Thesis defense: On the optimal control of linear complementarity systems

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Thesis defense
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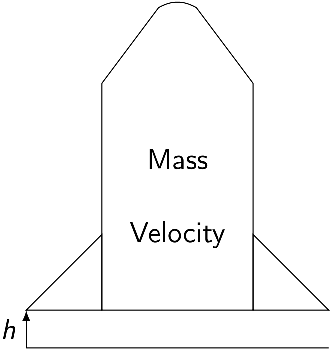
Optimal control:
control the way a **system** is changing/moving/transformed **with time**, while **maximizing/minimizing** a quantity.

Example: Goddard's Rocket Problem (1910)

How to send a rocket as high up in the air as possible in t_f seconds?
Maximize height at time t_f: $h(t_f)$.
You control: the evolution of the velocity, the mass,
using: the force of the motor (and some gas),
but the force of the motor and the gas are **limited**!



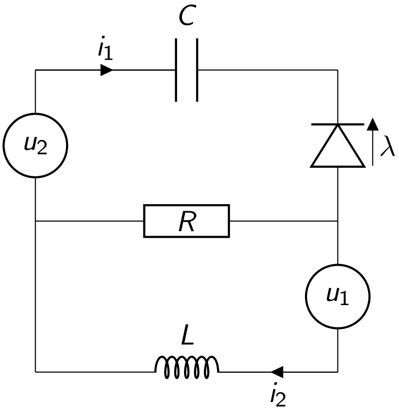
Introduction

Example: Circuit with ideal diode

One wants to **control** this kind of electrical system via the inputs, where the diode is supposed to be ideal.

At the beginning, the capacitor is charged, and there is no current through the inductance. One wants, in 1 second, to run flat the capacitor.

How can one do this while **minimizing the input energy**?

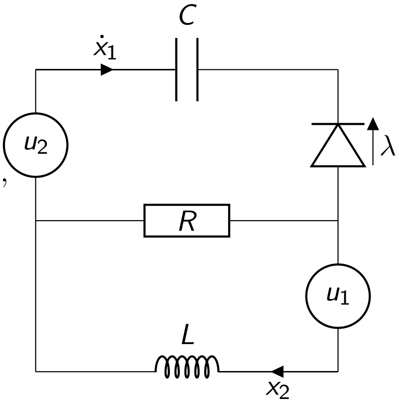


This problem is hardly solved.

Introduction

$$\min \ \text{Energy}(x,u)$$

$$\text{s.t.} \begin{cases} \dot{x}_1(t) = \frac{-1}{RC}x_1(t) + x_2(t) - \frac{1}{R}\lambda(t) + \frac{1}{R}u_2(t), \\ \dot{x}_2(t) = \frac{-1}{LC}x_1(t) - \frac{1}{L}\lambda(t) + \frac{1}{L}(u_2(t) - u_1(t)), \\ 0 \leq \lambda(t) \perp \frac{1}{RC}x_1(t) - x_2(t) \\ \qquad + \frac{1}{R}\lambda(t) - \frac{1}{R}u_2(t) \geq 0, \\ x(0) = (200C, 0A) \\ x(1) = (50C, 0A) \end{cases}$$



Introduction

Optimal control of Linear Complementarity Systems

$$\text{Minimize } C(T, x, u)$$

subject to:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp w(t) = Cx(t) + Dv(t) + Eu(t) \geq 0 \\ \text{Some boundary conditions on } x(0),\ x(T). \end{cases}$$

Two problems presented in this talk:

1. the quadratic optimal control problem,
2. the minimal time problem.

Content

Quadratic cost

Problem:

$$\min \int_0^T \left(x(t)^\intercal Q x(t) + u(t)^\intercal U u(t) + v(t)^\intercal V v(t)\right) dt$$

such that:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp w(t) = Cx(t) + Dv(t) + Eu(t) \geq 0 \\ x(0) = x_0, \; x(T) \text{ free} \end{cases}$$

where $T > 0$, $x : [0, T] \to \mathbb{R}^n$ absolutely continuous, $v : [0, T] \to \mathbb{R}^m$, $u : [0, T] \to \mathbb{R}^{m_u}$, A, B, C, D, E, F, Q, V and U matrices of according dimensions, U supposed symmetric positive definite, Q and V positive semi-definite.

