Discrete Systolic Inequalities and Decompositions of Triangulated Surfaces*

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Abstract

How much cutting is needed to simplify the topology of a surface? We provide bounds for several instances of this question, for the minimum length of topologically non-trivial closed curves, pants decompositions, and cut graphs with a given combinatorial map in triangulated combinatorial surfaces (or their dual cross-metric counterpart).

Our work builds upon Riemannian systolic inequalities, which bound the minimum length of non-trivial closed curves in terms of the genus and the area of the surface. We first describe a systematic way to translate Riemannian systolic inequalities to a discrete setting, and vice-versa. This implies a conjecture by Przytycka and Przytycki from 1993, a number of new systolic inequalities in the discrete setting, and the fact that a theorem of Hutchinson on the edge-width of triangulated surfaces and Gromov's systolic inequality for surfaces are essentially equivalent.

Then we focus on topological decompositions of surfaces. Relying on ideas of Buser, we prove the existence of pants decompositions of length $O(g^{3/2}n^{1/2})$ for any triangulated combinatorial surface of genus g with n triangles, and describe an $O(gn)$-time algorithm to compute such a decomposition.

Finally, we consider the problem of embedding a cut graph with a given combinatorial map on a given surface. Using random triangulations, we prove (essentially) that, for any choice of combinatorial map of cut graph, there are some surfaces on which any embedding has length superlinear in the number of triangles of the triangulated combinatorial surface.

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1 Introduction

Shortest curves and graphs with given properties on surfaces have been much studied in the recent computational topology literature; a lot of effort has been devoted towards efficient algorithms for computing shortest curves that simplify the topology of the surface, or shortest topological decompositions of surfaces [7, 8, 20–24, 37] (refer also to the recent surveys [13, 19]). These objects provide "canonical" simplifications or decompositions of surfaces, which turn out to be crucial for algorithm design in the case of surface-embedded graphs, where making the graph planar is needed [6, 9, 11, 39]. These topological algorithms are also relevant in a number of applications that deal with surfaces with non-trivial topology, notably in computer graphics and mesh processing, to simplify the topology of a surface [30, 57], for approximation [12] and compression [2] purposes, and to split a surface into planar pieces, for texture mapping [40, 46], surface correspondence [41], parameterization [29], and remeshing [1].

In this paper, we study the worst-case length of such shortest curves and graphs with prescribed topological properties on combinatorial surfaces. An important parameter in topological graph theory is the notion of *edge-width* of an (unweighted) graph embedded on a surface [7, 53], which is the length of the shortest closed walk in the graph that is non-contractible on the surface (cannot be deformed to a single point on the surface). The model question that we study is the following: What is the largest possible edge-width, over all triangulations with n triangles, of a closed orientable surface of genus g? It was known that an upper bound is $O(\sqrt{n/g}\log g)$ [33], and we prove that this bound is asymptotically tight, namely, that some combinatorial surfaces (of arbitrarily large genus) achieve this bound. We also study similar questions for other types of curves (non-separating closed curves, null-homologous but non-contractible closed curves) and for decompositions (pants decompositions, and cut graphs with a prescribed combinatorial map), and give an algorithm to compute short pants decompositions.

We always assume that the surface has *no boundary*, that that the underlying graph of the combinatorial surface is a *triangulation*, and that its edges are *unweighted*; the curves and graphs we seek remain on the edges of the triangulation. Lifting any of these three restrictions transforms the upper bound above to a function with a linear dependency in n. In many natural situations, such requirements hold, such as in geometric modeling and computer graphics, where triangular meshes of closed surfaces are typical and, in many cases, the triangles have bounded aspect ratio (which immediately implies that our bounds apply, the constant in the $O(\cdot)$ notation depending on the aspect ratio).

Most of our results build upon or extend to a discrete setting some known theorems in *Riemannian systolic geometry*, the archetype of which is an upper bound on the systole (the length of shortest non-contractible closed curves—a continuous version of the edge-width) in terms of the square root of the area of a closed Riemannian surface (or more generally the dth root of the volume of an essential Riemannian d-manifold). Riemannian systolic geometry [28, 34] was pioneered by Loewner and Pu [52], reaching its maturity with the fantastic work of Gromov [27].

After the preliminaries (Section 2), we prove three independent results (Sections 3–5), which are described and related to other works below. This paper is organized so as to showcase the more conceptual results before the more technical ones. Indeed, the results of Section 3 exemplify the strength of the connection with Riemannian geometry, while the results in Sections 4 and 5 are perhaps a bit more specific, but feature deeper algorithmic and combinatorial tools.

Systolic inequalities for closed curves on triangulations. Our first result (Section 3) gives a systematic way of translating a systolic inequality in the Riemannian case to the case of triangulations, and vice-versa. This general result, combined with known results from systolic geometry, immediately implies bounds on the length of shortest curves with given topological properties: On a triangulation of genus g with n triangles, some non-contractible (resp., non-separating, resp., null-homologous but non-contractible) closed curve has length $O(\sqrt{n/g}\log g)$, and, moreover, this bound is best possible.

These upper bounds are new, except for the non-contractible case, which was proved by Hutchinson [33] with a worse constant in the $O(\cdot)$ notation. The optimality of these inequalities is also new. Actually, Hutchinson [33] had conjectured that the correct upper bound was $O(\sqrt{n/g})$; Przytycka and Przytycki refuted her conjecture, building, in a series of papers [49–51], examples that show a lower bound of $\Omega(\sqrt{n \log g/g})$. They conjectured in 1993 [50] that the correct bound was $O(\sqrt{n/g}\log g)$; here, we confirm this conjecture.

Short pants decompositions. A pants decomposition is a set of disjoint simple closed curves that split the surface into *pairs of pants*, namely, spheres with three boundary components. In Section 4, we focus on the length of the shortest pants decomposition of a triangulation. As in all previous works, we allow several curves of the pants decomposition to run along a given edge of the triangulation (formally, we work in the cross-metric surface that is dual to the triangulation).

The problem of computing a shortest pants decomposition has been considered by several authors [18,48], and has found satisfactory solutions (approximation algorithms) only in very special cases, such as the punctured Euclidean or hyperbolic plane [18]. Strikingly, no hardness result is known; the strong condition that curves have to be disjoint, and the lack of corresponding algebraic structure, makes the study of short pants decompositions hard [31, Introduction]. In light of this difficulty, it seems interesting to look for algorithms that compute short pants decompositions, even without guarantee compared the optimum solution.

Inspired by a result by Buser [5, Th. 5.1.4] on short pants decompositions on Riemannian surfaces, we prove that every triangulation of genus g with n triangles admits a pants decomposition of length $O(g^{3/2}n^{1/2})$, and we give an $O(gn)$-time algorithm to compute one. In other words, while pants decompositions of length $O(gn)$ can be computed for arbitrary combinatorial surfaces [15, Prop. 7.1], the assumption that the surface is unweighted and triangulated allows for a strictly better bound in the case where $g = o(n)$ (it is always true that $g = O(n)$).

On the lower bound side, some surfaces have no pants decompositions with length $O(n^{7/6-\varepsilon})$, as proved recently by Guth et al. [31] using the probabilistic method: They show that polyhedral surfaces obtained by gluing triangles randomly have this property.

Shortest embeddings of combinatorial maps. Finally, in Section 5, we consider the problem of decomposing a surface using a short cut graph with a prescribed combinatorial map. To build a homeomorphism between two surfaces, a natural approach is to cut both surfaces along a cut graph, and put both disks in correspondence. For this approach to work, however, cut graphs *with the same combinatorial map* are needed. In this direction, Lazarus et al. [38] proved that every surface has a *canonical systems of loops* (a specific combinatorial map of a cut graph with one vertex) with length $O(gn)$, which is worst-case optimal, and gave an $O(gn)$-time algorithm to compute one.

There is, however, no strong reason to focus on canonical systems of loops: It is fairly natural to expect that other combinatorial maps will always have shorter embeddings (in particular, by allowing several vertices on the cut graph instead of just one). However, we prove (essentially) that, for any choice of combinatorial map of a cut graph, there exist triangulations with n triangles on which all embeddings of that combinatorial map have a *superlinear* length, actually $\Omega(n^{7/6-\varepsilon})$ (since n may be $O(g)$, there is no contradiction with the result by Lazarus et al. [38]). In particular, some edges of the triangulation are traversed $\Omega(n^{1/6-\varepsilon})$ times. This result translates to the case of polyhedral surfaces obtained by gluing together n equilateral triangles: In this model, some edges are intersected $\Omega(n^{1/6-\varepsilon})$ times. From the case of cut graphs, we can also deduce the same results for all cellular graph embeddings with prescribed combinatorial maps.

