A class of linear hyperbolic systems of conservation laws with multiple time scales which are modeled by a perturbation parameter is considered in this paper. By setting the perturbation parameter to zero, two subsystems, the reduced subsystem standing for the slow dynamics and the boundary-layer subsystem representing the fast dynamics, are computed. It is first proved that the exponential stability of the full system implies the stability of both subsystems. Secondly, a counterexample is given to indicate that the stability of the two subsystems does not ensure the full system’s stability. Moreover a new Tikhonov theorem for this class of infinite dimensional systems is stated. The solution of the full system can be approximated by that of the reduced subsystem, and this is proved by Lyapunov techniques. An application to boundary feedback stabilization of gas transport model is used to illustrate the results.

1. Introduction

The early interests in singular perturbation techniques arose from the physical problems exhibiting both fast and slow dynamics, such as DC-motors in Kokotović, Khalil, and O'Reilly (1986) where the inductance in the armature circuit plays the role of a perturbation parameter or semiconducting diodes in Smith (1985) where the Debye length is seen as a perturbation parameter. The decomposition of a singularly perturbed system into lower order subsystems, namely the reduced subsystem and the boundary-layer subsystem, provides a powerful tool for stability analysis. From late 1980s, singularly perturbed partial differential equations (PDEs) have been considered in research works. This kind of systems describes numerous phenomenon in various fields in physics and engineering, such as fluid dynamics, chemical-reactor, and aerodynamics (see Kadalbajoo and Patidar (2003)).

In this paper, we consider a class of linear hyperbolic systems of conservation laws with a small perturbation parameter. The principal motivation for this paper is the model of gas transport through a constant cross section tube represented by Euler equations (see Winterbone (2000, Chapt. 2)). Two time scales for propagation speed exhibit in this model, which can thus be described by a singularly perturbed hyperbolic system.

A first contribution of this paper concerns the stability analysis between the full system and its two subsystems. The stability analysis for hyperbolic systems of conservation laws has been considered by many researchers. For instance, a stability criterion for linear hyperbolic systems by characteristics method has been given in Li (1994) and the stability condition considered in Hale and Lunel (1993) relies on the frequency domain. In Coron, Bastin, and d’Andréa-Novel (2008), a stability condition for the quasilinear systems of conservation laws is introduced by Lyapunov method. In this paper, the first proposition gives a sufficient stability condition for both subsystems. Moreover a counterexample is used to illustrate that the exponential stability of the two subsystems does not guarantee the stability of the full system. This shows a major difference with what is well known for linear finite dimensional systems (e.g. Kokotović et al. (1986, Chapt. 2)).

A second contribution of our work relates to a new Tikhonov theorem for such singularly perturbed linear hyperbolic systems. Tikhonov theorem describes the limiting behavior of solutions of the perturbed system. It is a powerful tool for analysis of singular perturbation systems. This theorem has been studied in many works for finite dimensional systems (Khalil, 1996, Chapt. 9; Verhulst, 2007). The approximation of the full system by the reduced subsystem on the finite interval is based on the exponential stability of the boundary-layer subsystem. Furthermore, the approximation on the infinite interval is achieved by the exponential stability of both subsystems. In the present paper the infinite dimensional
case on the infinite interval is stated. We show that if the full system is stable then its slow dynamics can be approximated by the reduced subsystem. This method of approximation can lead to new boundary control strategies for linear hyperbolic systems. For instance, in Tang, Prieur, and Girard (2014) boundary control synthesis has been studied based on the singular perturbation method. More precisely, in this latter reference, the boundary condition for the reduced subsystem is firstly chosen as zero, which makes the slow dynamics converge to the equilibrium in finite time $T$. The boundary condition for the full system is chosen so as to guarantee stability. Then the full system reaches a neighborhood of the origin of size proportional to the perturbation parameter at time $T$.

The paper is organized as follows. The linear singularly perturbed system of conservation laws under consideration is presented in Section 2. The reduced and boundary-layer subsystems are computed in the same section. In Section 3, we link the exponential stability of the full system with the stability of each subsystem. Section 4 gives a Tikhonov theorem for linear hyperbolic systems of conservation laws. In Section 5, an application to a gas transport model is studied to illustrate the main results. Finally, concluding remarks end the paper.

**Notation.** For a positive integer $n$, $I_n$ is the identity matrix in $\mathbb{R}^{n \times n}$. Given a matrix $A$ in $\mathbb{R}^{m \times n}$, $A^{-1}$ and $A^T$ represent the inverse and the transpose matrix of $A$ respectively. For a symmetric matrix $B$ in $\mathbb{R}^{n \times n}$, $\lambda_{\min}(B)$ is the minimum eigenvalue of the matrix $B$. The symbol $\cdot$ in partitioned symmetric matrix stands for the symmetric block. $| \cdot |$ denotes the usual Euclidean norm in $\mathbb{R}^n$ and $\| \cdot \|$ is associated with the usual 2-norm of matrices in $\mathbb{R}^{n \times n}$. $\| \cdot \|_2$ denotes the associated norm in $L^2(0, 1)$, space, defined by $\| \xi \|_2^2 = \left( \int_0^1 (\xi(x))^2 \, dx \right)^{\frac{1}{2}}$ for all functions $\xi \in L^2(0, 1)$. Similarly, the associated norm in $H^2(0, 1)$ is space is denoted by $\| \cdot \|_2^2$, defined for all functions $\xi \in H^2(0, 1)$, by $\| \xi \|_{22}^2 = \left( \int_0^1 \| \xi \|_2^2 + \| \xi' \|_2^2 \right)^{\frac{1}{2}}$. Given a real interval $I$ and a normed space $J$, $C^k(I, J)$ denotes the set of continuous functions from $I$ to $J$. Let us adopt the following notation introduced in Coron et al. (2008), for all matrices $G \in \mathbb{R}^{n \times n}$, $\rho_1(G) = \inf \{ \| \Delta G \Delta^\top \|_2 : \Delta \in D_{n, +} \}$, where $D_{n, +}$ denotes the set of diagonal positive matrix in $\mathbb{R}^{n \times n}$.

2. **Linear singularly perturbed hyperbolic systems of conservation laws**

The linear singularly perturbed hyperbolic systems of conservation laws under consideration are given by,

\[
\begin{align*}
y_t(t, x) + A_1 y_x(t, x) &= 0, \quad \epsilon z_t(t, x) + A_2 z_x(t, x) &= 0, \\
y(x, 0) + A_1 y_0(x) &= 0, \quad z(x, 0) + A_2 z_0(x) &= 0,
\end{align*}
\]

where $x \in [0, 1], t \in [0, +\infty), y : [0, 1] \times [0, +\infty) \to \mathbb{R}^n, z : [0, 1] \times [0, +\infty) \to \mathbb{R}^m, A_1$ is a diagonal positive matrix in $\mathbb{R}^{n \times n}, A_2$ is a diagonal positive matrix in $\mathbb{R}^{m \times m}$, the perturbation parameter $\epsilon$ is a small positive value. Moreover we consider the following boundary condition

\[
\begin{align*}
y(0, t) &= G_y y(1, t), \quad \epsilon z(0, t) = G_z z(1, t),
\end{align*}
\]

where $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ is a constant matrix in $\mathbb{R}^{(n+m) \times (n+m)}$ with the matrices $G_{11}$ in $\mathbb{R}^{n \times n}, G_{12}$ in $\mathbb{R}^{n \times m}, G_{21}$ in $\mathbb{R}^{m \times n}$ and $G_{22}$ in $\mathbb{R}^{m \times m}$.