A difficult problem

$$0 \leq v(t) \perp Cx(t) + Dv(t) + Eu(t) \geq 0$$

- Existence of optimal solution not proved (classical Fillipov theory does not apply here due to lack of convexity). Cesari (2012), Theorem 9.2i and onwards
- Special cases arise when $E = 0$ and D P-matrix : switching modes are activated when the state reaches some threshold defined by the complementarity conditions. Georgescu et al. (2012), Passenberg et al. (2013)
- Since u is also involved $\implies$ mixed constraints; relies on some Constraint Qualifications (CQ). Complementarity constraints violate most classical qualifications. Clarke and De Pinho (2010)

Stationarity

How can one find stationarity results, a.k.a. first order necessary optimality conditions?

Two reasons for that:

- Useful for analyzing the solution (continuity, sensitivity...)
- Useful for designing numerical methods.

Stationarity

A first hint comes with a method to compute numerically an approximate solution: the direct method.

$$\min \sum_{i=0}^{N} x_i^\intercal Q x_i + u_i^\intercal U u_i + v_i^\intercal V v_i$$

$$\text{s.t.} \begin{cases} x_{i+1} = x_i + h(Ax_i + Bv_i + Eu_i) \\ 0 \leq v_i \perp Cx_i + Dv_i + Eu_i \geq 0, \quad \forall i \in \{0, ..., N\} \\ x_0 \text{ fixed} \end{cases}$$

$\implies$ Mathematical Program with Equilibrium Constraints (MPEC).

Stationarity

How can one find stationarity results, a.k.a. first order necessary optimality conditions?

$\implies$ Mathematical Program with Equilibrium Constraints (MPEC).

MPEC have their own tailored CQ, and several concepts of stationarity (while classical optimisation problems know only one stationary characterisation). Two of them will be important in this presentation:

- W(eak) stationarity
- S(trong) stationarity

Knowing the type of stationarity $\implies$ good numerical approach.

Stationarity

Really general necessary conditions were obtained in [1]. But as such, they are not really practical (complicated hypothesis, really general equations...).

Can it be enhanced in the case of LCS?

[1] L. Guo and J. J. Ye. Necessary optimality conditions for optimal control problems with equilibrium constraints (2016).

Weak stationarity

Define $S = \{(x,u,v) | 0 \leq v \perp Cx + Dv + Eu \geq 0\}$ and the partition of $\{1, ..., m\}$:

$$I_t^{0+}(x,u,v) = \{i \mid v_i(t) = 0 < (Cx(t) + Dv(t) + Eu(t))_i\}$$

$$I_t^{+0}(x,u,v) = \{i \mid v_i(t) > 0 = (Cx(t) + Dv(t) + Eu(t))_i\}$$

$$I_t^{00}(x,u,v) = \{i \mid v_i(t) = 0 = (Cx(t) + Dv(t) + Eu(t))_i\}$$

Theorem

Let (x^*, u^*, v^*) be a local minimizer of radius R. Suppose $\mathrm{Im}(C) \subseteq \mathrm{Im}(E)$. Then there exist an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ and measurable functions $\lambda^G : \mathbb{R} \to \mathbb{R}^m$, $\lambda^H : \mathbb{R} \to \mathbb{R}^m$ such that the following conditions hold:

1. the transversality condition: $p(T) = 0$

Weak stationarity

Theorem

2. the Weierstrass condition for radius R: for almost every $t \in [t_0, t_1]$,

$$(x^*(t), u, v) \in S, \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u^*(t) \\ v^*(t) \end{pmatrix} \right\| < R(t)$$

$$\Longrightarrow \langle p(t), Ax^*(t) + Bv + Fu) \rangle - \frac{1}{2}(x^*(t)^\intercal Q x^*(t) + u^\intercal U u + v^\intercal V v)$$

$$\leq \langle p(t), Ax^*(t) + Bv^*(t) + Fu^*(t)) \rangle$$

$$- \frac{1}{2}(x^*(t)^\intercal Q x^*(t) + u^*(t)^\intercal U u^*(t) + v^*(t)^\intercal V v^*(t))$$

Weak stationarity

Theorem

3. the Euler adjoint equation: for almost every $t \in [0, T]$,

$$\begin{aligned}
\dot{p}(t) &= -A^\intercal p(t) + Qx^*(t) - C^\intercal \lambda^H(t) \\
0 &= F^\intercal p(t) - Uu^*(t) + E^\intercal \lambda^H(t) \\
0 &= B^\intercal p(t) + \lambda^G(t) + D^\intercal \lambda^H(t) \\
0 &= \lambda_i^G(t), \ \forall i \in I_t^{+0}(x^*(t), u^*(t), v^*(t)) \\
0 &= \lambda_i^H(t), \ \forall i \in I_t^{0+}(x^*(t), u^*(t), v^*(t))
\end{aligned}$$

Remark : These are called W(eak) stationary conditions.