Our proof uses the probabilistic method in the same spirit as the aforementioned article of Guth et al. [31]: We show that combinatorial surfaces obtained by gluing triangles randomly satisfy this property asymptotically almost surely. This also sheds some light on the geometry of these "random surfaces", which have been heavily studied recently [25,42] because of connections to quantum gravity [47] and Belyi surfaces [3]

Another view of our result is via the following problem: Given two graphs G_1 and G_2 cellularly embedded on a surface S, is there a homeomorphism $\varphi : S \to S$ such that G_1 does not cross the image of G_2 too many times? Our result essentially says that, if G_1 is fixed, for most choices of trivalent graphs G_2 with n vertices, for any φ, there will be $\Omega(n^{7/6-\varepsilon})$ crossings between G_1 and $\varphi(G_2)$. This is related to recent preprints [26, 43], where upper bounds are proved for the number of crossings for the same problem, but with sets of disjoint curves instead of graphs. During their proof, Matoušek et al. [43] also encountered the following problem (rephrased here in the language of this paper): For a given genus g, does there exist a *universal* combinatorial map cutting the surface of genus g into a genus zero surface (possibly with several boundaries), and with a linear-length embedding on every such surface? We answer this question in the negative for cut graphs.

2 Preliminaries

2.1 Topology for Graphs on Surfaces

We only recall the most important notions of topology that we will use, and refer to Stillwell [56] or Hatcher [32] for details. We denote by $S_{g,b}$ the (orientable) surface of ***genus*** g with b ***boundaries***, which is unique up to homeomorphism. The surfaces $S_{0,0}$, $S_{0,1}$, $S_{0,2}$, and $S_{0,3}$ are respectively called the ***sphere***, the ***disk***, the ***annulus***, and the ***pair of pants***. Surfaces are assumed to be connected, compact, and orientable unless specified otherwise. The notation ∂S denotes the boundary of S.

A ***path***, respectively a ***closed curve***, on a surface S is a continuous map $p : [0,1] \to S$, respectively $\gamma : \mathbb{S}^1 \to S$. Paths and closed curves are ***simple*** if they are one-to-one. A ***curve*** denotes a path or a closed curve. We refer to Hatcher [32] for the usual notions of homotopy (continuous deformation) and homology. A closed curve is ***contractible*** if it is null-homotopic, i.e., it cannot be continuously deformed to a point. A simple closed curve is contractible if and only if it bounds a disk.

All the graphs that we consider in this paper are multigraphs, i.e., loops are allowed and vertices can be joined by multiple edges. An ***embedding*** of a graph G on a surface S is, informally, a crossing-free drawing of G on S. A graph embedding is ***cellular*** if its faces are homeomorphic to open disks. Euler's formula states that $v - e + f = 2 - 2g - b$ for any graph with v vertices, e edges, and f faces cellularly embedded on a surface S with genus g with b boundaries. A ***triangulation*** of a surface S is a cellular graph embedding such that every face is a triangle. A graph G cellularly embedded on a surface S yields naturally a ***combinatorial map*** M, which stores the combinatorial information of the embedding G, namely, the cyclic ordering of the edges around each vertex; we also say that G is an ***embedding*** of M on S. Two graphs embedded on S have the same combinatorial map if and only if there exists a self-homeomorphism of S mapping one (pointwise) to the other.

A graph G embedded on a surface S is a ***cut graph*** if the surface obtained by cutting S along G is a disk. A ***pants decomposition*** of S is a family of disjoint simple closed curves Γ such that cutting S along all curves in Γ gives a disjoint union of pairs of pants. Every surface $S_{g,b}$ except the sphere, the disk, the annulus, and the torus admits a pants decomposition, with $3g + b - 3$ closed curves.

2.2 Combinatorial and Cross-Metric Surfaces

We now briefly recall the notions of combinatorial and cross-metric surfaces, which define a discrete metric on a surface; see Colin de Verdière and Erickson [14] for more details. In this paper, all edges of the combinatorial and cross-metric surfaces are unweighted.

A ***combinatorial surface*** is a surface S together with an embedded graph G, which will always be a triangulation in this article. In this model, the only allowed curves are walks in G, and the length of a curve c, denoted by $|c|_G$, is the number of edges of G traversed by c, counted with multiplicity.

However, it is often convenient (Sections 4 and 5) to allow several curves to traverse a same edge of G, while viewing them as being disjoint (implicitly, by "spreading them apart" infinitesimally on the surface). This is formalized using the dual concept of ***cross-metric surface***: Instead of curves in G, we consider curves in ***regular*** position with respect to the dual graph G^*, namely, that intersect the edges of G^* transversely and away from the vertices; the length of a curve c, denoted by $|c|_{G^*}$, is the number of edges of G^* that c crosses, counted with multiplicity. Since, in this article, G is always a triangulation, G^* is always *trivalent*, i.e., all its vertices have degree three. Curves and graph embedded on cross-metric surfaces can be manipulated efficiently [14]. The different notions of systoles are easily translated for both combinatorial and cross-metric surfaces.

Once again, we emphasize that, in this paper, unless otherwise noted, **all combinatorial surfaces are triangulated (each face is a disk with three sides) and unweighted (each edge has weight one)**.

Dually, **all cross-metric surfaces are trivalent (each vertex has degree three) and unweighted (each edge has crossing weight one)**.

2.3 Riemannian surfaces and systolic geometry

We will use some notions of Riemannian geometry, referring the interested reader to standard textbooks [16, 36]. A ***Riemannian surface*** (S, m) is a surface S equipped with a metric m, defined by a scalar product in the tangent space of every point. For example, smooth surfaces embedded in some Euclidean space $\mathbb{R}^d$ are naturally Riemannian surfaces (conversely, every Riemannian surface can be isometrically embedded in some $\mathbb{R}^d$ [44, 45]). The length of a (rectifiable) curve c is denoted by $|c|_m$. The ***Gaussian curvature*** $\boldsymbol{\kappa_p}$ of S at a point p is the product of the eigenvalues of the scalar product at p. By the Bertrand–Diquet–Puiseux theorem [55, Chapter 3, Prop. 11], the area of the ball $\boldsymbol{B(p, r)}$ of radius r centered at p equals $\pi r^2 - \kappa_p \pi r^4 + o(r^4)$. We now collect the results from systolic geometry that we will use; for a general presentation of the field, see, e.g., Gromov [28] or Katz [34].

Theorem 2.1 ([4, 27, 28, 35, 54])**.** *There are constants $c, c', c'', c''' > 0$ such that, on any Riemannian surface with genus g and area A:*

1. *some non-contractible closed curve has length at most $c\sqrt{A/g}\log g$, where $c \leq 1/\sqrt{\pi}$;*
2. *some non-separating closed curve has length at most $c'\sqrt{A/g}\log g$;*
3. *some null-homologous non-contractible closed curve has length at most $c''\sqrt{A/g}\log g$.*

Furthermore,

4. *for an infinite number of values of g, there exist Riemannian surfaces of constant curvature -1 (hence area $A = 4\pi(g-1)$) and systole larger than $\frac{2}{3\sqrt{\pi}}\sqrt{A/g}\log g - c'''$. In particular, the three previous inequalities are tight up to constant factors.*

Indeed, for (1), the existence of c is due to Gromov [27], and the fact that $c \leq 1/\sqrt{\pi}$ is due to Katz and Sabourau [35]. (2) is due to Gromov [28]. (3) is due to Sabourau [54]. (4) is due to Buser and Sarnak [4, p. 45].

3 A Two-Way Street

In this section, we prove that any systolic inequality regarding closed curves in the continuous (Riemannian) setting can be converted to the discrete (triangulated) setting, and vice-versa.