Given two functions $y^0 : [0, 1] \to \mathbb{R}^n$ and $z^0 : [0, 1] \to \mathbb{R}^m$, the initial conditions are

\[
\begin{align*}
y(0, x) &= y^0(x), \quad z(0, x) = z^0(x),
\end{align*}
\]

where $x \in [0, 1]$.

**Remark 1.** Let us recall the existence of the solutions to the Cauchy problem (1)--(3). According to Section 2.1 in Coron (2007), for all $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in L^2(0, 1)$, there exists a unique weak solution $\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \in C^0([0, +\infty), L^2(0, 1))$ for the Cauchy problem (1)--(3).

By Proposition 2.1 in Coron et al. (2008), for every $\begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \in H^2(0, 1)$ satisfying the following compatibility conditions:

\[
\begin{align*}
y_0(0) &= G_y y_0(1), \quad z_0(0) = G_z z_0(1), \\
y_0(0) &= (A_1 y_0(0))', \quad z_0(0) = (A_2 z_0(0))',
\end{align*}
\]

the Cauchy problem (1)--(3) has a unique maximal classical solution $\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \in C^0([0, +\infty), H^2(0, 1))$.

**Remark 2.** In Perrollaz and Rosier (2014), a $2 \times 2$ quasilinear hyperbolic system is considered in $C^0([0, T] \times [0, 1], \mathbb{R}^2)$ instead of $H^2(0, 1)$, avoiding so strong compatibility conditions. However, in the following of the present paper, the $H^2$ convergence of a system is mandatory thus the compatibility conditions should be considered.

Considering infinite dimensional systems described by partial differential equations (PDEs), let us compute the reduced and the boundary-layer subsystems for (1)--(3). Inspired by the approach for finite dimensional systems described by ordinary differential equations (ODEs) in Khalil (1996, Chapt. 9) and Saberi and Khalil (1984), the two subsystems are formally calculated as follows. Setting $\epsilon = 0$ in (1) yields

\[
\begin{align*}
y_t(t, x) + A_1 y_x(t, x) &= 0, \\
z_t(t, x) &= 0,
\end{align*}
\]

Substituting (5b) into the second line of the boundary condition (2) and assuming $(I_m - G_{22})^{-1}$ invertible we obtain

\[
\begin{align*}
y(0, t) &= (G_{11} + G_{12}(I_m - G_{22})^{-1} G_{21}) y(1, t), \\
y(0, t) &= (G_{11} + G_{12}(I_m - G_{22})^{-1} G_{21}) y(1, t).
\end{align*}
\]

The reduced subsystem is computed as follows:

\[
\begin{align*}
y_t(t, x) + A_1 y_x(t, x) &= 0, \quad x \in [0, 1], \quad t \in [0, +\infty),
\end{align*}
\]

with the boundary condition

\[
\begin{align*}
y(0, t) &= G_y y(1, t), \quad t \in [0, +\infty),
\end{align*}
\]

where $G_r = G_{11} + G_{12}(I_m - G_{22})^{-1} G_{21}$, whereas the initial condition is chosen as the same as for the full system

\[
\begin{align*}
y(0, x) &= y^0(x) = y^0(x), \quad x \in [0, 1],
\end{align*}
\]

To compute the boundary-layer subsystem, let us first perform the following change of variable

\[
\begin{align*}
z(t, x) &= (I_m - G_{22})^{-1} G_{21} y(1, t).
\end{align*}
\]

This shifts the equilibrium of $z$ to the origin. Let us use a new time variable $\tau = \frac{t}{\epsilon}$. In the $\tau$ time scale, $y(1, t)$ in (10) is considered as a fixed parameter with respect to time. Then, the boundary-layer subsystem is calculated as

\[
\begin{align*}
\tau = \tau(\tau, x) + A_1 z_x(\tau, x) &= 0, \quad x \in [0, 1], \quad \tau \in [0, +\infty),
\end{align*}
\]

with the boundary condition

\[
\begin{align*}
z(0, \tau) &= G_{22} z(1, \tau), \quad \tau \in [0, +\infty),
\end{align*}
\]

whereas the initial condition is given by

\[
\begin{align*}
z(0, x) &= \bar{z}(x) = z^0(x) - (I_m - G_{22})^{-1} G_{21} y^0(1), \quad x \in [0, 1].
\end{align*}
\]
3. Stability of reduced and boundary-layer subsystems

In this section, our aim is to show how the stability of the full system (1)–(3) is related to the stability of the reduced subsystem (7)–(9) and the boundary-layer subsystem (11)–(13).

The following definition is adopted for the exponential stability of the linear singularly perturbed system of conservation laws (1)–(3) in $L^2$-norm.

**Definition 1.** The linear singularly perturbed hyperbolic system of conservation laws (1)–(3) is exponentially stable to the origin in $L^2$-norm if there exist $\gamma_1 > 0$ and $C_2 > 0$, such that for every $\left( \begin{array}{c} \rho \\ \theta \end{array} \right) \in L^2(0, 1)$, the solution to the system (1)–(3) satisfies

$$\left\| \left( \begin{array}{c} \rho(t) \\ \theta(t) \end{array} \right) \right\|_{L^2} \leq C_1 e^{-\gamma_1 t} \left\| \left( \begin{array}{c} \rho(0) \\ \theta(0) \end{array} \right) \right\|_{L^2}, \quad t \in [0, +\infty).$$

Similarly the exponential stability of system (1)–(3) in $H^2$-norm is defined as follows.

**Definition 2.** The linear singularly perturbed hyperbolic system of conservation laws (1)–(3) is exponentially stable to the origin in $H^2$-norm if there exist $\gamma_2 > 0$ and $C_2 > 0$, such that for every $\left( \begin{array}{c} \rho \\ \theta \end{array} \right) \in H^2(0, 1)$ satisfying the compatibility conditions (4), the solution to the system (1)–(3) satisfies

$$\left\| \left( \begin{array}{c} \rho(t) \\ \theta(t) \end{array} \right) \right\|_{H^2} \leq C_2 e^{-\gamma_2 t} \left\| \left( \begin{array}{c} \rho(0) \\ \theta(0) \end{array} \right) \right\|_{H^2}, \quad t \in [0, +\infty).$$

Similarly, we can define the exponential stability in $L^2$-norm and $H^2$-norm for the reduced and boundary-layer subsystems.

