Optimal control of parabolic equations by spectral decomposition*

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Parabolic equation

- $\mathcal{H}$: separable Hilbert space
- $\mathcal{A}: \mathcal{H} \to \mathcal{H}$ closed, positive semidefinite, self-adjoint linear operator with dense domain $D(\mathcal{A})$ and compact resolvent
- given the initial datum $u \in \mathcal{H}$, $\mathcal{S}_t u := y(t)$ where

$$(\mathcal{E}) \qquad \begin{cases} \frac{d}{dt} y(t) = -\mathcal{A} y(t) & \text{for } t > 0 \\ y(0) = u. \end{cases}$$

Remark

$\{\mathcal{S}_t\}_{t \geq 0} : \mathcal{H} \to \mathcal{H}$ is a strongly continuous semigroup of non-expansive linear operators

Main example: heat equation

- $\mathcal{H} = L^2(\Omega)$, for Ω bounded domain;
- $\mathcal{A} = -\Delta$ with $D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega)$ (Dirichlet Laplacian)

Then $\mathcal{S}_t u = y(t)$ where

$$\begin{cases} \frac{d}{dt}y(x,t) - \Delta y(x,t) & = & 0 & x \in \Omega, t > 0 \\ y(x,t) & = & 0 & x \in \partial\Omega, t > 0 \\ y(x,0) & = & u(x) & x \in \Omega, t = 0 \end{cases}$$

Definitions

Definition

Given a final target $y^T \in \mathcal{H}$, system $(\mathcal{E})$ is *exactly controllable* to y^T in time $T > 0$ if there exists an initial datum $u \in \mathcal{H}$ such that

$$\mathcal{S}_T u = y^T$$

Definition

System $(\mathcal{E})$ is *approximatately controllable* if for every final time $T > 0$, final target $y^T \in \mathcal{H}$ and tolerance $\varepsilon > 0$, there exists $u \in \mathcal{H}$ such that

$$\|S_T u - y^T\|_{\mathcal{H}} \leq \varepsilon \tag{1}$$

Exact controllability

Remark

In general, system $(\mathcal{E})$ is NOT exactly controllable to any state $y^T \in \mathcal{H}$

Example (Regularizing effect)

As in the case for the Dirichlet Laplacian in $L^2(\Omega)$,

if

- $\psi : \mathcal{H} \to \overline{\mathbb{R}}$ proper, differentiable and convex function;
- $\mathcal{A} = \nabla\psi$,

then

$$\mathcal{S}_t u \in D(\mathcal{A}) \quad \text{for all } t > 0 \text{ and all } u \in \mathcal{H};$$

in particular, no target state $y^T \in \mathcal{H} \setminus D(\mathcal{A})$ can be attained

Approximate controllability

Proposition

The range of $\mathcal{S}_t$ is dense in $\mathcal{H}$ for all $t > 0$; as a consequence, the system $(\mathcal{E})$ is approximately controllable

Proof: (orthogonal argument)

- as $\mathcal{A}$ is self-adjoint, $\mathcal{S}_t^* = \mathcal{S}_t$
- if $\langle y, \mathcal{S}_t x\rangle_{\mathcal{H}} = 0$ for all $x \in \mathcal{H}$, then for all x we have

$$\langle \mathcal{S}_t y, x\rangle_{\mathcal{H}} = \langle \mathcal{S}_t^* y, x\rangle_{\mathcal{H}} = \langle y, \mathcal{S}_t x\rangle_{\mathcal{H}} = 0$$

- then $\mathcal{S}_t y = 0$ and, by uniqueness, $y = 0$
- we have proven that

$$\langle y, z\rangle_{\mathcal{H}} = 0 \ \ \forall z \in \mathrm{Rank}\,(\mathcal{S}_t) \quad \Rightarrow \quad y = 0;$$

the statement follows from the Hahn-Banach theorem

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Formulation of the optimal control problem