3.1 From Continuous to Discrete Systolic Inequalities

Theorem 3.1. *Let (S, G) be a triangulated combinatorial surface of genus g, without boundary, with n triangles. Let $\delta > 0$ be arbitrarily small. There exists a Riemannian metric m on S with area n such that for every closed curve γ in (S, m) there exists a homotopic closed curve γ' on (S, G) with $|\gamma'|_G \leq (1+\delta)\sqrt[4]{3}\,|\gamma|_m$.*

This theorem, combined with known theorems from systolic geometry, immediately implies:

Corollary 3.2. *Let (S, G) be a triangulated combinatorial surface with genus g and n triangles, without boundary. Then, for some absolute constants c, c', and c'':*

1. *some non-contractible closed curve has length at most $c\sqrt{n/g}\log g$, for $c \leq \sqrt[4]{3/\pi^2}$;*
2. *some non-separating closed curve has length at most $c'\sqrt{n/g}\log g$;*
3. *some homologically trivial non-contractible closed curve has length at most $c''\sqrt{n/g}\log g$.*

We note that, by Euler's formula and double-counting, we have $n = 2v + 4g - 4$, where v is the number of vertices of G. Thus, on a triangulated combinatorial surface with $v \geq g$ vertices, the length of a shortest non-contractible closed curve is at most $\sqrt[4]{108/\pi^2} \cdot \sqrt{v/g}\log g < 1.82\sqrt{v/g}\log g$. This reproves a theorem of Hutchinson [33], except that her proof technique leads to the weaker constant 25.27. We also remark that, in (3), we cannot obtain a similar bound if we require the curve to be simple (and therefore to be *splitting* [10]), as Section A.1 in Appendix shows.

Proof of Corollary 3.2. The proof consists in applying Theorem 3.1 to (S, G), obtaining a Riemannian metric m. For each of the different cases, the appropriate Riemannian systolic inequality is known, which means that a short curve γ of the given type exists on (S,m) (Theorem 2.1(1–3)); by Theorem 3.1, there exists a homotopic curve γ' in (S,G) such that $|\gamma'|_G \le (1+\delta)\sqrt[4]{3}|\,\gamma|_m$, for any $\delta > 0$. □

Proof of Theorem 3.1. The first part of the proof is similar to Guth et al. [31, Lemma 5]. Define m_G to be the singular Riemannian metric given by endowing each triangle of G with the geometry of a Euclidean equilateral triangle of area 1 (and thus side length $2/\sqrt[4]{3}$): This is a genuine Riemannian metric except at a finite number of points, the set of vertices of G. The graph G is embedded on (S, m_G). Let γ be a closed curve $\gamma\colon \mathbb{S}^1 \to S$. Up to making it longer by a factor at most $\sqrt{1+\delta}$, we may assume that γ is piecewise linear and transversal to G. Now, for each triangle T and for every maximal part p of γ that corresponds to a connected component of $\gamma^{-1}(T)$, we do the following. Let x_0 and x_1 be the endpoints of p on the boundary of T. (If γ does not cross any of the edges of G, then it is contractible and the statement of the theorem is trivial.) There are two paths on the boundary of T with endpoints x_0 and x_1; we replace p with the shorter of these two paths. Since T is Euclidean and equilateral, elementary geometry shows that these replacements at most doubled the lengths of the curve. Now, the new curve lies on the graph G. We transform it with a homotopy into a no longer curve that is an actual closed walk in G, by simplifying it each time it backtracks. Finally, from a closed curve γ, we obtained a homotopic curve γ' that is a walk in G, satisfying $|\gamma'|_G = \sqrt[4]{3}/2\,|\gamma'|_{m_G} \le \sqrt{1+\delta}\,\sqrt[4]{3}\,|\gamma|_{m_G}$.

The metric m_G satisfies our conclusion, except that it has isolated singularities. However, it is easy to smooth and scale m_G to obtain a metric m, also with area n, that multiplies the length of all curves by at least $1/\sqrt{1+\delta}$ compared to m_G; see Appendix A.2. This metric satisfies the desired properties. □

3.2 From Discrete to Continuous Systolic Inequalities

Here we prove that, conversely, discrete systolic inequalities imply their Riemannian analogs. The idea is to approximate a Riemannian surface by the Delaunay triangulation of a dense set of points, and to use some recent results on intrinsic Voronoi diagrams on surfaces [17].

Theorem 3.3. *Let (S,m) be a Riemannian surface of genus g without boundary, of area A. Let $\delta > 0$. For infinitely many values of n, there exists a triangulated combinatorial surface (S,G) embedded on S with n triangles, such that every closed curve γ in (S,G) satisfies $|\gamma|_m \le (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{A/n}\,|\gamma|_G$.*

We have stated this result in terms of the number n of triangles; in fact, in the proof we will derive it from a version in terms of the number of vertices; Euler's formula and double counting imply that, for surfaces, the two versions are equivalent. Together with Hutchinson's theorem [33], this result immediately yields a new proof of Gromov's classical systolic inequality:

Corollary 3.4. *For every Riemannian surface (S,m) of genus g, without boundary, and area A, there exists a non-contractible curve with length at most $\frac{101.1}{\sqrt{\pi}}\sqrt{A/g}\log g$.*

Proof. Let $\delta > 0$, and let (S,G) be the triangulated combinatorial surface implied by Theorem 3.3 with $n \ge 6g-4$ triangles. Euler's formula implies that the number v of vertices of G is at least g, hence we can apply Hutchinson's result [33], which yields a non-contractible curve γ on G with $|\gamma|_G \le 25.27\sqrt{(\frac{n}{2}+2-2g)/g}\log g$. By Theorem 3.3, $|\gamma|_m \le \frac{101.08(1+\delta)}{\sqrt{\pi}}\sqrt{A/g}\log g$. □

On the other hand, using this theorem in the contrapositive together with the Buser–Sarnak examples (Theorem 2.1(4)) confirms the conjecture by Przytycka and Przytycki [50, Introduction]:

Corollary 3.5. *For any $\varepsilon > 0$, there exist arbitrarily large g and v such that the following holds: There exists a triangulated combinatorial surface of genus g, without boundary, with v vertices, on which the length of every non-contractible closed curve is at least $\frac{1-\varepsilon}{6}\sqrt{v/g}\log g$.*

Proof. Let $\varepsilon > 0$, let (S,m) be a Buser–Sarnak surface from Theorem 2.1(4), and let G be the graph obtained from Theorem 3.3 from (S,m), for some $\delta > 0$ to be determined later. Combining these two theorems, we obtain that every non-contractible closed curve γ in G satisfies

$$(1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{\frac{A}{n}}\,|\gamma|_G \ge \frac{2}{3\sqrt{\pi}}\sqrt{\frac{A}{g}}\log g - c''',$$

where $A = 4\pi(g-1)$. If δ was chosen small enough (say, such that $1/(1+\delta) \geq 1-\varepsilon/2$), and g was chosen large enough, we have $|\gamma|_G \geq \frac{1-\varepsilon}{3\sqrt{8}}\sqrt{\frac{n}{g}}\log g$. Finally, we have $n \geq 2v$ by Euler's formula. □

Proof of Theorem 3.3. Let η, $0 < \eta < 1/2$ be fixed, and $\varepsilon > 0$ to be defined later (depending on η). Let P be an ε-separated net on (S, m), that is, P is a point set such that any two points in P are at distance at least ε, and every point in (S, m) is at distance smaller than ε from a point in P. For example, if we let P be the centers of an inclusionwise maximal family of disjoint open balls of radius $\varepsilon/2$, then P is an ε-separated net. In the following we put P in general position by moving the points in P by at most $\eta\varepsilon$; in particular, no point in the surface is equidistant with more than three points in P.

Let $P = \{p_1, \ldots, p_v\}$, and let $V_i := \{x \in (S, m) \mid \forall j \neq i, d(x, p_i) \leq d(x, p_j)\}$ be the Voronoi region of p_i. Since every point of (S, m) is at distance at most $(1+\eta)\varepsilon$ from a point in P, each Voronoi region V_i is included in a ball of radius $(1+\eta)\varepsilon$ centered at p_i. Define the Delaunay graph of P to be the intersection graph of the Voronoi regions, and note that if $V_i \cap V_j \neq \emptyset$, then the corresponding neighboring points of the Delaunay graph are at distance at most $2(1+\eta)\varepsilon$.

It turns out that under these assumptions, and choosing ε smaller than $1/(1+\eta)$ times the so-called *strong convexity radius* of (S, m), the Delaunay graph, which we denote by G, can be embedded as a triangulation of S with shortest paths representing the edges; this follows from results by Dyer et al. [17], we refer the reader to Section A.3 in Appendix for further discussion.