Let us recall the following result for linear hyperbolic systems:

**Theorem 1** (Coron et al., 2008; Diagne, Bastin, and Coron, 2012). If $\rho_1(G) < 1$ (resp. $\rho_1(G) > 1$), then the full system (1)–(3) (resp. the reduced subsystem (7)–(9), the boundary-layer subsystem (11)–(13)) is exponentially stable to the origin in $L^2$-norm and $H^2$-norm.

With the above theorem, we are ready to give a proposition which is about the stability of the two subsystems.

**Proposition 1.** If $\rho_1(G) < 1$, then the reduced subsystem (7)–(9) and the boundary-layer subsystem (11)–(13) are exponentially stable to the origin in $L^2$-norm and $H^2$-norm.

**Proof.** Let us first prove the stability of the reduced subsystem (7)–(9). Let $\rho_1(G) \leq \alpha^* < 1$ and $\Delta \in D_{n+m,+}$ such that $\|\Delta^2G\|_{\alpha^*} \leq \alpha^*$. Let $\Delta_1 \in D_{n,+}$ and $\Delta_2 \in D_{m,+}$ be such that

$$\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}.$$ 

Consider $\tilde{Y} \in \mathbb{R}^{n+m}$ given by

$$\tilde{Y} = \begin{pmatrix} Y \\ \Delta_2(lm - G_{22})^{-1}G_{21}\Delta_1^{-1}Y \end{pmatrix},$$

where $Y$ is an arbitrary vector of $\mathbb{R}^n$. It follows directly

$$\Delta G \Delta^{-1} \tilde{Y} = \begin{pmatrix} \Delta_1(lm + G_{12}(l_{11} - G_{22})^{-1}G_{21})\Delta_1^{-1}Y \\ \Delta_2(lm - G_{22})^{-1}G_{21}\Delta_1^{-1}Y \end{pmatrix}.$$ (14)

Since $\|\Delta G \Delta^{-1}\| \leq \alpha^*$, it follows from (14)

$$|\Delta G \Delta^{-1} \tilde{Y}| \leq \alpha^* |\tilde{Y}|,$$

hence

$$|\Delta_1(lm + G_{12}(l_{11} - G_{22})^{-1}G_{21})\Delta_1^{-1}Y| \leq \alpha^* |Y|.$$ 

The previous inequality holds, for all $Y \in \mathbb{R}^n$, therefore

$$\|\Delta_1(lm + G_{12}(l_{11} - G_{22})^{-1}G_{21})\Delta_1^{-1}Y\| \leq \alpha^* < 1,$$ (15)

which implies $\rho_1(G) < 1$. Therefore, due to Theorem 1, the reduced subsystem (7)–(9) is exponentially stable in $L^2$-norm and $H^2$-norm.

Next let us prove the stability of the boundary-layer subsystem (11)–(13). Similarly, consider $\tilde{Z} \in \mathbb{R}^{n+m}$ given by

$$\tilde{Z} = \begin{pmatrix} 0 \\ Z \end{pmatrix},$$

where $Z$ is an arbitrary vector of $\mathbb{R}^m$. It follows directly

$$\Delta G \Delta^{-1} \tilde{Z} = \begin{pmatrix} \Delta_1(l_{12}l_{21}^{-1})Z \\ \Delta_2(l_{12}l_{21}^{-1})Z \end{pmatrix}.$$ (16)

Since $\|\Delta G \Delta^{-1}\| \leq \alpha^*$, it follows from (16)

$$|\Delta G \Delta^{-1} \tilde{Z}| \leq \alpha^* |\tilde{Z}|,$$

hence

$$|\Delta_2(l_{12}l_{21}^{-1})Z| \leq \alpha^* |Z|.$$ (17)

The previous inequality holds, for all $Z \in \mathbb{R}^m$, therefore

$$\|\Delta_2(l_{12}l_{21}^{-1})\| \leq \alpha^* < 1,$$ (18)

which implies $\rho_2(G_{22}) < 1$. Therefore, due to Theorem 1, the boundary-layer subsystem (11)–(13) is exponentially stable in $L^2$-norm and $H^2$-norm. This concludes the proof of Proposition 1. \qed

**Counterexample:** the stability criterion $\rho_1(G) < 1$ is a sufficient condition for stability of both subsystems. On the other hand, the stability of the two subsystems does not guarantee the stability of the full system. Let us consider the following counterexample. Let $\Delta_1 = \Delta_2 = 1$ in (11) with $n = m = 1$. The boundary condition of the full system in (2) is chosen as $G_{11} = 2.5, G_{12} = -1, G_{21} = 1, G_{22} = 0.5$. The boundary condition of the reduced subsystem in (8) is computed as $G_{2} = 0.5$. It holds $\rho_1(G_{2}) < 1$. By Theorem 1 the reduced subsystem is exponentially stable in $L^2$-norm and $H^2$-norm. The boundary condition of the boundary-layer subsystem in (12) is $G_{22} = 0.5$. It holds $\rho_2(G_{22}) < 1$, due to Theorem 1, the boundary-layer subsystem is exponentially stable in $L^2$-norm and $H^2$-norm. Now let us check the stability condition $\rho_1(G) < 1$, which is equivalent to find a diagonal positive matrix $\Delta$ such that $\|\Delta G \Delta^{-1}\| < 1$ and it is in fact equivalent to $G^T \Delta^2 G < \Delta^2$ (see Coron et al. (2008) section 4). Let us consider a positive value $\nu$ which is a little bit larger than 1, there is no loss of generality to look for $\Delta = \begin{pmatrix} 1 & 0 \\ 0 & \nu \end{pmatrix}$ such that

$$G^T \Delta^2 G < \nu \Delta^2.$$ (18)

Straightforward computations show that there is no such matrix $\Delta$ which satisfies the condition (18), thus $\rho_1(G) > 1$. Note that, Proposition 3.7 in Coron et al. (2008) implies that $\rho_1(G) < 1$ is a necessary and sufficient condition for stability of linear hyperbolic systems with dimension 1 to 5. Since this example is a linear singularly perturbed system of two conservation laws, it is not exponentially stable neither in $L^2$-norm nor in $H^2$-norm, although the reduced and boundary-layer subsystems are both exponentially stable.

