.

Notation: $\mathbb{R}_+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ are the sets of non-negative reals, n-order vectors and matrices, respectively. The set of diagonal positive matrices in $\mathbb{R}^{n\times n}$ is denoted by $\mathcal{D}_+^n$. Given a matrix A, the transpose matrix is denoted as $A^\top$, $\lambda_{max}(A)$, $\rho(A)$ are the largest real parts of all eigenvalues and the spectral radius of A. $A < (\leq) B$ denotes $B - A$ is a positive definite (semi-definite) matrix. Given two real values t_1 and t_2, $t_1 \wedge t_2$ denotes the minimal value between t_1 and t_2. The Euclidean norm in $\mathbb{R}^n$ is denoted by $|\cdot|$ and the associated matrix norm is $\|\cdot\|$. Given a function $g : [0, 1] \to \mathbb{R}^n$, its L^2-norm is $\|g\|_{L^2(0,1)} = \sqrt{\int_0^1 |g(x)|^2 dx}$. We call $L^2(0, 1)$ the space of all measurable functions $g(x)$ for which $\|g\|_{L^2(0,1)} < \infty$.

2. Markov jump linear hyperbolic systems

Let $(\Omega, \mathcal{F}, \Pr)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_t; t \in \mathbb{R}_+\}$ satisfying the usual hypotheses, that is, a right-continuous filtration augmented by all null sets in the Pr-completion of $\mathcal{F}$.

We consider a homogeneous Markov process $\{\sigma(t); t \in \mathbb{R}_+\}$ adapted to the filtration $\{\mathcal{F}_t; t \in \mathbb{R}_+\}$, with right-continuous trajectories and taking values on the set $\mathcal{S} = \{1, 2, \ldots, N\}$, where N is a positive integer number. The infinitesimal generator $\Pi \in \mathbb{R}^{N\times N}$ of Markov process $\sigma(t)$ is given by

$$\Pr\{\sigma(t+\Delta t) = j|\sigma(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if} \quad i \neq j \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{if} \quad i = j \end{cases} \tag{1}$$

where $\Delta t > 0$ is constant (it is seen as a small time increment) and $o(\cdot)$ is a function satisfying $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} = 0$. Here $\pi_{ij} \geq 0$, for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t + \Delta t$, while

$$\pi_{ii} = -\sum_{j=1, j\neq i}^{N} \pi_{ij}.$$

Let $\{\tau_k; k = 0, 1, \ldots\}$ be the successive sojourn times between jumps, then $t_k = \sum_{l=0}^{k-1}\tau_l$, for $k = 1, 2, \ldots$, be the waiting time for the kth jump with $t_0 = 0$.

Stating in mode $\sigma(0) = i$, the process sojourns there for a duration of time that is exponentially distributed with parameter $-\pi_{ii}$. The process then jumps to mode $j \neq i$ with probability $-\frac{\pi_{ij}}{\pi_{ii}}$, and the sojourn time in mode j is exponentially distributed with parameter $-\pi_{jj}$, and so on. We further assume that the Markov process is irreducible. Under this condition, $\sigma(t)$ has a unique stationary probability distribution $\gamma = [\gamma_1 \ldots \gamma_N]^\top$, which can be determined by solving the following linear equation $\gamma^\top\Pi = 0$ subject to $\sum_{j=1}^N \gamma_j = 1$ and $\gamma_j > 0$, for all $j \in \mathcal{S}$ (Costa et al., 2013 Definition 2.9).

We consider the following Markov jump linear hyperbolic (MJLH for short) conservation laws of the form

$$\partial_t\xi(x, t) + \Lambda_{\sigma(t)}\partial_x\xi(x, t) = 0, \tag{2}$$

where $t \in \mathbb{R}_+$, $x \in [0, 1]$, $\xi : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}^n$ is the system state, and the Markov process $\sigma(t) : \mathbb{R}_+ \to \mathcal{S}$ is a stochastic switching signal deciding the current operation mode. For all $i \in \mathcal{S}$, the system matrix $\Lambda_i \in \mathbb{R}^{n\times n}$ is a diagonal matrix with non-zero diagonal entries such that

$$\Lambda_i = \mathrm{diag}\{\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i\},$$

with $\lambda_j^i < 0$ for $j \in \{1, \ldots, m_i\}$ and $\lambda_j^i > 0$ for the other $j \in \{m_i + 1, \ldots, n\}$.

According to the sign of each characteristic velocity λ_j^i, $j = \{1, \ldots, n\}$, $i \in \mathcal{S}$, we introduce the notation $\xi_-^i(\cdot) = [\xi_1(\cdot), \ldots, \xi_{m_i}(\cdot)]^\top$, $\xi_+^i(\cdot) = [\xi_{m_i+1}(\cdot), \ldots, \xi_n(\cdot)]^\top$, and thus $\xi = [\xi_-^i, \xi_+^i]^\top$, for all $i \in \mathcal{S}$.

For MJLH system (2), the boundary condition also is a stochastic process, corresponding to the Markov chain $\sigma(t)$, written as

$$\begin{bmatrix} \xi_-^{\sigma(t)}(1, t) \\ \xi_+^{\sigma(t)}(0, t) \end{bmatrix} = G_{\sigma(t)} \begin{bmatrix} \xi_-^{\sigma(t)}(0, t) \\ \xi_+^{\sigma(t)}(1, t) \end{bmatrix}, \tag{3}$$

where G_i is a matrix in $\mathbb{R}^{n\times n}$, $i \in \mathcal{S}$. Let us introduce the matrices G_{--}^i in $\mathbb{R}^{m_i\times m_i}$, G_{-+}^i in $\mathbb{R}^{m_i\times(n-m_i)}$, G_{+-}^i in $\mathbb{R}^{(n-m_i)\times m_i}$, and G_{++}^i in $\mathbb{R}^{(n-m_i)\times(n-m_i)}$ such that $G_i = \begin{bmatrix} G_{--}^i & G_{-+}^i \\ G_{+-}^i & G_{++}^i \end{bmatrix}$.

We consider the initial condition given by

$$\xi(x, 0) = \xi^0(x), \ x \in (0, 1), \tag{4}$$

for a given function $\xi^0(\cdot) \in L^2(0, 1)$ and a initial operation mode $\sigma(0) \in \mathcal{S}$.

For each mode $i \in \mathcal{S}$, as respective hyperbolic equation (2)–(4) holds a sojourn for a duration of time $\{\tau_k; k = 0, 1, \ldots\}$, the existence and uniqueness of solution in the set $C^0([0, \infty), H^1(0, 1)) \cap C^1([0, \infty), L^2(0, 1))$ with initial condition in $L^2(0, 1)$ is quite classical, see e.g. Bastin and Coron (2016, Theorem A.4). Recently, the notion of solutions for an initial–boundary value problem of switched hyperbolic systems has been developed within the usual Lebesgue almost everywhere equivalence class, see e.g. Amin et al. (2012), and Prieur et al. (2014, Proposition 3.1).

We now provide an existence and uniqueness result for the solutions of the MJLH system (2)–(4).