Consider a closed curve γ on G. Since neighboring points in G are at distance no greater than $2(1+\eta)\varepsilon$ on (S, m), we have $|\gamma|_m \leq 2(1+\eta)\varepsilon|\gamma|_G$. To obtain the claimed bound, there remains to estimate the number v of points in P. By compactness, the Gaussian curvature of (S, m) is bounded from above by a constant K. By the Bertrand–Diquet–Puiseux theorem, the area of each ball of radius $\frac{1-2\eta}{2}\varepsilon$ is at least $\pi(1-2\eta)^2\frac{\varepsilon^2}{4} - K\pi(1-2\eta)^4\frac{\varepsilon^4}{16} + o(\varepsilon^4) \geq \pi(1-2\eta)^3\frac{\varepsilon^2}{4}$ if $\varepsilon > 0$ is small enough. Since the balls of radius $(1-2\eta)\frac{\varepsilon}{2}$ centered at P are disjoint, their number v is at most $A/(\pi(1-2\eta)^3\frac{\varepsilon^2}{4})$. In other words, $\varepsilon \leq \frac{2}{\sqrt{\pi(1-2\eta)^3}}\sqrt{A/v}$. Putting together our estimates, we obtain that $|\gamma|_m \leq \frac{4(1+\eta)}{\sqrt{\pi(1-2\eta)^3}}\sqrt{\frac{A}{n/2-2g+2}}\,|\gamma|_G$, where n is the number of triangles of G. Thus, if $\varepsilon > 0$ is small enough, n can be made arbitrarily large, and the previous estimate implies, if η was chosen small enough (where the dependency is only on δ) that $|\gamma|_m \leq (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{\frac{A}{n}}\,|\gamma|_G$. □

4 Computing Short Pants Decompositions

Recall that the problem of computing a shortest pants decomposition for a given surface is open, even in very special cases. In this section, we describe an efficient algorithm that computes a short pants decomposition on a triangulation. Technically, we allow several curves to run along a given edge of the triangulation, which is best formalized in the dual cross-metric setting. If g is fixed, the length of the pants decomposition that we compute is of the order of the square root of the number of vertices:

Theorem 4.1. *Let (S, G^*) be an (unweighted, trivalent) cross-metric surface of genus $g \geq 2$, with n vertices, without boundary. In $O(gn)$ time, we can compute a pants decomposition $(\gamma_1, \ldots, \gamma_{3g-3})$ of S such that, for each i, the length of γ_i is at most $C\sqrt{gn}$ (where C is some universal constant).*

The inspiration for this theorem is a result by Buser [5], stating that in the Riemannian case, there exists a pants decomposition with curves of length bounded by $3\sqrt{gA}$. The proof of Theorem 4.1 consists mostly of translating Buser's construction into the discrete setting and making it algorithmic. The key difference is that for the sake of efficiency, unlike Buser, we cannot afford to shorten the closed curves in their homotopy classes, and we have to use contractibility tests in a careful manner.

Given closed curves Γ in general position on a (possibly disconnected) cross-metric surface (S, G^*), cutting S along Γ, and/or restricting to some connected components, gives another surface S', and restricting G^* to S' naturally yields a cross-metric surface that we denote by $(S', G^*_{|S'})$. Also, to simplify notation, we denote by $|c|$ (instead of $|c|_{G^*}$) the length of a curve c on a cross-metric surface (S, G^*).

The main tool is to cut off a pair of pants of a surface with boundary, while controlling the length of the boundary of the new surface:

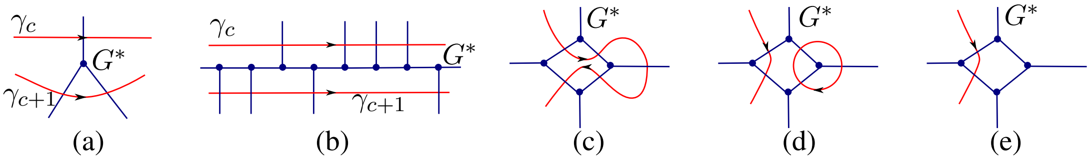

Figure 1: (a) Pushing a curve across a vertex. (b) The effect of a shifting step, if no self-tangency or tangency occurs. (c) A portion of a self-tangent curve. (d) The corresponding subcurves. (e) The curve after the removal of contractible subcurves.

Proposition 4.2. *Let (S, G^*) be a possibly disconnected cross-metric surface, such that every connected component has non-empty boundary and admits a pants decomposition. Let n be the number of vertices of G^* in the interior of S. Assume moreover that $|\partial S| \leq \ell$, where ℓ is an arbitrary positive integer.*

We can compute a family Δ of disjoint simple closed curves of (S, G^*) that splits S into one pair of pants, zero, one, or more annuli, and another possibly disconnected surface S' containing no disk, such that $|\partial S'| \leq \ell + 2n/\ell + 8$. The algorithm takes as input (S, G^*), outputs Δ and $(S', G^*_{|S'})$, and takes linear time in the complexity of (S, G^*).

We defer the proof of Theorem 4.1 to Section B.1 in Appendix: It relies on computing a good approximation of the shortest non-contractible closed curve, cutting along it, and applying Proposition 4.2 inductively.

Proof of Proposition 4.2. The idea is to *shift* the boundary components simultaneously until one boundary component *splits*, or two boundary components *merge*. This is analog to Morse theory on the surface with the function that is the distance to the boundary. However, in order to control the length of the decomposition, some backtracking is done before splitting or merging, as pictured in Figure 2.

Let $\Gamma = (\gamma_0^1, \ldots, \gamma_0^k)$ be (curves infinitesimally close to) the boundaries of S. Initially, let $\gamma^i = \gamma_0^i$. We orient each γ^i so that it has the surface to its right at the start. We will shift these curves to the right while preserving their simplicity and homotopy classes. We will only describe how Δ is computed, since one directly obtains S' by cutting along Δ and discarding the annuli and one pair of pants.

Shifting phase: We say that two simple closed curves on (S, G^*) are ***tangent*** if they both have a subpath in a common face of G^*. When a single closed curve has two subpaths in the same face of G^*, it will be called a ***self-tangent*** closed curve. The curves we handle in this phase are simple and homotopic to the γ^i. Since each such curve is separating, in a self-tangency, the two portions of a curve are oppositely oriented (Figure 1(c)). Therefore, "rewiring" such a curve at a self-tangency naturally splits it into two tangent closed curves, which we call its ***subcurves***, see Figure 1(d).

We define below how we shift a curve by one step to the right. The whole shifting phase consists of shifting the curves in a round robin way, i.e., we shift γ^1 by one step, then $\gamma^2, \ldots, \gamma^k$, and we reiterate. This phase is interrupted immediately whenever some tangency or self-tangency occurs, see below. To shift γ^i by one step, for every successive edge of G^* crossed by γ^i, in the order induced by γ^i, we push γ^i across the vertex adjacent to the edge (Figure 1(a)). The result of a shifting step is shown in Figure 1(b). Since G^* is trivalent, tangencies appear one at a time, determined by only two portions of curves. As soon as there is one (including before the very first step), we do the following:

- If γ^i is self-tangent, we test the two resulting subcurves for contractibility. If one of them is contractible, we discard it (Figure 1(e)) and continue the shifting process with the other one. Otherwise, both are non-contractible, and we go to the splitting phase below.
- If γ^i is tangent to γ^j for some $j \neq i$, we go to the merging phase below.

This finishes the description of the shifting phase. Let r be the integer such that each curve has been shifted between r and $r + 1$ steps to the right. For each i, $1 \leq i \leq k$, and each c, $1 \leq c \leq r$, let γ_c^i be the curve γ^i shifted by c steps. At every step of the shifting phase, we also maintain the sum of the lengths of the current curves. Then, at the end we denote by s the largest $c \leq r$ such that $\sum_{i=1}^k |\gamma_c^i| \leq \ell$. (Remember that this is the case for $c = 0$ by hypothesis.)

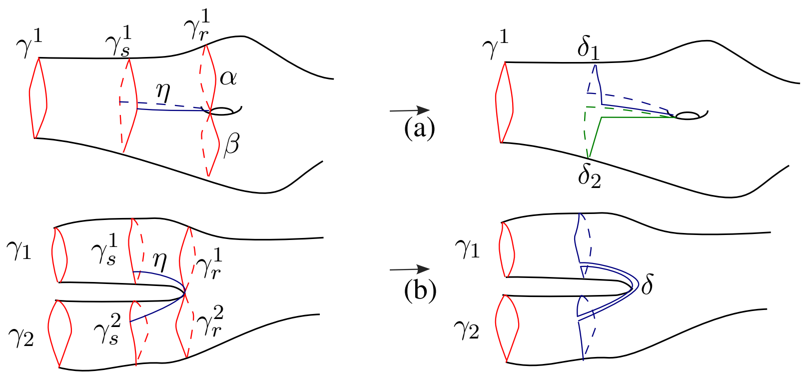

Figure 2: (a) Splitting phase. (b) Merging phase.