4. Approximation theorem for linear singularly perturbed system of conservation laws

A Tikhonov theorem is given in this section. It presents how solutions to the full system (1)–(3) can be approximated by solutions to the reduced subsystem (7)–(9).
Theorem 2. Consider the linear singularly perturbed system of conservation laws (1)–(3). Assuming the boundary condition matrix $G$ satisfies $\rho_1(G) < 1$, then for any initial condition $y^0 \in H^2(0, 1)$ satisfying the compatibility conditions $y^0(0) = G_1 y^0(1), A_1 y^0(0) = G, A_1 y^0(1)$ with $y^0 = y^0$, and for any $z^0 \in L^2(0, 1)$, there exist positive values $\varepsilon^*, C, C'$ and $\omega$ such that for all $0 < \varepsilon < \varepsilon^*$ and for all $t \geq 0$,

$$\|y(., t) - \bar{y}(., t)\|^2 \leq \varepsilon e^{-\omega t}$$

with

$$\int_0^\infty \|z(., t) - (l_m - G_{22})^{-1}G_{21}\bar{y}(1, t)\|^2 dt \leq C'$$

$$\times \left(\|z^0 - (l_m - G_{22})^{-1}G_{21}^\top y^0(1)\|^2 + \|y^0\|^2_{H^1}\right).$$

Lemma 1. If the boundary condition satisfies $\rho_1(G) < 1$, then there exist a positive value $\mu$ and diagonal positive matrices $P$ and $Q$ such that for all $\varepsilon > 0$ and $\kappa > 0$, it holds

$$\dot{V}_\varepsilon(\eta, \delta) \leq -\mu \int_0^1 e^{-\mu\varepsilon} \eta^\top Q A_1 \eta dx$$

$$- \left(\mu - \frac{\varepsilon}{\kappa} \|P(l_m - G_{22})^{-1}G_{21}\|\right) \int_0^1 e^{-\mu\varepsilon} \delta^\top P A_2 \delta dx$$

$$+ \varepsilon \|P(l_m - G_{22})^{-1}G_{21}\| \|\bar{y}_x(1, t)\|^2.$$
By Young’s inequality, for all \( \kappa > 0 \), it holds
\[
N_3 \leq e \kappa \| P(I_m - G_{22})^{-1} G_{21} \| \int_0^1 e^{-\mu \kappa} |\delta|^2 \, dx \\
+ e \kappa \| P(I_m - G_{22})^{-1} G_{21} \| \int_0^1 e^{-\mu \kappa} |\tilde{y}_x(1, t)|^2 \, dx.
\]
Combining (26), (30) and (31), we obtain
\[
\dot{V}_\kappa(\eta, \delta) \leq -\mu \int_0^1 e^{-\mu \kappa} \eta^T Q A_1 \eta \, dx \\
- \left( \mu - e \kappa \| P(I_m - G_{22})^{-1} G_{21} \| \right) \int_0^1 e^{-\mu \kappa} \delta^T P A_2 \delta \, dx \\
+ e \kappa \| P(I_m - G_{22})^{-1} G_{21} \| \int_0^1 e^{-\mu \kappa} |\tilde{y}_x(1, t)|^2 \, dx.
\]
This concludes the proof of Lemma 1.

By Poincaré inequality \( |\tilde{y}_x(1, t)| \) is bounded as follows:
\[
|\tilde{y}_x(1, t)| = \left| \int_0^1 (\tilde{y} x + \tilde{y}) \right| \, dx \\
\leq \int_0^1 (|\tilde{y} x| + |\tilde{y}_x| + |\tilde{y}|) \, dx \\
\leq \sqrt{3} \|\tilde{y}(., t)\|_{\mu^2}.
\]
In order to bound the term \( |\tilde{y}_x(1, t)| \), it is necessary to analyze the stability of the reduced subsystem in \( H^2 \)-norm.

Since we choose the initial condition of the reduced subsystem \( \tilde{y}^0 = \tilde{y}^0, \) then \( \tilde{y}^0 \in H^2(0, 1) \) the compatibility conditions \( \tilde{y}^0(0) = G_1 \tilde{y}^0(1) \) and \( \tilde{y}^0_\kappa(0) = G_1 \tilde{y}^0_\kappa(1) \) are satisfied. The estimate of \( \|\tilde{y}(., t)\|_{\mu^2} \) is given in the following Lemma 2.

**Lemma 2.** If \( \mu_1(G) < 1 \), let \( \mu \) as in Lemma 1, there exists a positive value \( C_0 \) such that for any initial condition \( \tilde{y}^0 \in H^2(0, 1) \) satisfying the compatibility conditions \( \tilde{y}^0(0) = G_1 \tilde{y}^0(1) \) and \( \tilde{y}^0_\kappa(0) = G_1 \tilde{y}^0_\kappa(1) \), it holds for all \( t > 0 \),
\[
\|\tilde{y}(., t)\|_{\mu^2} \leq C_0 e^{-\mu \lambda_\text{min}(A_1^\kappa)} \|\tilde{y}^0\|_{\mu^2}.
\]

**Proof.** The initial condition \( \tilde{y}^0 \in H^2(0, 1) \) and the compatibility conditions imply that \( \tilde{y}(., t) \in H^2(0, 1) \) for all \( t > 0 \). Therefore we consider the following candidate Lyapunov function for the reduced subsystem (7)–(9)
\[
V_1(\tilde{y}) = \int_0^1 e^{-\mu \kappa} (\tilde{y}^T Q_1 \tilde{y} + \tilde{y}^T Q_3 \tilde{y}_x + \tilde{y}_x^T Q_3 \tilde{y}_xx) \, dx,
\]
where \( Q_1, Q_3, Q_3 \) are diagonal positive matrices in \( \mathbb{R}^{n \times n} \). We rewrite \( V_1(\tilde{y}) \) as
\[
V_1(\tilde{y}) = V_{10} + V_{11} + V_{12},
\]
with
\[
V_{10} = \int_0^1 e^{-\mu \kappa} \tilde{y}^T Q_1 \tilde{y} \, dx,
\]
\[
V_{11} = \int_0^1 e^{-\mu \kappa} \tilde{y}^T Q_3 \tilde{y}_x \, dx,
\]
\[
V_{12} = \int_0^1 e^{-\mu \kappa} \tilde{y}_x^T Q_3 \tilde{y}_xx \, dx.
\]
Computing the time derivative of \( V_{10} \) along (7) and performing an integration by parts we have
\[
\dot{V}_{10} = -\left[ e^{-\mu \kappa} \tilde{y}^T Q_1 A_1 \tilde{y} \right]_{t=0}^{t=1} - \mu \int_0^1 e^{-\mu \kappa} \tilde{y}^T Q_1 A_1 \tilde{y} \, dx.
\]
Using the boundary condition (8), it follows from (35)
\[
\dot{V}_{10} = -\left[ e^{-\mu \kappa} \tilde{y}^T (Q_1 A_1 \tilde{y}) - (G_1 \tilde{y}(1))^T (Q_1 A_1 (G_1 \tilde{y}(1))) \right] \\
- \mu \int_0^1 e^{-\mu \kappa} \tilde{y}^T Q_1 A_1 \tilde{y} \, dx.
\]
Developing and reorganizing (36) we get
\[
\dot{V}_{10} = -\tilde{y}^T (1) \left( e^{-\mu \kappa} Q_1 A_1 - G_1^T Q_1 A_1 G_1 \right) \tilde{y}(1) \\
- \mu \int_0^1 e^{-\mu \kappa} \tilde{y}^T Q_1 A_1 \tilde{y} \, dx.
\]
Similar to the proof of Proposition 1, we show that \( \|\Delta G A^\kappa\| < 1 \) implies \( \|\Delta A_1^\kappa A_1^\kappa\| < 1 \) which is equivalent to \( \Delta A_1^\kappa - G_1^T \Delta A_1^\kappa G_1 > 0 \).