Sufficiency of weak stationarity

Definition

Let (x, u, v) and (x^*, u^*, v^*) be two admissible trajectories (associated with $w = Cx + Dv + Eu$ and w^*, defined the same way). We say that they have the same history on $[0, T]$ if the following condition holds for almost every $t \in [0, T]$ and for all $i \in \{1, ..., m\}$:

$$[v_i(t) = 0 \iff v_i^*(t) = 0] \text{ and } [w_i(t) = 0 \iff w_i^*(t) = 0]$$

Theorem

Suppose that (x^*, u^*, v^*) is an admissible W-stationary trajectory. Then, (x^*, u^*, v^*) is a minimizer among all admissible trajectories having the same history.

Euler equation

How can one solve the following BVP?

$$
\begin{aligned}
\dot{x} &= Ax + Bv + Fu \\
\dot{p} &= -A^{\mathsf{T}} p + Qx - C^{\mathsf{T}} \lambda^{H} \\
0 &= F^{\mathsf{T}} p - Uu + E^{\mathsf{T}} \lambda^{H} \\
0 &= B^{\mathsf{T}} p + \lambda^{G} + D^{\mathsf{T}} \lambda^{H} \\
0 &= \lambda_i^{G}(t),\ \forall i \in I_t^{+0}(x(t), u(t), v(t)) \\
0 &= \lambda_i^{H}(t),\ \forall i \in I_t^{0+}(x(t), u(t), v(t)) \\
x_0 &= x(0), \\
0 &= p(T)
\end{aligned}
$$

Euler equation

How can we solve the following BVP?

$$
\begin{aligned}
\dot{x} &= Ax + Bv + Fu \\
\dot{p} &= -A^\intercal p + Qx - C^\intercal \lambda^H \\
0 &= F^\intercal p - Uu + E^\intercal \lambda^H \rightarrow \text{ isolate } u \\
0 &= B^\intercal p + \lambda^G + D^\intercal \lambda^H \rightarrow \text{ isolate } \lambda^G
\end{aligned}
$$

$$
\boxed{\begin{aligned}
0 &= \lambda_i^G(t), \ \forall i \in I_t^{+0}(x(t), u(t), v(t)) \\
0 &= \lambda_i^H(t), \ \forall i \in I_t^{0+}(x(t), u(t), v(t))
\end{aligned}}
$$

$$
\boxed{\begin{aligned}
x_0 &= x(0), \\
0 &= p(T)
\end{aligned}}
$$

Strong stationarity

$$0 = \lambda_i^G(t), \ \forall i \in I_t^{+0}(x(t), u(t), v(t))$$
$$0 = \lambda_i^H(t), \ \forall i \in I_t^{0+}(x(t), u(t), v(t))$$

We miss a piece of information: what happens on I_t^{00} ?

Proposition

Let (x^*, u^*, v^*) be a local minimizer and suppose E invertible. Then (x^*, u^*, v^*) is strongly stationary, meaning:

$$\lambda_i^G(t) \geq 0, \lambda_i^H(t) \geq 0, \ \forall i \in I_t^{00}(x(t), u(t), v(t))$$

Strong stationarity

$$
\boxed{
\begin{aligned}
0 &= \lambda_i^G(t), && \forall i \in I_t^{+0}(x(t),u(t),v(t))\\
0 &= \lambda_i^H(t), && \forall i \in I_t^{0+}(x(t),u(t),v(t))\\
\lambda_i^G(t) \geq 0, &\lambda_i^H(t) \geq 0, && \forall i \in I_t^{00}(x(t),u(t),v(t))
\end{aligned}
}
$$

Almost like a linear complementarity problem!