Splitting phase: When a curve becomes self-tangent, we do a splitting, as is pictured on the top of Figure 2. For simplicity, let γ^1 denote the curve that became self-tangent during the shifting phase. First, for every $i \neq 1$, we add γ^i_s to the family Δ. During the shifting phase, the closed curve γ^1 split into two non-contractible closed curves α and β. Let η be the shortest path with endpoints on γ^1_s that goes between α and β. This path can be computed in linear time by shifting back, at the end of the shifting phase, γ^1 to γ^1_s, and adding pieces of η at every step. The path η cuts γ^1_s into two subpaths μ and ν, one of them being possibly empty. We denote by δ_1 the concatenation of μ and η, and by δ_2 the concatenation of ν and η. Then we add δ_1 and δ_2 to the family Δ and we are done.

Merging phase: When two shifted curves are tangent, we do a merging (Figure 2, bottom), by computing a curve δ homotopic to their concatenation. For simplicity, let us denote by γ^1 and γ^2 two curves that became tangent during the shifting phase. First, for every $i \neq 1, 2$, we add γ^i_s to the family Δ. Let η be the shortest path from γ^1_s and γ^2_s, which we can, similarly as above, compute in linear time. The curve δ is defined by the concatenation $\eta^{-1} \cdot \gamma^1_s \cdot \eta \cdot \gamma^2_s$. Now, we simply add δ to Δ and we are done.

Analysis: After joining or merging, we added curves to Δ that cut the surface into an additional pair of pants, (possibly) some annuli, and the remaining surface S'. We first observe that we did not add any contractible closed curve to Δ; thus, S' has no connected component that is a disk. The proof that $|\partial S'| \leq \ell + 2n/\ell + 8$ is deferred to Section B.2 in Appendix; we only provide the intuition. The subtlety is the way the value of s was chosen: If s was equal to r (perhaps the most natural strategy), the boundary of S' would contain (at least) one curve γ^i_r, and we would have no control on its length. On the opposite, if we had chosen $s = 0$, we would have no control on the lengths of the arcs η involved in the merge or the split. The choice of s gives the right tradeoff inbetween: the lengths of the curves γ^s_i are controlled by this threshold, while the lengths of the arcs are controlled by the area of the annulus between γ^s_i and γ^r_i.

Complexity: The complexity of the splitting phase or the merging phase is clearly linear in n. The complexity of outputting the new surface $(S', G^*_{|S'})$ is linear in the complexity $\partial S'$, which is, by construction, also linear in n. To conclude, it suffices to prove that the shifting phase takes linear time. Essentially, it boils down to bounding the complexity of the contractibility tests. Doing them in tandem yields the claimed complexity; we defer this result to Section B.3 in Appendix. □

5 Lower Bounds for the Length of Cellular Graphs with Prescribed Combinatorial Map

In this section, we essentially prove that, for any combinatorial map M of any cellular graph embedding (in particular, of any cut graph) of genus g, there exists an (unweighted, trivalent) cross-metric surface S with n vertices such that any embedding of M on S has length $\Omega(n^{7/6})$. We are not able to get this result in full generality, but are able to prove that it holds for infinitely many values of g. On the other hand, the result is stronger since it holds "asymptoticaly almost surely" with respect to the uniform distribution on unweighted trivalent cross-metric surfaces with given genus and number of vertices.

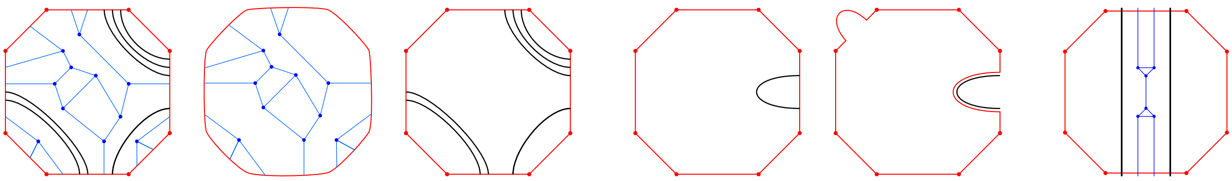

Figure 3: (a) The graph H, obtained after cutting S open along C. The vertices in B (on the outer face) and the vertices of G^* (not on the outer face) are shown. The chords are in thick black lines. (b) The graph H_1. (c) The graph H_2. (d), (e): The exchange argument to prove (i). (f): Two chords violating (ii).

Let (S, G^*) be a cross metric surface without boundary, and M a combinatorial map on S. The ***M*-systole** of (S, G^*) is the minimum among the lengths of all graphs embedded in (S, G^*) with combinatorial map M. Given g and n, we consider the set $\boldsymbol{\mathcal{S}(g,n)}$ of trivalent unweighted cross-metric surfaces of genus g, without boundary, and with n vertices, where we regard two cross-metric surfaces as equal if some self-homeomorphism of the surface maps one to the other (note that vertices, edges, and faces are unlabelled). (This refines the model introduced by Gamburd and Makover [25]) Here is our precise result:

Theorem 5.1. *Given strictly positive real numbers p and ε, and integers n_0 and g_0, there exist $n \geq n_0$ and $g \geq g_0$ such that, for any combinatorial map M of a cellular graph embedding with genus g, with probability at least $1-p$, a cross-metric surface chosen uniformly at random from $\mathcal{S}(g,n)$ has M-systole at least $n^{7/6-\varepsilon}$.*

We can obtain a similar result in the case of polyhedral triangulations, obtained by gluing n equilateral triangles with sides of unit length. Indeed, any short cut graph in a polyhedral triangulation leads to a short cut graph in the corresponding cross-metric surface; we defer the details to Section C.1.

The general strategy is inspired by Guth et al. [31], proving a related bound for pants decompositions, but the details of the method are rather different. The main tool is the following proposition.

Proposition 5.2. *Given integers g, n, and L, and a combinatorial map M of a graph embedding of genus g, at most $f(g,n,L) = 2^{O(n)} L\,(L/g+1)^{12g-9}$ cross-metric surfaces in $\mathcal{S}(g,n)$ have M-systole at most L.*

Proof. First, note that it suffices to prove the result for cut graphs with minimum degree at least three. Indeed, one can transform any cellular graph embedding into such a cut graph by removing edges, removing degree-one vertices with their incident edges, and *dissolving* degree-two vertices, namely, removing them and replacing the two incident edges with a single one. So let M be the combinatorial map of such a cut graph of genus g; let (S, G^*) be a cross-metric surface in $\mathcal{S}(g,n)$, and let C be an embedding of M of length at most L. Euler's formula and double-counting immediately imply that C has at most $4g-2$ vertices and $6g-3$ edges.

Let H' be the graph that is the overlay of G^* and C. Cutting S along C yields a topological disk D, and transforms H' into a connected graph H (Figure 3(a)) embedded in the plane, where the outer face corresponds to the copies of the vertices and edges of the cut graph C. The set B of vertices of degree two on the outer face of H exactly consists of the copies of the vertices of C; there are at most $12g-6$ of these. A *side* of H is a path on the boundary of D that joins two consecutive points in B.

Given the combinatorial map of H in the plane, we can (almost) recover the combinatorial maps corresponding to H' and to (S, G^*). Indeed, the set B of vertices of degree two on the outer face of H determines the sides of D. The correspondence between each side of D and each edge of the combinatorial map M is completely determined once we are given the correspondence between a single half-edge on the outer face of H and a half-edge of C; in turn, this determines the whole gluing of the sides of H and completely reconstructs H' with C distinguished. Finally, to obtain G^*, we just "erase" C. Therefore, one can reconstruct the combinatorial map corresponding to the overlay H' of G^* and C, just by distinguishing one of the $O(L)$ half-edges on the outer face of H.

A *chord* of H is an edge of H that is not incident to the outer face but connects to vertices incident to the outer face. Two chords are *parallel* if their endpoints lie on the same pair of sides of D. We claim that we can assume the following:

(i) no chord has its endpoints on the same side of H (Figure 3(d));

and that (at least) one of the two following conditions holds:

(ii) the subgraph of H between any two parallel chords only consists of other parallel chords (Figure 3(f) shows an example not satisfying this property), or

(ii') there are two parallel chords such that the subgraph of H between them contains all the interior vertices of H.

Indeed, without loss of generality, we can assume that our cut graph C has minimum length among all cut graphs of (S, G^*) with combinatorial map M. If a chord violates (i), one could shorten the cut graph by sliding a part of the cut graph over the chord (Figure 3(d–e)), which is a contradiction. The proof that either (ii) or (ii') holds uses a similar argument and is deferred to Section C.2 in Appendix.

We now estimate the number of possible combinatorial maps for H, by "splitting" it into two connected plane graphs H_1 and H_2, estimating all possibilities of choosing each of these graphs, and estimating the number of ways to combine them.