We search for an initial condition $u \in \mathcal{H}$ such that (1) is satisfied and a given functional $J : \mathcal{H} \to \mathbb{R}$ is minimized. More precisely, given a final time $T > 0$, a tolerance $\varepsilon > 0$ and a final target $y^T \in \mathcal{H}$, find

$$(\mathcal{P}) \qquad \hat{u} \in \underset{u \in \mathcal{H}}{\arg\min} \left\{ J(u) : \quad \|\mathcal{S}_T u - y^T\|_{\mathcal{H}} \leq \varepsilon \right\},$$

where

$$J(u) = \frac{\alpha}{2}\|u\|_{\mathcal{H}}^2 + \frac{1}{2}\int_0^T \beta(t)\,\|\mathcal{S}_t u - y^d(t)\|_{\mathcal{H}}^2 \, dt$$

- $\alpha > 0$: cost-of-the-control parameter;
- $\beta \in L^\infty(0, T; \mathbb{R}_+)$: regulation parameter;
- $y^d \in L^2(0, T; \mathcal{H})$: desidered trajectory

Classical approach (HUM) vs Trajectory regulation

- If $\beta \equiv 0$, problem $(\mathcal{P})$ can be solved by the Hilbert Uniqueness Method (HUM), which relies on the Fenchel-Rockafellar dual problem for convex optimisation:

$$(\mathcal{D}) \qquad -\min_{v\in\mathcal{H}}\left\{\frac{1}{\alpha}\|\mathcal{S}_T^* v\|_{\mathcal{H}}^2 + \langle v,\ y^T\rangle_{\mathcal{H}} + \varepsilon\|v\|_{\mathcal{H}}\right\};$$

- For general β, the numerical treatment of the dual problem is too complex: we propose a direct, explicit formula for the solution in terms of the given problem data.

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Reformulation

Define the ball $\bar{B} = \overline{B_\varepsilon\left(y^T\right)}$ (non-empty, convex and close) and the *indicator function*

$$I_{\bar{B}}\left(y\right) = \begin{cases} 0 & \text{if } y \in \bar{B} \\ +\infty & \text{else.} \end{cases}$$

Then $(\mathcal{P})$ is equivalent to

$$\hat{u} \in \arg\min_{u \in \mathcal{H}} \left\{ J\left(u\right) + I_{\bar{B}}\left(\mathcal{S}_T u\right) \right\}$$

Proposition

There exists a unique solution of $(\mathcal{P})$

Proof: $J + I_{\bar{B}} \circ \mathcal{S}_T$ is proper, strongly-convex and lower-semicontinuos

Optimality conditions

By (generalized) Fermat rule, Moreau-Rockafellar theorem and chain rule:

$$0 \in \partial \left(J + I_{\bar{B}} \circ \mathcal{S}_T \right) (\hat{u}) = \alpha \hat{u} + \underbrace{\int_0^T \beta(t) \mathcal{S}_t^* \left(\mathcal{S}_t \hat{u} - y^d \right) \, dt}_{M\hat{u} - f} + \mathcal{S}_T^* \mathcal{N}_{\bar{B}} \left(\mathcal{S}_T \hat{u} \right),$$

where $\mathcal{N}$ is the classical normal cone of convex analysis; for $\hat{y} = \mathcal{S}_T \hat{u}$,

$$\boxed{\exists \hat{g} \in \mathcal{N}_{\bar{B}} (\hat{y}) : \quad \alpha \hat{u} + M \hat{u} = f - \mathcal{S}_T^* \hat{g}}$$

Remark

$\hat{y} \in \partial \bar{B}$; so $\hat{g} \in \mathcal{N}_{\bar{B}} (\hat{y})$ has the form $\hat{g} = \hat{\lambda} \left(\hat{y} - y^T \right)$ for some $\hat{\lambda} > 0$

Analysis in the final-state space

Define the sublevel sets of $\psi = J \circ \mathcal{S}_{-T}$:

$$W_c = \{y \in \mathcal{H} : \; y = \mathcal{S}_T u \text{ for some } u \in \mathcal{H} \text{ with } J(u) \leq c\}$$

Remark

Let $\tilde{u}$ denote the (unique) solution of the *unconstrained problem* and $\tilde{y} = \mathcal{S}_T \tilde{u}$ the corresponding final state. Then