By selecting \( \tilde{Q}_1 = \Delta A_1^\kappa A_1^\kappa \), it is deduced from (37)
\[
\dot{V}_{10} = -\tilde{y}^T (1) \left( e^{-\mu \kappa} \tilde{Q}_1 A_1 - G_1 \tilde{Q}_1 A_1 G_1 \right) \tilde{y}(1) \\
- \mu \int_0^1 e^{-\mu \kappa} \tilde{y}^T \tilde{Q}_1 A_1 \tilde{y} \, dx.
\]
Differentiating system (7) with respect to \( x \) yields
\[
\tilde{y}_x + A_1 \tilde{y}_xx = 0,
\]
differentiating (8) with respect to \( t \) and using (7), the boundary condition is calculated as
\[
\tilde{y}_x(0, t) = A_1 \tilde{y}_xx(1, t).
\]
Computing the time derivative of \( V_{11} \) along (39) and performing an integration by parts we have
\[
\dot{V}_{11} = -\left[ -\left( e^{-\mu \kappa} \tilde{y}^T Q_2 A_1 \tilde{y} \right) \right]_{t=0}^{t=1} - \mu \int_0^1 e^{-\mu \kappa} \tilde{y}_x^T Q_2 A_1 \tilde{y}_x \, dx.
\]
Using the boundary condition (40), it follows from (41)
\[
\dot{V}_{11} = \left[ -\left( e^{-\mu \kappa} \tilde{y}^T Q_2 A_1 \tilde{y} \right) \right]_{t=0}^{t=1} - \mu \int_0^1 e^{-\mu \kappa} \tilde{y}_x^T Q_2 A_1 \tilde{y}_x \, dx.
\]
Developing and reorganizing (42) we get
\[
\dot{V}_{11} = -\left[ -\left( e^{-\mu \kappa} \tilde{y}^T Q_2 A_1 \tilde{y} \right) \right]_{t=0}^{t=1} - \mu \int_0^1 e^{-\mu \kappa} \tilde{y}_x^T Q_2 A_1 \tilde{y}_x \, dx.
\]
Similarly, by selecting \( \tilde{Q}_2 = A_1 A_1^\kappa \), it is deduced from (43)
\[
\dot{V}_{11} = -\left[ -\left( e^{-\mu \kappa} \tilde{y}^T Q_2 A_1 \tilde{y} \right) \right]_{t=0}^{t=1} - \mu \int_0^1 e^{-\mu \kappa} \tilde{y}_x^T Q_2 A_1 \tilde{y}_x \, dx.
\]
Differentiating system (39) with respect to \( x \) yields
\[
\tilde{y}_xx + A_1 \tilde{y}_xx = 0,
\]
differentiating (40) with respect to \( t \) and using (39), the boundary condition is calculated as
\[
\tilde{y}_xx(0, t) = A_1^2 G_1 \tilde{y}_xx(1, t).
\]
Computing the time derivative of $V_{12}$ along (45) and performing an integration by parts, we have
\[
\dot{V}_{12} = -\left[ e^{-\mu t} y_{\text{xx}}^T Q_2 A_1 \dot{y}_{\text{xx}} + \mu \int_0^1 e^{-\mu s} y_{\text{xx}}^T Q_2 A_1 \dot{y}_{\text{xx}} ds \right].
\]
(47)
Using the boundary condition (46), it follows from (47)
\[
\dot{V}_{12} = -\left( e^{-\mu t} y_{\text{xx}}^T (1) Q_2 A_1 \dot{y}_{\text{xx}} + \mu \int_0^1 e^{-\mu s} y_{\text{xx}}^T Q_2 A_1 \dot{y}_{\text{xx}} ds \right).
\]
(48)
Developing and reorganizing (48) we get
\[
\dot{V}_{12} = -\left( e^{-\mu t} y_{\text{xx}} (1) A_1^T Q_2 A_1^{-1} - G r A_1^{-2} Q_2 A_1^{-1} G r \right) y_{\text{xx}}^T (1)
- \mu \int_0^1 e^{-\mu s} y_{\text{xx}}^T Q_2 A_1 \dot{y}_{\text{xx}} ds.
\]
(49)
By selecting $Q_2 = A_1^T A_1 A_1^{-1}$, it is deduced from (49)
\[
\dot{V}_{12} \leq -\mu \int_0^1 e^{-\mu s} y_{\text{xx}}^T Q_2 A_1 \dot{y}_{\text{xx}} ds.
\]
(50)
Combining (38), (44) and (50), $\dot{V}_1(y)$ follows
\[
\dot{V}_1(y) \leq -\mu \lambda_{\text{min}}(A_1) V_1(y).
\]
(51)
Therefore, there exists a positive value $C_0$ such that
\[
\|y(\cdot, t)\|_2^2 \leq C_0 e^{-\mu \lambda_{\text{min}}(A_1) t} \|y_0\|_2^2.
\]
This concludes the proof of Lemma 2. □

Proof. Combining (24) in Lemma 1, (32) and (33) in Lemma 2 yields
\[
\dot{V}_e(\eta, \delta) \leq -\mu \int_0^1 e^{-\mu s} \eta^T Q_1 A_1 \eta ds
- \left( -\mu \kappa \frac{\lambda_{\text{min}}(P A_2)}{\lambda_{\text{min}}(P A_2)} \right) \int_0^1 e^{-\mu s} \delta^T P A_2 \delta ds
+ \frac{3 \epsilon C_0}{\kappa} \int_0^1 e^{-\mu s} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2 \eta^T \| \dot{y}_0 \|_2^2 ds.
\]
(52)
By choosing $\kappa = \frac{\mu \lambda_{\text{min}}(P A_2)}{2 \epsilon C_0 \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2}$, we get
\[
\dot{V}_e(\eta, \delta) \leq -\frac{\mu \beta}{2} V_e(\eta, \delta)
+ \frac{6 \epsilon^2 C_0}{\mu \lambda_{\text{min}}(P A_2)} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2
\times e^{-\mu \lambda_{\text{min}}(A_1) t} \| y_0 \|_2^2.
\]
(53)
where $\beta = \min(\lambda_{\text{min}}(A_1), \lambda_{\text{min}}(A_2))$.