Strong stationarity

Theorem

Let (x^*, u^*, v^*) be a local minimizer and suppose E invertible. Fix an arbitrary $r > 0$. Then there exist an arc p and measurable functions $\beta : [0, T] \to \mathbb{R}^m$, $\zeta : [0, T] \to \mathbb{R}$ such that, $u^*(t) = U^{-1}\left(F^\intercal p(t) + E^\intercal \beta(t) - (\zeta(t) + r)E^\intercal v^*(t)\right)$ and:

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ p \end{pmatrix} + \mathcal{B} \begin{pmatrix} \beta \\ v^* \end{pmatrix}$$

$$\begin{cases} 0 \leq \begin{pmatrix} \beta \\ v^* \end{pmatrix} \perp \mathcal{D} \begin{pmatrix} \beta \\ v^* \end{pmatrix} + \mathcal{C} \begin{pmatrix} x \\ p \end{pmatrix} \geq 0 \\ \beta \geq r v^* \\ x(0) = x_0, \; p(T) = 0 \end{cases}$$

How to solve a BVP LCS

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ p \end{pmatrix} + \mathcal{B} \begin{pmatrix} \beta \\ v \end{pmatrix}$$

$$\begin{cases} 0 \leq \begin{pmatrix} \beta \\ v \end{pmatrix} \perp \mathcal{D} \begin{pmatrix} \beta \\ v \end{pmatrix} + \mathcal{C} \begin{pmatrix} x \\ p \end{pmatrix} \geq 0 \\ \beta \geq rv \\ \boxed{x(0) = x_0, \ p(T) = 0} \end{cases}$$

Numerically, we usually do shooting: find the good $p(0) = p_0$ such that the computed solution $p(t; p_0)$ complies with $p(T; p_0) = 0$: nonsmooth Newton method.

- Need for an initial guess close enough
- How to compute a sensitivity matrix for $p(T; \cdot)$?

The hybrid method

Since it is based on a Newton method: two steps.

1. One solves, roughly, the optimal control problem with the Direct method $\Longrightarrow$ rough idea of the solution.
2. One refines the solution by solving the necessary conditions.

Direct method + stationarity conditions = Hybrid method

A 1D example

$$\boxed{\begin{aligned}&\dot{x} = ax + bv + fu\\ &0 \leq v \perp dv + eu \geq 0\\ &x(0) = x_0\end{aligned}}$$

We can show that the (strong) stationary solution in this case is given by:

$$p(t) = \left[\cosh(\sqrt{\gamma}t) - \frac{a}{\sqrt{\gamma}}\sinh(\sqrt{\gamma}t)\right]p(0) + \frac{\sinh(\sqrt{\gamma}t)}{\sqrt{\gamma}}x(0)$$

$$p(0) = -\frac{\sinh(\sqrt{\gamma}T)}{\sqrt{\gamma}\cosh(\sqrt{\gamma}T) - a\sinh(\sqrt{\gamma}T)}x(0).$$

$$u(t) = \begin{cases} fp(t) & \text{if } efp(0) \geq 0,\\ \left(f - \frac{eb}{d}\right)p(t) & \text{if } efp(0) \leq 0.\end{cases}$$

$$x(t) = \dot{p}(t) + ap(t).$$

A 1D example

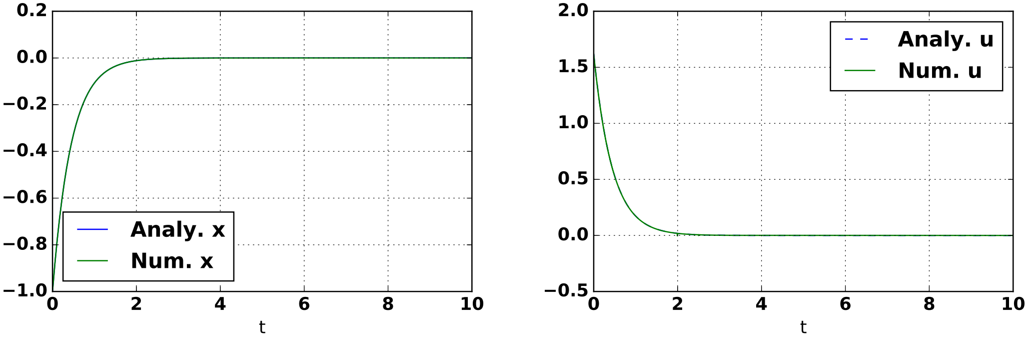


Figure: Solution via indirect method : state x and control u, on $[0, 10]$. $a = 1, b = 0.5, d = 1, e = -2, f = 3, x(0) = -1$. Initial guess with direct method and 300 nodes. Hybrid method with 10 000 nodes and 20 intervals of shooting. Obtained in 54s. (In order to have this same precision with the direct method : 453s.)