- For $c < \tilde{c} = J(\tilde{u})$, W_c is empty;
- $(W_c)_{c \geq \tilde{c}}$ is a nested family of nonempty closed convex sets centered at $\tilde{y}$ and incresing with c;
- for $\hat{c} = J(\hat{u})$ (*cost of the optimal control*), the target ball is hit for the first time by $W_{\hat{c}}$ and the intersection is the optimal final state:

$$W_{\hat{c}} \cap \bar{B} = \{\hat{y}\}$$

Final configuration space

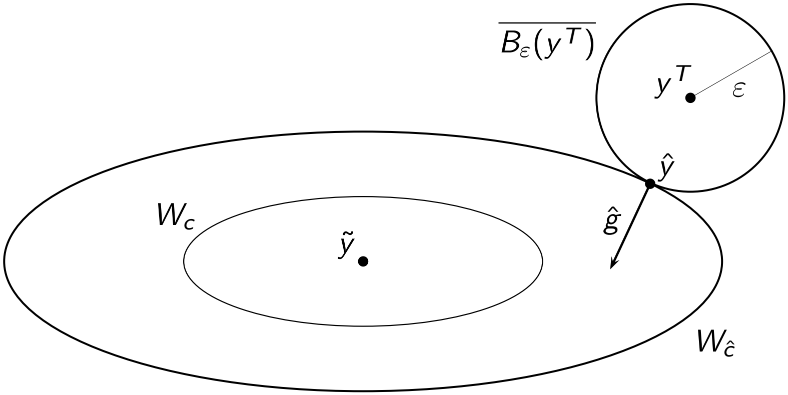

Figure: Optimal final state $\hat{y} = \Pi_{\hat{c}}\left(y^T\right)$ and normal $\hat{g} = \hat{\lambda}\left(\hat{y} - y^T\right)$.

Geometrical interpretation (1)

Recall that $\tilde{u}$ denote the (unique) solution of the *unconstrained problem*

$$(\tilde{\mathcal{P}}) \qquad \tilde{u} \in \arg\min_{u \in \mathcal{H}} \; J(u)$$

and $\tilde{y} = \mathcal{S}_T \tilde{u}$ the corresponding final state

Proposition

- *If* $\|\tilde{y} - y^T\|_{\mathcal{H}} \leq \varepsilon$, *then* $\hat{y} = \tilde{y}$;
- *Otherwise,* $\hat{y} \in \partial B$

So from now on we will suppose that $\|\tilde{y} - y^T\|_{\mathcal{H}} > \varepsilon$

Geometrical interpretation (2)

If $\Pi_{\hat{c}}$ is the projection operator onto $W_{\hat{c}}$,

$$\hat{y} = \Pi_{\hat{c}}\left(y^T\right)$$

Then, for some $\hat{\gamma} > 0$,

$$\hat{y} - y^T = -\hat{\gamma}\nabla\psi\left(\hat{y}\right)$$

Proposition

For $\varepsilon < \|\tilde{y} - y^T\|_{\mathcal{H}}$, the pair $(\hat{y}, \hat{\gamma}) \in \mathcal{H} \times \mathbb{R}_{++}$ verifies the system

$$\begin{cases} \hat{y} - y^T = -\hat{\gamma}\nabla\psi\left(\hat{y}\right); \\ \|\hat{y} - y^T\|_{\mathcal{H}} = \varepsilon \end{cases}$$

Spectral decomposition

Lemma

Under our hypothesis, there exists $(\varphi_n, \lambda_n)_{n\in\mathbb{N}}$ *sequence of eigenfunction - eigenvalue for* $\mathcal{A}$ *such that*

- $(\varphi_n)_{n\in\mathbb{N}}$ *is an orthonormal basis of* $\mathcal{H}$*;*
- $(\lambda_n)_{n\in\mathbb{N}}$ *is nonnegative, nondecreasing and* $\lim_{n\to+\infty} \lambda_n = +\infty$

We propose a *direct (primal) method* to characterize the solution of $(\mathcal{P})$ based on spectral decomposition. This allows us to reduce the optimal control problem in infinite dimension to the study of a scalar equation.