Proposition 1. *The MJLH system (2)–(3) admits a unique solution $\xi = \xi(\cdot, t)$, $t \in \mathbb{R}_+$, such that $\mathrm{E}\left\{\|\xi(\cdot, t)\|_{L^2(0,1)}\right\} < \infty$, for any initial condition $\xi^0 \in L^2(0, 1)$ and any initial operation mode $\sigma(0) \in \mathcal{S}$, where $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation.*

Proof. Recall that almost every sample path of stochastic process $\sigma(t), t \geq 0$, is a right-continuous step function with a finite number of jumps in any finite time interval. Then there exists a sequence $\{t_k; k = 0, 1, \ldots\}$ of stopping times such that $t_0 = 0$, $\lim_{k\to\infty} t_k = \infty$, and $\sigma(t) = \sigma(t_k)$ on $t_k \leq t < t_{k+1}$ for any $k \geq 0$.

We then build iteratively the solution between successive stopping times. Let $T \in \mathbb{R}_+$ be arbitrary, we first consider the MJLH system (2)–(3) on the time interval $t \in [0, t_1 \wedge T]$ which becomes

$$\partial_t\xi(x, t) + \Lambda_{\sigma(0)}\partial_x\xi(x, t) = 0, \tag{5}$$

for all $x \in (0, 1)$ with the boundary condition of the form

$$\begin{bmatrix} \xi_-^{\sigma(0)}(1, t) \\ \xi_+^{\sigma(0)}(0, t) \end{bmatrix} = G_{\sigma(0)} \begin{bmatrix} \xi_-^{\sigma(0)}(0, t) \\ \xi_+^{\sigma(0)}(1, t) \end{bmatrix}, \tag{6}$$

and the initial condition $\xi^0 \in L^2(0, 1)$. For any initial mode $\sigma(0) \in \mathcal{S}$, by Bastin and Coron (2016, Theorem A.4), the initial–boundary

value problem (5)–(6) has a unique solution $\xi(\cdot,t) \in L^2(0,1)$, which satisfies

$$\|\xi(\cdot,t)\|_{L^2(0,1)} \le C_1\|\xi^0(\cdot)\|_{L^2(0,1)}, \tag{7}$$

for all $t \in [0, t_1 \wedge T]$ with $C_1 > 0$, then almost surely we have

$$\mathrm{E}\left\{\|\xi(\cdot, t_1 \wedge T)\|_{L^2(0,1)}\right\} < \infty.$$

Setting $\xi^1(\cdot) = \xi(\cdot, \tau_1 \wedge T)$, we next consider the MJLH system (2)–(3) on the time interval $t \in [t_1 \wedge T, t_2 \wedge T]$, which becomes

$$\partial_t \xi(x,t) + \Lambda_{\sigma(t_1)}\partial_x \xi(x,t) = 0, \tag{8}$$

for all $x \in (0,1)$, with the corresponding boundary condition

$$\begin{bmatrix}\xi_-^{\sigma(t_1)}(1,t)\\ \xi_+^{\sigma(t_1)}(0,t)\end{bmatrix} = G_{\sigma(t_1)}\begin{bmatrix}\xi_-^{\sigma(t_1)}(0,t)\\ \xi_+^{\sigma(t_1)}(1,t)\end{bmatrix}, \tag{9}$$

and the redefined initial condition $\xi^1 \in L^2(0,1)$. Again applying Bastin and Coron (2016, Theorem A.4), for the initial–boundary value problem (8)–(9) of the mode $\sigma(t_1) \in \mathcal{S}$, as $t \in [t_1 \wedge T, t_2 \wedge T]$, there also exists a unique classical solution $\xi(\cdot,t) \in L^2(0,1)$, which satisfies

$$\|\xi(\cdot,t)\|_{L^2(0,1)} \le C_2\|\xi^1(\cdot)\|_{L^2(0,1)}, \tag{10}$$

with $C_2 > 0$, then it holds $\mathrm{E}\left\{\|\xi(\cdot, t_2 \wedge T)\|_{L^2(0,1)}\right\} < \infty$. Repeating above procedure, we see that the MJLH system (2)–(3) has a unique solution $\xi(\cdot,t)$ for any $t \in [0,T]$, and satisfies $\mathrm{E}\left\{\|\xi(\cdot,T)\|_{L^2(0,1)}\right\} < \infty$. Since time T is arbitrary, this concludes the proof of Proposition 1. □

3. Stochastic stability of Markov jump linear hyperbolic systems

Stochastic stability is an important issue in the analysis of stochastic systems. For the MJL systems, various definitions have been introduced in Costa et al. (2013) and Fang and Loparo (2002).

We next start this section by defining the exponential mean-square stability for MJLH systems.

Definition 1. System (2)–(4) is said to be exponentially mean-square stable, if there exist $\nu > 0$ and $C > 0$, such that any solution $\xi(\cdot,t)$ to (2)–(4) satisfies

$$\mathrm{E}\left\{\|\xi(\cdot,t)\|^2_{L^2(0,1)}\right\} \le Ce^{-\nu t}\|\xi^0(\cdot)\|^2_{L^2(0,1)}, \tag{11}$$

for all $t \in \mathbb{R}_+$, any initial condition $\xi^0(\cdot) \in L^2(0,1)$ and any initial operation mode $\sigma(0) \in \mathcal{S}$.

A sufficient condition for the exponential mean-square stability of MJLH systems is obtained using the Lyapunov function method. To do that, let us consider the following candidate stochastic Lyapunov function for the MJLH system (2)–(4)

$$V(\xi,\sigma(t)) = \int_0^1 \xi(x,t)^\top P(x,\sigma(t))\xi(x,t)dx, \tag{12}$$

with $P(x,\sigma(t) = i) = P_i(x)$, $i \in \mathcal{S}$. We define $P_i(x) = \mathrm{diag}\left\{e^{2\mu_i x}P_i^-, e^{-2\mu_i x}P_i^+\right\}$, $x \in [0,1]$, where $\mu_i \in \mathbb{R}$, $P_i^- \in \mathcal{D}_+^{m_i\times m_i}$ and $P_i^+ \in \mathcal{D}_+^{(n-m_i)\times(n-m_i)}$ are diagonal positive definite matrices with corresponding dimensions.

Definition 2. The infinitesimal generator of the solution process $(\xi(\cdot,t),\sigma(t))$ of the MJLH system (2)–(4), acting on the Lyapunov function $V(\xi,\sigma(t))$ at the point $(t,\xi(\cdot,t),\sigma(t) = i)$, $i \in \mathcal{S}$, is defined by, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}&\mathcal{L}V(\xi(\cdot,t),\sigma(t)=i)\\ &\triangleq \lim_{\Delta t\to 0^+}\frac{1}{\Delta t}[\mathrm{E}\{V(\xi(\cdot,t+\Delta t),\sigma(t+\Delta t))|\\ &\qquad \xi(\cdot,t),\sigma(t)=i\} - V(\xi(\cdot,t),\sigma(t)=i)].\end{aligned} \tag{13}$$

As the point t is clear in the context, we denote $V(\xi,i) = V(\xi,\sigma(t)=i)$ for simplicity.