Let H_1 be the graph (see Figure 3(b)) obtained from H by removing all chords and dissolving all degree-two vertices (which are either in B or endpoints of a chord). H_1 is connected, trivalent, and has at most n vertices not incident to the outer face, so $O(n)$ vertices in total. There are thus $2^{O(n)}$ possible choices for the combinatorial map of this planar trivalent graph H_1 [31, Lemma 4].

On the other hand, let H_2 be the graph (see Figure 3(c)) obtained from H by removing internal vertices together with their incident edges and dissolving all degree-two vertices not in B. A simple computation, deferred to Section C.3 in Appendix, shows that the number of possibilities for H_2 is at most $2^{O(g)}\left(\frac{e(L+12g-9)}{12g-9}\right)^{12g-9}$, by (i) and since the total number of chords is at most L.

Finally, in how many ways can we combine given H_1 and H_2 to form H? Let us first assume that (ii) holds; the parallel chords joining the same pair of sides are consecutive, so choosing the position of a single chord fixes the position of the other chords parallel to it. Therefore, given H_1, we need to count in how many ways we can insert the $O(g)$ copies of B on H_2 into H_1, and similarly the $O(g)$ intervals where endpoints of chords can occur, respecting the cyclic ordering. After choosing the position of a distinguished vertex of H_2, we have to choose $O(g)$ positions on the edges of the boundary of H_1, possibly with repetitions, which leaves us with $\binom{O(n+g)}{O(g)} \leq 2^{O(n+g)} = 2^{O(n)}$ possibilities. In case (ii') holds, a very similar argument gives the same result. The claimed bound follows by multiplying the number of all possible choices above. □

Proof of Theorem 5.1. Let g_0, n_0, p, ε be as indicated. Euler's formula implies that a cross-metric surface with n vertices has genus $g \leq (n+2)/4$. Proposition 5.2 implies, after a routine computation (deferred to Section C.4 in Appendix), that, if n is large enough, $\sum_{g=g_0}^{(n+2)/4} f(g, n, n^{7/6-\varepsilon}) \leq n^{(1-\varepsilon)n/2}$ (*).

Furthermore, let $h(g, n) = |\mathcal{S}(g, n)|$ be the number of cross-metric surfaces with genus g and n vertices. We have $\sum_{g=0}^{(n+2)/4} h(g, n) \geq e^{Cn} n^{n/2}$ if n is large enough and even, for some absolute constant C [31, Lemma 3] (see Section C.5 for details). But, if g is fixed, $h(g, n) = O(e^{C'n})$ for some constant C' [31, Lemma 4]. Thus, since g_0 is fixed, there is a constant C'' such that, for n large enough and even, $\sum_{g=g_0}^{(n+2)/4} h(g, n) \geq e^{C''n} n^{n/2}$ (**).

Choose any (even) $n \geq n_0$ such that $n^{-\varepsilon n/2} e^{-C''n} \leq p$ and such that (*) and (**) hold. This implies that, for some $g \geq g_0$, we have $f(g, n, n^{7/6-\varepsilon})/h(g, n) \leq n^{(1-\varepsilon)n/2}/(e^{C''n} n^{n/2}) \leq p$ (and the denominator is non-zero). In other words, among all $h(g, n)$ cross-metric surfaces with genus g and n vertices, for any combinatorial map M of a cellular graph embedding of genus g, a fraction at most p of these surfaces have an embedding of M with length at most $n^{7/6-\varepsilon}$. □

Finally, we remark that a tighter estimate on the number $h(g, n)$ of triangulations with n triangles of a surface of genus g could lead to the same result *for any large enough* g, instead of *for infinitely many values of* g.

Acknowledgment

We would like to thank Ramsay Dyer for pointing out the existing results on Voronoi diagrams of Riemannian surfaces [17].

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A Omitted Proofs for Section 3

A.1 Splitting Closed Curves are Longer than Homologically Trivial Non-Contractible Closed Curves

Figure 4 shows that the minimum length of a shortest homologically trivial, non-contractible closed curves can become much larger if we additionally request the curve to be simple (and thus splitting).

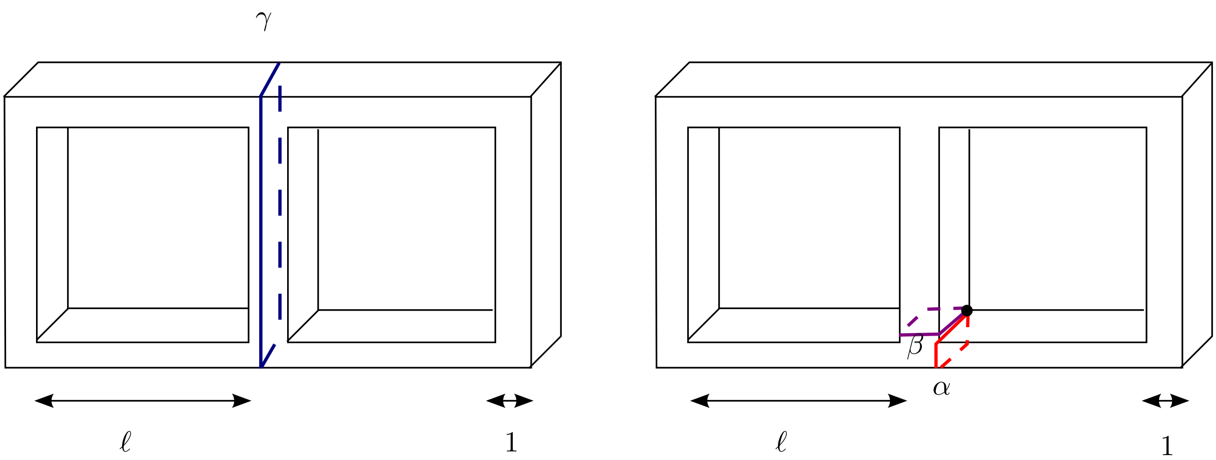

Figure 4: A piecewise linear double torus with area A such that the length of a shortest splitting closed curve is $\Omega(A)$ (left), but the length of a shortest homologically trivial non-contractible curve, concatenation of $\alpha\beta\alpha^{-1}\beta^{-1}$, has length $\Theta(1)$.

A.2 Smoothing Riemannian Surfaces

Lemma A.1. *With the notations of the proof of Theorem 3.1, there exists a smooth Riemannian metric m on S, also with area n, such that any closed curve γ in S satisfies $|\gamma|_m \geq |\gamma|_{m_G}/\sqrt{1+\delta}$.*

Proof. The idea is to smooth out each vertex v of G to make m_G Riemannian, as follows.

On the ball $B(v, 2\varepsilon)$, consider a Riemannian metric m_v with area at most $\delta/3$ such that any path in that ball is longer under m_v than under m_G. This is certainly possible provided ε is small enough: For example, build a diffeomorphism from $B(v, 2\varepsilon)$ onto the unit disk in the plane in the natural way (v being mapped at the center of the disk, and the trace of the edges of G being mapped to line segments forming equal angles); endow the disk with a metric just large enough so that the corresponding metric on $B(v, 2\varepsilon)$ is larger than m_v. If ε is taken small enough, the area that is needed for the new metric can be made as small as we want.

We now use a partition of unity to define a smooth metric $\hat{m}$ that interpolates between m_G and the metrics m_v, in the sense that:

- outside the balls of radius 2ε, we have $\hat{m} = m_G$;
- inside a ball $B(v, \varepsilon)$, we have $\hat{m} = m_v$;
- in $B(v, 2\varepsilon) \setminus B(v, \varepsilon)$, the metric $\hat{m}$ is a convex combination of m_G and m_v.

The area of $\hat{m}$ is at most the sum of the areas of m_G and the m_v's, which is at most $n(1 + \delta)$. Moreover, for any curve γ, we have $|\gamma|_{\hat{m}} \geq |\gamma|_{m_G}$.

Finally, we scale $\hat{m}$ to obtain the desired metric m with area n; for any curve γ, we indeed have $|\gamma|_m \geq |\gamma|_{\hat{m}}/\sqrt{1 + \delta}$. □

A.3 Delaunay Triangulations on Riemannian Surfaces

The *strong convexity radius* at a point in a Riemannian surface (S, m) is an invariant that refines the well-known injectivity radius. It is the supremum of the radius ρ_x such that for every $r < \rho_x$ the ball of radius r centered at x is strongly convex, that is, for any $p, q \in B(x, r)$ there is a unique shortest path in (S, m) connecting p and q, this shortest path lies entirely within $B(x, r)$, and moreover no other geodesic connecting p and q lies within $B(x, r)$, see Klingenberg [36, Def. 1.9.9]. The strong convexity radius is positive at every point, and its value on the surface is continuous (see also Dyer et al. [17, Sect. 3.2.1]). It follows that for every compact Riemannian surface (S, m), there exists a strictly positive lower bound on the strong convexity radius of every point. We use the following lemma, which is a result of of Dyer et al. [17, Corollary 2].