Content

Minimal time

Let us review a different problem:

$$\min \; T^*$$
$$\text{s.t.} \; \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp Cx(t) + Dv(t) + Eu(t) \geq 0, \\ u(t) \in \mathcal{U} \\ x(0) = x_0, \; x(T^*) = x_f. \end{cases}$$

where $\mathcal{U}$ is a finite union of polyhedral compact convex sets, and D is a P-matrix. Since u is now constrained: no more possibility to have strong stationarity and do the same manipulations. One still could have a weaker result, but not really useful as is.

A bang-bang property

$$\min \; T^*$$
$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t) \\ 0 \leq v(t) \perp Dv(t) + Eu(t) \geq 0, \\ u(t) \in \mathcal{U} \\ x(0) = x_0, \; x(T^*) = x_f. \end{cases}$$

Denote by $\Omega = \{(u, v) \in \mathcal{U} \times \mathbb{R}^m | 0 \leq v \perp Dv + Eu \geq 0\}$, and $\mathrm{Acc}_\Omega(x_0, t)$ the accessible set at time t, starting from x_0 with controls having values in Ω.

A bang-bang property

Notations: For an index set $\alpha \subseteq \{1, ..., m\}$,
$\mathbb{R}^m_\alpha = \{q \in \mathbb{R}^m \mid q_\alpha \geq 0, q_{\overline{m}\setminus\alpha} \leq 0\}$
$E^{-1}\mathbb{R}^m_\alpha = \{\tilde{u} \in \mathbb{R}^m | E\tilde{u} \in \mathbb{R}^m_\alpha\}$ (E is not necessarily invertible)

Theorem

For a certain $\alpha \subseteq \{1, ..., m\}$, denote by $\mathcal{E}_\alpha$ the set:

$$\mathcal{E}_\alpha = \left\{ (u, v) \in \mathrm{Ext}\left(\mathcal{U} \cap E^{-1}\mathbb{R}^m_\alpha\right) \times \mathbb{R}^m \middle| \begin{array}{l} v_\alpha = 0, \;\; D_{\overline{\alpha}\bullet} v + E_{\overline{\alpha}\bullet} u = 0, \\ v \geq 0, Dv + Eu \geq 0 \end{array} \right\}$$

and by $\mathcal{E}$ the set $\mathcal{E} = \bigcup_{\alpha \subseteq \overline{m}} \mathcal{E}_\alpha$. Then, for all $t > 0$ and all $x_0 \in \mathbb{R}^n$,

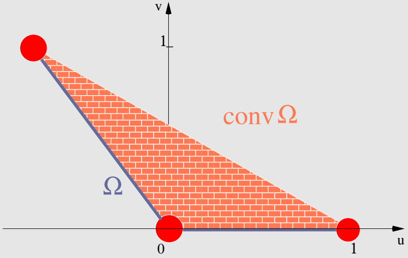
$$\mathrm{Acc}_\Omega(x_0, t) = \mathrm{Acc}_\mathcal{E}(x_0, t)$$

A bang-bang property

Example

$$\min \ T^*$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = ax(t) + bv(t) + fu(t) \\ 0 \leq v(t) \perp v(t) + u(t) \geq 0 \\ u(t) \in \mathcal{U} = [-1, 1] \\ (x(0), x(T^*)) = (x_0, x_f), \end{cases}$$



In this case, $\mathcal{E} = \{(-1,1), (0,0), (1,0)\}$. We can therefore search for the optimal solution with controls (u, v) in $\mathcal{E}$.

One can prove that, under complete controllability of the system: $(u^*, v^*) \equiv (-1, 1)$ or $(1, 0)$.

Hamilton-Jacobi-Bellman

Remark

One can show that the optimal time T^* is characterized by the HJB equation:

$$\begin{cases} z + H(x, \nabla z) = 1 & \text{in } \mathbb{R}^n \backslash \{x_f\}, \\ z = 0 & \text{on } \{x_f\}, \end{cases}$$

where

$$H(x,p) = \sup_{(u,v)\in\Omega} \langle -p, Ax + Bv + Fu \rangle.$$

Further details can be found in the manuscript.