With $\lambda_{\text{min}}(A_1) > \frac{\beta}{2}$, it follows from (53)
\[
V_e(\eta, \delta) \leq e^{-\mu \lambda_{\text{min}}(P A_2) t} V_e(\eta_0, \delta_0)
+ \frac{6 \epsilon^2 C_0}{\mu \lambda_{\text{min}}(P A_2)} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2
\times \| y_0 \|_2^2.
\]
(54)
Since $V_e(\eta, \delta)$ is lower and upper bounded by
\[
e^{-\mu \lambda_{\text{min}}(Q) \| \eta \|_2^2} + \epsilon e^{-\mu \lambda_{\text{min}}(P) \| \delta \|_2^2}
\leq V_e(\eta, \delta)
\leq \| Q \| \| y_0 \|_2^2 + \epsilon \| P \| \| \delta \|_2^2.
\]
(55)
it follows that
\[
\| \eta(\cdot, t) \|_2^2 \leq \frac{e^{\mu t}}{\lambda_{\text{min}}(Q)} V_e(\eta, \delta)
\leq \frac{e^{\mu t}}{\lambda_{\text{min}}(Q)} \left( e^{-\mu t} V_e(\eta, \delta) + \frac{\epsilon}{2} \right)
+ \frac{6 \epsilon^2 C_0}{\mu \lambda_{\text{min}}(P A_2)} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2
\times e^{-\mu t} \| \dot{y}_0 \|_2^2
\leq \frac{e^{\mu t}}{\lambda_{\text{min}}(Q)} \left( e^{-\mu t} V_e(\eta, \delta) + \frac{\epsilon}{2} \right)
+ \frac{6 \epsilon^2 C_0}{\mu \lambda_{\text{min}}(P A_2)} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2
\times e^{-\mu t} \| \dot{y}_0 \|_2^2.
\]
due to $y_0 = \bar{y}_0$, i.e. $\gamma_0 = 0$, therefore
\[
\| \eta(\cdot, t) \|_2^2 \leq C_e e^{-\frac{\mu}{2} \lambda_{\text{min}}(P A_2)} \| \delta(\cdot, t) \|_2^2
\times e^{-\mu \lambda_{\text{min}}(A_1) t} \| \bar{y}_0 \|_2^2.
\]
(56)
Performing a time integration of both sides of (56) and using $\lim_{t \to +\infty} V_e(\eta, \delta) = 0$, yield
\[
0 - V_e(\eta_0, \delta_0) \leq -\frac{\mu}{2} \| \delta(\cdot, t) \|_2^2 dt
+ \frac{6 \epsilon^2 C_0}{\mu \lambda_{\text{min}}(P A_2)} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2
\times e^{-\mu \lambda_{\text{min}}(A_1) t} \| \bar{y}_0 \|_2^2.
\]
(57)
Reorganizing (57), we obtain
\[
\int_0^\infty \| \delta(\cdot, t) \|_2^2 dt \leq \frac{2 e^{\mu t}}{\mu \lambda_{\text{min}}(P A_2)} \left( V_e(\eta_0, \delta_0)
+ \frac{6 \epsilon^2 C_0}{\mu \lambda_{\text{min}}(P A_2)} \| P (I_m - G_{22})^{-1} G_{21} A_1 \|_2^2
\| \bar{y}_0 \|_2^2.
\]
Therefore with (55) and $\gamma_0 = 0$, it follows
\[
\int_0^\infty \| \delta(\cdot, t) \|_2^2 dt \leq C_{\delta_1} \| \delta_0 \|_2^2 + \epsilon^2 C_{\delta_2} \| \bar{y}_0 \|_2^2,
\]
where $C_{\delta_1}$ and $C_{\delta_2}$ are positive constants. (20) is proved. This concludes the proof of Theorem 2. □
Corollary 1. There exists $C'' > 0$ such that if $z^0$ is the equilibrium point, that is $z^0 = (I_m - G_{22})^{-1}G_{21}y^0(1)$, then for any $\delta^0 \in H^2(0, 1)$ satisfying the compatibility conditions $y^0(0) = G_yy^0(1)$, $y^0(0) = G_yy^0(1)$ and for $t \geq 0$, the following estimate holds,

$$\|z(..., t) - (I_m - G_{22})^{-1}G_{21}y(1, t)\|_{L^2}^2 \leq C'e^{-\alpha t}\|y^0\|_{H^2}^2.$$  \hfill (58)

Proof. The proof of this corollary is similar to that of Theorem 2. Due to (54) and (55), since $\eta^0 = 0$ and $\delta^0 = 0$, therefore

$$\|\delta(..., t)\|_{L^2}^2 \leq 6C_0C_b\|P(I_m - G_{22})^{-1}G_{21}A_1\|^2 \|e^{-\frac{\mu t}{\tau}}\|y^0\|_{H^2}^2,$$

where $C_b > 0$. This concludes the proof of Corollary 1.  \hfill \Box

5. Application to a gas transport model

5.1. System description

In this section, we consider a gas transport model, which is an example of linear singularly perturbed hyperbolic system of conservation laws, to illustrate the results in Sections 3 and 4.

The gas dynamics through a constant cross section tube, where all the friction losses and heat transfers are neglected, are usually modeled by the following Euler equations as considered in Castillo, Wittrant, and Dugard (2012) and Winterbone (2000):

$$W_t + Aw = 0,$$

with $W = \begin{pmatrix} u \\ \rho \\ 0 \\ u(t) \end{pmatrix}, A = \begin{pmatrix} u & 0 & 1 \\ \rho & u & 0 \\ 0 & 0 & u \end{pmatrix}.$$

and

- $u = u(x, t)$ stands for the gas velocity at location $x$ in $[0, 1]$, (we assume that the length of the tube equals $1$) and at time $t$,
- $\rho = \rho(x, t)$ represents the gas density,
- $p = p(x, t)$ is the gas pressure,
- $a$ is sound speed in ideal gas.

System (59) admits a steady-state $(u^*, \rho^*, p^*)$. The deviations of the state $(u, \rho, p)$ with respect to the steady-state are defined as

$$\begin{align*}
\hat{u} &= u - u^*, \\
\hat{\rho} &= \rho - \rho^*, \\
\hat{p} &= p - p^*.
\end{align*}$$

The linearization of system (59) at this equilibrium is given by

$$\hat{W}_t + A\hat{W} = 0,$$

with $\hat{W} = \begin{pmatrix} \hat{u} \\ \hat{\rho} \end{pmatrix}, A = \begin{pmatrix} u^* & 0 & 1 \\ 0 & \rho^* & 0 \\ a^* & 0 & u^* \end{pmatrix}.$$

Let us rewrite the above system in Riemann coordinates

$$S_t + AS_x = 0,$$

with

$$A = \begin{pmatrix} u^* & 0 & 0 \\ 0 & u^* - a^* & 0 \\ 0 & 0 & u^* + a^* \end{pmatrix},$$

and

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{\rho^*}{a^*} & \frac{\rho^*}{a^*} \\ 0 & -a^* \rho^* & a^* \rho^* \end{pmatrix}^{-1}\hat{W}.$$  \hfill (62)

Performing the change of spatial variable, $S_1(1 - x, t) = S_2(x, t)$, we may assume without loss of generality that the matrix $A > 0$.

Assuming the propagation speed of gas is much slower than the sound speed, i.e., $u \ll a$, let us define a small positive value $\epsilon = \frac{u}{a^*}$ and a new time scale $\hat{t} = u't$. Then (61) can be approximated by a linear singularly perturbed hyperbolic system of conservation laws

$$\begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}_t + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\epsilon} & 0 \\ 0 & 0 & \frac{1}{\epsilon} \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}_x = 0,$$  \hfill (63)

where $S_1$ stands for the slow dynamics and $S_2, S_3$ stand for the fast dynamics.