Lemma A.2. *Let (S, m) be a Riemannian surface, let $\rho > 0$ be smaller than the strong convexity radius of any point in (S, m), and let P a point set in general position such that balls of radius ρ centered at P cover S. Then the Delaunay graph of P is a triangulation of S.*

To apply this result to our proof, we choose ε small enough so that $(1 + \delta)\varepsilon \leq \rho$.

B Omitted Proofs for Section 4

B.1 Proof of Theorem 4.1

In this appendix, we finish the proof of Theorem 4.1, the main theorem of our algorithm to compute pants decomposition.

Proof of Theorem 4.1. To prove Theorem 4.1, we consider our cross-metric surface without boundary (S, G^*), and we start by computing a simple non-contractible curve γ whose length is at most twice the length of the shortest non-contractible closed curve. Such a curve can be computed in $O(gn)$ time [7, Prop. 9] (see also Erickson and Har-Peled [21, Corollary 5.8]) and has length at most $C\sqrt{n}$, where C is a universal constant, see Section 3. This gives a surface $S^{(1)}$ with two boundary components.

The end of the proof just consists of applying Proposition 4.2 inductively: We start with $S^{(1)}$, and applying it to $S^{(k)}$ gives another surface $S^{(k)\prime}$, in which we remove all the pair of pants. We denote the resulting surface by $S^{(k+1)}$ and apply Proposition 4.2 again. We apply this induction until we obtain a surface $S^{(m)}$ that is empty. Note that, for every k, $S^{(k)}$ contains no disk, annulus, or pair of pants, and that every application of Proposition 4.2 gives another pair of pants. Therefore, we obtain a pants decomposition of S by taking the initial curve γ together with all the curves in Δ in all the applications of Proposition 4.2 and, when there are homotopic curves, by removing all of them except the shortest one. Therefore, the number of applications of Proposition 4.2 is bounded by the maximum size of a pants decomposition of S, i.e., $3g - 3$. The length of the pants decomposition is at most the sum, over k, of $\ell_k = |\partial S^{(k)}|$. The sequence ℓ_k satisfies the induction $\ell_{k+1} \leq \ell_k + 2n/\ell_k + 8$, with $\ell_1 \leq C\sqrt{n}$. A small computation gives that $\ell_k \leq C\sqrt{kn}$ for C larger than 16 and $k \leq 3n$, which proves the bound on the lengths since $k \leq 3g - 3 \leq 3n$. The total complexity of this algorithm is $O(gn)$ since we applied $O(g)$ times Proposition 4.2, which takes linear complexity. □

B.2 Analysis of the lengths of the curves

After the joining or the merging phase, we added curves in Δ that cut the surface into a new pair of pants, some annuli, and a new subsurface S'. We prove here that the length of the boundary S' satisfies $|\partial S'| \leq \ell + 2n/\ell + 8$.

Lengths after the splitting phase: After a splitting phase with the curve γ^1, the boundary $\partial S'$ of S' consists of all the other curves γ_s^i in Γ, and of the two new curves, whose sum of the lengths is bounded by $|\gamma_s^1| + 2|\eta|$. Hence $|\partial S'| \leq |\gamma_s^1| + 2|\eta| + \sum_{i=2}^{k} |\gamma_s^i|$, which is at most $\ell + 2|\eta|$ by the choice of s. Furthermore, by construction, $|\eta| \leq 2(r - s + 1)$.

Lengths after the merging phase: After a merging phase with the curves γ^1 and γ^2, the boundary $\partial S'$ of S' consists of all the other curves γ_s^i of Γ, and of the new closed curve, whose length is bounded by $|\gamma_s^1| + |\gamma_s^2| + 2|\eta|$. Hence similarly, $|\partial S'| \leq \ell + 2|\eta|$. Furthermore, by construction, $|\eta| \leq 2(r - s + 1)$.

Final analysis: Thus, after either the splitting or the merging phase, we proved that $|\partial S'| \leq \ell + 4(r - s + 1)$. To conclude the proof, there only remains to prove that $r - s \leq \frac{n}{2\ell} + 1$.

Let $c \in \{s, \ldots, r - 1\}$. The curves γ_c^i and γ_{c+1}^i bound an annulus K_c^i. The number $A(K_c^i)$ of vertices in the interior of this annulus, its *area*, is at least $|\gamma_c^i| + |\gamma_{c+1}^i|$ (see Figure 1(b)—this is where we use, in a crucial way, the fact that G^* is trivalent), because we may only have added vertices in the annulus when we discarded contractible curves.

For $c \in \{s, \ldots, r - 1\}$ and $i \in \{1, \ldots, k\}$, the annuli K_c^i have disjoint interiors, so the sum of their areas is at most n. By the above formula, this sum is at least $U_s + U_r + 2\sum_{c=s+1}^{r-1} U_c \geq 2\sum_{c=s+1}^{r-1} U_c$, where $U_c = \sum_{i=1}^{k} |\gamma_c^i|$. On the other hand, we have $U_c \geq \ell$ if $s + 1 \leq c \leq r$, by definition of s. Putting all together, we obtain $n \geq 2(r - s - 1)\ell$, so $r - s \leq \frac{n}{2\ell} + 1$.

B.3 Complexity Analysis of the Shifting Phase

To prove that the shifting phase takes linear time, it suffices to prove that the contractibility tests take linear time in total. We now show how to achieve this. To perform a contractibility test on two subcurves α and β, we perform a tandem search on the surfaces bounded by α and β, and stop as soon as we find a disk. If we find one, the complexity in the tandem search is at most twice the complexity of this disk, which is immediately discarded and never visited again. If we do not, the complexity is linear in n, but the shifting phase is over. Therefore, the total complexity of the contractibility tests is linear in the number of vertices swept by the shifting phase or in the disks, until the very last contractibility test, which takes time linear in n. In the end, the shifting phase takes time linear in n, which concludes the complexity analysis.

C Omitted Proofs for Section 5

C.1 A Polyhedral Version

We first note that an element of $\mathcal{S}(g, n)$ naturally corresponds to a polyhedral triangulation by gluing equilateral triangles of unit side length on the vertices. The notion of M-systole is defined similarly in this setting, and we now prove that Theorem 5.1 implies an analogous result for polyhedral triangulations:

Theorem C.1. *Given strictly positive real numbers p and ε, and integers n_0 and g_0, there exist $n \geq n_0$ and $g \geq g_0$ such that, for any combinatorial map M of a cellular graph embedding with genus g, with probability at least $1 - p$, a polyhedral triangulation chosen uniformly at random from $\mathcal{S}(g, n)$ has M-systole at least $n^{7/6-\varepsilon}$.*

Proof. As in the proof of Theorem 5.1, it suffices to prove the result for maps M that are cut graphs with minimum degree three, which have at most $4g - 2$ vertices and $6g - 3$ edges. Let G be the vertex-edge graph of a polyhedral triangulation on a surface S with genus g. Assume that M has an embedding C of length $O(n^{7/6-\varepsilon})$ on that polyhedral surface. We prove that M has an embedding of length $O(n^{7/6-\varepsilon})$ in the dual cross-metric surface (S, G^*). Since, by Theorem 5.1, the proportion of such surfaces is arbitrarily small, this implies the theorem.

Without loss of generality, we assume that C is piecewise-linear, and in general position with respect to G. We consider a tubular neighborhood N of G (Figure 5(a)), obtained by first building a small *disk* around each vertex of G, and then building a rectangular *strip* containing each part of edge not covered by a disk. The disks are pairwise disjoint, the strips are pairwise disjoint; each strip intersects only the

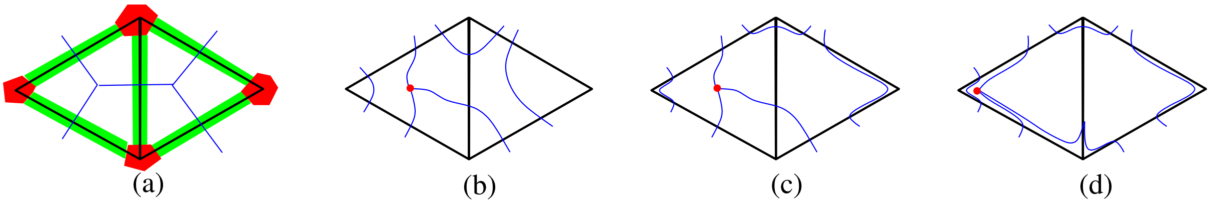

Figure 5: Illustration of the proof of Theorem C.1. (a): Two triangles of the graph G, the corresponding part of the tubular neighborhood N, made of disks and strips, and the dual cross-metric graph G^*, whose traces on the strips constitute the paths P_s. (b): A part of C. (c): Pushing the pieces not incident to vertices of C into N. (d): Pushing the vertices of C.