Lemma 1. *Given the stochastic Lyapunov function (12) and the MJLH system (2)–(4), the infinitesimal generator of Lyapunov function defined in (13), satisfies*

$$\mathcal{L}V(\xi,\sigma(t)=i) = \frac{d}{dt}V(\xi,i) + \sum_{j=1}^N \pi_{ij}V(\xi,j), \tag{14}$$

where derivative $\frac{d}{dt}V(\xi,i)$ stands for $\frac{\partial}{\partial\xi}V(\xi,i)\partial_t\xi$ at the point $(t,\xi(\cdot,t),\sigma(t)=i)$ for any solution to (2)–(4).

Proof. The weighting matrix $P(x,\sigma(t))$ of the Lyapunov function (12) is also a Markov process and is equipped with the transition probabilities

$$\begin{aligned}\Pr\{P(\cdot,\sigma(t+\Delta t)) &= P_j(\cdot)|\xi(\cdot,t),\sigma(t)=i\}\\ &= \pi_{ij}\Delta t + o(\Delta t), \qquad \text{if } i \ne j\end{aligned} \tag{15}$$

$$\begin{aligned}\Pr\{P(\cdot,\sigma(t+\Delta t)) &= P_i(\cdot)|\xi(\cdot,t),\sigma(t)=i\}\\ &= 1 + \pi_{ii}\Delta t + o(\Delta t), \ \text{if } i = j\end{aligned} \tag{16}$$

It follows from (1) and (2) that

$$\begin{aligned}\Pr\{\xi(\cdot,t+\Delta t) &= \xi(\cdot,t) - \Lambda_j\partial_x\xi(\cdot,t)\Delta t + o(\Delta t)\\ &\quad |\xi(\cdot,t),\sigma(t)=i\}\\ &= \pi_{ij}\Delta t + o(\Delta t), \ \text{if } i \ne j\end{aligned} \tag{17}$$

$$\begin{aligned}\Pr\{\xi(\cdot,t+\Delta t) &= \xi(\cdot,t) - \Lambda_i\partial_x\xi(\cdot,t)\Delta t + o(\Delta t)\\ &\quad |\xi(\cdot,t),\sigma(t)=i\}\\ &= 1 + \pi_{ii}\Delta t + o(\Delta t), \ \text{if } i = j\end{aligned} \tag{18}$$

Using the properties of the conditional expectation, we have

$$\begin{aligned}&\mathrm{E}\{V(\xi(\cdot,t+\Delta t),\sigma(t+\Delta t))|\xi(\cdot,t),\sigma(t)=i\}\\ &= [1+\pi_{ii}\Delta t]V(\xi(\cdot,t+\Delta t),i)\\ &\qquad + \sum_{j=1,j\ne i}^N [\pi_{ij}\Delta t]V(\xi(\cdot,t+\Delta t),j) + o(\Delta t)\\ &= V(\xi(\cdot,t+\Delta t),i)\\ &\qquad + \sum_{j=1}^N [\pi_{ij}\Delta t]V(\xi(\cdot,t+\Delta t),j) + o(\Delta t).\end{aligned} \tag{19}$$

Hence, in mode $i \in \mathcal{S}$, when the system state is changing from $\xi(\cdot,t)$ to $\xi(\cdot,t+\Delta t)$, we have

$$\begin{aligned}&V(\xi(\cdot,t+\Delta t),i)\\ &= \int_0^1 \xi^\top(\cdot,t+\Delta t)P_i(\cdot)\xi(\cdot,t+\Delta t)dx\\ &= V(\xi(\cdot,t),i) - 2\Delta t\int_0^1 \xi^\top(\cdot,t)\Lambda_i P_i(\cdot)\partial_x\xi(\cdot,t)dx\\ &\qquad + o(\Delta t)\\ &= V(\xi(\cdot,t),i) + \Delta t\int_0^1 [\partial_t\xi^\top(\cdot,t)P_i(\cdot)\xi(\cdot,t)\\ &\qquad + \xi^\top(\cdot,t)P_i(\cdot)\partial_t\xi(\cdot,t)]dx + o(\Delta t)\\ &= V(\xi(\cdot,t),i) + \Delta t\frac{d}{dt}V(\xi(\cdot,t),i) + o(\Delta t).\end{aligned} \tag{20}$$

From the definition of function $\mathcal{L}V$ (13), it follows that

$$\begin{aligned}
&\mathcal{L}V(\xi(\cdot,t),\sigma(t)=i)\\
&=\lim_{\Delta t\to 0^+}\frac{1}{\Delta t}\Bigg\{V(\xi(\cdot,t+\Delta t),i)+\sum_{j=1}^{N}[\pi_{ij}\Delta t]\\
&\qquad [V(\xi(\cdot,t+\Delta t),j)-V(\xi(\cdot,t),i)]+o(\Delta t)\Bigg\}\\
&=\lim_{\Delta t\to 0^+}\frac{1}{\Delta t}[V(\xi(\cdot,t+\Delta t),i)-V(\xi(\cdot,t),i)]\\
&\qquad+\sum_{j=1}^{N}\pi_{ij}V(\xi,j)\\
&=\frac{d}{dt}V(\xi,i)+\sum_{j=1}^{N}\pi_{ij}V(\xi,j).
\end{aligned}$$

This concludes the proof of Lemma 1. □

The following Theorem 1 captures the idea that seeking a balance, between the boundary conditions of hyperbolic modes and the transition probability of the Markov process, is essential for the exponential mean-square stability of MJLH systems. Let $|\Lambda_i|=\mathrm{diag}\{|\lambda_1^i|,\dots,|\lambda_n^i|\}$, $i\in\mathcal{S}$, and the matrices

$$G_{i-}=\begin{bmatrix}G_{--}^i & G_{-+}^i\\ 0 & I_{n-m_i}\end{bmatrix},\quad G_{i+}=\begin{bmatrix}I_{m_i} & 0\\ G_{+-}^i & G_{++}^i\end{bmatrix}.$$

Theorem 1. *Let us assume that there exist $\nu>0$, $\mu_i\in\mathbb{R}$, $i\in\mathcal{S}$, and diagonal positive definite matrices $P_i^-\in\mathcal{D}_+^{m_i\times m_i}$ and $P_i^+\in\mathcal{D}_+^{(n-m_i)\times(n-m_i)}$ such that $P_i(x)=\mathrm{diag}\left\{e^{2\mu_i x}P_i^-,e^{-2\mu_i x}P_i^+\right\}$, defined for each $x\in[0,1]$, satisfies the following matrix inequalities:*

$$-2\mu_i|\Lambda_i|P_i(x)+\sum_{j=1}^{N}\pi_{ij}P_j(x)\le-\nu P_i(x),\tag{21}$$

$$G_{i+}^\top\Lambda_iP_i(0)G_{i+}-G_{i-}^\top\Lambda_iP_i(1)G_{i-}\le 0.\tag{22}$$

Then, there exists $C>0$ such that (11) holds and the MJLH system (2)–(4) is exponentially mean-square stable.