5.2. Boundary conditions

The plant is equipped with two fans located at each end of the tube, the rotation speed of each fan is seen as control action. The gas mass density at the output and the gas pressure at both ends of the tube are assumed to be measured. Precisely, we consider here the following boundary conditions for system (59):

1. The first boundary condition describes the operation of the inflow fan (see the fan specification map in Wittrant and Johansson (2008)). It controls the input flow:

$$u(0, \hat{t}) = g_0(\hat{t})(p(0, \hat{t}) - p_0),$$  \hfill (64)

where $\alpha$ is the constant cross section of the tube, $g_0(\hat{t})$ denotes the rotation speed which is a constant input, and $p_0$ is a constant pressure before the inflow fan.

2. The second boundary condition is given by the outflow fan which is used to control the output flow:

$$u(1, \hat{t}) + g_1(\hat{t})(p_{out} - p(1, \hat{t})),\hat{t})$$  \hfill (65)

the control input is noted $c_1(\hat{t})$ and $p_{out}$ is a constant pressure behind the outflow fan.

3. The third boundary condition is a physical constraint, noting that the change of gas density through the inflow fan is small because the pressure at the boundary $x = 0$ is near atmospheric pressure, (see Castillo et al. (2012)):

$$\rho(0, \hat{t}) = \rho',\hat{t})$$  \hfill (66)

where $\rho'$ is a constant value.

After the linearization of the above three boundary conditions, we have the following boundary conditions for system (60):

$$\begin{align*}
\hat{u}(0, \hat{t}) &= g_0(\hat{t})(p^* - p_{in}) + c_0^*(\hat{p}(0, \hat{t})), \\
\hat{u}(1, \hat{t}) &= g_0(\hat{t})(\bar{c}_{21} + c_1(\hat{t})) - c_1(\hat{t})(\bar{c}_{21} + c_1(\hat{t})), \\
\hat{\rho}(0, \hat{t}) &= 0,
\end{align*}$$

where $c_0^*$, $c_1^*$ are the constant control actions at the equilibrium $(u^*, \rho^*, p^*)$, $\bar{c}_{21}(\hat{t}) = c_0(\hat{t}) - c_0^*$ and $\bar{c}_{21}(\hat{t}) = c_1(\hat{t}) - c_1^*$.

For a suitable choice of the control inputs $\bar{c}_0(\hat{t})$ and $\bar{c}_1(\hat{t})$, and for constant values $g_{21}, g_{22} \neq 1$, $g_{32} \neq 1$ in $\mathbb{R}$, the following boundary conditions for system (63) are equivalent to (67)-(69):

$$\begin{align*}
S_1(0, \hat{t}) &= \rho'\left(1 - \frac{g_{32}}{a^*}\right)S_2(1, \hat{t}), \\
S_2'(0, \hat{t}) &= g_{21}S_1(1, \hat{t}) + g_{22}S_2(1, \hat{t}), \\
S_3(0, \hat{t}) &= g_{22}S_2(1, \hat{t}).
\end{align*}$$

More precisely from (62) and (72), under the condition $g_{32} - 1 \neq 0$, $\hat{u}(0, \hat{t})$ is computed as follows:

$$\hat{u}(0, \hat{t}) = \frac{1 + g_{32}(g_{32} - 1)}{a^* \rho^*} \hat{p}(0, \hat{t}).$$  \hfill (73)
and substituting (73) into (67), the control action at the boundary \( x = 0 \) is
\[
c_0(\tilde{t}) = c_0^* + \frac{a_1(1 + g_{23})}{g_{11}^* p^* (g_{22}^* - 1)} - c_0^* \tilde{p}(0, \tilde{t}).
\]  
(74)
Similarly, due to (62), (68) and (71), under the condition \( 1 - g_{32} \neq 0 \), the control action at the boundary \( x = 1 \) is given by
\[
c_1(\tilde{t}) = c_1^* + \frac{a_1(1 + g_{23}) - 2p^* g_{32}}{g_{11}^* p^* (1 - g_{32})} + \frac{c_1^*}{p_{\text{out}} - p^*} \tilde{p}(1, \tilde{t})
\]
\[+ \frac{2p^* g_{21}}{g_{11}^* g_{22}^*} \tilde{p}(1, \tilde{t}).
\]  
(75)
Since the outputs of (59) are the gas pressure at both ends of the tube and the gas mass density at the output, the control actions \( c_0(\tilde{t}) \) in (74) and \( c_1(\tilde{t}) \) in (75) are the feedback laws.

The boundary condition needed to be imposed in system (63) is written as
\[
\begin{pmatrix}
S_1(0, \tilde{t}) \\
S_1'(0, \tilde{t}) \\
S_2(0, \tilde{t})
\end{pmatrix} = G \begin{pmatrix}
S_1(1, \tilde{t}) \\
S_1'(1, \tilde{t}) \\
S_2(1, \tilde{t})
\end{pmatrix},
\]  
(76)
where
\[
G = \begin{pmatrix}
0 & \rho^* (1 - g_{32}) & 0 \\
\rho^* 0 & \rho^* a^* & 0 \\
0 & 0 & g_{32}^* 
\end{pmatrix}.
\]  
(77)
To ensure \( \rho_1(G) < 1 \), it is sufficient that \( \|G\| < 1 \) which corresponds to \( \Delta = I \). Straightforward calculations show that \( \|G\| < 1 \) holds if and only if \( g_{21}, g_{23} \) and \( g_{32} \) can be selected as
\[
\begin{align*}
g_{21}^2 + g_{23}^2 < 1, \\
\frac{\rho^* a^* (1 - g_{32})^2}{\sigma^2} + g_{32}^2 < 1.
\end{align*}
\]
Adopting the computations of the two subsystems in Section 2, the reduced subsystem for (63) and (76) is given by
\[
\begin{pmatrix}
\tilde{S}_1 \tilde{t} \\
\tilde{S}_1 \tilde{x}
\end{pmatrix} + \begin{pmatrix}
\tilde{S}_1 \\
\tilde{S}_2
\end{pmatrix} = 0,
\]  
(78)
with the boundary condition
\[
\begin{pmatrix}
\tilde{S}_1(0, \tilde{t}) = G_\epsilon \tilde{S}_1(1, \tilde{t}),
\end{pmatrix}
\]  
(79)
where \( G_\epsilon = \frac{a_{21}(1 - g_{32})}{g_{32}^* \sigma^2} \).

The boundary-layer subsystem is given by
\[
\begin{pmatrix}
\tilde{S}_2' \\
\tilde{S}_3
\end{pmatrix} + \begin{pmatrix}
\tilde{S}_2 \\
\tilde{S}_3
\end{pmatrix} = 0,
\]  
(80)
with the boundary condition
\[
\begin{pmatrix}
\tilde{S}_2(0, \tilde{t}) = G_{22} \tilde{S}_2(1, \tilde{t}) \\
\tilde{S}_3(0, \tilde{t}) = 0
\end{pmatrix},
\]  
(81)
where \( G_{22} = \begin{pmatrix} 0 & g_{32}^* \epsilon \\ g_{32}^* & 0 \end{pmatrix} \) and \( \tilde{t} = \tilde{t}/\epsilon \).