Fourier's representation

Change of variable: considering the Fourier coefficients of u and y, we know by *separation of variables* on equation $(\mathcal{E})$ that

$$y_n = u_n \; e^{-\lambda_n T} \quad \text{and so} \quad u_n = y_n \; e^{\lambda_n T}$$

Then we can re-write problem $(\mathcal{P})$ in the (decomposed) final-state space:

- Objective functional $J(u)$:

$$\psi(y) = \sum_n \left(a_n y_n^2 + b_n y_n + c_n \right),$$

  for some coefficients a_n, b_n, c_n (computable from data α, β, y_n^d)

- Constraint $\mathcal{S}_T u \in \partial B$:

$$\sum_n \left(y_n - y_n^T \right)^2 = \varepsilon^2$$

- From relation $\hat{y} - y^T = -\hat{\gamma}\nabla\psi(\hat{y})$ and the computation $(\nabla\psi(\hat{y}))_n = 2a_n\hat{y}_n + b_n$, we obtain

$$\hat{y}_n = \frac{y_n^T - \hat{\gamma} b_n}{1 + 2\hat{\gamma} a_n}$$

- The latter can be substituted in $\sum_n \left(\hat{y}_n - y_n^T\right)^2 = \varepsilon^2$ to obtain that $\hat{\gamma}$ is the solution of the scalar equation

$$G(\gamma) := \sum_n \left(\frac{\gamma \ \left(2a_n y_n^T + b_n\right)}{1 + 2\gamma a_n} \right)^2 = \varepsilon^2. \tag{2}$$

Proposition

- $G(0) = 0$;
- $\lim_{\gamma \to +\infty} G(\gamma) = \|\tilde{y} - y^T\|_{\mathcal{H}}^2$;
- $G'(\gamma) > 0$ *for every* $\gamma > 0$

In particular, (2) *has exactly one positive solution for* $\varepsilon \in \left(0, \|\tilde{y} - y^T\|_{\mathcal{H}}\right)$.

Main result

Theorem

Let $\varepsilon \in \big(0, \|\tilde{y} - y^T\|_{\mathcal{H}}\big)$. The solution of the optimal control problem $(\mathcal{P})$ is given by

$$\hat{u} = \sum_n e^{\lambda_n T} \hat{y}_n \, \varphi_n,$$

where the Fourier coefficients of the optimal final state $\hat{y}$ are given by

$$\hat{y}_n = \frac{y_n^T - \hat{\gamma} b_n}{1 + 2\hat{\gamma} a_n}$$

and the constant $\hat{\gamma} > 0$ is the unique solution of the scalar equation

$$G(\gamma) := \sum_n \left(\frac{\gamma \ \left(2 a_n y_n^T + b_n\right)}{1 + 2\gamma a_n} \right)^2 = \varepsilon^2.$$

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Truncation and algorithm

Obtained explicit formulas incorporate infinite series: truncation is required for numerics. Fixed $N \in \mathbb{N}$,

- compute (by bisection method) $\hat{\gamma}_N > 0$ as the unique solution of

$$G_N(\gamma) := \sum_{n=1}^{N} \left(\frac{\gamma \ \left(2a_n y_n^T + b_n\right)}{1 + 2\gamma a_n} \right)^2 = \varepsilon^2$$

- compute, for $n = 1, \ldots, N$,

$$\hat{y}_n^N = \frac{y_n^T - \hat{\gamma}_N b_n}{1 + 2\hat{\gamma}_N a_n}$$

- define the approximated solution $\hat{u}^N$ as

$$\hat{u}^N = \sum_{n=1}^{N} e^{\lambda_n T} \hat{y}_n^N \ \varphi_n$$

Approximation estimate

Denote the truncated series representation of the target final state and the reference trajectory as

- $y^{T,N} = \sum_{n=0}^{N} y_n^T \varphi_n$;
- $y^{d,N}(t) = \sum_{n=0}^{N} y_n^d(t)\varphi_n$.