Proof. Let us consider the Lyapunov function candidate $V(\xi,\sigma(t))$ given by (12) and assume that the diagonal positive matrix $P_i(x)$ satisfies the inequalities (21) and (22), for each $i\in\mathcal{S}$. Given any $t\in\mathbb{R}_+$ with the Markov process $\sigma(t)=i$, computing the time-derivative of $V(\xi,i)$ along the solutions to (2) yields the following: $\frac{d}{dt}V(\xi,i)=\int_0^1[\xi^\top P_i(x)\partial_t\xi+\partial_t\xi^\top P_i(x)\xi]dx=-\int_0^1[\xi^\top P_i(x)\Lambda_i\partial_x\xi+\partial_x\xi^\top\Lambda_iP_i(x)\xi]dx$. Using an integration by parts, and because $\partial_xP_i(x)\Lambda_i=-2\mu_iP_i(x)|\Lambda_i|$, we obtain

$$\begin{aligned}
&\frac{d}{dt}V(\xi,i)\\
&=-\left[\xi^\top P_i(x)\Lambda_i\xi\right]_0^1-\int_0^1 2\mu_i\xi^\top|\Lambda_i|P_i(x)\xi\,dx\\
&=\xi^\top(0,t)P_i(0)\Lambda_i\xi(0,t)-\xi^\top(1,t)P_i(1)\Lambda_i\xi(1,t)\\
&\qquad-2\int_0^1\xi^\top\mu_i|\Lambda_i|P_i(x)\xi\,dx\\
&=\partial_tV_1(\xi,i)+\partial_tV_2(\xi,i),
\end{aligned}\tag{23}$$

with

$$\begin{aligned}
\partial_tV_1(\xi,i)&\triangleq\xi^\top(0,t)P_i(0)\Lambda_i\xi(0,t)\\
&\qquad-\xi^\top(1,t)P_i(1)\Lambda_i\xi(1,t),\\
\partial_tV_2(\xi,i)&\triangleq-2\int_0^1\xi^\top\mu_i|\Lambda_i|P_i(x)\xi\,dx.
\end{aligned}$$

Under the boundary condition (3) for mode i, we have

$$\begin{aligned}
\partial_tV_1(\xi,i)&=\begin{bmatrix}\xi_-^i(0,t)\\ \xi_+^i(1,t)\end{bmatrix}^\top\Bigg(\begin{bmatrix}I_{m_i} & 0\\ G_{+-}^i & G_{++}^i\end{bmatrix}^\top\Lambda_iP_i(0)\\
&\qquad\begin{bmatrix}I_{m_i} & 0\\ G_{+-}^i & G_{++}^i\end{bmatrix}-\begin{bmatrix}G_{--}^i & G_{-+}^i\\ 0 & I_{n-m_i}\end{bmatrix}^\top\Lambda_iP_i(1)\\
&\qquad\begin{bmatrix}G_{--}^i & G_{-+}^i\\ 0 & I_{n-m_i}\end{bmatrix}\Bigg)\begin{bmatrix}\xi_-^i(0,t)\\ \xi_+^i(1,t)\end{bmatrix}\\
&=\begin{bmatrix}\xi_-^i(0,t)\\ \xi_+^i(1,t)\end{bmatrix}^\top\Big(G_{i+}^\top\Lambda_iP_i(0)G_{i+}\\
&\qquad-G_{i-}^\top\Lambda_iP_i(1)G_{i-}\Big)\begin{bmatrix}\xi_-^i(0,t)\\ \xi_+^i(1,t)\end{bmatrix}.
\end{aligned}\tag{24}$$

Combining the time-derivatives (23), (24) and inequality conditions (21), (22) and Lemma 1, the infinitesimal generator of $V(\xi,\sigma(t))$ satisfies

$$\begin{aligned}
\mathcal{L}V(\xi,\sigma(t)=i)&=\frac{d}{dt}V(\xi,i)+\sum_{j=1}^{N}\pi_{ij}V(\xi,j)\\
&=\partial_tV_1(\xi,i)+\partial_tV_2(\xi,i)+\sum_{j=1}^{N}\pi_{ij}V(\xi,j)\\
&\le\int_0^1\xi^\top[-2\mu_i|\Lambda_i|P_i(x)+\sum_{j=1}^{N}\pi_{ij}P_j(x)]\xi\,dx.\\
&\le-\nu V(\xi,i).
\end{aligned}\tag{25}$$

Now, we can define a new Lyapunov function $\tilde{V}(\xi,\sigma(t))$ as $\tilde{V}(\xi,\sigma(t)=i)=e^{\nu t}V(\xi,\sigma(t)=i)$, for all $i\in\mathcal{S}$. From Lemma 1 and (25), we have

$$\mathcal{L}\tilde{V}(\xi,\sigma(t)=i)=\mathcal{L}V(\xi,\sigma(t)=i)+\nu V(\xi,i)\le 0.\tag{26}$$

Based on the definition of $\mathcal{L}\tilde{V}$, for each $\sigma(t)=i\in\mathcal{S}$, $t\ge 0$, integrating (26) from zero to t, we get

$$\begin{aligned}
&\mathrm{E}\left\{e^{\nu t}V(\xi,\sigma(t))\right\}-V(\xi^0(x),\sigma(0))\\
&\qquad=\mathrm{E}\left\{\int_0^t\mathcal{L}\tilde{V}(\xi,\sigma(\tau))d\tau\right\}\le 0.
\end{aligned}\tag{27}$$

Therefore $\mathrm{E}\left\{V(\xi(x,t),\sigma(t))|\xi^0,\sigma(0)\right\}\le V(\xi^0,\sigma(0))e^{-\nu t}$ for any initial condition $\xi^0\in L^2(0,1)$ and any initial operation mode $\sigma(0)\in\mathcal{S}$. By remarking that there exist $\alpha>0$, $\beta>0$ (depending on μ_i, P_i^+ and P_i^-, for all $i\in\mathcal{S}$), such that

$$\alpha\|\xi(x,t)\|_{L^2(0,1)}^2\le V(\xi(x,t),\sigma(t))\le\beta\|\xi(x,t)\|_{L^2(0,1)}^2,$$

we obtain that, for all $t\in\mathbb{R}_+$,

$$\begin{aligned}
\alpha\mathrm{E}\left\{\|\xi(t,x)\|_{L^2(0,1)}^2\right\}&\le\mathrm{E}\left\{V(\xi(t,x),\sigma(t))\right\}\\
&\le\beta\|\xi^0(x)\|_{L^2(0,1)}^2e^{-\nu t}
\end{aligned}$$

which straightly implies

$$\mathrm{E}\left\{\|\xi(t,x)\|_{L^2(0,1)}^2\right\}\le\alpha^{-1}\beta e^{-\nu t}\|\xi^0(x)\|_{L^2(0,1)}^2.\tag{28}$$

Consequently, the MJLH system (2)–(4) is exponentially mean-square stable in L^2-norm by choosing $C=\alpha^{-1}\beta$. This completes the proof of Theorem 1. □

Remark 1. For the case of only one mode of operation in the MJLH system (2)–(4), $N=1$, that is no jumps in the Markov process, condition (21) in Theorem 1 implies the parameter $\mu_1>0$, and furthermore condition (22) in Theorem 1 is consistent with the dissipative boundary conditions for the linear hyperbolic systems of conservation laws by Coron et al. (2007). □

The condition (21) of Theorem 1 involves the spatial variable, then the number of inequality constraints is infinite. The following Corollary 1 presents a more easily checked matrix inequality condition by testing the upper bound of the involved spatial variable $x \in [0, 1]$. To do that, denote $P_i = \mathrm{diag}\{P_i^-, P_i^+\}$ and

$$I_i(x) = \mathrm{diag}\left\{e^{2\mu_i x} I_{m_i}, e^{-2\mu_i x} I_{n-m_i}\right\}, \tag{29}$$

which implies $P_i(x) = P_i I_i(x)$, for all $x \in [0, 1]$.