5.3 Numerical solutions

To numerically compute the solutions of the full system (63) and of the reduced subsystem (78), let us discretize them using a two-step variant of the Lax–Wendroff method which is presented in Shampine (2005b) and the solver on Matlab in Shampine (2005a). More precisely, we divide the space domain \([0, 1]\) into 100 intervals of identical length, and choose 3 as final time. We select a time-step \( dt = 0.05 \) such that satisfies the CFL condition for the stability and select the following initial functions,
\[
\begin{align*}
S_1^0 &= S_1^0 = \cos(4\pi x) - 1, \\
S_2^0 &= \sin(5\pi x), \\
S_3^0 &= - \sin(5\pi x),
\end{align*}
\]
such that the compatibility conditions are satisfied.

Let us choose \( g_{21} = 0.1, g_{23} = 0, g_{32} = 0.2, a^* = 340, \rho^* = 1.2, \) \( u^* = 0.5, \epsilon = \frac{u^*}{\sigma} \), the boundary condition matrix \( G \) in (77) is written as
\[
G = \begin{pmatrix}
0 & 0.003 & 0 \\
0.1 & 0 & 0 \\
0 & 0.2 & 0
\end{pmatrix}.
\]  
(82)
The boundary condition of the reduced subsystem (79) is \( G_\epsilon = 0.003 \). Considering \( \Delta = I \), it computed \( \|G_\epsilon A_\Delta^{-1}\| < 1 \). The condition \( \rho_1(G) < 1 \) holds, thus Proposition 1 and Theorem 2 apply.

In the following figures and tables, \( \eta, \delta_1, \) and \( \delta_2 \) denote:
\[
\begin{align*}
\eta &= S_1 - \tilde{S}_1, \\
\delta_1 &= S_2 - g_{21} \tilde{S}_1(1, \tilde{t}), \\
\delta_2 &= S_3 - g_{23} \tilde{S}_2(1, \tilde{t})
\end{align*}
\]

Fig. 1 shows the time evolution of the solution \( \tilde{S}_1 \) of the reduced subsystem (78) with \( G_\epsilon = 0.003 \). It is observed that \( \tilde{S}_1 \) converges to the origin as time increases, as expected from Proposition 1. Time evolution of \( \eta \), which is the error between the slow dynamics of the full system (63) with the boundary condition (82) and the reduced subsystem (78) with \( G_\epsilon = 0.003 \), is shown in Fig. 2. It is found that the value of the difference is small and close to 0 as time increases. Figs. 3 and 4 represent the time evolutions of \( \delta_1 \) and \( \delta_2 \), which are the differences between \( S_2', S_3 \) and their equilibrium points \( \tilde{S}_2(1, \tilde{t}), \tilde{S}_3(1, \tilde{t}) \) respectively. The differences decrease to 0 as time increases.

In the following tables, we choose \( u^* = [1.5, 1, 0.5] \) and \( \epsilon = \frac{u^{**}}{\sigma} \). Table 1 gives the estimates of \( \|\eta(t, \epsilon = 0.3)\|_2^2 \) for different values of \( \epsilon \). It indicates that the difference between the slow dynamics of the full system (63) with (82) and the reduced subsystem (78) with \( G_\epsilon = 0.003 \) is near zero, and it decreases as \( \epsilon \) decreases. The estimates of \( \int_t^0 \|\delta_1\|_2^2 \) and of \( \int_t^0 \|\delta_3\|_2^2 \) dt are shown in Table 2. It is observed that when \( \epsilon \) decreases, the values of the difference decrease and are close to zero, as expected from Theorem 2.

Let us define \( \epsilon \), which is the difference between the slow dynamics of (63) and of (61) with the same boundary conditions matrix (82). Table 3 shows the value of \( \|\epsilon(t, \epsilon = 0.3)\|_2^2 \) for different \( \epsilon \). It is observed that \( \epsilon \) is small and close to zero. This motivates the approximation of (61) by (63).

![Fig. 1](image-url)
applied to a gas transport setup which can be modeled by a singularly perturbed hyperbolic system by employing the fact that the propagation speed of gas is much slower than the sound speed.

This work leaves many open questions. It is natural to extend this work to systems of balance laws. Another interesting point is to consider some other physical applications, like open channels as considered in Dos Santos and Prieur (2008). The nonlinear case in Coron et al. (2008) will also be considered in the future works.

References


Table 1

<table>
<thead>
<tr>
<th>( \epsilon = \frac{a}{\tau} )</th>
<th>( | \eta(t, 0) |_{\ell_2}^2 | )</th>
<th>( 0.004 )</th>
<th>( 0.003 )</th>
<th>( 0.001 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>( 4.3 \times 10^{-11} )</td>
<td>( 1.1 \times 10^{-11} )</td>
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</table>

Table 2

<table>
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<th>( \epsilon = \frac{a}{\tau} )</th>
<th>( | \delta_1(t) |_{\ell_2}^2 | )</th>
<th>( 0.004 )</th>
<th>( 0.003 )</th>
<th>( 0.001 )</th>
</tr>
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<td>( | \delta_1(t) |_{\ell_2}^2 )</td>
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<td>( 2.3 \times 10^{-6} )</td>
<td>( 5.8 \times 10^{-7} )</td>
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<tr>
<td>( | \delta_2(t) |_{\ell_2}^2 | )</td>
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<td>( 6.5 \times 10^{-7} )</td>
<td>( 1.6 \times 10^{-7} )</td>
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</table>

Table 3

<table>
<thead>
<tr>
<th>( \epsilon = \frac{a}{\tau} )</th>
<th>( | \eta(t, 0) |_{\ell_2}^2 | )</th>
<th>( 0.004 )</th>
<th>( 0.003 )</th>
<th>( 0.001 )</th>
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<td>( | \eta(t, 0) |_{\ell_2}^2 )</td>
<td>( 1.9 \times 10^{-15} )</td>
<td>( 3.7 \times 10^{-16} )</td>
<td>( 2.3 \times 10^{-17} )</td>
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</table>

6. Conclusion

This work has dealt with a linear singularly perturbed hyperbolic system of conservation laws. The reduced subsystem and the boundary-layer subsystem have been formally computed by setting the perturbation parameter to zero. In Proposition 1, assuming \( \rho_1(G) < 1 \) (which implies that the full system is stable), it has been first shown that both subsystems are stable. Moreover, a counterexample has been given to illustrate that the stability of the two subsystems is not enough to guarantee the stability of the full system.

In Theorem 2 it has been shown that, under the stability condition \( \rho_1(G) < 1 \), the solution to the full system can be approximated by that to the reduced subsystem. This theorem has been
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