Conclusion

- First stationarity results, that we can use analytically and numerically.
- Numerical algorithms working fast, even with high precision.
- A code was developed for this and is available online (`https://gitlab.inria.fr/avieira/optLCS`).

What is left to be done: a thousand things!

- The stationarity LCS, even in this case, still is not entirely analysed.
- When the dimension of the complementarity becomes high: the numerical resolution fails.
- Get rid of some assumptions (E invertible, D non P-matrix, $C \neq 0$,...).
- ...

Bounded Slope Condition

There exists a positive measurable function k_S such that for almost every $t \in [0, T]$, the bounded slope condition holds:

$$(x, w) \in S_*^{\varepsilon,R}(t), (\alpha, \beta) \in \mathcal{N}_{S(t)}^P(x, w) \implies \|\alpha\| \leq k_S(t)\|\beta\|.$$

Strong stationarity

Define, for two scalars ζ and r:

$$\mathcal{A} = \begin{pmatrix} A & FU^{-1}F^\intercal \\ Q & -A^\intercal \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} FU^{-1}E^\intercal & B - (\zeta + r)FU^{-1}E^\intercal \\ -C^\intercal & (\zeta + r)C^\intercal \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} C & EU^{-1}F^\intercal \\ \zeta C & \zeta EU^{-1}F^\intercal - B^\intercal \end{pmatrix}$$

$$\mathcal{D} = \begin{pmatrix} EU^{-1}E^\intercal & D - (\zeta + r)EU^{-1}E^\intercal \\ \zeta EU^{-1}E^\intercal - D^\intercal & \zeta D + (\zeta + r)\left(D^\intercal - \zeta EU^{-1}E^\intercal\right) \end{pmatrix}$$

How to solve a BVP LCS

$$\boxed{\begin{array}{l} \dot{z} = \mathcal{A}z + \mathcal{B}\Lambda \\ 0 \leq \Lambda \perp \mathcal{D}\Lambda + \mathcal{C}z \geq 0 \end{array}}$$

Denote $\mathcal{T}_h(z)$ a linear Newton Approximation to the solution Λ of the LCP. Then, a linear Newton approximation for the solution map $z(T,\cdot)$ can be obtained by solving the DI in matrix function:

$$\dot{J}(t) \in \mathcal{A}J(t) + (\text{co } \mathcal{T}_h(z(t;\xi)))J(t),\ J(0) = I$$

... But it supposes that $\mathcal{B}$ SOL$(\mathcal{D},\mathcal{C}z)$ is a singleton for all $z \in \mathbb{R}^{2n}$ (which we can not prove).

JS Pang, D. Stewart, Solution dependence on initial conditions in differential variational inequalities (2009)

Compare direct and indirect method

Let us compare the time of computation using the Direct Method and the Hybrid Approach (rough direct + refinements with indirect) in this example:

$$\min \int_0^1 \left(\|x(t)\|_2^2 + 25\|u(t)\|_2^2\right) dt,$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} v(t) + \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} u(t), \\ 0 \leq v(t) \perp \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} x(t) + v(t) + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} u(t) \geq 0, \quad \text{a.e. on } [0,1] \\ x(0) = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix},\ x(T) \text{ free,} \end{cases}$$

Compare direct and indirect method

h_D	Time spent (s)
10^{-2}	1.31
10^{-3}	37.50
10^{-4}	400.65
10^{-5}	∞
10^{-6}	∞

Table: Time spent With Direct Method

Parameters	Time spent (s)
$h_D = 10^{-1}$, $h_I = 10^{-2}$, $n_S = 5$	1.39
$h_D = 10^{-1}$, $h_I = 10^{-3}$, $n_S = 10$	11.26
$h_D = 10^{-2}$, $h_I = 10^{-4}$, $n_S = 20$	97.56
$h_D = 10^{-3}$, $h_I = 10^{-5}$, $n_S = 50$	1 298.62
$h_D = 10^{-4}$, $h_I = 10^{-6}$, $n_S = 100$	32 163.36

Table: Time spent with Hybrid approach

Lavrentiev effect

Some examples show that the set of absolutely continuous functions might be too narrow.