Corollary 1. *Let us assume there exist $\mu_i \in \mathbb{R}$ and diagonal positive definite matrices $P_i \in \mathcal{D}_+^{n\times n}$ such that the following matrix inequalities hold, for all $i, j \in \mathcal{S}$*

$$(\pi_{ii} I_n - 2\mu_i|\Lambda_i|)P_i + e^{2|\mu_i|}\sum_{j\neq i}^{N} \pi_{ij} e^{2|\mu_j|} P_j < 0, \tag{30}$$

$$G_{i+}^\top \Lambda_i P_i G_{i+} - G_{i-}^\top \Lambda_i P_i I_i(1) G_{i-} \leq 0. \tag{31}$$

Then the MJLH system (2)–(4) is exponentially mean-square stable.

Proof. The matrix inequality (30) implies that there exists a common positive small number $\nu > 0$ (depends on matrices P_i, $|\Lambda_i|$, and $\mu_i, \pi_{ij}, i, j \in \mathcal{S}$), such that

$$(\pi_{ii} I_n - 2\mu_i|\Lambda_i|)P_i + e^{2|\mu_i|}\sum_{j\neq i}^{N} \pi_{ij} e^{2|\mu_j|} P_j \leq -\nu P_i. \tag{32}$$

Multiplying $I_i(x)$ on both sides of (32), we obtain $-2\mu_i|\Lambda_i|P_i(x) + \pi_{ii}P_i(x) + e^{2|\mu_i|}\sum_{j\neq i}^{N}\pi_{ij}e^{2|\mu_j|}P_j I_i(x) \leq -\nu P_i(x)$. Since $\pi_{ij} \geq 0$, and $e^{2|\mu_i|}I_i(x) \geq 1$, we have $\sum_{j=1, j\neq i}^{N}\pi_{ij}P_j(x) \leq e^{2|\mu_i|}\sum_{j\neq i}^{N}\pi_{ij}e^{2|\mu_j|}P_j I_i(x)$, thus the condition (21) of Theorem 1 holds. This concludes the proof of Corollary 1. □

Compared with the results of Theorem 1, the sufficient conditions (30)–(31) of Corollary 1 remain nonlinear in the unknown variables μ_i and P_i, $i \in \mathcal{S}$. However, since parameter μ_i is a scalar variable, one may use a line search algorithm to solve inequalities (30) and (31) (see Section 4).

4. Numerical computation for stochastic stability conditions

In this section, a bounded real interval of the parameter μ_i is obtained by a trade-off between the condition (21) and (22) of Theorem 1. We define, for each $i \in \mathcal{S}$,

$$\mu_i^\pi = \lambda_{max}\left(\frac{\pi_{ii}}{2}|\Lambda_i|^{-1}\right), \tag{33}$$

$$\mu_i^G = -\ln \rho(G_i). \tag{34}$$

Proposition 2. *For a given mode $i \in \mathcal{S}$, conditions (21) and (22) of Theorem 1 are both satisfied, only if $\mu_i^\pi < \mu_i \leq \mu_i^G$.*

Proof. For a given parameter $\mu_i \in \mathbb{R}$, we first prove that $\mu_i > \mu_i^\pi$. On the one hand, for all $x \in [0, 1]$, condition (21) may be developed as $(-\mu_i|\Lambda_i|)^\top P_i(x) + P_i(x)(-\mu_i|\Lambda_i|) + \sum_{j=1}^{N}\pi_{ij}P_j(x) \leq -\nu P_i(x)$. By letting $x = 0$, condition (21) includes a finite number of constraints and describes the stochastic stability of the following continuous-time MJL system

$$\dot{y}(t) = -\mu_i|\Lambda_i|y(t). \tag{35}$$

Let $A_i = -\mu_i|\Lambda_i|$, $\otimes$ denotes the Kronecker product and $\oplus$ denotes the Kronecker sum, i.e., $A \oplus B = A \otimes I + B \otimes I$, and just recalling the result for the exponentially mean-square stable of MJLs (Fang & Loparo, 2002), the following $n^2N \times n^2N$ matrix Γ is Hurwitz,

$$\Gamma = \mathrm{diag}\{A_1 \oplus A_1, \ldots, A_N \oplus A_N\} + \Pi^\top \otimes I_{n^2}. \tag{36}$$

Note that $\pi_{ij} > 0$ for all $i \neq j$, $i, j \in \mathcal{S}$, then the matrix $A_i + \frac{\pi_{ii}}{2}I_n$ is Hurwitz which requires the largest real part of all eigenvalues being non-positive. Hence, requiring the parameter $\mu_i > \lambda_{max}(\frac{\pi_{ii}}{2}|\Lambda_i|^{-1})$ is necessary to satisfy the condition (21) of Theorem 1.

On the other hand, condition (22) of Theorem 1 may be described as

$$Q_i = \begin{bmatrix} Q_{--}^i & Q_{-+}^i \\ Q_{+-}^i & Q_{++}^i \end{bmatrix} \leq 0, \tag{37}$$

with $Q_{--}^i = (G_{+-}^i)^\top \Lambda_i^+ P_i^+ G_{+-}^i + e^{2\mu_i}(G_{--}^i)^\top|\Lambda_i^-|P_i^- G_{--}^i - |\Lambda_i^-|P_i^-$, $Q_{+-}^i = (G_{+-}^i)^\top \Lambda_i^+ P_i^+ G_{++}^i + e^{2\mu_i}(G_{--}^i)^\top|\Lambda_i^-|P_i^- G_{-+}^i$, $Q_{-+}^i = (Q_{+-}^i)^\top$, $Q_{++}^i = (G_{++}^i)^\top \Lambda_i^+ P_i^+ G_{++}^i + e^{2\mu_i}(G_{-+}^i)^\top|\Lambda_i^-|P_i^- G_{-+}^i - e^{-2\mu_i}\Lambda_i^+ P_i^+$, where $|\Lambda_i^-| = \mathrm{diag}\{|\lambda_1^i|, \ldots, |\lambda_{m_i}^i|\}$, and $\Lambda_i^+ = \mathrm{diag}\{\lambda_{m_i+1}^i, \ldots, \lambda_n^i\}$.

Then the matrix Q_i in (37) can be rewritten as

$$Q_i = (e_i^{\mu} G_i)^\top \mathrm{diag}\left\{|\Lambda_i^-|P_i^-, e^{-2\mu_i}\Lambda_i^+ P_i^+\right\} e_i^{\mu} G_i - \mathrm{diag}\left\{|\Lambda_i^-|P_i^-, e^{-2\mu_i}\Lambda_i^+ P_i^+\right\}. \tag{38}$$

It implies that the discrete-time system

$$y(k+1) = e^{\mu_i} G_i y(k), \quad k = 0, 1, \ldots \tag{39}$$

is marginally stable.