In the next, we provide a criterion to choose N, given a desired precision on the optimal final state:

Theorem

Let $\varepsilon < \|\tilde{y} - y^T\|_{\mathcal{H}}$ and let $\hat{y}^N$ be a truncated approximation of the optimal final state. Then the following estimate holds:

$$\|\hat{y}^N - \hat{y}\|_{\mathcal{H}}^2 \leq 4\|y^T - y^{T,N}\|_{\mathcal{H}}^2 + \frac{4\|\beta\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_N T}}\|y^d - y^{d,N}\|_{L^2(0,T;\mathcal{H})}^2.$$

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Heat equation with Dirichlet boundary condition

We consider $\mathcal{S}_t u = y(t)$, where y is the solution of the PDE

$$\begin{cases} \frac{d}{dt} y - \Delta y = 0 & \Omega \times (0, T) \\ y = 0 & \Omega \times (0, T) \\ y(0) = u & \Omega . \end{cases}$$

We will show two examples:

- the first one, in 1D, concerns not just the energy minimization of the initial state but also the regulation of the trajectory during the time evolution;
- the second one is, on the other hand, a numerical experiment on energy minimization in 2D

Trajectory regulation in 1D

For $\Omega = (0, L)$ and $\mathcal{H} = L^2(0, L)$, the eigenfunctions-eigenvalues of the Dirichlet Laplacian are

$$\varphi_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right), \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2; \qquad k = 1, 2, \ldots$$

We set

- $L = \pi$ and $T = 0.01$;
- $\alpha = 10^{-4}$;
- $\beta(t) = \mathbb{1}_{[t_1, t_2]}(t)$, with $t_1 = T/3$ and $t_2 = 2T/3$;
- $y^d(x, t) = \mathbb{1}_{[x_1^d, x_2^d]}(x)$, with $x_1^d = \pi/5$ and $x_2^d = 2\pi/5$;
- $y^T(x) = \mathbb{1}_{[x_1^T, x_2^T]}(x)$, with $x_1^T = 3\pi/5$ and $x_2^T = 4\pi/5$.

The target trajectory and the final state are illustrated in next Figure.

Data

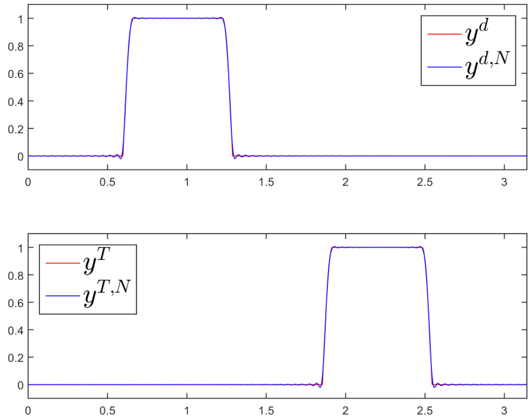

Figure: In red, reference trajectory y^d for the distributed cost (top) and target final state y^T (bottom). In blue, their reconstructions after Fourier decomposition.

Numerical setting

By the Approximation Theorem, to guarantee a priori fixed precision of $\rho = 10^{-2}$ on $\|\hat{y}_N - \hat{y}\|_{L^2}$, we need to use $N = 137$ Fourier coefficients.

For the numerical experiments, we used three different values of ε:

$$\varepsilon^2 = [0.02, 0.5, 0.98] \cdot \varepsilon_N^2,$$

where $\varepsilon_N = \|\tilde{y}^N - y^{T,N}\|_{L^2(0,L)}$.

Execution time: less than 0.3 seconds.

Function G_N

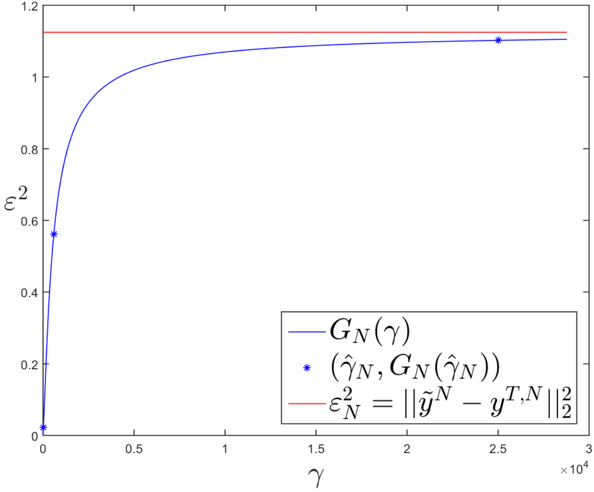

Figure: The function G_N, the chosen values of ε, and the corresponding $\hat{\gamma}_N$.