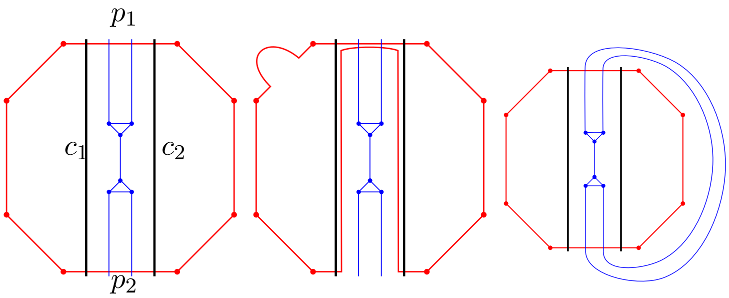

Figure 6: The exchange argument to prove (ii) or (ii'). Left: Two chords violating (ii). Middle: The exchange argument, in case p_1 and p_2 have different perturbed lengths. Right: A schematic view of the situation, in case p_1 and p_2 have the same perturbed length.

disks corresponding to the incident vertices of the corresponding edge, along paths. We first push C into N as follows. First consider the maximal pieces of edges C that lie inside a triangle, but do not contain a vertex of C. It is easy, using elementary geometry in equilateral triangles, to prove that one can push, by an isotopy, all such pieces, without moving their endpoints, into C, while at most doubling their total length (Figure 5(b–c)). Finally, we push the $O(g)$ vertices of C into the disks, thereby pushing also the incident pieces into N; this adds $O(g)$ to the length of C (Figure 5(d)).

For each strip s, draw a shortest path P_s with endpoints on its boundary, that separates the two sides touching disks. If a piece of C inside s crosses P_s, it forms a bigon with P_s; by flipping innermost bigons, without increasing the length of C, we can assume that each piece of C inside s crosses P_s at most once.

Now we extend the paths P_s to form the graph G^* (Figure 5(a)). By the paragraph above, each crossing of a path P_s corresponds to a piece of a path of C that crosses the strip containing P_s, and thus has length at least $1 - \delta$, for $\delta > 0$ arbitrarily close to zero. Therefore, the length of C on the cross-metric surface (S, G^*) is at most $(1 - \delta)$ times that of the length of C on the polyhedral triangulated surface. ☐

C.2 Proof That (ii) or (ii') Holds

Lemma C.2. *With the notations in the proof of Proposition 5.2, we can assume that at least one of the following conditions holds:*

(ii) the subgraph of H between any two parallel chords only consists of other parallel chords (Figure 6, left), or

(ii') there are two parallel chords such that the subgraph of H between them contains all the interior vertices of H.

Proof. The basic idea is to use a similar exchange argument as to prove (i), but we need a perturbation argument as well. Specifically, let us temporarily perturb the crossing weights of the edges of G^* as follows: The weight of each edge e of G^* becomes $1 + w_e$, where the w_e's are i.i.d. real numbers strictly between 0 and $1/L$. Let C be a shortest embedding of M under this perturbed metric.

It is easy to see that C is also a shortest embedding of M under the unweighted metric: Indeed, two cut graphs C_1 and C_2 with respective (integer) lengths $\ell_1 < \ell_2 \leq L$ in the unweighted metric have respective lengths $\ell'_1 < \ell'_2$ in the perturbed metric, since the perturbation increases the length of each edge by less than $1/L$.

We claim that either (ii) or (ii') holds for this choice of C. Assume that (ii) does not hold; we prove that (ii') holds. So the region R of D between two parallel chords c_1 and c_2 of D contains internal vertices; without loss of generality (by (i)), assume that the region R contains no other chord in its interior. Let p_1 and p_2 be the two subpaths of the cut graph on the boundary of R. If p_1 and p_2 have different lengths under the perturbed metric, e.g., p_1 is shorter, then we can push the part of p_2 to let it run along p_1 and shorten the cut graph, which is a contradiction. Therefore, p_1 and p_2 have the same length under the perturbed metric, which implies with probability one that they cross exactly the same set S of edges of G^*. (We exclude from S the edges on the endpoints of p_1 and p_2.) Since none of the edges in S are chords, all the endpoints of the edges in S belong to the region of D bounded by p_1, p_2, c_1, and c_2, which implies (ii'). ☐

C.3 A Bound on the Number of Possibilities for H_2

Lemma C.3. *With the notations in the proof of Proposition 5.2, the number of different possible combinatorial maps for H_2 is at most $2^{O(g)} \left(\frac{e(L+12g-9)}{12g-9}\right)^{12g-9}$.*

Proof. Since the chords are non-crossing and connect distinct sides of D, the pairs of sides connected by at least one chord form a subset of a triangulation of the polygon having one vertex per side of D. To describe H_2, it therefore suffices to describe a triangulation of this polygon with at most $12g - 6$ edges, which makes $2^{O(g)}$ possibilities, and to describe, for each of the $12g - 9$ edges of the triangulation, the number of parallel chords connecting the corresponding pair of sides. Since there are at most L chords, the number of possibilities for the latter numbering is at most the area of the simplex $\{(x_1, \dots, x_{12g-9}) \mid x_i \geq 0, \sum_i x_i \leq L + 12g - 9\}$ (since this simplex contains all the copies of the unit cube translated by the non-negative integer points $(x_1, \dots, x_{12g-9})$ with total sum at most L), which is, using Stirling's formula,

$$\frac{1}{(12g-9)!}(L + 12g - 9)^{12g-9} \leq \left(\frac{e(L + 12g - 9)}{12g - 9}\right)^{12g-9}. \qquad \square$$

C.4 An Upper Bound on the Number of Surfaces with Short Map Embeddings

Lemma C.4. *If n is large enough, we have $\sum_{g=g_0}^{(n+2)/4} f(g, n, n^{7/6-\varepsilon}) \leq n^{(1-\varepsilon)n/2}$.*

Proof. This is routine computation. We have $f(g, n, n^{7/6-\varepsilon}) \leq 2^{C_0 n} \left(n^{7/6-\varepsilon}/g + 1\right)^{12g-9}$ for some constant C_0. We need to sum up these terms from $g = g_0$ to $(n + 2)/4$. For n large enough, the largest term in this sum is for $g = (n + 2)/4$. Thus the desired sum is bounded from above by $n 2^{C_0 n}. \left(4n^{1/6-\varepsilon} + 1\right)^{12(n+2)/4-9}$, which is at most $2^{C_1 n}.n^{(1/6-\varepsilon).3n}$ (for n large enough, for some constant C_1), which in turn is at most $n^{(1-\varepsilon)n/2}$ for n large enough. ☐

C.5 The Number of Connected Cross-Metric Surfaces

Lemma C.5. *The number of (unweighted, trivalent)* connected *cross-metric surfaces with n vertices is, for n even large enough, at least $e^{Cn} n^{n/2}$ for some absolute constant C.*

Proof. By duality, this is equivalent to counting triangulations with n triangles. Guth et al. [31, Lemma 3] prove that, for $n \geq 2$ even and large enough, the number of *possibly disconnected* triangulations with n triangles is between $e^{Kn}.n^{n/2}$ and $e^{K'n}.n^{n/2}$, where K and K' are absolute constants. Like us, they actually need to prove such bounds for *connected* surfaces. We shall fill this gap here.

Every disconnected triangulation with n triangles can be expressed as the disjoint union of two (possibly disconnected) triangulations with k and $n-k$ triangles, respectively. Therefore, the number of disconnected triangulations with n triangles is bounded from above by

$$\sum_{\substack{2\leq k\leq n/2 \\ k \text{ even}}} e^{K'n}.k^{k/2}(n-k)^{(n-k)/2}.$$

This sum is dominated by its first term, so the number of disconnected triangulations with n triangles is $O\left(e^{K'n}.(n-2)^{(n-2)/2}\right)$. Therefore, the number of connected triangulations with n triangles is at least $e^{Kn}.n^{n/2} - K''e^{K'n}.(n-2)^{(n-2)/2}$ for some constant K'', which is $\Omega\left(e^{Kn}.n^{n/2}\right)$, as desired. $\square$