Hence, for the system matrix $e^{\mu_i}G_i$ of (39), we have $e^{\mu_i}\rho(G_i) \leq 1$. Thus, the inequality $\mu_i \leq -\ln\rho(G_i)$ is necessary to satisfy the condition (22) of Theorem 1. This concludes the proof of Proposition 2. □

Remark 2. Following the proof of Corollary 1, if the conditions (30) and (31) of Corollary 1 are satisfied, then, by Proposition 2, it holds $\mu_i \in (\mu_i^\pi, \mu_i^G]$. Therefore, when using a branch and bound algorithm for finding the parameters $\mu_i, i \in \mathcal{S}$, one can further limit the search range of μ_i among the initial interval $(\mu_i^\pi, \mu_i^G]$. □

5. Application to boundary control of freeway traffic

5.1. Aw–Rascle traffic flow model

The traffic dynamic in a freeway section of length L is governed by a quasi-linear hyperbolic system including two equations, the so-called Aw–Rascle traffic flow model (Aw & Rascle, 2000). That is

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0 \end{cases} \tag{40}$$

where $\rho(x, t)$ is vehicle density, $v(x, t)$ is average speed, $x \in [0, L]$, $t \geq 0$, and the function $p(\rho)$ means the traffic pressure which is supposed to be increasing against density. In Zhang (2002), pressure function $p(\rho)$ is given as

$$p(\rho) = v_f - V(\rho), \tag{41}$$

where v_f is the free (maximal) speed. Typically, using the Greenshields fundamental diagram (Greenshields, 1935), define

$$V(\rho) = v_f\left(1 - \frac{\rho}{\rho_m}\right), \tag{42}$$

where ρ_m is the maximal density. Thus, we get a linear pressure function $p(\rho) = a\rho$ with $a = \frac{v_f}{\rho_m}$.

Let $z = v$, and $w = v + a\rho$, the Aw–Rascle equation (40) may be rewritten in the characteristic Riemann coordinates as

$$\partial_t \xi(x, t) + \Lambda(\xi)\partial_x \xi(x, t) = 0, \tag{43}$$

with the system matrix $\Lambda(\xi) = \mathrm{diag}\{2z - w, z\}$ and the state $\xi = [z, w]^\top$. Eq. (43) is assumed to be strictly hyperbolic as the

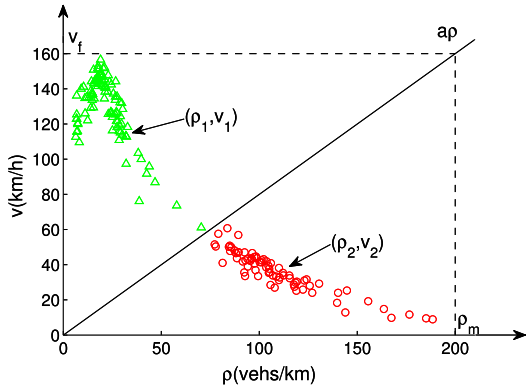

Fig. 1. Relation of the speed and density. The experiment data have been recorded on I80 freeway in Emeryville, CA, 4:00–4:30 pm, June 1, 2015, by PeMS (2004).

characteristic velocities $\lambda_1 = 2z - w = v - a\rho$, $\lambda_2 = z = v$ are different and nonzero for all $x \in [0, L]$, $t \geq 0$.

In (43), the sign of the first characteristic velocity $\lambda_1 > 0$, (or < 0), indicates the transfer direction of traffic information, such as the average speed z (or v), from the freeway upstream to the downstream, or inverse. It is natural as the feature to determine the freeway traffic lies in the free-flow traffic mode or in the congestion traffic mode (Kerner, 2009).

5.2. MJLH traffic flow model and linearization

Motivated by the development of stochastic traffic flow models (Boel & Mihaylova, 2006; Zhang & Mao, 2015), we deduce the quasi-linear hyperbolic equation (40) to a MJLH system.

Firstly, we choose two typical traffic states $\xi_1 = (\rho_1, v_1)$ and $\xi_2 = (\rho_2, v_2)$ from the separated regions of the fundamental diagram, with $v_1 - a\rho_1 > 0$ and $v_2 - a\rho_2 < 0$, respectively, to represent the characteristics of the free-flow and the congestion modes, as shown in Fig. 1.

In Riemann coordinates, the system matrices of the MJLH system (2) for free-flow or congestion modes are given respectively by $\Lambda_1 = \Lambda(\xi_1) = \mathrm{diag}\{2z_1 - w_1, z_1\}$, $\Lambda_2 = \Lambda(\xi_2) = \mathrm{diag}\{2z_2 - w_1, z_2\}$. We further assume the probabilistic conditions of mode switching follows a Markov process $\sigma(t)$ under a prior determined infinitesimal generator $\Pi \in \mathbb{R}^{2\times 2}$ (Sumalee et al., 2011).

Hence, the MJLH traffic flow model becomes

$$\partial_t \xi(x,t) + \Lambda_{\sigma(t)} \partial_x \xi(x,t) = 0, \tag{44}$$

with $\sigma(t) \in \{1, 2\}$, in which $\sigma(t) = 1$ means that the freeway traffic lies in the free-flow mode, and $\sigma(t) = 2$ means the congestion mode.

A steady-state of freeway traffic is a constant traffic state (ρ^*, v^*) which satisfies the flux conservation condition at the left boundary

$$p_{in} + r_0 = \rho^* v^*, \tag{45}$$

where p_{in} and r_0 are constant flux of driving-in vehicles through the mainline and the on-ramp of the freeway section, respectively. Then, the associated steady-state $\xi^* = (z^*, w^*)^\top$ in Riemann coordinates is given as $z^* = v^*$ and $w^* = v^* + a\rho^*$. We define the deviations of traffic state $(\rho(x,t), v(x,t))$ with respect to the steady-state (ρ^*, v^*) as $\widetilde{\rho} = \rho - \rho^*$, and $\widetilde{v} = v - v^*$. The MJLH system (44) around the steady-state ξ^* is derived as

$$\partial_t \widetilde{\xi}(x,t) + \Lambda_{\sigma(t)} \partial_x \widetilde{\xi}(x,t) = 0, \tag{46}$$

where the deviation $\widetilde{\xi} = \xi - \xi^*$ includes two entries $\widetilde{z} = z - z^*$ and $\widetilde{w} = w - w^*$.

Fig. 2. On-ramp metering and variable speed limit on a freeway section.

5.3. Boundary feedback control

We now are going to show how Corollary 1 may be applied to analyze the stochastic stability of the MJLH model (44). The boundary control applies the proportional feedback by combining the on-ramp metering $u(t)$ and the speed limit $v(0,t)$, which are all stored at the upstream boundary. Traffic measurements are density $\rho(L,t)$ and average speed $v(L,t)$ at the downstream boundary, see Fig. 2.

Precisely, we introduce the boundary feedback control:

$$\begin{cases} u(t) = r_0 - k_\rho(\rho(L,t) - \rho^*) \\ v(0,t) = v^* - k_v(v(L,t) - v^*) \end{cases}, \tag{47}$$

where $k_\rho \in \mathbb{R}$ and $k_v \in \mathbb{R}$ are proportional gains to be designed. In order to get the boundary conditions corresponding to the Riemann invariant (46), two cases are discussed separately.

Case 1: Free-flow mode for the system matrix Λ_1.