Numerical results

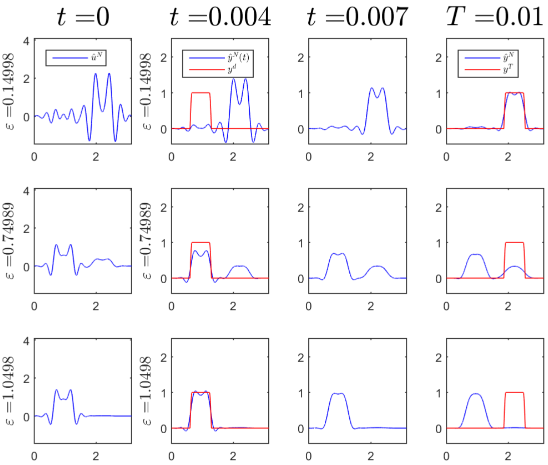

Figure: Computed solution $\mathcal{S}_t \hat{u}^N$ compared with $y^d(t)$ at times $t = 0$, $t = 0.004$ and $t = 0.007$ and the computed final state $\hat{y} = \mathcal{S}_T \hat{u}^N$ compared with y^T.

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Setting

We set

- $\Omega = (0, L_1) \times (0, L_2)$;
- $L_1 = L_2 = 1$ and $T = 0.001$;
- $\alpha = 1$ and $\beta \equiv 0$;
- $\varepsilon^2 = \left[10^{-6}, 0.99\right] \cdot \|y^T\|^2_{L^2}$;
- $N_1 = N_2 = 15$, for a total of 225 Fourier coefficients.

Then

$$(\mathcal{P}) \qquad \hat{u} \in \underset{u \in L^2((0,L_1)\times(0,L_2))}{\arg\min} \left\{ \|u\|_{L^2} : \; \mathcal{S}_T u \in \overline{B_\varepsilon\left(y^T\right)} \right\}$$

Remark

In particular, the unconstrained problem has the trivial solution $\tilde{y} = \tilde{u} = 0$.

Idea of the experiment

Given a reachable final target y^T, we want to investigate the differences between the initial datum u^T that generates the target and the solution of the optimal control problem for various values of ε.

Data

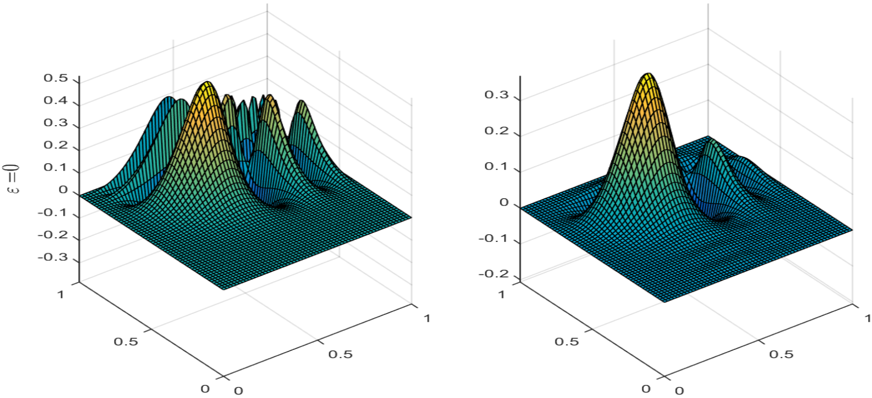
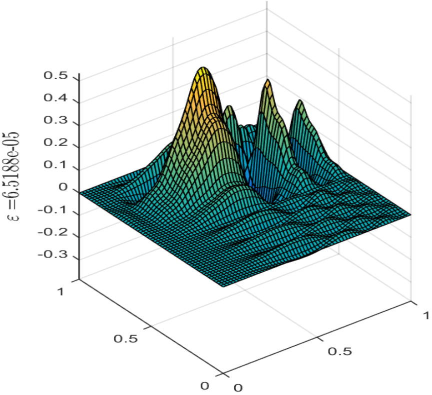


Figure: LEFT: the data u^T that generates the target (corresponding to $\varepsilon = 0$); RIGHT: the final target y^T.