$$\min \int_0^{10} \left(\|x(t)\|_2^2 + u(t)^2\right) dt,$$

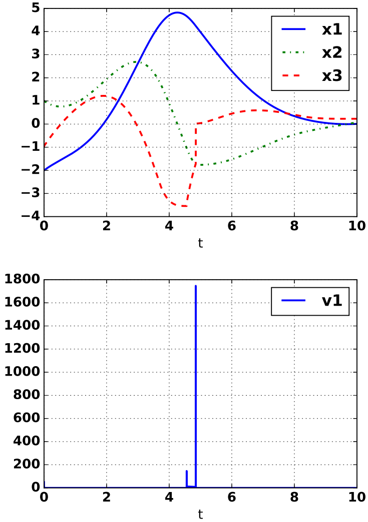
$$\text{s.t.} \begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ v(t) + u(t) \end{pmatrix}, \\ 0 \leq v(t) \perp \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) + u(t) \geq 0, \qquad \text{a.e. on } [0, 10] \\ x(0) = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \\ x(T) \text{ free}, \end{cases}$$

Lavrentiev effect

$$\min \int_0^{10} \left(\|x(t)\|_2^2 + u(t)^2\right) dt,$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ v(t) + u(t) \end{pmatrix}, \\ 0 \leq v(t) \perp \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x(t) + u(t) \geq 0, \\ x(0) = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \\ x(T) \text{ free}, \end{cases}$$

Resolution with Direct Method and relaxation of the complementarity. Library used: IPOPT and CasADI. $h = 10^{-3}$.



Minimal time

Theorem

Suppose:

- either $C = 0$,
- or D is a diagonal matrix with positive entries.

Let (x^*, u^*, v^*) be a local minimizer for the minimal time problem. Then (x^*, u^*, v^*) is W-stationary; i.e. there exist an arc $p : [0, T^*] \to \mathbb{R}^n$, a scalar $\lambda_0 \in \{0, 1\}$ and multipliers $\lambda^G, \lambda^H : [0, T^*] \to \mathbb{R}^m$ such that:

$$(\lambda_0, p(t)) \neq 0 \ \forall t \in [0, T^*]$$

and:

Minimal time

Theorem

$$\dot{p}(t) = -A^\intercal p(t) - C^\intercal \lambda^H(t)$$

$$0 = B^\intercal p + D^\intercal \lambda^H + \lambda^G$$

$$0 \in -F^\intercal p - E^\intercal \lambda^H + \mathcal{N}^C_{\mathcal{U}}(u^*(t))$$

$$\lambda_i^G(t) = 0, \ \forall i \in I_t^{+0}(x^*, u^*, v^*)$$

$$\lambda_i^H(t) = 0, \ \forall i \in I_t^{0+}(x^*, u^*, v^*)$$

Minimal time

Also, since the system is linear, we know there exist a second set of multipliers η^G and η^H such that:

$$0 = B^\intercal p + D^\intercal \eta^H + \eta^G$$

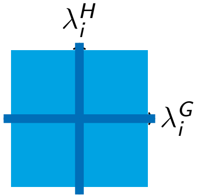
$$0 \in -F^\intercal p - E^\intercal \eta^H + \mathcal{N}_{\mathcal{U}}^C(u^*(t))$$

$$\eta_i^G(t) = 0, \ \forall i \in I_t^{+0}(x^*, u^*)$$

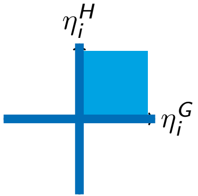
$$\eta_i^H(t) = 0, \ \forall i \in I_t^{0+}(x^*, u^*)$$

$$\eta_i^G \eta_i^H = 0 \text{ or } \eta_i^G > 0, \ \eta_i^H > 0, \ \forall i \in I_t^{00}(x^*, u^*)$$

... But they can be different from the corresponding λ^G and λ^H on a subset of $[0, T^*]$ of positive measure.



(a) W-stationarity



(b) M-stationarity

A bang-bang property

This second example, close to the previous one, suggests that the hypothesis of D P-matrix can possibly be relaxed.

Example

$$\min \; T^*$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t), \\ 0 \leq v(t) \perp -v(t) + u(t) \geq 0 \\ u(t) \in \mathcal{U} = [-1, 1] \\ (x(0), x(T^*)) = (x_0, x_f), \end{cases}$$

In this case, one can prove that $\mathcal{E} = \{(0,0),(1,0),(1,1)\}$ also works for covering the entire accessibility set.