At the left boundary of the freeway section, *i.e.*, $x = 0$, the driving-in flux conservation equation holds

$$u(t) + p_{in} = \rho(0,t)v(0,t). \tag{48}$$

After the linearization of boundary condition (48) with integrating the feedback control law (47), we have the following boundary condition:

$$\widetilde{\rho}(0,t) = \rho^* \frac{k_v}{v^*} \widetilde{v}(L,t) - \frac{k_\rho}{v^*} \widetilde{\rho}(L,t). \tag{49}$$

Rewrite condition (49) in Riemann coordinates as

$$\begin{aligned} \widetilde{w}(0,t) = &-\frac{k_\rho}{v^*} \widetilde{w}(L,t) \\ &+ \left[a\rho^* \frac{k_v}{v^*} + \frac{k_\rho}{v^*} - k_v \right] \widetilde{z}(L,t). \end{aligned} \tag{50}$$

Since $\widetilde{v}(0,t) = -k_v \widetilde{v}(L,t)$, and $\widetilde{v} = \widetilde{z}$, the boundary condition that needs to be imposed for the free flow mode can be written as

$$\begin{bmatrix} \widetilde{z}(0,t) \\ \widetilde{w}(0,t) \end{bmatrix} = G_1 \begin{bmatrix} \widetilde{z}(L,t) \\ \widetilde{w}(L,t) \end{bmatrix}, \tag{51}$$

with $G_1 = \begin{bmatrix} -k_v & 0 \\ a\rho^* \frac{k_v}{v^*} + \frac{k_\rho}{v^*} - k_v & -\frac{k_\rho}{v^*} \end{bmatrix}$.

Case 2: Congestion mode for the system matrix Λ_2.

In this case, using the same feedback control (47), we have $\widetilde{v}(L,t) = -\frac{1}{k_v}\widetilde{v}(0,t)$. Similarly, in the Riemann coordinates, it holds

$$\widetilde{w}(0,t) = -\frac{k_\rho}{v^*} \widetilde{w}(L,t) + \left[1 - \frac{a\rho^*}{v^*} - \frac{k_\rho}{v^* k_v} \right] \widetilde{z}(0,t). \tag{52}$$

Hence, the boundary condition that needs to be imposed for the congestion mode is written as

$$\begin{bmatrix} \widetilde{z}(L,t) \\ \widetilde{w}(0,t) \end{bmatrix} = G_2 \begin{bmatrix} \widetilde{z}(0,t) \\ \widetilde{w}(L,t) \end{bmatrix}, \tag{53}$$

with $G_2 = \begin{bmatrix} -\frac{1}{k_v} & 0 \\ 1 - \frac{a\rho^*}{v^*} - \frac{k_\rho}{v^* k_v} & -\frac{k_\rho}{v^*} \end{bmatrix}$.

By selecting the boundary feedback gains in (47) with $|k_\rho| < v^*$ and $|k_v| < 1$, it holds $\rho(G_1) < 1$ and $\rho(G_2) > 1$. Then the free-flow mode with the boundary condition matrix G_1 is exponentially stable in L^2-norm, and the congestion mode with G_2 is unstable (Coron et al., 2007). MeanWhile, the overall MJLH system (44) might be exponentially mean-square stable or not, in L^2-norm, under the Markov process $\sigma(t)$.

5.4. Simulations

The developed boundary feedback strategy (47) and the stability conditions are now tested with numerical simulations based on the MJLH system (46) presented above. To this end, we consider a local freeway section whose road traffic parameters are given, respectively, as $\rho_m = 200$ veh./h, $v_f = 160$ km/h, $a = 0.8$, $p_{in} = 6000$ veh./h, $r_0 = 2000$ veh./h, and the total road length is 1 km, *i.e.*, $x \in [0, 1]$.

After clustering the historical traffic data, such as the speed–density records in Fig. 1, we represent the free-flow and the congestion modes with two typical traffic states $(\rho_1, v_1) = (25, 120)$, and $(\rho_2, v_2) = (75, 30)$, respectively. Then, the system matrices of the MJLH system (44) are given, respectively, as

$$\Lambda_1 = \begin{bmatrix} 100 & 0 \\ 0 & 120 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} -30 & 0 \\ 0 & 30 \end{bmatrix}. \tag{54}$$

We consider the steady-state $(\rho^*, v^*) = (80, 100)$ which satisfies the flux conservation condition (48), and choose the control gains in (47) as $k_\rho = 20$ and $k_v = 0.9$. The associated boundary condition matrices G_1 and G_2 with respect to the system matrices Λ_1 and Λ_2 are calculated, respectively, as

$$G_1 = \begin{bmatrix} -0.9 & 0 \\ -0.1240 & -0.2 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1.1111 & 0 \\ 0.1378 & -0.2 \end{bmatrix}.$$

Firstly, we assume that the transition probability of two traffic modes follows the generator $\Pi_1 = \begin{bmatrix} -1 & 1 \\ 7 & -7 \end{bmatrix}$. Using (33)–(34) the upper and lower bounds of parameters μ_1 and μ_2 are computed as $\mu_1^\pi = -0.0042$, $\mu_1^G = 0.1054$, $\mu_2^\pi = -0.1167$ and $\mu_2^G = -0.1054$, respectively. Solving conditions (30)–(31) of Corollary 1, we obtain $\mu_1 = 0.0971$, $\mu_2 = -0.1072$, $P_1 = \begin{bmatrix} 0.3107 & 0 \\ 0 & 0.0175 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 4.6235 & 0 \\ 0 & 0.2101 \end{bmatrix}$. To numerically compute the solutions to the MJLH system (46), let us discretize them using a WENO scheme (Jiang & Shu, 1996) in Matlab. The initial deviations from the steady-state (ρ^*, v^*) are given as $\tilde{v}(x, 0) = 4\sin(0.5\pi x)$ and $\tilde{\rho}(x, 0) = 5\sin(0.5\pi x)$, which satisfies the compatibility conditions and the initial mode is selected as $\sigma(0) = 1$.

Fig. 3 depicts a path of Markov process with generator Π_1. Figs. 4 and 5 show the time-evolution of v and ρ. It is observed that the traffic states clearly converge to their steady-state $\rho^* = 80$ veh./km and $v_1^* = 100$ km/h, respectively, as time increases, as expected from Corollary 1. The simulation results illustrate an interesting phenomenon that a fixed boundary control, just using the traffic information from the downstream, might stabilize freeway traffic even though some congestion (unstable) modes occur occasionally.

In the other simulation, just only the Markov generator is changed as $\Pi_2 = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix}$. In this case, since $\mu_2^\pi = -0.0667$ and $\mu_2^G = -0.1054$, we have $\mu_2^\pi > \mu_2^G$. The conditions (30) and (31) of Corollary 1 do not hold. Fig. 6 depicts a path of the Markov process with generator Π_2. We can find that the sojourn time of the congestion mode has greatly increased. These factors finally overturn the balance between the boundary conditions G_1, G_2, and the transition probability Π_2. Figs. 7 and 8 show the unstable behaviors for a sample path of $v(x, t)$ and $\rho(x, t)$ under Π_2 using the same initial condition.