Numerical results: ε small



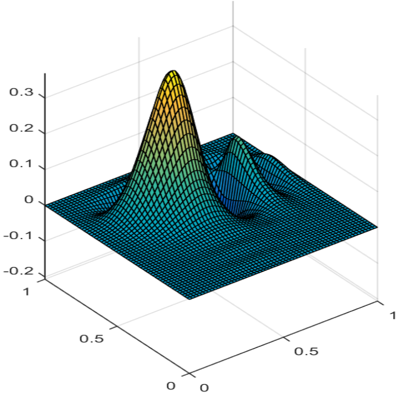


Figure: LEFT: the solution of the optimal control problem $\hat{u}^N$ for $\varepsilon = 10^{-6} \cdot \|y^T\|^2_{L^2}$. RIGHT: corresponding final state $\hat{y}^N$.

Numerical results: ε big

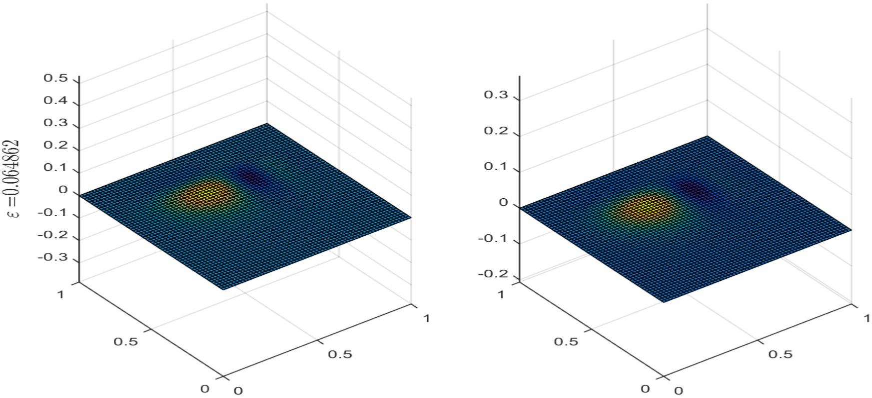


Figure: LEFT: the solution of the optimal control problem $\hat{u}^N$ for $\varepsilon = 0.99 \cdot \|y^T\|^2_{L^2}$. RIGHT: corresponding final state $\hat{y}^N$.

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Conclusions (1)

We presented a new numerical method to determine the initial datum such that

- the final state lies within a prescribed distance to a given target;
- it minimises a cost functional J, comprising an energy term on the control and a regulation term on the trajectory

Approach: *spectral decomposition* of the operator $\mathcal{A}$ governing the parabolic evolution $(\mathcal{E})$

Conclusions (2)

- PRO: one-shot (not iterative) algorithm that reduces the problem to finding the unique zero of a bounded, strictly increasing function in $\mathbb{R}$
- CONTRA: it requires the knowledge of eigenfunctions/eigenvalues.

Then this technique is particularly efficient for *offline-online* procedures:

A) Offline - spectral decomposition of $\mathcal{A}$;

B) Online - almost instantaneous solution of many instances for different data, but with the same operator

Future developments

We are actually working at the extension of the method to

- *wave equation*;
- *distributed control problem:*

$$(\mathcal{P}) \qquad \hat{u} \in \underset{u \in L^2(0,T;\mathcal{H})}{\arg\min} \left\{ J(u) : \quad \| y(T) - y^T \|_{\mathcal{H}} \leq \varepsilon \right\},$$

where

$$(\mathcal{E}) \qquad \begin{cases} \frac{d}{dt} y(t) + \mathcal{A} y(t) &= \mathcal{B}_t u(t) \qquad \text{for } t > 0 \\ y(0) &= 0 \end{cases}$$

and

$$J(u) = \frac{1}{2} \int_0^T \left[\alpha(t) \| u(t) \|_{\mathcal{H}}^2 + \beta(t) \| y(t) - y^d(t) \|_{\mathcal{H}}^2 \right] dt$$

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Thank you for your attention!