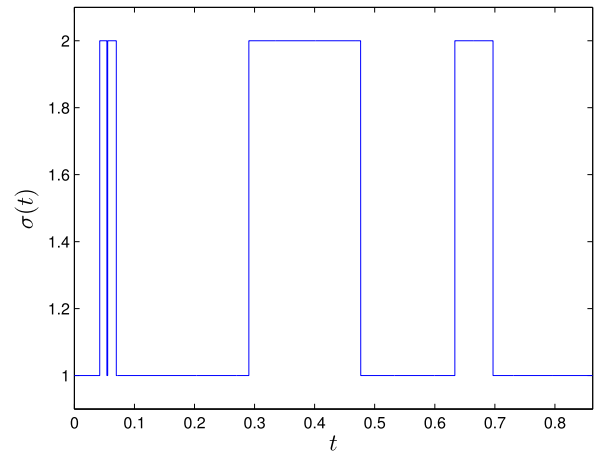

Fig. 3. Simulation of a path of the Markov process with generator Π_1.

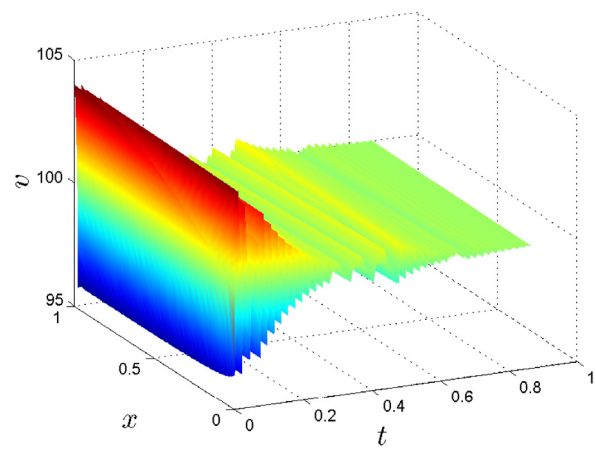

Fig. 4. The evolution of v of the MJLH traffic flow model with generator Π_1.

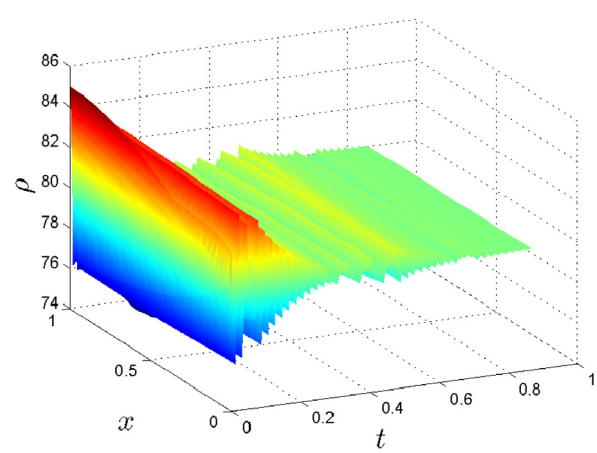

Fig. 5. The evolution of ρ of the MJLH traffic flow model with generator Π_1.

Also with the generator Π_2, after adjusting the proportional gains in control (47) as $k_\rho = 10$, $k_v = 0.95$, we can get a new set of the upper and lower bounds as $\mu_1^\pi = -0.0042$, $\mu_1^G = 0.0513$, $\mu_2^\pi = -0.0667$ and $\mu_2^G = -0.0513$. In this case, conditions (30)–(31) of Corollary 1 hold and we compute $\mu_1 = 0.0473$,

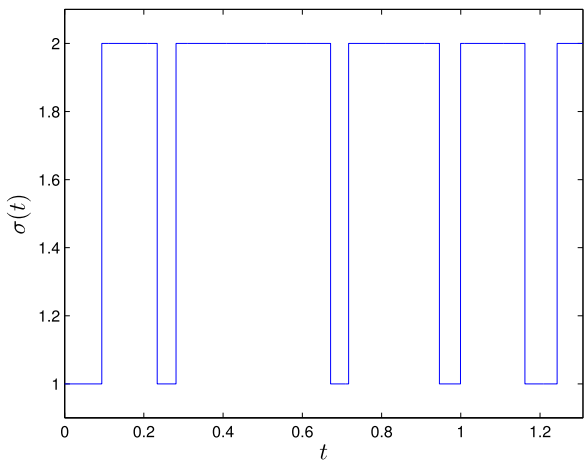

Fig. 6. Simulation of a path of the Markov process with generator Π_2.

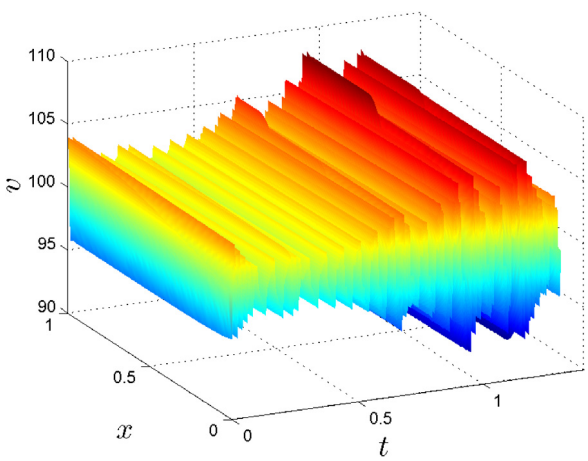

Fig. 7. The evolution of v of the MJLH traffic flow model with generator Π_2.

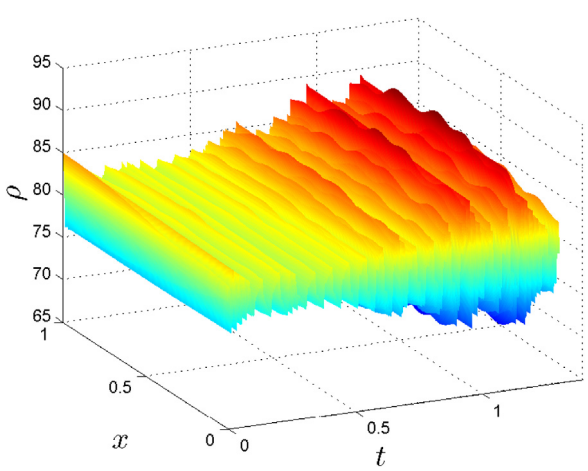

Fig. 8. The evolution of ρ of the MJLH traffic flow model with generator Π_2.

$\mu_2 = -0.0553$, $P_1 = \begin{bmatrix} 0.5261 & 0 \\ 0 & 0.0255 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 4.1475 & 0 \\ 0 & 0.1744 \end{bmatrix}$. Therefore, the MJLH traffic flow model (44) becomes exponentially mean-square stable again under the new boundary feedback control (47). For simplicity, we omit the figures of this case.

6. Conclusion

We present a class of switched systems, namely the MJLH system, in which the logical decision is Markov chain and subsystems are linear hyperbolic conservation laws. By means of a stochastic Lyapunov function, exponentially mean-square stability are derived depending on the balance between the boundary conditions and the transition probability. Theoretical contribution was applied to stabilize freeway traffic by integrating the on-ramp metering with the speed limit in the distributed control action. This work leaves many open questions. It is natural to extend the theoretical results, such as Theorem 1 and Corollary 1 to more general MJLH systems with time-varying velocities. Some effort will also devoted to the study of more general MJLH systems governed by an unobservable Markov chain.

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